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Dynamics of a delayed SEIQ epidemic model

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Abstract

In this work we consider an epidemic model that contains four species susceptible, exposed, infected and quarantined. With this model, first we find a feasible region which is invariant and where the solutions of our model are positive. Then the persistence of the model and sufficient conditions associated with extinction of infection population are discussed. To show that the system is locally asymptotically stable, a Lyapunov functional is constructed. After that, taking the delay as the key parameter, the conditions for local stability and Hopf bifurcation are derived. Further, we estimate the properties for the direction of the Hopf bifurcation and stability of the periodic solutions. Finally, some numerical simulations are presented to support our analytical results.

Keywords: SEIQ model; Delay; Boundedness; Lyapunov functional; Persistence; Hopf bifurcation; Periodic solution

1 Introduction

Since the principles to the mathematical model of epidemics, as the susceptible–infected–susceptible (SIS) model and the susceptible–infected–removed (SIR) model, were presented in [1, 2], the mathematical investigation of disease transmission has developed quickly. As is well known, many epidemic diseases such as HIV/AIDS [3], H1N1 [4], H5N1 [5] and SARS [6], are harmful to individual health and to the stability of our society. It is an increasingly urgent issue to control the prevalence of epidemic diseases. Mathematical epidemiology, which describes the prevalence of epidemic diseases by building and analyzing mathematical models, has been one of the major areas of biology. In recent years, mathematical models have become one of the important tools in the investigation of the prevalence and control of epidemic diseases since the pioneering work of Kermack and Mckendrick [7, 8]. For example, the SIRS (Susceptible–Infectious–Recovered–Susceptible) epidemic model [9–12], the SEIS (Susceptible–Exposed–Infectious–Susceptible) epidemic model [13–16], SEIR (Susceptible–Exposed–Infectious–Recovered) epidemic model [17–20], the SEIRS (Susceptible–Exposed–Infectious–Recovered–Susceptible) epidemic model [21–25] and epidemic models with vaccination [26–28].

In real world, some people can be quarantined once they are found to have been infected with epidemic diseases in the exposed state or the infectious state. Based on this consider-

Table 1 Parameters and their meanings in this paper

Parameter	Description
A	the constant recruitment rate of the population
β	The infection rate of the susceptible population
μ	The natural mortality rate of all populations
c	The rate that the infected recovers and comes into the susceptible class
ε	The rate at which some exposed people become infective
α	The mortality rate of the infected and quarantined population due to disease
σ_1	The quarantined rate of the exposed population
σ_2	The quarantined rate of the infected population
γ_1	The recovery rate of the exposed population
γ_2	The recovery rate of the infected population
γ_3	The recovery rate of the quarantined population

ation, recently, Chen et al. [29] proposed the following epidemic model with quarantine:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + cI(t), \\ \frac{dE(t)}{dt} = \beta S(t)I(t) - (\mu + \varepsilon + \sigma_1 + \gamma_1)E(t), \\ \frac{dI(t)}{dt} = \varepsilon E(t) - (\mu + \alpha + c + \sigma_2 + \gamma_2)I(t), \\ \frac{dQ(t)}{dt} = \sigma_1 E(t) + \sigma_2 I(t) - (\mu + \alpha + \gamma_3)Q(t), \end{cases} \tag{1}$$

where $S(t)$, $E(t)$, $I(t)$ and $Q(t)$ are the numbers of the susceptible, the exposed, the infected and the quarantined individuals at time t, respectively. The meanings of the parameters are listed in Table 1.

With this model they [29] investigated the local and the global stability of system (1), and they also estimated the domain of attraction of system (1).

As stated in [30], most infectious diseases evolve by infection, and then there appear some symptoms needing a period of time (namely the incubation period). Therefore, if an epidemic model considers time delay, then it is more consistent with the actual situation [11, 12, 14, 22, 23]. Compared with ordinary differential equations, delay differential equations exhibit more complicated dynamics, such as the loss of stability, oscillations and periodic solutions. Recently, there appeared some work about epidemic models with time delay. In [31], Bai and Wu studied the traveling waves of a delayed SIR epidemic model with nonlinear incidence. In [32], Liu et al. investigated the asymptotic properties of a stochastic delayed SIR epidemic model with temporary immunity. Liu et al. [33, 34] analyzed the Hopf bifurcation of different SIRS epidemic model with time delay. Liu et al. [35] studied the global attractiveness and persistence of a delayed SIRS epidemic model on a scale-free network. Liu and Wang [36] investigated the Hopf bifurcation of an SIRS epidemic model with delays and stage structure. Sharma et al. [37, 38] analyzed the impact of time delay on the dynamics of different SEIR epidemic model. Jiang et al. [39] considered an SEIRS system with two delays and the general nonlinear incidence rate and dealt with the global Hopf bifurcation and permanence of the model. Liu and Wang [40] studied the Hopf bifurcation of an SIQR epidemic model with two delays and a nonlinear incidence rate. Obviously, most of the epidemic models with time delay above neglect the effect of quarantine. Based on this consideration and motivated by the work above, we incorporate the latent delay into system (1) and study the following delayed system:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + cI(t), \\ \frac{dE(t)}{dt} = \beta S(t)I(t) - (\mu + \sigma_1 + \gamma_1)E(t) - \varepsilon E(t-\tau), \\ \frac{dI(t)}{dt} = \varepsilon E(t-\tau) - (\mu + \alpha + c + \sigma_2 + \gamma_2)I(t), \\ \frac{dQ(t)}{dt} = \sigma_1 E(t) + \sigma_2 I(t) - (\mu + \alpha + \gamma_3)Q(t), \end{cases} \tag{2}$$

subject to the initial conditions

$$\begin{aligned} &S(\theta) = \phi_1(\theta) > 0, \\ &E(\theta) = \phi_2(\theta) > 0, \\ &I(\theta) = \phi_3(\theta) > 0, \\ &Q(\theta) = \phi_4(\theta) > 0, \quad \theta \in [-\tau, 0), \phi_i(0) > 0, i = 1, 2, 3, 4, \end{aligned} \tag{3}$$

where the meanings of the parameters are given in Table 1 and they are assumed to be positive and τ is the latent delay of the disease.

The organization of the paper is as follows. In the next section, it is shown that the solution of (2) is positive and bounded in a feasible region $\bar{R}$, which is invariant. Also, the persistence of the proposed model and some sufficient conditions associated with extinction of infective population are discussed. In Sect. 3, the condition for local asymptotical stability is examined by constructing a suitable Lyapunov functional. By taking the latent delay τ as the bifurcation parameter, the conditions for the occurrence of Hopf bifurcation are derived in Sect. 4. Further, the direction of Hopf bifurcation and the stability of the periodic solution are examined in Sect. 5. Some numerical results are carried out for our expository results in Sect. 6. Finally, the paper ends with the conclusion of the work.

2 The boundedness, persistence and extinction of infected population

2.1 The boundedness

In this section we shall discuss about the positivity and boundedness of solution of system (2).

For this purpose, we assume the function V to be

$$V(t) = S(t) + E(t) + I(t) + Q(t). \tag{4}$$

Taking the derivative of (4) and using (2) we get

$$\dot{V}(t) = A - \mu S(t) - (\mu + \gamma_1)E(t) - (\mu + \alpha + \gamma_2)I(t) - (\mu + \alpha + \gamma_3)Q(t), \tag{5}$$

where $S(t) > 0$ and $E(t), I(t), Q(t) \geq 0$.

If $E(t) = 0$, $I(t) = 0$ and $Q(t) = 0$ from (5) we get

$$\lim_{t\to\infty} \sup V(t) \leq \frac{A}{\mu}. \tag{6}$$

Also, if $V(t) > \frac{A}{\mu}$ then $\dot{V}(t) < 0$. Therefore, we get $0 < V \leq \frac{A}{\mu}$, i.e., we get a feasible region $\bar{R}$:

$$\bar{R} = \left\{ \big(S(t), E(t), I(t), Q(t)\big) \in R^4 : 0 < S(t) + E(t) + I(t) + Q(t) \leq \frac{A}{\mu} \right\}.$$

Thus we see that the solution of system (2) is bounded and independent of the initial condition. So the feasible region $\bar{R}$ is an invariant set. Also, as $A > 0$, $\mu > 0$, $\frac{A}{\mu} > 0$, i.e., the feasible region $\bar{R}$ is positive.

