Geophys. J. R. astr. Soc. (1978) **52**, 65–96

Dynamics in prestressed media with moving phase boundaries: a continuum theory of failure in solids

C. B. Archambeau *CIRES, University of Colorado/NOAA, Boulder, Colorado 80309, USA*

J. B. Minster *California Institute of Technology, Pasadena, California 91125, USA*

Received 1977 June 20; in original form 1976 April 19

Summary. Spontaneous failure in a solid medium is described as a localized transition of the material from one physical state to another, characterized in part by contrasting rheological properties and density. Such a process is viewed as a local disordering of the relatively ordered structure of the solid due to any variety of causes, such as massive microfracturing or shear melting, and can be confined to a very thin zone, but nevertheless of finite volume such that a volumetric transition energy can be defined. This leads to the description of failure as a generalized phase transition in a prestressed continuum, with instability and transition zone growth being driven by the energy contributions from the relaxation of stress in the surrounding medium. Direct application of mass, momentum and energy conservation to such a generalized phase transition leads to 'jump' conditions specified on the growing boundary surface of the transition zone, that relate the rupture growth to discontinuous changes in the dynamic field variables across the failure zone boundary. These field discontinuities are, in turn, related to the localized changes in physical properties induced by failure. Dynamical conditions for rapid spontaneous failure growth in a stressed medium are investigated in some detail, and we find that the failure boundary growth can be simply expressed in terms of energy 'failure condition' and a dynamic growth condition specifying the rupture velocity. These results imply that the integral energy change associated with earthquakes is in the range 10^4–10^6 erg/g. Further the failure growth rate is shown to be expressible in terms of the rheological properties of the material before and after failure. For shear melting resulting in a low viscosity fluid, for example, the rupture velocity will be near the shear velocity of the original material. A general Green's function solution for the radiation due to stress relaxation in the medium surrounding the growing failure zone is given and provides the basis for detailed computations of the strain or displacement field changes due to

spontaneous failure processes. In particular, it is shown that the jump conditions for the growing transition zone boundary appear naturally as surface integral terms over the boundary. Since these boundary conditions contain the failure rate explicitly, then these terms include effects that have not been represented in previous integral representations of the radiation field resulting from failure. Further, we show that the formal Green's integral representation for the dynamical wave field can be used with known, simple Green's functions to generate approximate solutions for complex failure processes occurring in media with inhomogeneous material properties and prestress.

Definitions of principal symbols

Symbol	Definition	First introduced
$F(\mathbf{x}, t)$	Any function dependent on the flow variables through the particle coordinates (a general function of the flow, or deformation in solids).	Equation (1)
$\Sigma(\mathbf{x}, t)$	Surface of discontinuity across which the flow may be discontinuous.	Equation (3)
$\mathbf{U}(\mathbf{x}, t)$	Velocity of a surface of discontinuity Σ.	Equation (3)
$\mathbf{V}(\mathbf{x}, t)$	Material velocity field.	Equation (3)
$[\![F]\!]_\Sigma$	Jump in the value of a function of the flow, F, across Σ, i.e. $[\![F]\!]_\Sigma = F(\Sigma_1) - F(\Sigma_2)$.	Equation (3)
$\mathscr{F}$	Current density of F per unit time per unit area (flux vector).	Equation (9)
$\mathscr{F}^*$	Relative current density or flux vector for the field F across a moving boundary Σ.	Equation (10)
ρ	Material density.	Table 1
T_{kl}	Cauchy Stress Tensor.	Table 1
u	Internal energy density of the material.	Table 1
$\mathbf{b}$	Body force density.	Table 1
ϕ	Body force potential, $\mathbf{b} = -\nabla\phi$.	Table 1
$\mathbf{q}$	Heat Flux Vector.	Table 1
E	Energy density $\rho E = \rho u + \rho/2\, V_l V_l$.	Table 1
$\mathscr{E}$	Total Energy Density $\rho\mathscr{E} = \rho E + \rho\phi$	Table 1
h	Heat source density	Table 1
U_R	Normal component of the velocity of a discontinuity surface Σ, measured *relative* to the particle velocity $\mathbf{V}$ in the medium into which the normal to the surface points. (Used as the definition of 'rupture velocity'.)	Equation (20)

Symbol	Definition	First introduced
L	The change in internal energy, u, across a surface of discontinuity: $L = [\![u]\!]_\Sigma$ (used as a transition energy characteristic of a physical process of failure in a given material.)	Equation (24)
e_{kl}	Elastic strain measured relative to the relaxed state of the solid material (free from external forces and surface tractions).	Equations (34) and (35)
$\mathbf{u}$	Elastic displacement field measured relative to the relaxed state of the material.	Equations (34) and (35)
C_{ijkl}	Elastic modulus tensor (Latin indices take on the values 1, 2, 3).	Equation (33)
$C_{\alpha\beta\gamma\delta}$	Elastic-Inertial Tensor (Greek indices take on the values 1, 2, 3, 4).	Equation (45)
$\tau_{\alpha\beta}$	Four Dimensional Inertial-Stress Tensor.	Equation (48)
$L_{\alpha\gamma}$	Space-time 'elastic operator' $L_{\alpha\gamma} \equiv \dfrac{\partial}{\partial x_\beta}\left(C_{\alpha\beta\gamma\delta}\dfrac{\partial}{\partial x_\delta}\right)$.	Equation (49)
η_α	Four dimensional space-time normal vector defined for a moving surface of discontinuity.	Equation (51)
$G_\alpha^\beta(\mathbf{x}, \mathbf{x}_0)$	Two point elastic Green's tensor (α component of the displacement at $\mathbf{x}$ due to an impulse force component in the β direction at $\mathbf{x}_0$).	Equation (53)
$\Delta_\alpha^\beta(\mathbf{x}, \mathbf{x}_0)$	Generalized Delta Function.	Equation (53)
$\mathcal{G}_{\alpha\beta}^\mu(\mathbf{x}, \mathbf{x}_0)$	Two point elastic Green's stress tensor generated from the (displacement) Green's tensor G_α^β.	Equation (75)
u_k^*	Equilibrium elastic displacement field in the medium.	Equation (92)

Introduction

A general description of material failure under stress loading is obtained within the frame of continuum mechanics by considering the failure process to be a generalized phase transition (Archambeau 1969). In this description the failure zone occupies a finite volume, the boundary of which corresponds to a moving phase boundary in a prestressed medium. The growth of the failure zone is then controlled by the energy balance at the 'phase transition'. That is, we can define, as a macroscopic equivalent, a 'latent heat' of transition, associated with the energy required for microfracture and/or partial melting along grain boundaries or fracture surfaces. This energy term can be used as an intrinsic characteristic of the material, and describes the energetics of the irreversible processes of failure.

From this point of view it is not necessary to specify the microscopic details of the failure process, which may involve brittle fracture, plastic flow, melting, etc...., but only the composite irreversible energy required for these processes, as a function of temperature, pressure and material type.

The dynamical evolution of the failure zone is then controlled by the laws of conservation of mass, momentum, and energy, in a continuum including a moving phase boundary. Because of the discontinuous behaviour of the medium at the failure boundary, the momentum and energy equations are coupled through boundary conditions at the failure surface. However, one can still construct a Green's tensor representation theorem for the radiated field, in which the coupling appears as a term involving rupture velocity in the boundary surface integral over the failure surface. Since the radiation field arises from the relaxation of stress in the surrounding medium, the elastic field representation corresponds to that for relaxation source theory (e.g. Archambeau 1964, 1968, 1972; Minster 1973).

In this paper we shall give a concise account of the theoretical development of the basic conservation conditions that must hold in a continuum with localized discontinuous properties. We shall then focus on the case where the dynamical deformation of the surrounding medium induces rapid growth of a failure zone, since it corresponds to observed tectonic failure resulting in earthquakes. (The theory is, of course, also applicable to a variety of physical processes, including shock waves in solids and ordinary phase changes.) In this regard we will investigate the constraints placed on the rate of failure by the conservation relations ('jump conditions') that must hold on the rupture boundary.

Finally, the representation of the radiation field which arises from stress relaxation around the failure zone is expressed in terms of a general Green's function solution. In order to obtain such a solution, the usual linearizing approximations are needed (e.g. the assumption of infinitesimal strains in the medium exterior to the failure zone), as well as some approximations that are unique to this problem. We will, however, develop an integral representation of the radiated field with the minimum number of assumptions in order to obtain general results.

2 Transport theorems and conservation equations

In a general continuum representation of material flow the conservation of a function of the flow F may be expressed as*

$$\frac{d\mathscr{F}}{dt} \equiv \frac{d}{dt}\int_{V(t)} F(\mathbf{x}, t)\, d^3x = \int_{V(t)} k(\mathbf{x}, t)\, d^3x, \tag{1}$$

where the source density has the general form

$$k(\mathbf{x}, t) = K(\mathbf{x}, t) + \nabla\cdot \mathbf{J}(\mathbf{x}, t), \tag{2}$$

with $\mathbf{J}$ the flux of interacting fields and K corresponding to the intrinsic production of F in the volume $V(t)$.

Let $S(t)$ be the boundary of the material volume $V(t)$, let $\mathbf{V}(\mathbf{x}, t)$ be the material velocity field, and assume that $V(t)$ is cut by a surface of discontinuity $\Sigma(t)$; moving with the velocity $\mathbf{U}(\mathbf{x}, t)$, as illustrated in Fig. 1. Let $[\![G]\!]_\Sigma$ denote the jump of a quantity G across Σ. Then the Reynolds transport theorem takes the form

$$\frac{d}{dt}\int_{V(t)} F(\mathbf{x}, t)\, d^3x = \int_{V(t)\ominus\{V(t)\cap\Sigma(t)\}} \left\| \frac{\partial F}{\partial t} + \nabla_x\cdot(F\mathbf{V}) \right\| d^3x - \int_{V(t)\cap\Sigma(t)} [\![F(\mathbf{U}-\mathbf{V})\cdot\hat{n}]\!]\, da \tag{3}$$

* We will use an Eulerian description of deformation and flow throughout.

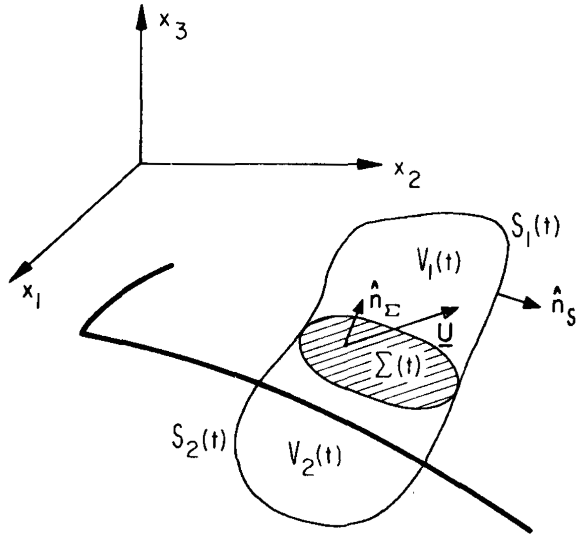

Figure 1. Case of a propagating discontinuity. $\Sigma(t)$ is the surface of discontinuity moving with velocity $\mathbf{U}$. $S_1(t) + S_2(t)$ is a material surface, moving with the medium.

and Gauss's theorem takes the form

$$\int_{V\ominus\{\Sigma\cap V\}} \nabla\cdot\mathbf{J}d^3x + \int_{\Sigma\cap V} [\![\mathbf{J}\cdot\hat{n}]\!]da = \int_{S\ominus\{S\cap\Sigma\}} \mathbf{J}\cdot\hat{n}da, \tag{4}$$

where Θ and $\cap$ denote the set theoretic difference and intersection respectively. A simplified proof of these results can be found, for example, in Eringen (1975). Edelen (1962) states a similar result to (4), but neglects to exclude the points of Σ from the volume integral, so that the term in $\mathbf{J}\cdot\hat{n}$ is effectively included twice in his result. Note that (3) and (4) hold only as long as none of the functions F, $\mathbf{v}$, $\mathbf{J}$ has a singularity on Σ stronger than a mere discontinuity (Minster 1973). A detailed proof of these results can be obtained using the theory of distributions. If generalized functions are used, it is easy to see that (4) may, in fact, be written in the usual form

$$\int_V \nabla\cdot\mathbf{J}d^3x = \int_S \mathbf{J}\cdot\hat{n}da. \tag{5}$$

Using (3), (4) in (1), (2), and noting that (1) holds for any volume $V(t)$, we have away from discontinuities

$$\frac{\partial F}{\partial t} + \nabla_x\cdot(F\mathbf{V} - \mathbf{J}) = K, \tag{6}$$

which is the differential conservation equation. Since this has to hold at all points away from Σ, (1) reduces to

$$\int_{V(t)\cap\Sigma(t)} [\![F(\mathbf{V} - \mathbf{U})\cdot\hat{n}]\!]\,da = \int_{V(t)\cap\Sigma(t)} [\![\mathbf{J}\cdot\hat{n}]\!]\,da \tag{7}$$

so that, since $V(t)$ is arbitrary, we must have at all points on Σ

$$[\![F(\mathbf{V}-\mathbf{U})\cdot\hat{n}]\!]_\Sigma = [\![\mathbf{J}\cdot\hat{n}]\!]_\Sigma, \tag{8}$$

which is the boundary condition to be satisfied on Σ for conservation of F.

A result of similar form has also been given by Freund (1970) in a somewhat different context. In a study by Snoke (1976), motivated by results equivalent to (8) given by Minster (1973), an attempt was made to obtain general 'jump conditions' such as (8) by a method of transformation of the equation of motion to a moving coordinate frame. The approach does not contain the essence of the 'moving boundary problem', however, namely the discontinuous behaviour of the field variables, and yields results which are not conservation relations on the moving boundary surface. Burridge (1976) applies equations of the form of (8) in circumstances similar to those to be investigated here. However, he starts from relations that are approximations to equation (15)–(17) of the next section, these being generated by choice of F in (8) as the density, momentum and energy respectively. His equation for energy does not, however, agree with our own or other results (e.g. Eringen 1975, vol. II, p. 460). Further, he takes the density jump across Σ to vanish and this may be physically incompatible with the application of the other boundary conditions involving momentum and energy jumps. His final result can, nevertheless, be obtained using results given in equation (32), if it is assumed that $[\![V_k]\!]_\Sigma$, the particle velocity change across Σ, is very small.

If we define $\mathbf{V}^* = \mathbf{V} - \mathbf{U}$ as the velocity of the material, relative to the boundary Σ, and introduce the current density of F per unit time per unit area (flux vector): $\mathscr{F} = F\mathbf{V} - \mathbf{J}$, and the relative flux vector through Σ: $\mathscr{F}^* = F\mathbf{V}^* - \mathbf{J}$; then the conservation laws have the form

$$\frac{\partial F}{\partial t} + \frac{\partial \mathscr{F}_l}{\partial x_l} = K \tag{9}$$

$$[\![\mathscr{F}_l^* n_l]\!]_\Sigma = 0 \tag{10}$$

where K is, as before, the rate of production of F per unit time per unit volume.

We observe that if the boundary Σ is a material surface, or if $\mathbf{V}^*\cdot\hat{n} = 0$, then (10) becomes

$$[\![\mathbf{J}\cdot\hat{n}]\!]_\Sigma = 0 \tag{11}$$

which is the usual boundary condition when the boundaries move with material flow or deformation.

Conservation equations apply in general to tensor components, and it is straightforward to show that the general forms for (9) and (10) are

$$\left.\begin{aligned} &\frac{\partial F_{i\ldots j}}{\partial t} + \frac{\partial}{\partial x_m}[\mathscr{F}_{i\ldots jm}] = K_{i\ldots j} \\ &[\![\mathscr{F}^*_{i\ldots jm} n_m]\!]_\Sigma = 0 \end{aligned}\right\}. \tag{12}$$

Table 1 furnishes a summary of the most common conservation equations. Additional conservation equations are also possible, such as those investigated by Fletcher (1974) for hyperelastic media using Noether's theorem. The additional relations involve internal angular momentum for polar media, and since we are only concerned with nonpolar media, conservation of angular momentum only requires that the stress tensor T_{ij} be symmetric.

Table 1

Conservation law	F	$\mathscr{F}$	$\mathscr{F}^*$	K
Mass	ρ	$\rho \mathbf{V}$	$\rho \mathbf{V}^*$	0
Momentum	ρV_i	$\rho V_i V_j - T_{ij}$	$\rho V_i V_j^* - T_{ij}$	ρb_i
Energy	$\rho E = \rho u + \rho/2 V_l V_l$	$\rho E V_i - V_k T_{ki} + q_i$	$\rho E V_i^* - V_k T_{ki} + q_i$	$\rho b_k V_k + \rho h$
Total energy	$\rho \mathscr{E} = \rho E + \rho\phi$	$\rho \mathscr{E} V_i - V_k T_{ki} + q_i$	$\rho \mathscr{E} V_i^* - V_k T_{ki} + q_i$	ρh

where ρ = density; b_i = body force density; q_i = heat flux vector; ϕ = body force potential: $\mathbf{b} = - \nabla\phi$; T_{ij} = Cauchy stress tensor; u = internal energy density; h = heat source density

Thus, equations (12), together with Table 1, provide the complete set of conservation relations for the media of importance in this study.

An alternative expression of the conservation equations in four dimensional form is also possible. We find such a compact representation to be useful when we consider integral representations of the linearized equations of motion for the continuum in a following section. Here we simply display the forms for the general (nonpolar) case.

In particular, the conservation relations can be put in a compact explicit form that expresses conservation of mass and momentum by introducing fields $\mathscr{F}_{\alpha\beta}$ and K_β with the Greek indices running over the range 1–4. Thus, with space-time coordinates represented by x_α we have* for the momentum equation

$$\frac{\partial \mathscr{F}_{\alpha\beta}}{\partial x_\alpha} = K_\beta; \quad \alpha, \beta = 1, 2, 3, 4 \tag{13a}$$

$$[\![\mathscr{F}^*_{\alpha\beta} n_\alpha]\!]_\Sigma = 0 \tag{13b}$$

where K_β and n_α are the 'space-like' variables given by

$$\left.\begin{aligned} K_\beta &= (\rho b_1, \rho b_2, \rho b_3, 0) \equiv (\rho b_j, 0) \\ n_\alpha &= (n_1, n_2, n_3, 0) \equiv (n_j, 0) \end{aligned}\right. \tag{13c}$$

with Latin indices, such as j, running over the (space) indices 1, 2, 3 only. The vector **b** denotes the body force density, ρ the medium density, and the n_j are the components of the normal to any spatial surface in the medium. The fields $\mathscr{F}_{\alpha\beta}$ and $\mathscr{F}^*_{\alpha\beta}$ are given by:

$$\mathscr{F}_{\alpha\beta}: \begin{cases} \mathscr{F}_{ij} = \rho V_i V_j - T_{ij}; & i, j = 1, 2, 3 \\ \mathscr{F}_{4j} = F_{j4} = \rho V_j; & j = 1, 2, 3 \\ \mathscr{F}_{44} = \rho \end{cases} \tag{13d}$$

and

$$\mathscr{F}^*_{\alpha\beta}: \begin{cases} \mathscr{F}^*_{ij} = \rho V_i V_j^* - T_{ij}; & i, j = 1, 2, 3 \\ \mathscr{F}^*_{4j} = F^*_{j4} = \rho V_j^*; & j = 1, 2, 3 \\ \mathscr{F}^*_{44} = \rho \end{cases} \tag{13e}$$

*Here we retain a non-relativistic description, and merely consider time t as a fourth coordinate x_4 in the Newtonian sense. The summation convention over all repreated indices applies to both Greek and Roman indices.

Similarly, the energy conservation equation, defined in terms of $\mathscr{E}$ the total density function for the medium may be expressed as

$$\partial \mathscr{E}_{\alpha\beta}/\partial x_\alpha = H_\beta \tag{14a}$$

where

$$\mathscr{E}_{\alpha\beta}: \begin{cases} \mathscr{E}_{ij} = 0, & i, j = 1, 2, 3 \\ \mathscr{E}_{4j} = 0; & j = 1, 2, 3 \\ \mathscr{E}_{i4} = \rho \mathscr{E} V_i - V_j T_{ij} + q_i; & 1 = 1, 2, 3 \\ \mathscr{E}_{44} = \rho \mathscr{E} \end{cases} \tag{14b}$$

and

$$H_\beta = (0, 0, 0, \rho h). \tag{14c}$$

The associated condition on Σ is:

$$[\![\mathscr{E}^*_{\alpha\beta} n_\alpha]\!]_\Sigma = 0 \tag{14d}$$

with

$$\mathscr{E}^*_{\alpha\beta}: \begin{cases} \mathscr{E}^*_{ij} = 0; & i, j = 1, 2, 3 \\ \mathscr{E}^*_{4j} = 0; & j = 1, 2, 3 \\ \mathscr{E}^*_{i4} = \rho \mathscr{E} V^*_i - V_j T_{ij} + q_i; & i = 1, 2, 3. \\ \mathscr{E}^*_{44} = \rho \mathscr{E} \end{cases} \tag{14e}$$

It is easy to verify that the conservation relations given by (13) and (14) are the same as those obtained from equation (12) and Table 1.

3 Conservation relations on Σ: Consequences for failure growth

The boundary conditions to be satisfied on Σ, which may be any boundary surface, are

$$[\![\rho V_i^* n_i]\!]_\Sigma = 0 \tag{15}$$

$$[\![(\rho V_k V_i^* - T_{ki}) n_i]\!]_\Sigma = 0 \tag{16}$$

$$[\![(\rho \mathscr{E} V_i^* - V_k T_{ki} + q_i) n_i]\!]_\Sigma = 0 \tag{17}$$

where the last equation can be trivially modified to include non-conservation body force fields using the results given in Table 1 along with equations (10) or (12).

These conditions may be recast in a more useful general form if one notes that $[\![U_i n_i]\!]_\Sigma = 0$ and uses it in (10) to obtain

$$[\![F]\!]_\Sigma U_i n_i = [\![F_i n_i]\!]_\Sigma .$$

From this equation we obtain, provided F has a nonzero jump across Σ

$$U_i n_i = ([\![F V_i n_i]\!]_\Sigma - [\![J_i n_i]\!]_\Sigma)/([\![F]\!]_\Sigma) \tag{18}$$

The special condition of no growth of the failure zone is that no material transport occurs across Σ, in which case $\mathbf{V}^* \cdot \hat{n} = 0$. We have seen earlier that this leads to equation (11).

In this case (15) is satisfied identically, while from (16) and (17) we obtain, respectively

$$[\![T_{ki}n_i]\!]_\Sigma = [\![t_k]\!]_\Sigma = 0$$

$$[\![V_k T_{ki} n_i]\!]_\Sigma = [\![V_k]\!]_\Sigma t_k = [\![q_i n_i]\!]_\Sigma$$

which are the well-known boundary conditions at (ordinary) material boundaries. Note that the tangential velocity may be discontinuous on Σ inasmuch as slip along Σ is allowable.

More generally, investigation of (19) shows that slow boundary propagation is characterized by low values of the fluxes $[\![J_i n_i]\!]_\Sigma$ and/or large values of $[\![F]\!]_\Sigma$.

We are, however, most interested in 'fast' processes. From the conservation of mass (15), we have*

$$U_i n_i = [\![\rho V_i n_i]\!]/[\![\rho]\!]. \tag{19}$$

Let the unit normal to Σ point away from the failure zone, and, with reference to Fig. 1, let

$$\rho^{(1)} \equiv \rho(\Sigma^{(1)}),\quad \rho^{(2)} \equiv \rho(\Sigma^{(2)}),\quad [\![\rho]\!] \equiv \rho^{(1)} - \rho^{(2)}$$

where $\Sigma^{(1)}$ and $\Sigma^{(2)}$ denote Σ approached, in the limit, from region 1 and 2 respectively. With similar definitions for the other field variables, then (19) may be written as

$$U_\mathrm{R} \equiv (U_i - V_i^{(1)})n_i = \frac{\rho^{(2)}}{[\![\rho]\!]}[\![V_i n_i]\!] \tag{20}$$

where U_R is the failure propagation velocity relative to the untransformed material. A large class of rapid processes of interest to us can be characterized by small fractional density jumps, since we expect material failure, involving macroscopic and microscopic fracture and cracking, plastic flow, grain boundary melting, etc.... to result in very small change in the average density. This kind of transition is precisely of greatest geophysical interest and in this case $[\![\rho]\!]$ is very small and U_R may be very large as a consequence.

We also observe that the momentum equation (16) may be written in the form

$$-U_\mathrm{R}[\![\rho V_k]\!] + \rho^{(1)}V_k^{(1)}[\![V_i n_i]\!] - [\![\rho V_k]\!]\,[\![V_i n_i]\!] = [\![T_{ki}n_i]\!].$$

Eliminating the jump $[\![V_i n_i]\!]$ by use of (20), we have

$$\rho^{(1)}[\![V_k]\!]U_\mathrm{R} = -[\![t_k]\!] \tag{21}$$

and this relation is exact. When the density jump $[\![\rho]\!]$ is small, then we have $\rho^{(1)} \simeq \rho^{(2)}$ and so, as an approximation

$$[\![\rho V_k]\!]\,U_R = -[\![t_k]\!]. \tag{22}$$

Precisely the same approximate result is obtained from (16) if we consider $U_i n_i \gg V_i n_i$, which serves as the definition of a fast transition process. Then $V_i n_i$ can be neglected in (16), and we get (22) directly, with $U_\mathrm{R} \simeq U_i n_i$. Equation (22) will be used extensively in later sections where we consider a 'representation theorem' for the radiated elastic field associated with a growing failure zone.

We note that typical inferred values for (shear) stress drops and rupture velocities for earthquakes are of the order 10^8 dyne/cm^2 and 3×10^5 cm/s respectively. With a density

*In the following we will often simply write $[\![\ \ ]\!]$ for $[\![\ \ ]\!]_\Sigma$, where it is understood that the jump notation always applies to a surface Σ.

near 3.5 g/cm^3, then (21) gives $[\![V_k]\!] \sim 10^2$ cm/s. Thus, (21) implies tangential particle velocity jumps near 1 m/s across the failure boundary.