Hence all solutions of (2) will enter the field $\bar{R}$ and will remain in $\bar{R}$.

2.2 The persistence and extinction of infection species

In this section, we will consider the ultimate state of infection, that is, the disease will be either persistent or extinct ultimately.

Since the variable Q does not appear explicitly in the first three equations in system (2), we need only to consider the dynamics of a subsystem consisting of the first three equations in system (2). We have

$$\begin{aligned} \frac{dS(t)}{dt} &= A - \beta S(t)I(t) - \mu S(t) + cI(t), \\ \frac{dE(t)}{dt} &= \beta S(t)I(t) - (\mu + \sigma_1 + \gamma_1)E(t) - \varepsilon E(t-\tau), \\ \frac{dI(t)}{dt} &= \varepsilon E(t-\tau) - (\mu + \alpha + c + \sigma_2 + \gamma_2)I(t), \end{aligned} \tag{7}$$

From the first equation in system (7), we have $\frac{ds}{dt} \leq \mu(\frac{A}{\mu} - S)$; it implies that $\lim_{t\to\infty} \sup S(t) \leq \frac{A}{\mu}$; therefore, the set $\omega = \{(S, E, I) \in R^3_+ : S \leq \frac{A}{\mu}\}$ is positively invariant under system (7). Thus, we only consider the dynamical behavior of system (7) on the set ω.

When $R_0 < 1$, define the function

$$V_1 = \rho E + I, \tag{8}$$

where $\rho \in (\frac{1}{\varepsilon}, \frac{\mu+\alpha+c+\sigma_2+\gamma_2}{\beta(A/\mu)})$, then the derivative of V_1 with respect to t along the solution of (7) on the set ω is given by

$$\frac{dV_1}{dt} = -\rho(\mu + \sigma_1 + \gamma_1)E(t) + (1 - \rho\varepsilon)E(t-\tau) + \left\{ \rho\beta\frac{A}{\mu} - (\mu + \alpha + c + \sigma_2 + \gamma_2) \right\} I(t). \tag{9}$$

Thus,

$$\frac{dV_1}{dt} \leq (1 - \rho\varepsilon)E(t-\tau) + \left\{ \rho\beta\frac{A}{\mu} - (\mu + \alpha + c + \sigma_2 + \gamma_2) \right\} I(t). \tag{10}$$

As $\rho \in (\frac{1}{\varepsilon}, \frac{\mu+\alpha+c+\sigma_2+\gamma_2}{\beta(A/\mu)})$, we have $1 - \rho\varepsilon < 0$, and $\rho\beta\frac{A}{\mu} - (\mu + \alpha + c + \sigma_2 + \gamma_2) < 0$. Therefore we consider a positive number σ, such that, $1 - \rho\varepsilon < \sigma\rho(R_0 - 1)$ and $1 - \rho\varepsilon < 0$, and $\rho\beta\frac{A}{\mu} - (\mu + \alpha + c + \sigma_2 + \gamma_2) < \sigma(R_0 - 1)$. Thus from (10) we can write

$$V_1(t) \leq V_1(0)\exp\big[\sigma(R_0 - 1)t\big], \tag{11}$$

where $V_1(0) = \rho E(0) + I(0)$, therefore for $R_0 < 1$ we have

$$\lim_{t\to\infty} V_1(t) = 0, \quad \text{i.e.,} \quad \lim_{t\to\infty} E(t) = 0 = \lim_{t\to\infty} I(t).$$

This shows that the disease will extinct if $R_0 < 1$.

In order to discuss the persistence of the disease, we first introduce some definitions; then we follow the steps in [41].

Assume that X is a locally compact metric space with metric d, and let F be a closed subset of X with boundary δF and interior $\operatorname{int} F$. Let π be a semidynamical system defined on F. We say that π is persistent if, for all $u \in \operatorname{int} F$, $\lim_{t\to+\infty} \inf d(\pi(u,t), \delta F) > 0$, and we say that π is uniformly persistent if there is $\xi > 0$ such that, for all $u \in \operatorname{int} F$, $\lim_{t\to+\infty} \inf d(\pi(u,t), \delta F) > \xi$.

In [41], Fonda gives a result about persistence in terms of repellers. A subset Σ of F is said to be a uniform repeller if there is an $\eta > 0$ such that, for each $u \in F \setminus \Sigma$, $\lim_{t\to+\infty} \inf d(\pi(u,t), \Sigma) > \eta$. A semiflow on a closed subset F of a locally compact metric space is uniformly persistent if the boundary of F is repelling in a suitable strong sense.

Lemma 1 *Let Σ be a compact subset of X such that $X \setminus \Sigma$ is positively invariant. A necessary and sufficient condition for Σ to be a uniform repeller is that there exist a neighborhood* $\mathbf{U}$ *of Σ and a continuous function $P : X \to R_+$ satisfying*

(1) *$P(u) = 0$ if and only if $u \in \Sigma$,*

(2) *for all $u \in \mathbf{U} \setminus \Sigma$ there is a $T_u > 0$ such that $P(\pi(u, T_u)) > P(u)$.*

For any $u_0 = (S_0, E_0, I_0) \in \omega$, there is a unique solution $\pi(u_0, t) = (S, E, I)(t; u_0)$ of system (7), which is defined in R_+ and satisfies $\pi(u_0, 0) = (S_0, E_0, I_0)$. Since ω is a positively invariant set of system (7), then $\pi(u_0, t) \in \omega$ for $t \in R_+$ and it is a semidynamical system in ω. Here, we will prove that, when $R_0 > 1$, $\Sigma = \{(S, E, I) \in \Sigma : I = 0\}$ is a uniform repeller, which implies that the semidynamical system π is uniformly persistent. Obviously, $I(t) > 0$ for $t > 0$ if $I(0) > 0$, then $\omega \setminus \Sigma$ is invariant to (7). Again the set Σ is a compact subset of ω.

Let $P : \omega \to R_+$ be defined by $P(S, E, I) = I$, and let $\mathbf{U} = \{(S, E, I) \in \omega : P(S, E, I) < \zeta_1\}$, where $\zeta_1 > 0$ is small enough so that

$$\begin{aligned} &\left\{A - \frac{\mu(\mu + \sigma_1 + \gamma_1 + \varepsilon)(\mu + \alpha + c + \sigma_2 + \gamma_2)}{\beta\varepsilon}\right\} \\ &\quad - \left\{\frac{(\mu + \sigma_1 + \gamma_1 + \varepsilon)(\mu + \alpha + c + \sigma_2 + \gamma_2)}{\varepsilon} - c\right\}\zeta_1 > 0. \end{aligned} \tag{12}$$

Since $R_0 > 1$, the positive number ζ_1 is sufficiently small to satisfy the inequality (12).

Assume that there is $\bar{u} \in \mathbf{U}$ ($\bar{u} = (\bar{S}, \bar{E}, \bar{I})$) such that for each $t > 0$ we have $P(\pi(\bar{u}, t)) < P(\bar{u}) < \zeta_1$, which implies that $I(t, \bar{u}) < \zeta_1$ for $t > 0$. From the first equation of (7) we have

$$\frac{dS}{dt} \ge A + c\zeta_1 - (\mu + \beta\zeta_1)S, \tag{13}$$

then

$$\lim_{t\to\infty} \inf S(t, \bar{u}) \ge \frac{A + c\zeta_1}{\mu + \beta\zeta_1}. \tag{14}$$

For sufficiently large number of $T > 0$ we have $S(t,\bar{u}) > \frac{A+c\zeta_1}{\mu+\beta\zeta_1}$ for $t \geq T$.