The energy equation (17) can also be put into a more appropriate form for the present application. In particular, eliminating $[\![V_i n_i]\!]$ using (20), gives

$$-\rho^{(1)} U_R [\![\mathscr{E}]\!] = [\![(V_k T_{ki} - q_i) n_i]\!]. \tag{23}$$

Here

$$[\![\mathscr{E}]\!] = [\![u]\!] + \tfrac{1}{2} [\![V_k V_k]\!] + [\![\phi]\!]$$

where u is the internal energy and ϕ the body force potential energy. The second term is the change in the kinetic energy density of the material across the boundary. We can realistically neglect the change in ϕ across a failure boundary compared with the change in the kinetic or internal energies.

The change in the internal energy may be viewed as the energy required for the transition process in the material since we may consider the medium on either side of Σ at a given time as two states of the same material. Viewed in this manner we observe that, while the transition is irreversible, we can nevertheless consider u to be a function of the ordering in the material, that is the entropy, and the strain energy density. In the transition process we would expect a decrease in the ordering upon failure, that is an increase in the disorder and in the entropy, while the strain energy density (i.e. the recoverable or reversible energy) should decrease. (Here, of course, the measure of strain is relative to the two distinctly different natural states of the material in question, while the difference in the ordering in these two states is measured by the change in the entropy.) Therefore we let

$$[\![u]\!] = L \tag{24}$$

with L the internal energy change associated with the specific process taking place in the material; where L is used to emphasize the fact that the quantity $[\![u]\!]$ is characteristic of the process for the material and hence a material property. We note that, because of the definition of $[\![u]\!] = u^{(1)} - u^{(2)}$, we expect the change in u across Σ due to disordering of the material to be negative, while the part of the change due to the strain energy positive. Thus the total change, being the sum of these, may in principle be positive or negative. We will show, however, that for a spontaneous process to occur along with rapid growth of the transition zone, then it is generally necessary that L be positive. In particular, if the spatial heat flux jump across Σ is neglected (i.e. if the change in the thermal gradient and possible changes in the thermal coefficient across Σ are neglected as small compared with the other effects), then L must always be positive. In general, for failure processes we expect terms like $[\![V_k t_k]\!]$ in (23) to be much larger in magnitude than the term $[\![q_k n_k]\!]$, and this size ordering also leads to the requirement that L be positive for non-zero growth rate of a spontaneous failure zone. A requirement that L be positive simply means that the change in the strain energy must generally be larger in magnitude than that part of the internal energy change due solely to disordering. This requires relatively high (nonhydrostatic) strain energy for such a process to occur.

To show that these statements follow from the condition (23), we can rewrite it as

$$U_R = -[[\![V_k t_k - q_k n_k]\!]]/[\rho(L + \tfrac{1}{2}[\![V_k V_k]\!])] \tag{25}$$

where we consider the case in which the denominator is non-zero, of course. Here we have suppressed the superscript on the density, $\rho^{(1)}$, and simply written ρ to denote this quantity.

Using (21) in this equation, after expanding both $[\![V_k t_k]\!]$ and $[\![V_k V_k]\!]$, we have the results

$$\begin{aligned} U_\mathrm{R} &= -\,[[\![V_k]\!] t_k^{(1)} - [\![q_k n_k]\!]] / [\rho(L + \tfrac{1}{2}[\![V_k]\!]\,[\![V_k]\!])] \\ &= -\,[[\![V_k]\!] t_k^{(2)} - [\![q_k n_k]\!]] / [\rho(L - \tfrac{1}{2}[\![V_k]\!]\,[\![V_k]\!])]. \end{aligned} \tag{26}$$

Using (21) again, in either of the expressions for U_R in (26), to eliminate terms in $[\![V_k]\!]$, we get

$$U_\mathrm{R}^2 - ([\![q_k n_k]\!] U_\mathrm{R})/(\rho L) - ([\![t_k t_k]\!])/(2\rho^2 L) = 0. \tag{27}$$

The roots of this quadratic in U_R are real provided

$$[\![q_k n_k]\!]^2 + 2L[\![t_k t_k]\!] \geqslant 0. \tag{28}$$

When this condition is not met, then U_R has complex (or imaginary) roots and this means that initiation and growth of the transition process cannot occur.

When the stresses, and hence the tractions, drop upon failure, then $[\![t_k t_k]\!] > 0$, and the scalar terms involving the jumps on Σ are all positive. In this case we must have

$$L \geqslant -([\![q_k n_k]\!]^2)/(2[\![t_k t_k]\!]) \leqslant 0. \tag{28a}$$

As was already noted, for a failure process we expect the heat flux term to be much smaller than the traction term so that, for all practical purposes, we must have

$$L \geqslant 0$$

If we neglect the heat flux term completely on the grounds of its relative size for a rapid spontaneous process, then (27) gives

$$U_\mathrm{R} \simeq [[\![t_k t_k]\!]/(2\rho^2 L)]^{1/2}. \tag{29}$$

Alternatively, under these circumstances

$$L \simeq [\![t_k t_k]\!]/(2\rho^2 U_\mathrm{R}^2). \tag{30}$$

We note that rapid growth of a tectonic failure zone involves $U_\mathrm{R} \simeq 3 \times 10^5$ cm/s and traction jumps at the front of the failure boundary which appear to be of the order of 10^8–10^9 dyne/cm^2 (e.g. Archambeau 1977). This implies that the internal energy change L is in the range from 10^4 to 10^6 erg/g.

We can also express L in terms of the particle velocity and traction changes across Σ when we neglect the heat flux term. That is, eliminating U_R^2 from (30), using (21), gives

$$L = \tfrac{1}{2}[\![V_k]\!]\,[\![V_k]\!] \left\{ \frac{[\![t_k t_k]\!]}{[\![t_k]\!]\,[\![t_k]\!]} \right\} \tag{31}$$

The ratio of the quadratics in traction in (31) will be near unity when the hydrostatic stress is maintained after failure (i.e. small specific volume change and density change) and when the deviatoric stresses nearly vanish upon failure – that is when there is nearly a complete loss of shear strength upon failure. It is quite possible that this situation would prevail for many, if not most, earthquakes. If this is the case then

$$L \simeq \tfrac{1}{2}[\![V_k]\!]\,[\![V_k]\!] \tag{31a}$$

With L in the range 10^4–10^6 erg/g then this implies a magnitude of from 1 to 10 m/s for particle velocity jumps across the failure surface and conversely.

If the heat flux term can be neglected, that is if $|[\![q_k n_k]\!]| \ll |[\![V_k t_k]\!]|$, then the energy relation (23) or (25) may be expressed as

$$U_R = -[[\![V_k t_k]\!]]/[\rho(L + \tfrac{1}{2}[\![V_k V_k]\!])].$$

If we use the expression (31) for L in this, then we find that this expression of energy conservation can be put in the form:

$$(\rho U_R [\![V_k]\!] - [\![t_k]\!])[\![t_k]\!] = 0.$$

In view of the relation (21) for U_R, we see that this equation is satisfied when momentum is conserved.

Therefore this shows that if $[\![q_k n_k]\!] \ll [\![V_k t_k]\!]$, then conservation of energy is insured when equation (31) (or (31a)) is satisfied along with momentum conservation – which is expressed by equation (21). Consequently, the equations for failure zone growth seem to be well approximated by the relations

$$\left.\begin{array}{l}\rho U_R [\![V_k]\!] = -[\![t_k]\!] \\ L[\![t_k]\!]\,[\![t_k]\!] = \tfrac{1}{2}[\![V_k]\!]\,[\![V_k]\!]\,\{[\![t_k t_k]\!]\}\end{array}\right\} \qquad (32)$$

If we treat L as a material property, then the second of these relations can be viewed as a condition for failure growth – that is when the particle velocities and tractions across the failure boundary are such that this relation is satisfied for a value of L appropriate to a specific failure mode of the material, then growth may occur; with the rate of growth then given by the first relation in (32). In at least some instances of rapid failure the shear tractions would be very small within the failure zone, and then, as previously noted, the second relation reduces to $L \simeq \tfrac{1}{2}[\![V_k]\!]\,[\![V_k]\!]$.

Thus we may view the second relation in (32) as a dynamic failure condition and the first relation as an equation for U_R, the dynamic growth rate.

The dependence of the rupture growth rate – or rupture velocity U_R – on the rheological properties of the material can be inferred from the first relation in (32) if we specify appropriate constitutive relations for the material in its two states, before and after failure has taken place. For example, a linear constitutive relation that could in many circumstances describe the local behaviour of the material *after* failure has the form* (e.g. Fung 1965)

$$T_{ij} = \sum_{\alpha} \int_0^t \exp[-\Omega_\alpha(t-\tau)]\, D_{ijkl}^{(\alpha)} e_{kl}(\tau) d\tau + C_{ijkl} e_{kl} + C'_{ijkl} \frac{\partial e_{kl}}{\partial t} \qquad (33)$$

where the term involving the sum over α arises from relaxation processes within the material, of which there may be N types each denoted by an index α. Here Ω_α is a relaxation frequency (reciprocal relaxation time), and the object $D_{ijkl}^{(\alpha)}$ is a time independent relaxation modulus (or 'relaxation strength') tensor.

Relaxation terms can be shown to result from a variety of processes, including the movement of interstitial atoms, and vacancies, twining, chemical reactions, crystalline thermal

*The more general case of a nonlinear rheology and finite strain for the material within the failure zone can also be addressed of course. However, we view the process of failure as a sudden (nonlinear) change in material properties, with the material afterwards having some new rheology such as that given by (33). The important characteristics of the process is the local sudden change in physical properties and the *effective* elastic characteristic of the material following this process since these properties will determine the stress relaxation and energy radiated by the stressed medium surrounding this failure zone, as is shown in the following sections.

currents and dislocation motion. Descriptions of typical physical processes of this type for solids and liquids are given, for example, by De Groot & Mazur (1969) and Mason (1966).

The second term in (33) is a linear elastic term, while the last term is a simple linear viscous term.

The strain e_{kl} is defined as an infinitesimal strain measured from the completely relaxed state* of the material *after* transition. Thus, in the infinitesimal strain approximation which we will adopt for the material after transition

$$e_{kl} = u^{(2)}_{(k,l)} \equiv \tfrac{1}{2}\left(\frac{\partial u_k^{(2)}}{\partial x_l} + \frac{\partial u_l^{(2)}}{\partial x_k}\right) \tag{34}$$

where $u_k^{(2)}$ is a displacement from the relaxed reference state.

A similar constitutive relation holds for the material outside the failure zone or for the material before failure. However, the relaxed state of this material is different from that for the failed material, so that the infinitesimal strain is defined as

$$e_{kl} = u^{(1)}_{(k,l)} \equiv \tfrac{1}{2}\left(\frac{\partial u_k^{(1)}}{\partial x_l} + \frac{\partial u_l^{(1)}}{\partial x_k}\right) \tag{35}$$

with $u_k^{(1)}$ the displacement measured relative to the relaxed state of the unfailed material.

Further, while the form of the constitutive relation may be the same for the material in the two states, the moduli $D^{(\alpha)}_{ijkl}$, C_{ijkl} and C'_{ijkl} would be very different. For the unfailed material we can, in the present context at least, neglect the anelastic moduli $D^{(\alpha)}_{ijkl}$ and viscous term C'_{ijkl} relative to the linear elastic term C_{ijkl} for the unfailed material. Thus for the region outside the failure zone we may use a constitutive relation of the form

$$T_{ij} = C_{ijkl}e_{kl} \tag{36}$$

with e_{kl} defined by (35). In any case, however, (33) can be used to represent the form of the constitutive equation in both the failed and unfailed material, with the coefficients and strains taking on different values in the two zones.

We can write (33) in the compact operational form

$$T_{ij} = M_{ijkl}e_{kl} \tag{37}$$

where M_{ijkl} is the integral operator

$$M_{ijkl} = \sum_{\beta=1} m^{(\beta)}_{ijkl}\int_0^t dt' \exp\left[-\Omega_\beta(t-t')\right]\left[(1-\delta_{1\beta})(1-\delta_{2\beta}) + \delta_{1\beta}\delta(t-t')\right.$$
$$\left. + \delta_{2\beta}\delta_1(t-t')\right] \tag{38}$$

with $\delta_{\alpha\beta}$ denoting the Kronecker delta, while $\delta(t-t')$ is a Dirac delta function and $\delta_1(t-t')$ its derivative. Here $m^{(\beta)}_{ijkl}$ is a time independent modulus tensor, equal to the elastic and viscous tensors for $\beta = 1, 2$ and equal to the set of relaxation moduli tensors for $\beta = 3, 4 \ldots$. We can also write (38) as

$$M_{ijkl} = \sum_{\beta} m^{(\beta)}_{ijkl} I_\beta \tag{39}$$

where I_β is the time-dependent integral operator in (38).

* By relaxed state we mean the equilibrium configuration taken by a piece of the material removed from the surrounding medium, so that it is free from external forces and surface tractions.

Now the momentum condition in (32) can be written as

$$\rho U_R [\![V_k]\!] = -[\![t_k]\!] = -\sum_\beta [\![m_{ijkl}^{(\beta)} I_\beta e_{ij}]\!] n_l. \tag{40}$$

Without undue loss of generality, we will consider the isotropic case. This gives

$$m_{ijkl}^{(\beta)} = \lambda_\beta \delta_{ij}\delta_{lk} + \mu_\beta(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

for the modulus tensor.