Now we define another function $V_2(t) = (1-\zeta_2)E(t) + \frac{\mu+\sigma_1+\gamma_1+\varepsilon}{\beta}I(t)$, where ζ_2 $(0 < \zeta_2 < 1)$ is a sufficiently small constant so that

$$\begin{aligned}&\left\{A - \frac{\mu(1-\zeta_2)(\mu+\sigma_1+\gamma_1+\varepsilon)(\mu+\alpha+c+\sigma_2+\gamma_2)}{\beta\varepsilon}\right\}\\ &\quad > \left\{\frac{(\mu+\sigma_1+\gamma_1+\varepsilon)(\mu+\alpha+c+\sigma_2+\gamma_2)}{\varepsilon} - c\right\}\zeta_1.\end{aligned} \tag{15}$$

Now we differentiate $V_2(t)$ along with $\pi(\bar{u},t)$ as follows:

$$\begin{aligned}\frac{dV_2}{dt} &\geq \frac{\zeta_2(\mu+\sigma_1+\gamma_1)}{\beta}E(t) + \left\{A - \frac{\mu(1-\zeta_2)(\mu+\sigma_1+\gamma_1+\varepsilon)(\mu+\alpha+c+\sigma_2+\gamma_2)}{\beta\varepsilon}\right.\\ &\quad \left. - \left(\frac{(\mu+\sigma_1+\gamma_1+\varepsilon)(\mu+\alpha+c+\sigma_2+\gamma_2)}{\varepsilon} - c\right)\zeta_1\right\}I(t), \end{aligned} \tag{16}$$

$$\frac{dV_2}{dt} > \kappa V_2, \tag{17}$$

where

$$\begin{aligned}\kappa = \min&\left\{\frac{\zeta_2(\mu+\sigma_1+\gamma_1)}{\beta(1-\zeta_2)},\right.\\ &\frac{\beta}{\mu+\sigma_1+\gamma_1}\left[A - \frac{\mu(1-\zeta_2)(\mu+\sigma_1+\gamma_1+\varepsilon)(\mu+\alpha+c+\sigma_2+\gamma_2)}{\beta\varepsilon}\right.\\ &\left.\left. - \left(\frac{(\mu+\sigma_1+\gamma_1+\varepsilon)(\mu+\alpha+c+\sigma_2+\gamma_2)}{\varepsilon} - c\right)\zeta_1\right]\right\} > 0.\end{aligned}$$

Thus we see that

$$\lim_{t\to\infty} V_2(t) = +\infty. \tag{18}$$

Therefore, this proof shows that, for each $u \in \omega \setminus \Sigma$ with u belonging to a suitable small neighborhood of Σ, there is some T_u such that $P(\pi(u,T_u)) > P(u)$. Therefore, it follows from Lemma 1 that $\Sigma = \{(S,E,I) \in \Sigma : I = 0\}$ is a uniform repeller when $R_0 > 1$, i.e., the infection is uniformly persistent. So we conclude that system (7) will be persistent for $R_0 > 1$ and infection will be extinct when $R_0 < 1$.

3 Stability analysis

Based on the analysis in [29], we know that if $R_0 > 1$, then system (2) has a unique endemic equilibrium $P^*(S^*,E^*,I^*,Q^*)$, where

$$\begin{aligned}S^* &= \frac{A}{\mu R_0}, I^* = \frac{\varepsilon E^*}{\mu+\alpha+c+\sigma_2+\gamma_2},\\ E^* &= \frac{A(R_0-1)(\mu+\alpha+c+\sigma_2+\gamma_2)}{R_0[(\mu+\sigma_1+\gamma_1)(\mu+\alpha+c+\sigma_2+\gamma_2)+\varepsilon(\mu+\alpha+\sigma_2+\gamma_2)]},\\ Q^* &= \frac{(\mu+\alpha+c+\sigma_2+\gamma_2)\sigma_1+\varepsilon\sigma_2}{(\mu+\alpha+\gamma_3)(\mu+\alpha+c+\sigma_2+\gamma_2)}E^*,\end{aligned}$$

$$R_0 = \frac{A\beta\varepsilon}{\mu(\mu+\varepsilon+\sigma_1+\gamma_1)(\mu+\alpha+c+\sigma_2+\gamma_2)}.$$

In this section the linear stability of system (2) is discussed by constructing a suitable Lyapunov functional given in (20). For this purpose, let $u_1(t) = S(t) - S^*$, $u_2(t) = E(t) - E^*$, $u_3(t) = I(t) - I^*$, $u_4(t) = Q(t) - Q^*$ then system (2) transforms into

$$\begin{cases} \frac{du_1(t)}{dt} = (-\beta I^* - \mu)u_1 + (c - \beta S^*)u_3, \\ \frac{dB_1(t)}{dt} = \beta I^* u_1 - (\mu+\sigma_1+\gamma_1+\varepsilon)u_2 + \beta S^* u_3, \\ \frac{dB_2(t)}{dt} = \varepsilon u_2 - (\mu+\alpha+c+\sigma_2+\gamma_2)u_3, \\ \frac{du_4}{dt} = \sigma_1 u_2 + \sigma_2 u_3 - (\mu+\alpha+\gamma_3)u_4, \end{cases} \tag{19}$$

where $B_1(t) = u_2 - \varepsilon\int_{t-\tau}^{t} u_2(s)\,ds$ and $B_2(t) = u_3 + \varepsilon\int_{t-\tau}^{t} u_2(s)\,ds$.

Now following the steps in [42, 43], we shall check the stability of the system by assuming a suitable Lyapunov function $w(v)(t)$ as follows:

$$\begin{aligned} w(u)(t) &= k_1 w_1(u)(t) + k_2 w_2(u)(t) + k_3 w_3(u)(t) + k_4 w_4(u)(t) + k_5 w_5(u)(t) \\ &\quad + k_6 w_6(u)(t) + k_7 w_7(u)(t) + k_8 w_8(u)(t) + k_9 w_9(u)(t) + k_{10} w_{10}(u)(t), \end{aligned} \tag{20}$$

where $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}$ are given in the Appendix and

$$\begin{aligned} w_1(u)(t) &= u_1^2(t), \\ w_2(u)(t) &= B_1^2(t) + \varepsilon\bigl(\mu+\sigma_1+\gamma_1+\varepsilon-\beta I^*-\beta S^*\bigr)\int_{t-\tau}^{t}\int_{s}^{t} u_2^2(l)\,dl\,ds, \\ w_3(u)(t) &= B_2^2(t) + \bigl\{\varepsilon^2 - \varepsilon(\mu+\alpha+c+\sigma_2+\gamma_2)\bigr\}\int_{t-\tau}^{t}\int_{s}^{t} u_2^2(l)\,dl\,ds, \\ w_4(u)(t) &= u_4^2(t), \\ w_5(u)(t) &= u_1(t)B_1(t) + \frac{\beta I^*\varepsilon + \mu\varepsilon - c\varepsilon + \beta S^*\varepsilon}{2}\int_{t-\tau}^{t}\int_{s}^{t} u_2^2(l)\,dl\,ds, \\ w_6(u)(t) &= u_1(t)B_2(t) + \frac{c\varepsilon - \beta S^*\varepsilon - \beta I^*\varepsilon - \mu\varepsilon}{2}\int_{t-\tau}^{t}\int_{s}^{t} u_2^2(l)\,dl\,ds, \\ w_7(u)(t) &= u_1(t)u_4(t), \\ w_8(u)(t) &= B_1(t)B_2(t) + \frac{\beta I^*\varepsilon - \sigma_1\varepsilon - \gamma_1\varepsilon - 2\varepsilon^2 + \beta S^*\varepsilon + \alpha\varepsilon + c\varepsilon + \sigma_2\varepsilon + \gamma_2\varepsilon}{2} \\ &\quad \times \int_{t-\tau}^{t}\int_{s}^{t} u_2^2(l)\,dl\,ds, \\ w_9(u)(t) &= B_1(t)u_4(t) + \frac{\mu\varepsilon + \alpha\varepsilon + \gamma_3\varepsilon - \sigma_1\varepsilon - \sigma_2\varepsilon}{2}\int_{t-\tau}^{t}\int_{s}^{t} u_2^2(l)\,dl\,ds, \\ w_{10}(u)(t) &= B_2(t)u_4 + \frac{\sigma_1\varepsilon + \sigma_2\varepsilon - \mu\varepsilon - \alpha\varepsilon - \gamma_3\varepsilon}{2}\int_{t-\tau}^{t}\int_{s}^{t} u_2^2(l)\,dl\,ds. \end{aligned}$$