If we now define a signal velocity C_S to be associated with the rate at which changes in the displacement fields are propagated, in the sense that displacements are causal with respect to such a velocity, then

$$u_k^{(n)} \equiv u_k^{(n)}(\tau) \tag{41}$$

where $n = 1, 2$ for the two material states, and $\tau \equiv t - r/C_S$ with $r \equiv (x_j x_j)^{1/2}$. Then we have

$$\frac{\partial u_k^{(n)}}{\partial x_l} = -\frac{1}{C_S}\left(\frac{x_l}{r}\right)\frac{\partial u_k^{(n)}}{\partial \tau}.$$

Since we can use the infinitesimal strain approximation for the material in its initial (unfailed) and final (failed) states, then the reduced time derivative of u_k is also the particle velocity. Thus

$$\left.\begin{aligned} \frac{\partial u_k^{(n)}}{\partial \tau} &= \frac{\partial u_k^{(n)}}{\partial t} = V_k^{(n)} \\ \frac{\partial u_k^{(n)}}{\partial x_l} &= -\frac{1}{C_S}\left(\frac{x_l}{r}\right) V_k^{(n)} \end{aligned}\right\}. \tag{42}$$

Using (42) in the momentum condition yields

$$\rho U_R [\![V_k]\!] = \sum_\beta I_\beta \left\{ \left[\!\!\left[\frac{\lambda_\beta}{C_S} V_i\right]\!\!\right] \frac{x_i}{r} n_k + \left[\!\!\left[\frac{\mu_\beta}{C_S}\left\{V_k\left(\frac{x_l}{r}\right) + V_l\left(\frac{x_k}{r}\right)\right\} n_l\right]\!\!\right]\right\} \tag{43}$$

for the isotropic case. The magnitude of U_R is thus directly proportional to the changes in the material moduli upon failure.

For example, suppose the material has the properties of a perfect fluid after transition. If the jump in the normal component of velocity across the transition boundary is small compared with the jump in the tangential components of the velocity (i.e. small changes in density and compressibility) then we have as an approximation to the maximum magnitude of U_R

$$|U_R| \simeq \left[\!\!\left[\frac{\mu}{\rho C_S} V_k\right]\!\!\right] \Big/ [\![V_k]\!]$$

considering the failure process to be driven by a shear wave, so that $C_S \simeq V_S$ with $V_S = \sqrt{\mu/\rho}$ the shear velocity in the medium before failure and since $\mu = 0$ for an ideal fluid, then:

$$|U_R| \simeq V_S.$$

In conclusion then, it is evident that the form of (43) indicates a dependence of the failure rate on the transition process in terms of the changes in material moduli. In general we see that consideration of an expression like (40) is required in order to obtain a precise estimate of the magnitude and space-time dependence of U_R. Nevertheless, simplified expressions such as (43) and approximate results, such as that for the ideal fluid transition, are quite useful and can provide a basis for interpretation of observed failure rates.

4 Linearization of the coupled conservation equations

We will now focus our attention on the medium outside the failure zone and express the conservation relations in this region in linearized form, using the usual approximation of infinitesimal strain for this region. This will give linear equations of motion for this part of the medium when we take the material to be elastic and specify the rheology according to (34).

We note that we can also describe the dynamical behaviour of the medium within the failure zone itself *after* failure using linear theory, if we adopt (33) as an adequate description of the rheology. In this case we would simply repeat the arguments leading to linearized equations and obtain an integral Green's function solution in a manner analogous to that for the region outside the failure zone. The two integral representations of the medium response would then be connected by the boundary conditions on the failure surface – as given by the equations (32) in the previous section. In this development, however, we will only consider the exterior region. This does not result in any loss of generality in the analytical description of the radiation field since the dynamical behaviour of the interior medium manifests itself directly in the boundary conditions for the exterior problem, and these can be treated in a formal manner with the tractions and the particle velocity of the interior zone material at the failure boundary being left completely arbitrary. The resulting general integral representation for the exterior radiation field can then be evaluated for particular cases when the rheological properties of the material after transition are independently specified. Then the resulting dynamical variation of the boundary surface tractions and the particle velocity are determined by either solving for the interior deformation fields using the Green's function representation or by choosing a simple, yet adequate, rheological description for which the dynamical behaviour in the interior is known with sufficient accuracy *a priori*. An example of the latter would be the assumption of melting and the production of an essentially inviscid fluid, so that the shear tractions in the interior would vanish and the particle velocity would only reflect any accompanying density change.

To treat the exterior region we first observe that, in the present context, it may be viewed as containing an elastic material. In this case the energy equations are trivially satisfied in the medium away from the failure boundary if we neglect heat transport and transport terms in the energy equation. On the failure boundary, however, energy is absorbed by the failure process and energy conservation is expressed, approximately, by the second relation in (32).

The linearized equations of motion for the exterior medium can be written in the form

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u_i}{\partial t}\right)=\frac{\partial}{\partial x_j}\left[C_{ijkl}\frac{\partial u_k}{\partial x_l}\right]+\rho b_i \tag{44}$$

where u_i is the elastic displacement, and both the density ρ and the elastic tensor C_{ijkl}

are treated as functions of the spatial coordinates alone*. In obtaining (43) we have used symmetry relations (e.g. Landau & Lifschitz 1965)

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

Because of the form of (44), we can write these linearized equations of motion in the same four-dimensional form as was used to express all the conservation relations earlier. Specifically, we define an 'elastic-intertial tensor' as

$$C_{\alpha\beta\gamma\delta}: \begin{cases} C_{\alpha\beta\gamma\delta} = -C_{ijkl}; & \alpha, \beta, \gamma, \delta = 1, 2, 3 \\ C_{i4k4} = C_{4i4k} = \rho\delta_{ik}; & i, k = 1, 2, 3 \\ C_{\alpha\beta\gamma\delta} = 0; & \text{otherwise} \end{cases} \tag{45}$$

where $C_{\alpha\beta\gamma\delta}$ is seen to have the symmetry properties

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta} = C_{\alpha\beta\delta\gamma} = C_{\gamma\delta\alpha\beta} \tag{46}$$

with the Greek indices ranging from 1 to 4. If we also define the spacelike fields

$$\left.\begin{aligned} u_\alpha &\equiv (u_1, u_2, u_3, 0) \\ f_\alpha &\equiv (b_1, b_2, b_3, 0) \end{aligned}\right\} \tag{47}$$

then (44) can be written in the form

$$\left.\begin{aligned} \tau_{\alpha\beta,\beta} &= \rho f_\alpha \\ \tau_{\alpha\beta} &= C_{\alpha\beta\gamma\delta}\, u_{\gamma,\delta} \end{aligned}\right\}. \tag{48}$$

Alternately, in a form convenient for a solution for the displacement field u_γ, the equations of motion may be expressed in the operator form:

$$L_{\alpha\gamma} u_\gamma = \rho f_\alpha \tag{49}$$

where the 'elastic operator' is given by†

$$L_{\alpha\gamma} \equiv \frac{\partial}{\partial x_\beta}\left(C_{\alpha\beta\gamma\delta}\frac{\partial}{\partial x_\delta}\right) \tag{50}$$

The boundary conditions can also be expressed in terms of the elastic-inertial tensor defined in (45) and the spacelike displacement field u_γ if we define a four dimensional space-time 'normal' as

$$\eta_\alpha = (n_1, n_2, n_3, -\mathbf{U}^* \cdot \hat{n}) \tag{51}$$

with

$$U_l^* = U_l - V_l$$

* In what follows, we use the common notation

$$u_{k,l} \equiv \frac{\partial u_k}{\partial x_l};\ u_{k,lm} \equiv \frac{\partial^2 u_k}{\partial x_l \partial x_m}; \text{etc.}$$

for partial derivatives.

† Kupradze (1963) defines the 'elastic operator' in an analogous fashion in the frequency domain.

where n_j is the regular spacelike normal to any spatial surface in the medium, in particular the surface Σ, and $\mathbf{U}^* \cdot \hat{n} = U_R$ is the projection of the spacelike relative velocity vector for the surface Σ along the normal to the surface. Now we observe that in view of the properties of the elastic-inertial tensor

$$[\![\tau_{\alpha\beta}\eta_\beta]\!]_\Sigma = [\![C_{\alpha\beta\gamma\delta}\, u_{\gamma,\delta}\, \eta_\beta]\!]_\Sigma = -\,[\![C_{klmn}u_{m,n}n_l + \rho u_{k,4}U_l^* n_l]\!]_\Sigma$$

or

$$[\![\tau_{\alpha\beta}\eta_\beta]\!]_\Sigma = [\![(\rho V_k V_l^* - T_{kl})n_l]\!]_\Sigma$$

but the right side expresses conservation of momentum on Σ, as is verified from (16), and vanishes. Thus we have that

$$[\![\tau_{\alpha\beta}\eta_\beta]\!]_\Sigma = 0$$

for all α and β.

Thus the linearized equations for the exterior region are most naturally expressed in a four vector form, as

$$\left.\begin{aligned} L_{\alpha\gamma}u_\gamma &= \tau_{\alpha\beta,\beta} = \rho f_\alpha \\ [\![\tau_{\alpha\beta}\eta_\beta]\!]_\Sigma &= 0 \end{aligned}\right\} \qquad (52)$$

with the inertial-stress tensor $\tau_{\alpha\beta}$ given by (48) and where the elastic operator $L_{\alpha\gamma}$ is defined by (50). In the following we will linearize the boundary condition in (52) to the extent that we will use U_l instead of U_l^*, which amounts to the neglect of V_l compared with U_l, as is justified for failure processes.

5 Green's tensor equations for the radiation field in the elastic zone: elastodynamic representation theorems

The differential operator $L_{\alpha\gamma}$ applies to space-like tensors (i.e. tensors without time-like components) with the property,

$$w_{\alpha\beta\,\ldots\,\lambda} = w_{\alpha\beta\,\ldots\,\lambda}\,(1-\delta_{\alpha 4})\ldots(1-\delta_{\lambda 4}).$$

The contraction $L_{\mu\alpha}w_{\alpha\beta\,\ldots\,\lambda}$ is another space-like tensor. The operator has been defined in a four dimensional space-time, and it is useful to consider an arbitrary four volume Ω in this space over which an inner product of space-like tensors is defined. Thus, given two space-like tensors $g_{\alpha\,\ldots\,\lambda}$ and $h_{\beta\,\ldots\,\mu}$, we define an inner product in Ω by

$$(g_{\alpha\,\ldots\,\lambda}, h_{\beta\,\ldots\,\mu})_\Omega \equiv \int_\Omega g_{\alpha\,\ldots\,\lambda}, h_{\beta\,\ldots\,\mu}\, d^4x.$$

Here $(g_{\alpha\,\ldots\,\lambda}, g_{\alpha\,\ldots\,\lambda})$ is positive and vanishes only when $g_{\alpha\,\ldots\,\lambda} \equiv 0$. Let $G_\alpha^\beta(\mathbf{x},\mathbf{x}')$ denote a two-point tensor, a function of the independent coordinates $\mathbf{x}$ and $\mathbf{x}'$ in Ω, then we define the inner products*

$$(G_\alpha^\beta, u_\alpha)_\Omega^{\mathbf{x}'} \equiv \int_\Omega G_\alpha^\beta(\mathbf{x},\mathbf{x}')u_\alpha(\mathbf{x}')d^4x'$$

$$(G_\alpha^\beta, u_\alpha)_\Omega^{\mathbf{x}} \equiv \int_\Omega G_\alpha^\beta(\mathbf{x},\mathbf{x}')u_\alpha(\mathbf{x})d^4x$$

* We will use, as a notational convenience, both superscript and subscript indices for the tensors appearing here. However, all tensors are Cartesian and there is no distinction between covariant or contravariant tensors.

which give a vector function of x in the first case or a vector function of x′ in the second. If G_α^β is symmetric in x and x′, then the two inner product forms are entirely equivalent.

The Green's functions associated with the operator $L_{\alpha\gamma}$ are two point tensors that are of special importance in the present development. In particular, we define the two point Green's tensor associated with $L_{\alpha\gamma}$ to ge given by

$$L_{\alpha\gamma} G_\gamma^\beta(\mathbf{x}, \mathbf{x}_0) = \Delta_\alpha^\beta(\mathbf{x}, \mathbf{x}_0) \tag{53}$$

where the space-like tensor G_γ^β is a fundamental solution of the operator $L_{\alpha\gamma}$ with a pole at $\mathbf{x}_0$; with the space-like tensor Δ_α^β defined to be the generalized delta function

$$\Delta_\alpha^\beta \equiv 4\pi\delta_{\alpha\beta}(1-\delta_{\alpha 4})(1-\delta_{\beta 4})\delta(\mathbf{x}-\mathbf{x}_0)$$

Here $\delta(\mathbf{x}-\mathbf{x}_0)$ is the (four-dimensional) Dirac delta distribution. Clearly G_γ^β defined by (53) will have a singularity at $\mathbf{x} = \mathbf{x}_0$ due to the presence of the Dirac delta distribution and therefore will satisfy (53) in a distributional sense (see, e.g. Stakgold 1968). In the present context $G_\gamma^\beta(\mathbf{x}, \mathbf{x}_0)$ can be viewed as the γ component of the displacement field in the continuum at x due to an impulsive force in the β direction located at $\mathbf{x}_0$. Consequently $\mathbf{x}_0$ can be thought of as the source point and x as the receiver or observer point.