All the parameters are assumed to be positive and chosen in such a way that $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $k_4 > 0$, $k_5 > 0$, $k_6 > 0$, $k_7 > 0$, $k_8 > 0$, $k_9 > 0$, $k_{10} > 0$ and $w(u)(t) > 0$. Taking the

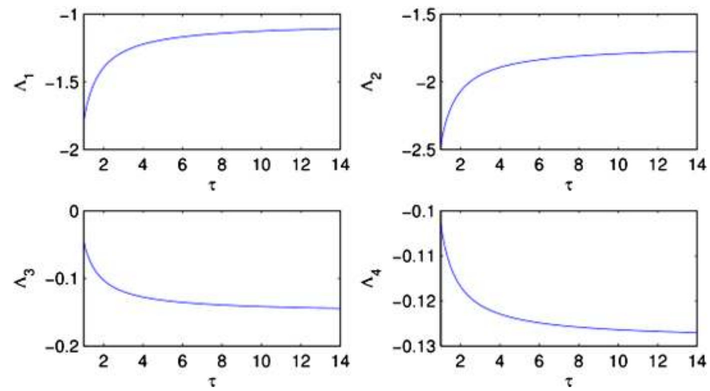

Figure 1 τ versus Λ_1, Λ_2, Λ_3, Λ_4 have been plotted, where all the Λ_1, Λ_2, Λ_3, Λ_4 are negative for $\tau \leq \tau_0 = 13.9858$

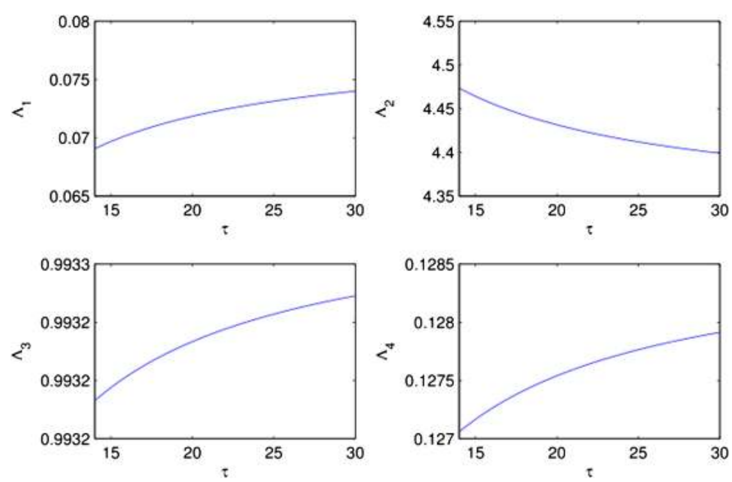

Figure 2 τ versus Λ_1, Λ_2, Λ_3, Λ_4 have been plotted, where all the Λ_1, Λ_2, Λ_3, Λ_4 are positive and $\tau \geq \tau_0 = 13.9858$

derivative of (20), and using (19) we get

$$\frac{d}{dt} w(u)(t) \leq \Lambda_1 u_1^2 + \Lambda_2 u_2^2 + \Lambda_3 u_3^2 + \Lambda_4 u_4^2, \tag{21}$$

where the expressions for Λ_1, Λ_2, Λ_3, Λ_4 are given in the Appendix.

Theorem 1 *If the value of the delay τ satisfy the conditions $\Lambda_1 < 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$, $\Lambda_4 < 0$ then the interior equilibrium point $P^*(S^*, E^*, I^*, Q^*)$ of (2) is locally asymptotically stable (Fig. 1). Otherwise if any one of the Λ_i become positive then the system will be unstable (Fig. 2)*

Proof Let $\Lambda = \max\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$. Then, for $t > T$, from (21) we get

$$w(u)(t) + \Lambda \int_T^t \bigl(u_1^2(s) + u_2^2(s) + u_3^2(s) + u_4^2(s)\bigr)\, ds \leq w(u)(T),$$

for $t \geq T$, implies $u_1^2 + u_2^2 + u_3^2 + u_4^2 \in L_1[T, \infty]$. It is easy to conclude from (19) and the boundedness of $u(t)$ that $u_1^2(t) + u_2^2(t) + u_3^2(t) + u_4^2(t)$ is uniformly continuous. Using Barbalat's lemma [44], we can say that

$$\lim_{t \to \infty} \{u_1^2 + u_2^2 + u_3^2 + u_4^2\} = 0. \tag{22}$$

So the internal solution of (19) and the solutions of (2) are asymptotically stable, i.e., the positive equilibrium P^* of (2) is locally asymptotically stable. Hence, this completes the proof. □

Remark As Λ_1, Λ_2, Λ_3, Λ_4 depends on the delay τ and the local stability condition for P^* of system (2) is preserved for small τ satisfying $\Lambda_1 < 0$, $\Lambda_2 < 0$, $\Lambda_3 < 0$, $\Lambda_4 < 0$. For a set of parameters, Λ_1, Λ_2, Λ_3, Λ_4 have been plotted in Fig. 1, it shows that all the values of Λ are negative within an interval of τ, which implies the stability of the system. But, for increased values of the latent delay τ, all the values of Λ are positive (see Fig. 2), which shows that the system is unstable.

4 Linear stability and Hopf-bifurcation analysis

Here we shall discuss the condition for linear stability and then taking τ as bifurcation parameter the condition for Hopf bifurcation is discussed. The characteristic equation of system (2) is

$$\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} = 0, \tag{23}$$

where

$$\begin{aligned}
m_0 &= a_1a_4a_6a_9, \\
m_1 &= -[a_1a_4a_6 + a_9(a_1a_4 + a_1a_6 + a_4a_6)], \\
m_2 &= a_1a_4 + a_1a_6 + a_4a_6 + a_9(a_1 + a_4 + a_6), \\
m_3 &= -(a_1 + a_4 + a_6 + a_9), \\
n_0 &= a_1a_9(a_6b_1 - a_5b_2) + a_2a_3a_9b_2, \\
n_1 &= a_5b_2(a_1 + a_9) - a_2a_3b_2 - b_1(a_1a_6 + a_1a_9 + a_6a_9), \\
n_2 &= b_1(a_1 + a_6 + a_9) - a_5b_2, \qquad n_3 = -b_1,
\end{aligned}$$

and

$$\begin{aligned}
&a_1 = -(\beta I^* + \mu), \qquad a_2 = c - \beta S^*, \qquad a_3 = \beta I^*, \\
&a_4 = -(\mu + \sigma_1 + \gamma_1), \qquad a_5 = \beta S^*, \\
&a_6 = -(\mu + \alpha + c + \sigma_2 + \gamma_2), \\
&a_7 = \sigma_1, \qquad a_8 = \sigma_2, a_9 = -(\mu + \alpha + \gamma_3), \qquad b_1 = -\varepsilon, \qquad b_2 = \varepsilon.
\end{aligned}$$

Theorem 2 *For system* (2), *if* $R_0 > 1$ *and the conditions* (H_1)–(H_2) *hold, then the endemic equilibrium* $P^*(S^*, E^*, I^*, Q^*)$ *is locally asymptotically stable when* $\tau \in [0, \tau_0)$; *system* (2) *undergoes a Hopf bifurcation at the endemic equilibrium $P^*(S^*, E^*, I^*, Q^*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from the endemic equilibrium $P^*(S^*, E^*, I^*, Q^*)$. The conditions* (H_1) *and* (H_2) *are described in the following.*

Proof The proof proceeds by using some lemmas.

Lemma 2 ([29]) *When $R_0 > 1$, the unique endemic equilibrium $P^*(S^*, E^*, I^*, R^*)$ is locally asymptotically stable when $\tau = 0$ for system* (2).

For $\tau > 0$, let $\lambda = i\omega$ ($\omega > 0$) be the root of Eq. (23), then

$$\begin{cases} (n_1\omega - n_3\omega^3)\sin\tau\omega + (n_0 - n_2\omega^2)\cos\tau\omega = m_2\omega^2 - \omega^4 - m_0, \\ (n_1\omega - n_3\omega^3)\cos\tau\omega - (n_0 - n_2\omega^2)\sin\tau\omega = m_3\omega^3 - m_1\omega, \end{cases} \tag{24}$$

which leads to

$$\omega^8 + l_3\omega^6 + l_2\omega^4 + l_1\omega^2 + l_0 = 0, \tag{25}$$

where

$$\begin{aligned} l_0 &= m_0^2 - n_0^2, \\ l_1 &= m_1^2 - 2m_0m_2 + 2n_0n_2 - n_1^2, \\ l_2 &= m_2^2 + 2m_0 - 2m_1m_3 - n_2^2 + 2n_1n_3, \\ l_3 &= m_3^2 - n_3^2 - 2m_2. \end{aligned}$$

Let $\omega^2 = \nu$, then Eq. (25) becomes

$$\nu^4 + l_3\nu^3 + l_2\nu^2 + l_1\nu + l_0 = 0. \tag{26}$$

Define

$$f(\nu) = \nu^4 + l_3\nu^3 + l_2\nu^2 + l_1\nu + l_0. \tag{27}$$

Thus,

$$f'(\nu) = 4\nu^3 + 3l_3\nu^2 + 2l_2\nu + l_1. \tag{28}$$

Set

$$4\nu^3 + 3l_3\nu^2 + 2l_2\nu + l_1 = 0. \tag{29}$$

Let $y = \nu + \frac{3l_3}{4}$. Then Eq. (29) becomes

$$y^3 + r_1y + s_1 = 0, \tag{30}$$

where

$$r_1 = \frac{l_2}{2} - \frac{3}{16}l_3^2, \qquad s_1 = \frac{l_3^3}{32} - \frac{l_2 l_3}{8} + l_1.$$

Denote

$$D = \left(\frac{s_1}{2}\right)^2 + \left(\frac{r_1}{3}\right)^3, \qquad \sigma = \frac{-1+\sqrt{3}i}{2},$$
$$y_1 = \sqrt[3]{-\frac{s_1}{2}+\sqrt{D}} + \sqrt[3]{-\frac{s_1}{2}-\sqrt{D}},$$
$$y_2 = \sqrt[3]{-\frac{s_1}{2}+\sqrt{D}}\sigma + \sqrt[3]{-\frac{s_1}{2}-\sqrt{D}}\sigma^2,$$
$$y_3 = \sqrt[3]{-\frac{s_1}{2}+\sqrt{D}}\sigma^2 + \sqrt[3]{-\frac{s_1}{2}-\sqrt{D}}\sigma,$$
$$v_i = y_i - \frac{3l_3}{4}, \quad i = 1,2,3.$$

Based on the discussion of the distribution of roots of Eq. (26) in Lemma 2.1 and Lemma 2.2 in [45], we have the following results.

Lemma 3 *For Eq.* (26), *we have*

(H_1) *If* $l_0 < 0$, *Eq.* (26) *has at least one positive root*;
(H_2) *If* $l_0 \geq 0$ *and* $D \geq 0$, *Eq.* (26) *has positive roots if and only if* $v_1 > 0$ *and* $f(v_1) < 0$;
(H_3) *If* $l_0 \geq 0$ *and* $D < 0$, *Eq.* (26) *has positive roots if and only if there exists at least one* $v_* \in \{v_1, v_2, v_3\}$ *such that* $v_* > 0$ *and* $f(v_*) \leq 0$.

In what follows, we assume (H_1): the coefficients in $f(v)$ satisfy one of the following conditions in (a)–(c).

(a) $l_0 < 0$;
(b) $l_0 \geq 0$, $D \geq 0$, $v_1 > 0$ and $f(v_1) < 0$;
(c) $l_0 \geq 0$, $D < 0$, and there exists at least one $v_* \in \{v_1, v_2, v_3\}$ such that $v_* > 0$ and $f(v_*) \leq 0$.

If the condition (H_1) holds, then Eq. (26) has positive root v_0 such that Eq. (23) has a pair of imaginary roots $\pm i\omega_0 = \pm i\sqrt{v_0}$. Further, we have

$$\tau_0 = \frac{1}{\omega_0}\arccos\left\{\frac{(n_2 - m_3 n_3)\omega_0^6 + (m_1 n_3 + m_3 n_1 - m_2 n_2)\omega_0^4 + (m_0 n_2 + m_2 n_0 - m_1 n_1)\omega_0^2 - m_0 n_0}{(n_1\omega_0 - n_3\omega_0^3)^2 + (n_0 - n_2\omega_0^2)^2}\right\}. \tag{31}$$

Differentiating both sides of Eq. (23) with respect to τ yields

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1}{\lambda(\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0)} + \frac{3n_3\lambda^2 + 2n_2\lambda + n_1}{\lambda(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)} - \frac{\tau}{\lambda}.$$

Further, we have

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_0}^{-1} = \frac{f'(v_{**})}{(n_1\omega_0 - n_3\omega_0^3)^2 + (n_0 - n_2\omega_0^2)^2},$$

where $f(v) = v^4 + l_3 v^3 + l_2 v^2 + l_1 v + l_0$ and $v_{**} = \omega_0^2$.

Clearly, if the condition (H_2): $f'(\omega_0^2) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau]^{-1}_{\tau=\tau_0} \neq 0$. Therefore, by the Hopf-bifurcation theorem that determines the existence of a Hopf bifurcation for a delayed dynamical system in [44], we can obtain the results described in Theorem 2. The proof is completed. □

5 Direction of the Hopf bifurcation and stability of the periodic solutions

Let $u_1(t) = S(t) - S^*$, $u_2(t) = E(t) - E^*$, $u_3(t) = I(t) - I^*$, $u_4(t) = Q(t) - Q^*$, and normalize the delay with $t \to (t/\tau)$. Let $\tau = \tau_0 + \varrho$ ($\varrho \in R$), then $\varrho = 0$ is the Hopf-bifurcation value of system (2). And system (2) can be transformed into a functional differential equation in $C = C([-1,0], R^4)$ as follows:

$$\dot{u}(t) = L_\varrho(u_t) + F(\varrho, u_t), \tag{32}$$

where $u(t) = (u_1, u_2, u_3, u_4)^T \in C = C([-1,0], R^4)$ and L_ϱ: $C \to R^4$ and F: $R \times C \to R^4$ are given, respectively, by

$$L_\varrho\phi = (\tau_0 + \varrho)\bigl(A_{\max}\phi(0) + B_{\max}\phi(-1)\bigr),$$

and

$$F(\varrho, \phi) = \begin{pmatrix} -\beta\phi_1(0)\phi_3(0) \\ \beta\phi_1(0)\phi_3(0) \\ 0 \\ 0 \end{pmatrix},$$

with

$$A_{\max} = \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ a_3 & a_4 & a_5 & 0 \\ 0 & 0 & a_6 & 0 \\ 0 & a_7 & a_8 & a_9 \end{pmatrix}, \qquad B_{\max} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3

(i) *If* $\mu_2 > 0$ ($\mu_2 < 0$), *then the Hopf bifurcation is supercritical* (*subcritical*);
(ii) *if* $\beta_2 < 0$ ($\beta_2 > 0$, *then the bifurcation periodic solutions are stable* (*unstable*);
(iii) *if* $T_2 > 0$ ($T_2 < 0$), *then the bifurcating periodic solutions increase* (*decrease*).