With these definitions of the inner product in Ω for space-like tensors we can generate integral equations that are equivalent to the differential equations and associated boundary conditions which specify the displacement field u_γ in a continuum. To do so we will make use of the special properties of the Green's function.

In particular, the Green's theorem in Ω is obtained by considering the inner product

$$(L\mathbf{w}, \mathbf{v})_\Omega \equiv \int_\Omega v_\alpha L_{\alpha\gamma} w_\gamma d^4x \tag{54}$$

where the operator L is simply the matrix operator with components given in (49), while w and v are fields satisfying either of the operator equations (48) or (53) and hence may be single or two point space-like tensors of first or second order.

Now using the representation of the components of L given in (49) we have

$$(L\mathbf{w}, \mathbf{v})_\Omega \equiv \int_\Omega v_\alpha (C_{\alpha\beta\gamma\delta}\, w_{\gamma,\delta}),_\beta d^4x. \tag{55}$$

A result similar to the ordinary Green's theorem is obtained by first observing that the integrand can be expressed in terms of an identity as:

$$v_\alpha (C_{\alpha\beta\gamma\delta}\, w_{\gamma,\delta})_{,\beta} = (v_\alpha C_{\alpha\beta\gamma\delta}\, w_{\gamma,\delta})_{,\beta} - v_{\alpha,\beta} C_{\alpha\beta\gamma\delta}\, w_{\gamma,\delta}\,.$$

However the final term can be rewritten by using the symmetry properties of $C_{\alpha\beta\gamma\delta}$, so that

$$-v_{\alpha,\beta} C_{\alpha\beta\gamma\delta}\, w_{\gamma,\delta} = w_\alpha (C_{\alpha\beta\gamma\delta}\, v_{\gamma,\delta})_{,\beta} - (w_\alpha C_{\alpha\beta\gamma\delta}\, v_{\gamma,\delta})_{,\beta}.$$

Using these results in (55) we get

$$(L\mathbf{w}, \mathbf{v})_\Omega^{\mathbf{x}} = \int_\Omega [w_\alpha (C_{\alpha\beta\gamma\delta}\, v_{\gamma,\delta})_{,\beta}]\, d^4x + \int_\Omega [(v_\alpha C_{\alpha\beta\gamma\delta}\, w_{\gamma,\delta} - w_\alpha C_{\alpha\beta\gamma\delta}\, v_{\gamma,\delta})_{,\beta}]\, d^4x. \tag{56}$$

We can now define a formal adjoint operator L^*, where

$$L^*_{\alpha\gamma} \equiv \frac{\partial}{\partial x_\beta}\left(C_{\alpha\beta\gamma\delta}\frac{\partial}{\partial x_\delta}\right) \tag{57}$$

are the matrix elements of L^* and further define the bilinear concomitant J_β of **w** and **v** where

$$J_\beta \equiv v_\alpha C_{\alpha\beta\gamma\delta}\, w_{\gamma,\delta} - w_\alpha C_{\alpha\beta\gamma\delta}\, v_{\gamma,\delta} \tag{58}$$

in order to write (56) in the form

$$(L\mathbf{w},\mathbf{v})^{\mathbf{x}}_\Omega = (\mathbf{w}, L^*\mathbf{v})^{\mathbf{x}}_\Omega + \int_\Omega J_{\beta,\beta}\, d^4x. \tag{59}$$

This is the generalized Green's theorem for the operator L in Ω.

We observe from comparison of (57) with (49) that

$$L^* \equiv L$$

and hence that L is formally self adjoint.

In deriving (59) we have considered the operation of L on fields specified at **x** and have formed inner products on Ω with respect to the coordinates **x**. We may form inner products with respect to fields specified at the coordinates $\mathbf{x}_0$, which in the case of two point tensor fields, such as $G^\beta_\gamma(\mathbf{x}, \mathbf{x}_0)$, means integrating over the 'source coordinates' $\mathbf{x}_0$ rather than the 'observer coordinates' **x**. In formal terms we may generate a completely parallel result to (59). Consider the operator $L^{(0)}$ defined by

$$L^{(0)}_{\alpha\gamma} \equiv \frac{\partial}{\partial x^0_\beta}\left(C_{\alpha\beta\gamma\delta}\frac{\partial}{\partial x^0_\delta}\right) \tag{60}$$

in component form. Now

$$L^{(0)}_{\alpha\gamma}u_\gamma = \frac{\partial}{\partial x^0_\beta}\left(C_{\alpha\beta\gamma\delta}(\mathbf{x}_0)\frac{\partial u_\gamma}{\partial x^0_\delta}\right)$$

is just the operation of L on **u** at $\mathbf{x}_0$. Here of course u_γ is expressed as a function of the coordinates $\mathbf{x}_0$. In case the operand is a two point tensor G^β_γ

$$L^{(0)}_{\alpha\gamma}G^\beta_\gamma(\mathbf{x}_0;\mathbf{x}) = \frac{\partial}{\partial x^0_\beta}\left[C_{\alpha\beta\gamma\delta}(\mathbf{x}_0)\frac{\partial G^\beta_\gamma(\mathbf{x}_0;\mathbf{x})}{\partial x^0_\delta}\right].$$

Clearly we can now form the inner product

$$(L^{(0)}\mathbf{w},\mathbf{v})^{\mathbf{x}_0}_\Omega \equiv \int_\Omega v_\alpha L^{(0)}_{\alpha\gamma} w_\gamma\, d^4x^0. \tag{61}$$

By exactly the same formal manipulations used to obtain the previous result, we have the complementary Green's theorem in Ω for $L^{(0)}$

$$(L^{(0)}\mathbf{w},\mathbf{v})^{\mathbf{x}_0}_\Omega = (\mathbf{w}, L^{(0)*}\mathbf{v})^{\mathbf{x}_0}_\Omega + \int_\Omega J^{(0)}_{\beta,\beta}\, d^4x^0 \tag{62}$$

where $L^{(0)*}$ is the adjoint of $L^{(0)}$ and where

$$L^{(0)*} \equiv L^{(0)}$$

so that $L^{(0)}$ is formally self adjoint, as is obvious since L was self adjoint. Here also $J_\beta^{(0)}$ is formally identical with J_β, with the coordinates x_β replaced by the coordinates x_β^0.

If **w** and **v** are one-point tensor fields then (59) and (62) are clearly identical results. However if one or both of **w** and **v** are two point tensor fields of **x** and $\mathbf{x}_0$, then (59) yields relations between function of $\mathbf{x}_0$ while (52) yields relations involving functions of **x** which may be different.

The integral relations obtained take a physical meaning when the fields **v** and **w** are specified. In particular we will take **w** to be the displacement field **u** satisfying the equations of motion and boundary conditions for the region Ω surrounding a closed failure region. In this region we take the failure zone to have a volume V_2 with a closed boundary surface Σ while the remaining spatial volume is denoted by V_1 with an external boundary S, as shown on Fig. 2. Thus the boundary of V_1 is $\Sigma \oplus S$ while the boundary of V_2 is Σ. The space-time volume Ω is the four volume occupied by the medium external to Σ as shown in Fig. 2. The external boundary of Ω is generated by parallel displacement of S in the direction of the x_4 (time) axis, since S does not change with time. The internal boundary of Ω, however, has generators that are not parallel to the x_4 axis, reflecting the growth of the failure volume with time.

In addition to the space-like boundary conditions expressed by the boundary conditions in (52), which apply on the spatial boundary Σ, there may be conditions on the fields that apply on the time-like boundary of Ω. These may be connected with the space-like conditions in (52) in that they may result from the change in Σ as a function of time, and in any case the conditions along the time-like surface of Ω would correspond to 'generalized initial conditions'. These general time-like conditions are in fact already contained in the boundary conditions (52) provided we recognize that Σ is a function of time and that the fields may be discontinuous in time as well as in space. We will later show how the time-like discontinuous behaviour gives rise to relaxation phenomena that account for the stress wave radiation accompanying the growth of the failure zone in an initially stressed medium.

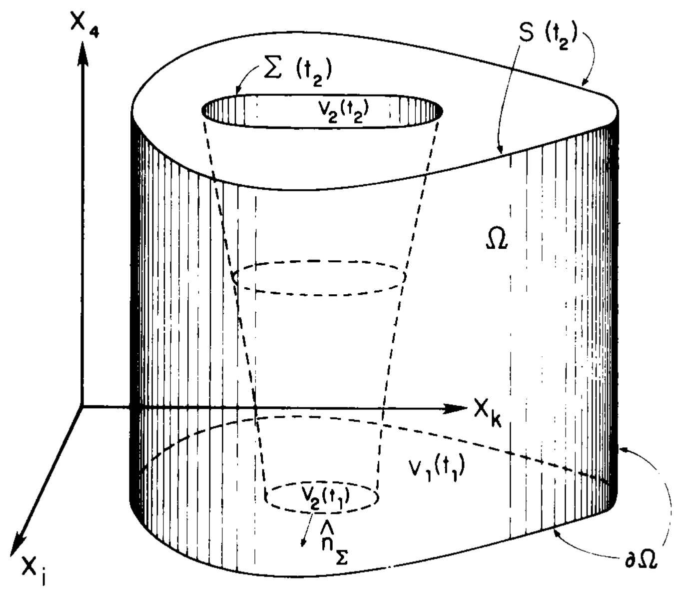

Figure 2. Geometry of the four volume Ω with x_i and x_k spatial coordinates, x_4 the time coordinate. A spatial section through the four-dimensional volume Ω at time t_1 is the ordinary spatial volume $V_1(t_1)$, external to the failure region.

To complete the description of the physical problem we consider the impulse response of the medium to a simple delta 'function' point source. This response is of course provided by the Green's function appearing in equation (53), the role of Green's integral theorem being simply to provide the means of superposing or adding together weighted Green's functions in such a way as to represent the complex source we are dealing with.

Therefore we will take the field **v** appearing in the Green's theorem formulae (59) and (62) to be the two point tensor field $G^{\beta}_{\gamma}(\mathbf{x}, \mathbf{x}_0)$. We shall require that causality is satisfied, that is (e.g. Morse & Feshbach 1953)

$$G^{\beta}_{\gamma}(\mathbf{x}, \mathbf{x}_0) = 0, \quad \text{for } x_4 \leqslant x_4^0 \tag{63}$$

and hence that

$$G^{\beta}_{\gamma}(x_k, x_4; x_k^0, x_4^0) = G^{\beta}_{\gamma}(x_k^0, -x_4^0; x_k, -x_4). \tag{64}$$

In addition to the differential operator equation and the causality condition, G^{β}_{α} is only fully determined by specification of boundary conditions to be fulfilled and the choice of boundary conditions appropriate for G^{β}_{γ} depends upon the physical problem to be solved. In the present case, the problem is represented by the operator equations:

$$\left.\begin{aligned} L\mathbf{u} &= \rho \mathbf{f}; \quad \mathbf{x} \epsilon \Omega \\ B\mathbf{u} &= \bar{\mathbf{b}}; \quad \mathbf{x} \epsilon \partial\Omega \end{aligned}\right\} . \tag{65}$$

The second equation simply represents the boundary conditions in (52) in compact operational form. In particular B is a matrix operator with components

$$B_{\alpha\gamma} \equiv C_{\alpha\beta\gamma\delta}\, \eta_\beta \frac{\partial}{\partial x_\delta}$$

so that $B_{\alpha\gamma} u_\gamma = \tau_{\alpha\beta} \eta_\beta$. Further, on Σ, $\bar{\mathbf{b}}_\alpha$ is the four vector equal to the value of $\tau_{\alpha\beta}\eta_\beta$ arising from the action of the material within the failure zone enclosed by Σ, which we may take as independently specified. For a physical system governed by operator equations of this form, then $\mathbf{G} \equiv (G^{\beta}_{\gamma})$ is taken to satisfy the homogeneous system (e.g. Stakgold 1968)

$$\left.\begin{aligned} L\mathbf{G} &= \mathbf{\Delta}; \quad \mathbf{x} \epsilon \Omega \\ B\mathbf{G} &= 0; \quad \mathbf{x} \epsilon \partial\Omega \end{aligned}\right\} \tag{66}$$

with L and B being identical operators in (65) and (66). We note that the Green's theorems expressed by (59) and (62) involve the adjoint operators L^* and $L^{(0)*}$. Consequently we also define a system adjoint to (66) as

$$\left.\begin{aligned} L^*\mathbf{G}^* &= \mathbf{\Delta}; \quad \mathbf{x} \epsilon \Omega \\ B^*\mathbf{G}^* &= 0; \quad \mathbf{x} \epsilon \partial\Omega \end{aligned}\right\} \tag{67}$$

where $\mathbf{G}^*$ is defined as the adjoint Green's function, L^* the operator adjoint to L and $B^*\mathbf{G}^* = 0$ as adjoint boundary conditions defined to be such that

$$(L\mathbf{G}, \mathbf{G}^*)^{\mathbf{x}}_{\Omega} = (\mathbf{G}, L^*\mathbf{G}^*)^{\mathbf{x}}_{\Omega} \tag{68}$$

Referring to the results (59)–(62) we observe, for the system represented by the differential operator and boundary conditions of (65) and the associated systems (66) and (67), that B^* is determined by the condition that the bilinear concomitant of **G** and **G*** vanish.