The expressions of μ_2, β_2 *and* T_2 *are described in the following.*

Proof By the Riesz representation theorem, there exists a function $\eta(\theta, \varrho)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \varrho)\phi(\theta), \quad \text{for } \phi \in C. \tag{33}$$

In fact, we choose

$$\eta(\theta, \varrho) = (\tau_0 + \varrho)\bigl(A_{\max}\delta(\theta) + B_{\max}\delta(\theta + 1)\bigr),$$

where $\delta(\theta)$ is the Dirac delta function.

For $\phi \in C([-1,0],R^4)$, define

$$A(\varrho)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^{0} d\eta(\theta,\varrho)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\varrho)\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\varrho,\phi), & \theta = 0. \end{cases}$$

Then system (32) is equivalent to

$$\dot{u}(t) = A(\varrho)u_t + R(\varrho)u_t. \tag{34}$$

For $\varphi \in C^1([0,1],(R^4)^*)$, the adjoint operator A^* of $A(0)$ is defined as follows:

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0. \end{cases}$$

Next, we define the bilinear inner form for A and A^*

$$\langle \varphi(s),\phi(\theta)\rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \tag{35}$$

where $\eta(\theta) = \eta(\theta,0)$.

Let $q(\theta) = (1,q_2,q_3,q_4)^T e^{i\tau_0\omega_0\theta}$ be the eigenvector of $A(0)$ corresponding to $+i\tau_0\omega_0$ and $q^*(s) = V(1,q_2^*,q_3^*,q_4^*)^T e^{i\tau_0\omega_0 s}$ be the eigenvector of $A^*(0)$ corresponding to $-i\tau_0\omega_0$, respectively. Based on the definition of $A(0)$ and A^*, one can obtain

$$\begin{aligned} &q_2 = \frac{a_3 + a_5 q_3}{i\omega_0 - a_4 - b_1 e^{-i\tau_0\omega_0}}, \qquad q_3 = \frac{i\omega_0 - a_1}{a_2}, \qquad q_4 = \frac{a_7 q_2 + a_8 q_3}{(i\omega_0 - a_9)}, \\ &q_2^* = -\frac{i\omega_0 + a_1}{a_3}, \qquad q_4^* = -\frac{a_2 + a_5 q_2 + (a_6 + i\omega_0)q_3}{a_8}, \\ &q_3^* = \frac{a_7(a_2 + a_5 q_2) - a_8(i\omega_0 + a_4 + b_1 e^{i\tau_0\omega_0})q_2}{a_8 b_2 e^{i\tau_0\omega_0} - a_7(a_6 + i\omega_0)}. \end{aligned} \tag{36}$$

From Eq. (35), we can obtain

$$\bar{V} = \bigl[1 + q_2 q_2^* + q_3\bar{q}_3^* + q_4\bar{q}_4^* + \tau_0 e^{-i\tau_0\omega_0} q_2\bigl(b_1\bar{q}_2{}^* + b_2\bar{q}_3^*\bigr)\bigr]^{-1}$$

such that $\langle q^*,q\rangle = 1$ and $\langle q^*,\bar{q}\rangle = 0$.

Next, according to the algorithms in [44] and a similar computation process to that in [46], we can obtain the expressions of g_{20}, g_{11}, g_{02} and g_{21} as follows:

$$\begin{aligned} &g_{20} = 2\beta\tau_0\bar{V}q_3\bigl(\bar{q}_3^* - 1\bigr), \\ &g_{11} = \beta\tau_0\bar{V}(q_3 + \bar{q}_3)\bigl(\bar{q}_3^* - 1\bigr), \end{aligned}$$

$$
\begin{aligned}
g_{02} &= 2\beta\tau_0\bar{V}\bar{q}_3\bigl(\bar{q}_3^* - 1\bigr),\\
g_{21} &= 2\beta\tau_0\bar{V}\bigl(\bar{q}_3^* - 1\bigr)\biggl(W_{11}^{(1)}(0)q_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{q}_2\\
&\quad + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\biggr),
\end{aligned}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_1e^{2i\tau_0\omega_0\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_2.
\end{aligned}
$$

E_1 and E_2 can be obtained by the following two equations:

$$
\begin{aligned}
E_1 &= 2\begin{pmatrix} 2i\omega_0 - a_1 & 0 & -a_2 & 0\\ -a_3 & 2i\omega_0 - a_4 - b_1e^{-2i\tau_0\omega_0} & -a_5 & 0\\ 0 & -b_2e^{-2i\tau_0\omega_0} & 2i\omega_0 - a_6 & 0\\ 0 & a_7 & -a_8 & 2i\omega_0 - a_9 \end{pmatrix}^{-1} \times \begin{pmatrix} \beta q_3\\ \beta q_3\\ 0\\ 0 \end{pmatrix},\\
E_2 &= -\begin{pmatrix} a_1 & 0 & a_2 & 0\\ a_3 & a_4 + b_1 & a_5 & 0\\ 0 & b_2 & a_6 & 0\\ 0 & a_7 & a_8 & a_9 \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta(q_3 + \bar{q}_3)\\ \beta(q_3 + \bar{q}_3)\\ 0\\ 0 \end{pmatrix}.
\end{aligned}
$$

Then one can obtain

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\tau_0\omega_0}\biggl(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\biggr) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}},\\
\beta_2 &= 2\operatorname{Re}\bigl\{C_1(0)\bigr\},\\
T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0}.
\end{aligned}
\tag{37}
$$

Thus, we can obtain the results described in Theorem 3. The proof is completed. □

6 Numerical simulation

In this section we shall perform some numerical scenario as the support of our obtained analytical results by choosing the suitable value of the parameters. For different values of delays we obtain different scenarios with $P^*(S^*, E^*, I^*, Q^*)$ as interior equilibrium point. The value of the parameters are taken as follows:

$$
\begin{aligned}
&A = 20, \qquad \beta = 0.5, \qquad \mu = c = 0.25, \qquad \varepsilon = 0.25,\\
&\sigma_1 = \gamma_1 = 0.125, \qquad \alpha = 0.5, \qquad \sigma_2 = \gamma_2 = 0.25, \qquad \gamma_3 = 0.5.
\end{aligned}
$$

The parameters are chosen in such a way that they satisfy the conditions obtained in the previous sections analytically. We have

$$\begin{cases} \frac{dS(t)}{dt} = 20 - 0.5S(t)I(t) - 0.25S(t) + 0.25I(t), \\ \frac{dE(t)}{dt} = 0.5S(t)I(t) - 0.5E(t) - 0.5E(t-\tau), \\ \frac{dI(t)}{dt} = 0.5E(t-\tau) - 1.5I(t), \\ \frac{dQ(t)}{dt} = 0.125E(t) + 0.25I(t) - 1.25Q(t). \end{cases} \quad (38)$$

With this set of parameters we get the basic reproduction number $R_0 = 13.3333 > 0$ and the unique endemic equilibrium $P^*(6, 20.1818, 6.7273, 3.3636)$. Biologically it shows that all the individuals coexist. First, in the absence of latent delay, i.e. $\tau = 0$ the dynamics of system (1) has been plotted in Fig. 3 and the dynamics is stable in the absence of delay (Lemma 1). But in Fig. 4 it is seen that in the presence of delay (very small value of τ) initially all the individuals are oscillating and after some time again it comes to a stable situation. Thus for a more increased value of the delay the oscillation for the individuals also increases. Hence, the interior equilibrium point $P^*(S^*, E^*, I^*, Q^*)$ is seen to be stable for $\tau < \tau_0$ (Fig. 5) and at the critical value of delay we get a stable periodic solution where the Hopf bifurcation occurs (Fig. 6). Finally for large value of delay $\tau > \tau_0$ the system loses

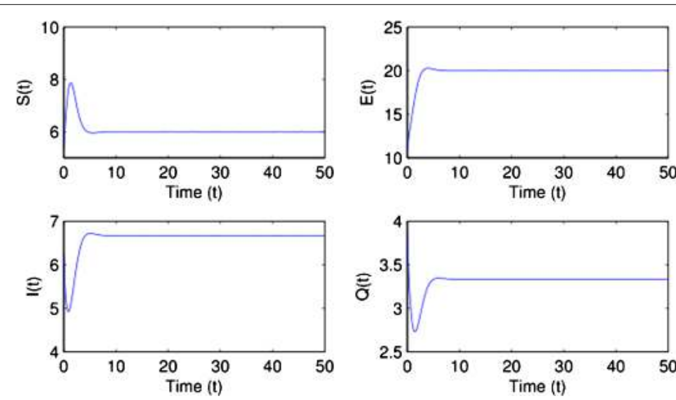

Figure 3 Time series plot for $\tau = 0$, which is stable

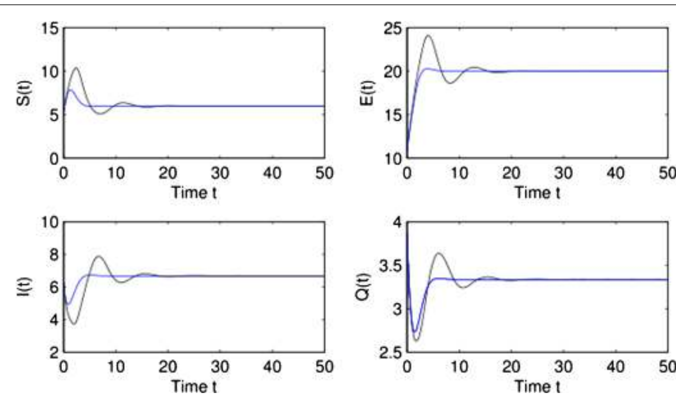

Figure 4 Time series plot for $\tau \neq 0$ but very small τ, which is stable

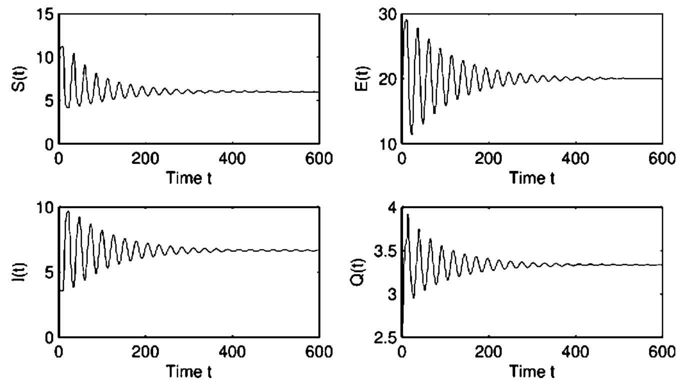

Figure 5 P^* is locally asymptotically stable when $\tau = 10.65 < \tau_0 = 13.9858$

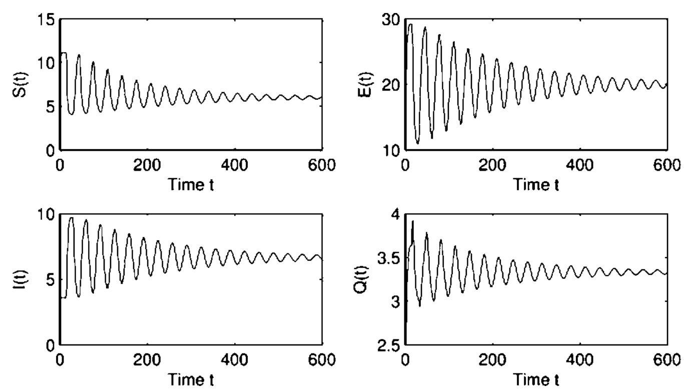

Figure 6 P^* is stable periodic when $\tau = \tau_0 = 13.9858$

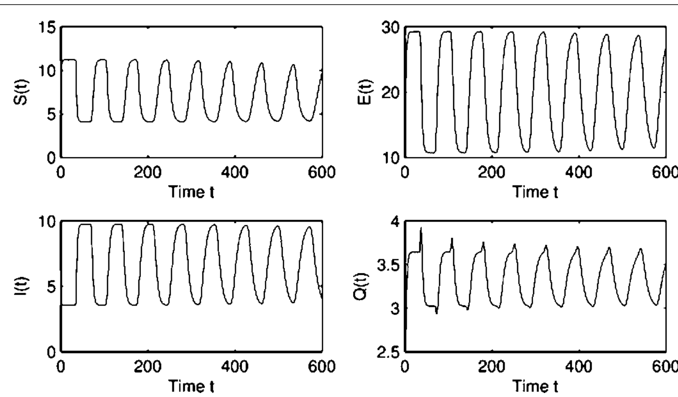

Figure 7 P^* loses its stability when $\tau = 28.85 > \tau_0 = 13.9858$

its stability (Fig. 7). This property can be also illustrated by the bifurcation diagram with respect to τ in Fig. 8.

7 Conclusions

In [29], a non-delayed SEIQ epidemic model has been investigated by the authors. But the disease models are not instantaneous, i.e., infected diseases start by infection and then

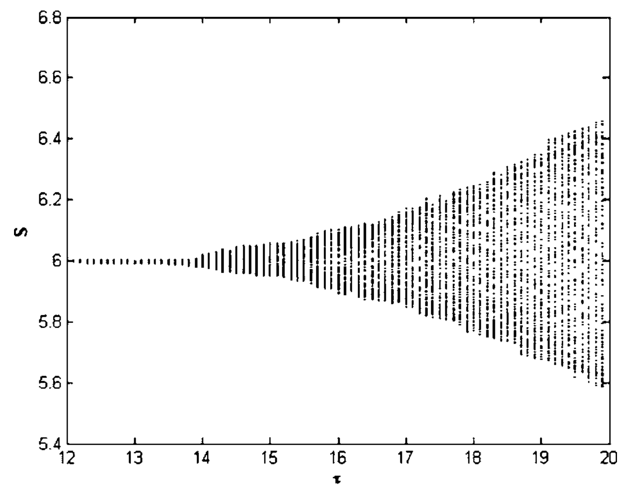

Figure 8 Bifurcation diagram with respect to time delay

some symptoms can be seen. Thus, in the incubation period it takes some time for a response to occur, i.e., delay is arising. Here, in this article we have assumed a delayed SEIQ epidemic model by incorporating the latent delay to the model proposed in [29]. Thus, compared with the model proposed in [29], the model we consider in the present paper is more general. We consider not only the effect of the time delay on the model, but also the boundedness, persistence and the properties of the Hopf bifurcation. The results obtained in the present paper are the complement of the research work in the literature [29].

For this model, a feasible region $\bar{R}$ is obtained with the appropriate choice of the parameters. It can be seen that all the solutions of (2) will remain in or tend to $\bar{R}$, i.e., the feasible region $\bar{R}$ is positive and invariant. If the basic reproduction number $R_0 > 1$, then the model has an endemic equilibrium point which is unique. Also if $R_0 > 1$ the system (2) will be persistent and for $R_0 < 1$, the infection will be extinct, i.e., system (2) becomes disease free. Next, we construct a suitable Lyapunov functional of the form (20) to check the stability of system (2). Using this Lyapunov functional the sufficient conditions for local asymptotic stability are given in Theorem 1. With the choice of the parameters given in the numerical section and for $\tau < \tau_0$ all the Λ are negative (Fig. 1), which satisfies the conditions obtained in Theorem 1. Next, the sufficient conditions for local stability of the endemic equilibrium of the model and the existence of a Hopf bifurcation are obtained by taking the delay as the bifurcating parameter. Also, the critical value of the latent delay is obtained. We can conclude that if the latent delay for system (2) is less than its critical value then the endemic equilibrium for the system gets in a stable situation but if it is greater than the critical value the endemic equilibrium for system (2) will lose its stability. Further, properties of the Hopf bifurcation such as direction and stability are studied by means of the center manifold and normal form theory. At the end, with a set of suitable parameters, some numerical computations are presented to justify our results obtained analytically and to see the effect of the latent delay on the stability of the system.