Noting that

$$L_{\alpha\gamma}G_\gamma^\beta(\mathbf{x};\ \mathbf{x}_0) = \Delta_\alpha^\beta(\mathbf{x};\ \mathbf{x}_0)$$
$$L_{\alpha\gamma}^* G_\gamma^{\beta *}(\mathbf{x};\ \mathbf{x}_1) = \Delta_\alpha^\beta(\mathbf{x};\ \mathbf{x}_1)$$

with $\mathbf{x}_0$ and $\mathbf{x}_1$ arbitrary source points in Ω, and inserting these expressions in the integral relationship (68) involving **G** and **G***, we have, using the properties of the delta functions

$$\left.\begin{aligned} G_\gamma^{\beta *}(\mathbf{x}_0;\ \mathbf{x}) &= G_\gamma^\beta(\mathbf{x};\mathbf{x}_0) \\ G_\gamma^{\beta *}(\mathbf{x};\ \mathbf{x}_0) &= G_\gamma^\beta(\mathbf{x}_0;\ \mathbf{x}) \end{aligned}\right\} \tag{69}$$

which is the reciprocity relation. Thus, from (64) and (69) it follows that the causal Green's tensor G_γ^β has the adjoint given by

$$\left.\begin{aligned} G_\gamma^{\beta *}(x_k^0, x_4^0; x_k, x_4) &= G_\gamma^\beta(x_k^0, -x_4^0; x_k, -x_4) \\ G_\gamma^{\beta *}(x_k, x_4; x_k^0, x_4^0) &= G_\gamma^\beta(x_k, -x_4; x_k^0, -x_4^0) \end{aligned}\right\}. \tag{70}$$

Now consider the operator $L^{(0)}$, defined to act on the source coordinates $\mathbf{x}_0$. Clearly, interchanging the roles of $\mathbf{x}$ and $\mathbf{x}_0$ in (66) gives

$$L_{\alpha\gamma}^{(0)} G_\gamma^\beta(\mathbf{x}_0;\ \mathbf{x}) = \Delta_\alpha^\beta(\mathbf{x}_0;\ \mathbf{x}) = \Delta_\alpha^\beta(\mathbf{x};\ \mathbf{x}_0)$$
$$B_{\alpha\gamma}^{(0)} G_\gamma^\beta(\mathbf{x}_0;\ \mathbf{x}) = 0.$$

By the reciprocity relation (69), then

$$\left.\begin{aligned} L_{\alpha\gamma}^{(0)} G_\gamma^{\beta *}(\mathbf{x};\ \mathbf{x}_0) &= \Delta_\alpha^\beta(\mathbf{x};\ \mathbf{x}_0) \\ B_{\alpha\gamma} G_\gamma^{\beta *}(\mathbf{x};\ \mathbf{x}_0) &= 0 \end{aligned}\right\} \tag{71}$$

Similarly,

$$\left.\begin{aligned} L_{\alpha\gamma}^{(0)*} G_\gamma^\beta(\mathbf{x};\ \mathbf{x}_0) &= \Delta_\alpha^\beta(\mathbf{x};\ \mathbf{x}_0) \\ B_{\alpha\gamma}^{(0)*} G_\gamma^\beta(\mathbf{x};\ \mathbf{x}_0) &= 0 \end{aligned}\right\}. \tag{72}$$

Now returning to the Green's theorem expressed by (62), where the integration is over the source coordinates $\mathbf{x}_0$, and using **G** for **v** and **u** for **w**, we have

$$(L^{(0)}\mathbf{u}, \mathbf{G})_\Omega^{\mathbf{x}_0} = (\mathbf{u}, L^{(0)*}\mathbf{G})_\Omega^{\mathbf{x}_0} + \int_\Omega J_{\beta,\beta}^\mu(\mathbf{u}, \mathbf{G})\, d^4x^0 \tag{73}$$

where J_β^μ is seen from (68) to be

$$J_\beta^\mu = G_\alpha^\mu(\mathbf{x};\ \mathbf{x}_0)\tau_{\alpha\beta}(\mathbf{x}_0) - u_\alpha(\mathbf{x}_0)\mathcal{G}_{\alpha\beta}^\mu(\mathbf{x};\ \mathbf{x}_0) \tag{74}$$

and where

$$\mathcal{G}_{\alpha\beta}^\mu(\mathbf{x};\ \mathbf{x}_0) \equiv C_{\alpha\beta\gamma\delta}(\mathbf{x}_0)\frac{\partial G_\gamma^\mu(\mathbf{x};\ \mathbf{x}_0)}{\partial x_\delta^0}. \tag{75}$$

From the equations for u_α in (65) we have

$$(L^{(0)}\mathbf{u}, \mathbf{G})_\Omega^{\mathbf{x}_0} = \int_\Omega \rho f_\alpha G_\alpha^\mu\, d^4x^0$$

and from (72) we have that

$$L^{(0)*}_{\alpha\gamma} G^{\mu}_{\gamma} = \Delta^{\mu}_{\alpha}$$

so

$$(\mathbf{u}, L^{(0)*}\mathbf{G})^{\mathbf{x}_0}_{\Omega} = \int_{\Omega} \Delta^{\mu}_{\alpha}(\mathbf{x};\ \mathbf{x}_0) u_{\alpha}(\mathbf{x}_0) d^4 x_0$$

$$= \begin{cases} 0, & \text{if } \mathbf{x} \notin \Omega \\ 4\pi u_{\mu}(\mathbf{x}), & \text{if } \mathbf{x} \in \Omega. \end{cases}$$

Therefore the Green's theorem in (73) yields

$$4\pi u_{\mu}(\mathbf{x}) = \int_{\Omega} \rho(\mathbf{x}_0) f_{\alpha}(\mathbf{x}_0) G^{\mu}_{\alpha}(\mathbf{x};\ \mathbf{x}_0) d^4 x^0$$

$$- \int_{\Omega} \frac{\partial}{\partial x^0_{\beta}} \{G^{\mu}_{\alpha}(\mathbf{x};\ \mathbf{x}_0) \tau_{\alpha\beta}(\mathbf{x}_0) - u_{\alpha}(\mathbf{x}_0) \mathcal{G}^{\mu}_{\alpha\beta}(\mathbf{x};\ \mathbf{x}_0)\}\, d^4 x^0. \tag{76}$$

Now formal application of the divergence theorem to the second integral on the right gives, with $\hat{N}$ the normal to the boundary of Ω:

$$\int_{\Omega} \frac{\partial}{\partial x^0_{\beta}} [G^{\mu}_{\alpha} \tau_{\alpha\beta} - u_{\alpha} \mathcal{G}^{\mu}_{\alpha\beta}]\, d^4 x^0 = \int_{\partial\Omega} [G^{\mu}_{\alpha} \tau_{\alpha\beta} - u_{\alpha} \mathcal{G}^{\mu}_{\alpha\beta}]\, N_{\beta} d^3 x^0. \tag{77}$$

The integral over the boundary $\partial\Omega$ however takes a special form because some of the fields, namely u_{α} and G^{μ}_{α}, are purely space-like, while both $\tau_{\alpha\beta}$ and $G^{\mu}_{\alpha\beta}$ have, by construction, time-like components. Hence, we can give more explicit form to the 'surface' integral in (77) by applying the divergence theorem to the expanded form of the integrand, where $\tau_{\alpha\beta}$ and $\mathcal{G}^{\mu}_{\alpha\beta}$ are written out in terms of the purely space-like fields used in their construction. From the definitions of $\tau_{\alpha\beta}$ and $\mathcal{G}^{\mu}_{\alpha\beta}$ we have

$$\int_{\Omega} \frac{\partial}{\partial x^0_{\beta}} [G^{\mu}_{\alpha} \tau_{\alpha\beta} - u_{\alpha} \mathcal{G}^{\mu}_{\alpha\beta}]\, d^4 x^0 = -\int_{\Omega} \frac{\partial}{\partial x^0_{\beta}} \left[G^m_k \left\{ C_{klij} \delta_{\beta l} \frac{\partial u_i}{\partial x^0_j} - \rho \delta_{ik} \delta_{\beta 4} \frac{\partial u_i}{\partial x^0_4} \right\} - u_k \left\{ C_{klij} \delta_{\beta l} \frac{\partial G^m_i}{\partial x^0_j} - \rho \delta_{ik} \delta_{\beta 4} \frac{\partial G^m_i}{\partial x^0_4} \right\} \right] d^4 x^0$$

where the Latin indices take only the values 1, 2, 3. Regrouping the terms and noting that the ordinary elastic stress is $-\tau_{kl}$ and similarly that $-\mathcal{G}^m_{kl}$ is the elastic stress associated with the displacement G^m_k, we have

$$\int_{\Omega} \frac{\partial}{\partial x^0_{\beta}} [G^{\mu}_{\alpha} \tau_{\alpha\beta} - u_{\alpha} \mathcal{G}^{\mu}_{\alpha\beta}]\, d^4 x^0 = \int_{\Omega} \frac{\partial}{\partial x^0_l} [G^m_k \tau_{kl} - u_k \mathcal{G}^m_{kl}]\, d^4 x^0$$

$$- \int_{\Omega} \frac{\partial}{\partial x^0_4} \left[u_k \left(\rho \frac{\partial G^m_k}{\partial x^0_4} \right) - G^m_k \left(\rho \frac{\partial u_k}{\partial x^0_4} \right) \right] d^4 x^0. \tag{78}$$

Now we observe that the first integral involves a purely spatial divergence of a space-like field, while the second integral involves the purely time-like part of the (four-dimensional) divergence of a spatial field. Thus we integrate over only spatial coordinates in the first and over the time coordinate in the second, in the latter case observing that Ω is an explicit function of time since the surface Σ is expanding. In particular, taking the initiation of the process resulting in a dynamic field to begin at $x_4^0 \equiv t_0 = 0$ and noting that G_k^m is causal, so that $G_k^m = 0$ when $x_4^0 > x_4$, then

$$\int_\Omega \frac{\partial}{\partial x_l^0}[G_k^m \tau_{kl} - u_k \mathcal{G}_{kl}^m]\, d^4x^0 = \int_0^{t^+} dt^0 \int_{\partial v_1} [G_k^m \tau_{kl} - u_k \mathcal{G}_{kl}^m]\, n_l da \tag{79}$$

by an ordinary application of the divergence theorem. Here ∂v_1 is the boundary of the spatial volume v_1 at time t, and n_l is the normal to this spatial boundary, as shown in Fig. 2. Also, t^+ denotes $t + \epsilon$, with $\epsilon > 0$ and infinitesimal. Here t^+ is used instead of t alone in order to avoid any (distributional) singularity of the Green's function at the integral limit. Further, we have replaced x_4^0 and x_4 by the equivalents t_0 and t.

The second integral in (78) demands special care in performing the integration over the time coordinate x_4^0 since v_1 is an explicit function of time, so that the integration over the spatial coordinates is to be performed before that over the time coordinate. Further, the resulting integrand for the time integration need not be continuous, as was noted earlier. The integral in question is

$$\int_\Omega \frac{\partial}{\partial x_4^0}\left[u_k\left(\rho\frac{\partial G_k^m}{\partial x_4^0}\right) - G_k^m\left(\rho\frac{\partial u_k}{\partial x_4^0}\right)\right] d^4x^0 = \int_0^{t^+} dt_0 \int_{v_1(t_0)} \frac{\partial}{\partial t_0} \times \left[u_k\left(\rho\frac{\partial G_k^m}{\partial t_0}\right) - G_k^m\left(\rho\frac{\partial u_k}{\partial t_0}\right)\right] dv^0$$

where we let $dv^0 \equiv dx_1^0 dx_2^0 dx_3^0$ denote the purely spatial volume element. Now we observe that, for a spatial integral of a function F of the medium deformation or flow, evaluated over a volume v_1 with v_1 a function of time t_0

$$\frac{d}{dt_0}\int_{v_1(t_0)} F dv^0 = \int_{v_1(t_0)} \frac{\partial F}{\partial t_0} dv^0 + \int_{\partial v_1(t_0)} F\mathbf{U}\cdot\hat{\mathbf{n}} da$$

which follows from the transport theorem of equation (3). Here the surface integral is due to movement of the boundary ∂v_1 with velocity $\mathbf{U}$. This form is valid no matter what the value of $\mathbf{U}$, that is whether the boundary of v_1 moves with the material or not. Hence it is immediately applicable to the integral being considered, where we may regard the integrand in brackets as corresponding to F, so that

$$\int_\Omega \frac{\partial}{\partial x_4^0}\left[u_k\left(\rho\frac{\partial G_k^m}{\partial x_4^0}\right) - G_k^m\left(\rho\frac{\partial u_k}{\partial x_4^0}\right)\right] d^4x^0$$
$$= \int_0^{t^+} d \int_{v_1(t_0)} \left[u_k\left(\rho\frac{\partial G_k^m}{\partial t_0}\right) - G_k^m\left(\rho\frac{\partial u_k}{\partial t_0}\right)\right] dv^0$$
$$- \int_0^{t^+} dt_0 \int_{\partial v_1(t_0)} \left[u_k\left(\rho\frac{\partial G_k^m}{\partial t_0}\right) - G_k^m\left(\rho\frac{\partial u_k}{\partial t_0}\right)\right] U_l n_l da^0 \tag{80}$$

where the first term on the right-hand side must be considered as a Stieljes integral (Archambeau 1972; Minster 1973).