It should be pointed out that system (2) undergoes a local Hopf bifurcation at the endemic equilibrium $P^*(S^*, E^*, I^*, Q^*)$ and a bifurcating periodic solution exists when τ near

τ_0. In other words, the existence of these periodic solutions remains valid only in a small neighborhood of the critical value τ_0. It is interesting to investigate whether these periodic solutions remain when the value of the time delay τ becomes large enough. We leave the existence of a global Hopf bifurcation of system (2) as our next work in the near future.

Appendix

We have

$$\begin{aligned}
\Lambda_1 &= -2k_1(\mu + \beta I^*) - k_2\beta I^*\varepsilon\tau + k_5\left\{\beta I^* + \frac{\beta I^*\varepsilon\tau + \mu\varepsilon\tau}{2}\right\} \\
&\quad + k_6\frac{-\mu\varepsilon\tau - \beta I^*\varepsilon\tau}{2} + k_8\frac{\beta I^*\varepsilon\tau}{2} + k_{10}\frac{\sigma_1\varepsilon\tau}{2}, \\
\Lambda_2 &= k_2\{-2\mu - 2\sigma_1 - 2\gamma_1 - 2\varepsilon + \varepsilon\tau(2\mu + 2\sigma_1 + 2\gamma_1 + 2\varepsilon - \beta I^* - \beta S^*)\} \\
&\quad + k_3\{2\tau\varepsilon^2 - \varepsilon\tau(\mu + \alpha + c + \sigma_2 + \gamma_2)\} + k_5\frac{\beta I^*\varepsilon\tau + \mu\varepsilon\tau - c\varepsilon\tau + \beta S^*\varepsilon\tau}{2} \\
&\quad + k_6\frac{-\tau\varepsilon(\beta I^* + \mu - c + \beta S^*)}{2} \\
&\quad + k_8\left\{\varepsilon + \frac{\tau\varepsilon}{2}(-4\varepsilon - \mu - 2\sigma_1 - 2\gamma_1 + \beta I^* + \beta S^* + \alpha + c + \sigma_2 + \gamma_2)\right\} \\
&\quad + k_9\frac{2\sigma_1 + \tau\varepsilon(\mu + \alpha + \gamma_3 - 2\sigma_1 - \sigma_2)}{2} + k_{10}\frac{\sigma_1\varepsilon\tau}{2}, \\
\Lambda_3 &= k_2(-\beta S^*\varepsilon\tau) - k_3\{2(\mu + \alpha + c + \sigma_2 + \gamma_2) + \tau\varepsilon(\mu + \alpha + c + \sigma_2 + \gamma_2)\} \\
&\quad + k_5\frac{\beta S^*\varepsilon\tau - c\varepsilon\tau}{2} + k_6\frac{2c - 2\beta S^* + c\varepsilon\tau - \beta\varepsilon\tau S^*}{2} \\
&\quad + k_8\frac{2\beta S^* + (\beta S^* + \mu + \alpha + c + \sigma_2 + \gamma_2)\varepsilon\tau +}{2} + k_9\frac{-\varepsilon\sigma_2\tau}{2} + k_{10}\frac{2\sigma_2 + \sigma_2\epsilon\tau}{2}, \\
\Lambda_4 &= -2k_4(\mu + \alpha + \gamma_3) + k_9\frac{(\mu + \alpha + \gamma_3)\tau\varepsilon}{2} + k_{10}\frac{(-\mu - \alpha - \gamma_3)\tau\varepsilon}{2}, \\
k_1 &= k_4 \\
&= \left(\gamma_3 S^* + c\tau + \frac{c\gamma_2\tau + 2\beta^2\tau + 2\mu\alpha}{2} + \frac{\sigma_1\sigma_2}{2} + 2\beta S^* + \frac{c - \beta S^*}{2} + \frac{-\beta S^*\varepsilon\tau}{2}\right. \\
&\quad \left. + \frac{-\beta S^*\varepsilon\tau - c\varepsilon\tau}{2} + \frac{\mu + \alpha + \gamma_3}{2}\right) \Big/ \left(2\varepsilon\left(-\gamma_1 - \beta + \frac{\beta\gamma_3\tau}{2} + \frac{\gamma_1 + \sigma_2\tau}{2} + \frac{\beta I^* + \mu}{2}\right.\right. \\
&\quad \left.\left. + \frac{\varepsilon\tau(-2\mu - \sigma_1 - \gamma_1 - \varepsilon)}{2} + \frac{(\mu + \alpha + c + \sigma_2 + \gamma_2)\tau}{2} + \frac{-2\mu + \gamma_1\gamma_2\tau + 2\sigma_1\tau}{2}\right)\right), \\
k_2 &= k_3 = 2k_8 \\
&= \left(\sigma_2 I^* + \sigma_2\gamma_2\tau + \frac{\sigma_1\gamma_3\tau + \beta I^* + \beta S^*\tau\varepsilon - c\tau\varepsilon}{2} + \frac{-\mu\tau\varepsilon - \beta\varepsilon I^*\tau}{2} + \frac{\beta\gamma_3\tau}{2}\right. \\
&\quad \left. + \frac{\varepsilon + \alpha + c}{2} + \frac{\sigma_1 + \alpha + c + 2\gamma_2}{2}\right) \Big/ \left(2\varepsilon\left(-\gamma_1 - \beta + \frac{\beta\gamma_3\tau}{2} + \frac{\gamma_1 + \sigma_2\tau}{2} + \frac{\beta I^* + \mu}{2}\right.\right. \\
&\quad \left.\left. + \frac{\varepsilon\tau(-2\mu - \sigma_1 - \gamma_1 - \varepsilon)}{2} + \frac{(\mu + \alpha + c + \sigma_2 + \gamma_2)\tau}{2} + \frac{-2\mu + \gamma_1\gamma_2\tau + 2\sigma_1\tau}{2}\right)\right), \\
k_5 &= k_{10}
\end{aligned}$$

$$
\begin{aligned}
&= 2(\mu+\alpha+\gamma_3) + \frac{\beta S^*\tau}{2} + \frac{\beta I^*\varepsilon\tau}{2} + 2\big(\mu+\beta I^*\big) + \frac{\alpha+c+\mu+\gamma_2\tau}{2} + 2\sigma_1 + \varepsilon\tau \\
&\quad + \frac{c-\beta S^*-\beta I^*-2\mu}{2} + \frac{\varepsilon\tau}{2}(2\mu+2\alpha+\gamma_3+c+\sigma_2+\gamma_2)\Big/\bigg(2\varepsilon\bigg(-\gamma_1-\beta+\frac{\beta\gamma_3\tau}{2} \\
&\quad + \frac{\gamma_1+\sigma_2\tau}{2} + \frac{\beta I^*+\mu}{2} + \frac{\varepsilon\tau(-2\mu-\sigma_1-\gamma_1-\varepsilon)}{2} \\
&\quad + \frac{(\mu+\alpha+c+\sigma_2+\gamma_2)\tau}{2} + \frac{-2\mu+\gamma_1\gamma_2\tau+2\sigma_1\tau}{2}\bigg)\bigg),
\end{aligned}
$$

$$
\begin{aligned}
k_6 &= k_7 = k_9 \\
&= \big(-2\mu-2\sigma_1-2\gamma_1-2\varepsilon+\tau\varepsilon\big(2\mu+2\sigma_1+2\gamma_1+2\varepsilon-\beta I^*-\beta S^*\big)\big)\Big/\bigg(2\varepsilon\bigg(-\gamma_1 \\
&\quad -\beta+\frac{\beta\gamma_3\tau}{2} + \frac{\gamma_1+\sigma_2\tau}{2} + \frac{\beta I^*+\mu}{2} + \frac{\varepsilon\tau(-2\mu-\sigma_1-\gamma_1-\varepsilon)}{2} \\
&\quad + \frac{(\mu+\alpha+c+\sigma_2+\gamma_2)\tau}{2} + \frac{-2\mu+\gamma_1\gamma_2\tau+2\sigma_1\tau}{2}\bigg)\bigg).
\end{aligned}
$$

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Availability of data and materials
All of the authors clare that all the data can be accessed in our manuscript in the numerical simulation section.

Competing interests
All the authors declare that they have no financial and non-financial competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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