In the theory of elastic continua and in linear continuum theory the second integral in (80) is neglected since the boundary movement is with the particle velocity and the integral term is small – a second-order effect due to transport which is negligible for solids. However, when the boundary is not simply a material boundary and may in fact move with a normal velocity $\mathbf{U}\cdot\hat{n}$ much larger than the particle velocity, then this term cannot be neglected since it may, in fact, be as large as the traction term at the boundary. We are dealing with precisely the case in which $\mathbf{U}$ may be large along that part of the boundary of v_1 corresponding to Σ and so the term will be retained for integration over the 'phase transition' boundary Σ. It can, however, be neglected on other boundaries corresponding to regular boundaries or external boundaries of the medium, since there $U_l n_l$ has the value of the particle velocity. For consistency with previous neglect of second order terms we are in fact forced to neglect material boundary position variations for which U_l in (80) is equal to the particle velocity. In expressing (80), however, we will retain ∂v_1 for the surface integral limit, but consider U_l to be zero on material boundaries, which in effect neglects this transport integral term for boundaries other than Σ. In addition, $\mathbf{U}$ is of course a spatial function so that its value may be negligible on parts of the surface Σ as well. Indeed, for spontaneous failure processes we would expect it to be of the order of the particle velocity over much of the failure surface except near the expanding 'edges' of the failure zone, which usually has a elongated shape. The magnitude of $\mathbf{U}$ on Σ is of course determined by the jump conditions in (32), so that $\mathbf{U}$ is not arbitrary but determined by the radiation field itself. The shape of the failure boundary is thus determined by the spatial variation of $\mathbf{U}$, so that some parts of Σ may move rapidly while other sections may simply move with the particles as a simple material boundary.

Now substituting (79) and (80) into (78) yields

$$\int_{\Omega}\frac{\partial}{\partial x^0_\beta}[G^\mu_\alpha\tau_{\alpha\beta}-u_\alpha\mathscr{G}^\mu_{\alpha\beta}]\,d^4x_0$$
$$=\int_0^{t^+}dt_0\int_{\partial v_1}\left[G^m_k\left(\tau_{kl}-\rho\frac{\partial u_k}{\partial t_0}U_l\right)-u_k\left(\mathscr{G}^m_{kl}-\rho\frac{\partial G^m_k}{\partial t_0}U_l\right)\right]n_l da^0$$
$$-\int_0^{t^+}d\left\{\int_{v_1(t_0)}\rho\left[u_k\frac{\partial G^m_k}{\partial t_0}-G^m_k\frac{\partial u_k}{\partial t_0}\right]dv^0\right\} \qquad (81)$$

From previous definitions of $\tau_{\alpha\beta}$ and $\mathscr{G}^\mu_{\alpha\beta}$ and from the definition of the elastic-inertial tensor we have

$$\tau_{\alpha\beta}\eta_\beta=\left[\tau_{kl}-\rho\frac{\partial u_k}{\partial t_0}U^*_l\right]n_l$$

$$\mathscr{G}^\mu_{\alpha\beta}\eta_\beta=\left[\mathscr{G}^m_{kl}-\rho\frac{\partial G^m_k}{\partial t_0}U^*_l\right]n_l.$$

However, as previously noted, we must neglect terms involving products of the particle velocities in comparison with traction terms or terms involving products of the particle velocity and $U_l n_l$. Therefore we can replace U^*_l appearing in these identities by U_l alone. Thus in this linearized theory we use

$$\left.\begin{aligned}\tau_{\alpha\beta}\eta_\beta&=\left[\tau_{kl}-\rho\frac{\partial u_k}{\partial t_0}U_l\right]n_l\\ \mathscr{G}^\mu_{\alpha\beta}\eta_\beta&=\left[\mathscr{G}^m_{kl}-\rho\frac{\partial G^m_k}{\partial t_0}U_l\right]n_l\end{aligned}\right\} \qquad (82)$$

and observe that these factors appear in (81). Further, we observe that, from (74)

$$J_4^{\mu} = \rho \left[u_k \frac{\partial G_k^m}{\partial t_0} - G_k^m \frac{\partial u_k}{\partial t_0} \right].$$

Using these identities in the integrals in (81), we get

$$\int_{\Omega} \frac{\partial}{\partial x_\beta^0} [G_\alpha^\mu \tau_{\alpha\beta} - u_\alpha G_{\alpha\beta}^\mu] \, d^4x_0 = \int_0^{t^+} dt_0 \int_{\partial v_1} J_\beta^\mu \eta_\beta da^0 - \int_0^{t^+} d \left[\int_{v_1} J_4^\mu dv^0 \right]. \tag{83}$$

Thus the basic representation theorem for $u_\gamma(\mathbf{X})$, expressed by equation (76), is

$$4\pi u_\mu(\mathbf{X}) = \int_{\Omega} \rho f_\alpha G_\alpha^\mu d^4x_0 - \int_0^{t^+} dt_0 \int_{\partial v_1} J_\beta^\mu \eta_\beta da^0 + \int_0^{t^+} d \left[\int_{v_1} J_4^\mu dv^0 \right]. \tag{84}$$

Here

$$J_\beta^\mu = G_\alpha^\mu \tau_{\alpha\beta} - u_\alpha \mathscr{G}_{\alpha\beta}^\mu$$

and

$$\eta_\beta = (n_1, n_2, n_3, -\mathbf{U} \cdot \hat{n}).$$

Written out explicitly in three-dimensional spatial components, this result takes the form

$$4\pi u_m(\mathbf{X}) = \int_0^{t^+} dt_0 \int_{v_1} \rho f_k G_k^m d^3x_0 + \int_0^{t^+} dt_0 \int_{\partial v_1} \left[G_k^m \left(C_{ijkl} \frac{\partial u_i}{\partial x_j^0} + \rho \frac{\partial u_k}{\partial t_0} U_l \right) - u_k \left(C_{ijkl} \frac{\partial G_i^m}{\partial x_j} + \rho \frac{\partial G_k^m}{\partial t_0} U_l \right) \right] n_l da^0 + \int_0^{t^+} d \int_{v_1} \rho \left[u_k \frac{\partial G_k^m}{\partial t_0} - G_k^m \frac{\partial u_k}{\partial t_0} \right] dv^0. \tag{85}$$

Here ρ and C_{ijkl} are functions of the spatial coordinates x_k^0 alone.

6 Approximate radiation field solutions for growing failure zones

The fields u_γ and G_γ^β satisfy the boundary conditions

$$\left. \begin{aligned} B_{\alpha\gamma} u_\gamma &= \bar{b}_\alpha; \quad \mathbf{x} \in \partial\Omega \\ B_{\alpha\gamma} G_\gamma^\beta &= 0; \quad \mathbf{x} \in \partial\Omega \end{aligned} \right\} \tag{86a}$$

or, equivalently

$$\left. \begin{aligned} \tau_{\alpha\beta}^{(1)} \eta_\beta &= \tau_{\alpha\beta}^{(2)} \eta_\beta; \quad \mathbf{x} \in \partial\Omega \\ \mathscr{G}_{\alpha\beta}^{(1)\mu} \eta_\beta &= 0; \quad \mathbf{x} \in \partial\Omega \end{aligned} \right\} \tag{86b}$$

where the superscripts (1) and (2) refer to the quantities evaluated by limits on the surface approached from the inside or from the outside of Ω, respectively. Thus the result in (84) reduces to

$$4\pi u_\mu(\mathbf{x}) = \int_{\Omega} \rho f_\alpha G_\alpha^\mu d^4x_0 - \int_0^{t^+} dt_0 \int_{\partial v_1} G_\alpha^\mu [\tau_{\alpha\beta}^{(2)} \eta_\beta] \, da^0 + \int_0^{t^+} d \left[\int_{v_1} J_4^\mu dv^0 \right] \tag{87}$$

with $\tau^{(2)}_{\alpha\beta}$ 'prescribed' on the boundary of v_1. The reduction of (84) to (87) is a consequence of the choice of the Green's function, which eliminates the integral involving the unknown displacement field u_α on the boundary of v_1, as well as from the fact that the 'generalized tractions' $\tau_{\alpha\beta}\eta_\beta$ appear explicitly in the boundary integral over ∂v_1 so that the 'jump condition' $[\![\tau_{\alpha\beta}\eta_\beta]\!] = 0$ can be used to introduce the action of the material external to v_1 at the boundary of v_1.

It is clear, however, that determination of the Green's function satisfying the boundary conditions in (86) is at best difficult and quite impractical for applications in view of the necessity of satisfying boundary conditions on the growing failure boundary as well as on the material boundaries of the medium. We can, instead, introduce a Green's function Γ^β_α defined on a space Ω' that includes the problem space Ω, i.e. $\Omega \subset \Omega'$. In particular we can take Ω' to be the space bounded by the same external boundaries as Ω but without the internal boundary created by the failure zone. Then Γ^β_γ is given by

$$\left.\begin{aligned} L_{\alpha\gamma}\Gamma^\beta_\gamma(\mathbf{x};\ \mathbf{x}_0) &= \Delta^\beta_\alpha(\mathbf{x};\ \mathbf{x}_0); \quad && \mathbf{x}\in\Omega' \\ B_{\alpha\gamma}\Gamma^\beta_\gamma(\mathbf{x};\ \mathbf{x}_0) &= 0; && \mathbf{x}\in\partial\Omega' \end{aligned}\right\} \tag{88}$$

where $L_{\alpha\beta}$ is defined throughout Ω' as if the whole medium were in its initial, unfailed state. The Green's function defined by (88) has the same properties as G^β_α and is also such that inner products over the subregion Ω of Ω' can be defined. Thus the Green's representation of (84) is reproduced, with G^μ_α replaced by Γ^μ_α. However the price paid for the presence of the simpler Green's function is that (84) cannot be reduced to the simple form of the solution given in (87). In particular, we get, from (84) and (88)

$$4\pi u_\mu(\mathbf{x}) = \int_\Omega \rho f_\alpha \Gamma^\mu_\alpha d^4x_0 - \int_0^{t^+} dt_0 \int_{\partial v_1} \Gamma^\mu_\alpha [\tau^{(2)}_{\alpha v}\eta_\beta]\, da^0$$
$$+ \int_0^{t^+} dt_0 \int_{\partial v_1 \ominus \partial v_1'} u_\alpha [\gamma^\mu_{\alpha\beta}\eta_\beta]\, da^0 + \int_0^{t^+} d\left[\int_{v_1} J^\mu_4\, dv^0\right] \tag{89}$$

where $\partial v_1 \ominus \partial v_1'$ is the difference between the spatial boundaries of Ω and Ω'. With Ω' defined as above, this is just the failure surface Σ. Here $\gamma^\mu_{\alpha\beta}$ is the 'generalized' Green's stress tensor' associated with Γ^μ_α, that is

$$\gamma^\mu_{\alpha\beta}(\mathbf{x};\ \mathbf{x}_0) = C_{\alpha\beta\gamma\delta}(\mathbf{x}_0)\,\frac{\partial\Gamma^\mu_\gamma}{\partial x^0_\delta}$$

and J^μ_4 is obtained from

$$J^\mu_\beta = \Gamma^\mu_\alpha \tau_{\alpha\beta} - u_\alpha \gamma^\mu_{\alpha\beta}.$$

Thus we get an extra integral in (89) involving u_α on the failure boundary Σ. Clearly this term arises from the fact that we have in effect relaxed the boundary conditions required for the Green's function.

The natural boundary conditions on Σ do not involve u_α directly, but instead specify conditions relating space and time derivatives of u_α across Σ, i.e. $[\![\tau_{\alpha\beta}\eta_\beta]\!]_\Sigma = 0$. Hence u_α on Σ is unknown and (95) is an integral equation for $u_\mu(\mathbf{x})$ rather than a solution.

Note that instead of the Green's function Γ^μ_γ, we can use the infinite space Green's function $\bar{\Gamma}^\mu_\gamma$. The latter affords a relatively simple closed form expression (e.g. Maruyama 1963). In that case, however, the boundary integral over $\partial v_1 \ominus \partial v_1'$ in the representation theorem (89) extends over all boundaries of v_1, including exterior boundaries as well as Σ. In addition, we must consider the process to occur in an isotropic and homogeneous space from the onset. This is quite an acceptable assumption, to first order at least, in the vicinity of the failure zone.

Successive approximations to the solution of (89) can be obtained by iteration. In this particular case we take the first iterate, $u_\mu^{\mathrm{I}}(\mathbf{x})$, to be the final term in (89), the 'relaxation', or 'initial value' term. Hence, using the infinite space Green's function we define

$$4\pi u_\mu^{\mathrm{I}}(\mathbf{x}) = \int_0^{t^+} d\left[\int_{v_1} \bar{J}_4^\mu\, dv^0\right] \tag{90}$$

A correction to this term, which yields the next iterate $u_\mu^{\mathrm{II}}(\mathbf{x})$ is defined by

$$4\pi\left[u_\mu^{\mathrm{II}}(\mathbf{x}) - u_\mu^{\mathrm{I}}(\mathbf{x})\right] = \int_0^{t^+} dt_0 \int_\Sigma u_\alpha^{\mathrm{I}}\left[\bar{\gamma}_{\alpha\beta}^\mu \eta_\beta\right] da^0 - \int_0^{t^+} dt_0 \int_\Sigma \bar{\Gamma}_{\alpha\beta}^\mu\left[\tau_{\alpha\beta}^{\mathrm{I}}\eta_\beta\right] da^0 \tag{91}$$

where the integration is only over the failure surface Σ. (Generally the volume integral over the body force density f_α is neglected as being small and is not included.) In the past we have referred to $\mathbf{u}^{\mathrm{I}}(\mathbf{x})$ as the 'transparent source approximation to the direct source field', since it neglects the scattering at the failure boundary. The field $\mathbf{u}^{\mathrm{II}}(\mathbf{x})$ will simply be referred to as the 'approximate direct source field'.

By adding a general eigenfunction expansion with adjustable coefficients, which is regular everywhere in the domain Ω (or Ω'); one can then satisfy boundary conditions on the other boundaries. For example, for a problem involving failure in the Earth, we could use a layered spherical model, and hence a vector spherical harmonic expansion for each layer zone. One could also express the Green's function Γ_α^μ in (89) as an eigenfunction expansion and use it in place of $\bar{\Gamma}_\alpha^\mu$ in (90) and (91) to obtain, directly, the approximate source fields in the domain Ω'.

It is clear that the relaxation or initial value term plays a dominant role in this procedure. In fact, we have pointed out that it represents the essentials of the elastic wave radiation attributable to the source. Therefore we will consider its expression in more explicit and simplified form for spontaneous processes such as failure under a static or quasi-static initial stress condition. Related investigations have also been considered by Archambeau (1968, 1972) and Minster (1973).

If we consider the field $u^{\mathrm{I}}(\mathbf{x})$ at any particular time t_1 we can represent its evolution at any later time t in terms of the initial value integral (e.g. Love 1944)

$$4\pi u_m^{\mathrm{I}}(\mathbf{x}, t; t_1) = \int_{v_1(t_1)} \rho\left[u_k \frac{\partial G_k^m}{\partial t_0} - G_k^m \frac{\partial u_k}{\partial t_0}\right]_{t_0=t_1} dv^0.$$

This expression is the radiation field due to all effects occurring up to time t_1, including prior rupture growth. It is therefore the radiation that would be observed for $t > t_1$ if the failure zone boundary were fixed for times $t > t_1$. If we now consider the same initial value representation of the field at $t_1 + \delta t_1$, where some additional change in the failure boundary has occurred in the time increment δt_1, then the difference in the radiation fields is given by

$$u_m^{\mathrm{I}}(\mathbf{x}, t; t_1 + \delta t_1) - u_m^{\mathrm{I}}(\mathbf{x}, t; t_1) = \frac{\delta t_1}{4\pi} \int_{v_1(t_1+\delta t_1)} \rho\left[\left(\frac{\partial u_k^*}{\partial t_0}\right)\frac{\partial G_k^m}{\partial t_0}\right]_{t_0=t_1} dv^0 + 0(\delta t_1^2)$$

where $(\partial u_k^*/\partial t_0)|_{t_0=t_1}\,\delta t_1$ is the incremental initial value, equal to the change in the equilibrium field caused by incremental growth of the failure zone, and where we have used the fact that accelerations remain finite so that the velocity is continuous.

Dividing through by δt_1, and taking the limit as δt_1 approaches zero gives

$$\frac{\partial u_m^{\mathrm{I}}(\mathbf{x}, t; t_1)}{\partial t_1} = \frac{1}{4\pi}\int_{v_1(t_1)} \left[\left(\frac{\partial u_k^*}{\partial t_0}\right)\frac{\partial G_k^m}{\partial t_0}\right]_{t_0=t_1} dv^0. \tag{92}$$

For failure occurring at a finite rate and of total duration τ, then the derivative with respect to t_1 vanishes for $t_1 > \tau$. Thus integration of (92) yields

$$u_m^{\mathrm{I}}(\mathbf{x}, t) = c(\mathbf{x}, t) + \frac{1}{4\pi}\int_0^{\min(t,\tau)} dt_1 \int_{v_1(t_1)} \left[\left(\frac{\partial u_k^*}{\partial t_0}\right)\frac{\partial G_k^m}{\partial t_0}\right]_{t_0=t_1} dv^0 \tag{93}$$

where the constant of integration is fixed by the choice of reference equilibrium state. If we chose to measure u_m relative to the final equlibrium state after the complete boundary has been formed, then

$$c(\mathbf{x}, t) = u_m^*(\mathbf{x}, t) - u_m^*(\mathbf{x}, \tau)$$

so that

$$u_m(\mathbf{x}, 0) = u_m^*(\mathbf{x}, 0) - u_m^*(\mathbf{x}, \tau).$$

If we refer the displacements to the initial equilibrium state then $c(\mathbf{x}, t) = 0$. In either case $c(\mathbf{x}, t)$ vanishes for $t > \tau$. This result is formally exact; if Γ_α^μ or $\bar{\Gamma}_\alpha^\mu$ is used, however, source boundary scattering is neglected. On the other hand, the equilibrium field changes expressed by $u_l^*(\mathbf{x}_0, t_0)$ account for both the spatial and temporal behaviour of the failure boundary, so that the only term neglected, as expressed approximately in (91) represents the dynamic interaction of the radiation field and the boundary, that is scattering. We expect this scattering to be significant only for wavelengths smaller than the characteristic source dimension. This is confirmed by the results of Burridge (1975), Koyama, Hariuchi & Hirasawa (1973) and Minster & Suteau (1976). These authors considered expanding spherical failure zones in an uniformly prestress infinite space. Scattering by Σ is shown to affect the spectrum only for frequencies greater than the source characteristic frequency $f_c = \bar{U}_R/L$, where L is the maximum source dimension and $\bar{U}_R$ the average failure rate, and the effects are not large. In the time domain, the 'transparency' approximation leads to a minor shortening of the far-field pulse and elimination of its small amplitude oscillatory coda.

7 Conclusions and consequences

Some of the conclusions that we have drawn from the results of the work described are:

(1) Conservation relations expressed on an expanding material transition surface (or failure zone boundary) show that traction and particle velocity jumps across such a boundary cannot be specified independently, as has been (implicitly) assumed, for example, in dislocation and stress pulse (crack theory) models for failure. In particular, it has been shown that the proportionality factor between particle velocity changes and traction changes across the failure boundary is ρU_R, with U_R the rupture rate and ρ the initial material density.

(2) Introduction of a transition energy function provides a means of characterizing failure processes in terms of energy requirements. This function can be determined from seismically estimated rupture variables – in particular from the stress drop (on the failure boundary) and the rupture velocity. Hence the transition energy (density) function can be obtained for natural failure processes and compared to laboratory determined values which should lead to a more complete quantitative description and subsequent understanding of failure processes in the Earth (see also Husseini *et al.* 1975).

(3) Since rupture rates have been shown to be directly proportional to stress drops on the failure boundary and also proportional to the change in the material moduli, it follows that observations of high rupture rates for earthquakes, near the shear velocity of the medium, imply near total loss of shear strength and probable melting during failure (see also Archambeau 1977).

(4) The Green's function solution for the radiation field exterior to the failure zone shows that the field depends explicitly on the failure boundary growth rate, whereas previous radiation field representations did not contain any such explicit dependence since dynamical boundary conditions were ignored in favour of heuristic kinematical representations of boundary growth effects. This suggests that source representations such as those generated by displacement dislocations and stress pulse equivalents, that are formulated entirely in terms of equivalent *applied* boundary displacements or equivalent *applied* boundary tractions, must be carefully reconsidered since, at present, they do not account for the boundary integral terms involving the surface growth rate (i.e. the surface integral terms with integrands involving U_l explicitly, as given for example in equation (85)).

(5) The Green's function solution shows that an appropriate first-order representation of the radiation field is obtained using the 'initial value' or 'relaxation' term in the Green's integral representation. The radiation field predictions based on this integral representation can be directly related to the physical parameters for the failure process (i.e. the transition energy and the material rheology moduli before and after failure as well as the rupture rate and stress level changes) and can be used for the representation of the radiation from complex failure phenomena in inhomogeneous media that is also inhomogeneously prestressed. Since strong prestress inhomogeneities are considered to be present for most if not all failure processes (especially for earthquakes) and to have a very important effect on the character of the radiated elastic wave field, then it follows that the ability to theoretically predict the field from such complex sources is critical to the advancement of understanding of the mechanics of failure processes, particularly in the Earth.

Acknowledgments

This research has been supported under Advanced Research Projects Agency, Department of Defense, Contracts F 44620-72-C-0078 and F 44620-72-C-0083 monitored by the Air Force Office of Scientific Research and the Air Force Cambridge Research Laboratory monitored contract No. F 19628-76-C-0061. A part of this work was also supported by the National Science Foundation under grants GI-44212 and DES 75-20137.

References

Archambeau, C. B., 1964. Elastodynamic source theory, *PhD thesis*, California Institute of Technology, Pasadena, California.

Archambeau, C. B., 1968. General theory of elastodynamic source fields, *Rev. Geophys.*, **16**, 241–288.

Archambeau, C. B., 1969. Elastodynamic representation theorems in prestress elastic media with moving boundaries, *Abstract, SSA Meeting,* St Louis, Missouri.

Archambeau, C. B., 1972. The theory of stress wave radiation from explosions in prestressed media, *Geophys. J. R. astr. Soc.,* **29,** 329–366. (Appendix, 1973. *Geophys. J. R. astr. Soc.,* **31,** 361–363).

Archambeau, C. B., 1977. Shear stress in the Earth's lithosphere: estimates based on the analysis of seismic radiation, *J. geophys. Res.,*

Burridge, R., 1975. The pulse shape and spectra of elastic waves generated when a cavity expands in an initial shear field, *J. geophys. Res.,* **80,** 2606–2607.

Burridge, R., 1976. Reply to Commentary by J. A. Snoke, *J. geophys. Res.,* **81,** 1037.

DeGroot, S. R. & Mazur, P., 1969. *Non-equilibrium thermodynamics,* North-Holland.

Edelen, G. B., 1962. *The structure of field space,* University of California Press.

Eringen, A. C., 1971, 1975. *Continuum physics,* vols I and II, Academic Press, New York.

Fletcher, D. C., 1974. The conservation laws of three dimensional linearized elasticity theory, *PhD thesis,* California Institute of Technology, Pasadena, California.

Freund, L. B., 1970. Conditions at a Propagation Interface of Two Media, *Transactions ASME, Applied Mechanics Division,* pp. 190–192.

Fung, Y. C., 1965. *Foundations of solid mechanics,* Prentice-Hall Series in Dynamics.

Husseini, M. I., Jovanavich, D. B., Randall, M. J. & Freund, L. B., 1975. The Fracture Energy of Earthquakes, *Geophys. J. R. astr. Soc.,* **43.**

Koyama, J., Hariuchi, S. E. & Hirasawa, T.; 1973. Elastic wave generated from sudden vanishing of rigidity in a spherical region, *J. Phys. Earth,* **21,** 213–226.

Kupradze, V. D., 1963. Dynamical problems in the theory of elasticity, in *Progress in solid mechanics,* vol. III, Eds Sneddon & Hill, North-Holland, Amsterdam.

Landau, L. D. & Lifshitz, E. M., 1965. *Theory of elasticity, course of theoretical physics, vol. 7,* Addison-Wesley Publishing Co.

Love, A. E. H., 1944. *A treatise on the mathematical theory of elasticity,* Dover.

Maruyama, T., 1963. On the force equivalents of dynamic elastic dislocations with reference to the earthquake mechanism, *Bull. Earthq. Res. Inst. Tokyo,* **41,** 467.

Mason, W. P., 1966. *Physical acoustics,* vol. III, Part A, *The effect of imperfections,* Academic Press.

Minster, J. B., 1973. Elastodynamics of failure in a continuum, *PhD thesis,* California Institute of Technology, Pasadena, California.

Minster, J. B. & Suteau, A. M., 1976. Far-field waveforms from an arbitrarily expanding, transparent spherical cavity in a prestressed medium, *Geophys. J. R. astr. Soc.,* **50,** 215–233.

Morse, P. M. & Feshbach, H., 1953. *Methods of theoretical physics,* Parts I and II, McGraw-Hill.

Snoke, J. A., 1976. Implications of moving boundaries in elastodynamics: comments on 'The pulse shapes and spectra of elastic waves generated when a cavity expands in an initial shear field' by Robert Burridge, *J. geophys. Res.,* **81,** 1035–1036.

Stakgold, I., 1968. *Boundary value problems of mathematical physics,* vols I and II, Macmillan Publishing Co.