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Dynamics of a system of rational third-order difference equation

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Abstract
In this paper, we study the dynamical behavior of positive solution for a system of a rational third-order difference equation

$$x_{n+1}=\frac{x_{n-2}}{B+y_{n-2}y_{n-1}y_n},\qquad y_{n+1}=\frac{y_{n-2}}{A+x_{n-2}x_{n-1}x_n},\qquad n=0,1,\ldots,$$

where $A, B \in (0,\infty)$, $x_{-2}, x_{-1}, x_0 \in (0,\infty)$; $y_{-2}, y_{-1}, y_0 \in (0,\infty)$.

MSC: 39A10

Keywords: difference equation; local behavior; unstable

1 Introduction

Rational difference equations that are the ratio of two polynomials are one of the most important and practical classes of nonlinear difference equations. Marwan Aloqeili [1] investigated the stability character, semicycle behavior of the solution of the difference equation $x_{n+1} = x_{n-1}/(a - x_{n-1}x_n)$. These difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, *etc.* [2, 11]. Also, Cinar [3] investigated the global behavior of all positive solutions of the rational second-order difference equation

$$x_{n+1}=\frac{x_{n-1}}{1+x_nx_{n-1}},\qquad n=0,1,\ldots.$$

Similarly Shojaei, Saadati, and Adibi [4] investigated the stability and periodic character of the rational third-order difference equation

$$x_{n+1}=\frac{\alpha x_{n-2}}{\beta+\gamma x_{n-2}x_{n-1}x_n},\qquad n=0,1,\ldots,$$

where the parameters α, β, γ, and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers. Related difference equations readers can refer to the references [5–7].

Papaschinopoulos and Schinas [8] studied the system of two nonlinear difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\qquad y_{n+1}=A+\frac{x_n}{y_{n-q}},\qquad n=0,1,\ldots,\tag{1}$$

where p, q are positive integers.

Clark and Kulenovic [9, 10] investigated the system of rational difference equations

$$x_{n+1} = \frac{x_n}{a + cy_n}, \qquad y_{n+1} = \frac{y_n}{b + dx_n}, \qquad n = 0, 1, \ldots, \tag{2}$$

where $a, b, c, d \in (0, \infty)$, and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

Our aim in this paper is to investigate the solutions, stability character, and asymptotic behavior of the system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n}, \qquad y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n}, \qquad n = 0, 1, \ldots, \tag{3}$$

where $A, B \in (0, \infty)$, and the initial conditions $x_{-2}, x_{-1}, x_0 \in (0, \infty)$; $y_{-2}, y_{-1}, y_0 \in (0, \infty)$.

2 Preliminaries

Let I_x, I_y be some intervals of real number and $f : I_x^3 \times I_y^3 \to I_x$, $g : I_x^3 \times I_y^3 \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -2, -1, 0)$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}), \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}), \end{cases} \qquad n = 0, 1, 2, \ldots, \tag{4}$$

has a unique solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$. A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of (4) if $\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y})$, $\bar{y} = g(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y})$, *i.e.*, $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 0$.

Let I_x, I_y be some intervals of real numbers; interval $I_x \times I_y$ is called invariant for system (4) if, for all $n > 0$,

$$x_{-2}, x_{-1}, x_0 \in I_x, \qquad y_{-2}, y_{-1}, y_0 \in I_y \quad \Rightarrow \quad x_n \in I_x, \qquad y_n \in I_y.$$

Definition 2.1 Assume that $(\bar{x}, \bar{y})$ be a fixed point of system (4). Then

(i) $(\bar{x}, \bar{y})$ is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -2, -1, 0)$, with $\sum_{i=-2}^{0} |x_i - \bar{x}| < \delta$, $\sum_{i=-2}^{0} |y_i - \bar{y}| < \delta$, implies $|x_n - \bar{x}| < \varepsilon$, $|y_n - \bar{y}| < \varepsilon$.

(ii) $(\bar{x}, \bar{y})$ is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_i) \in I_x \times I_y$ $(i = -2, -1, 0)$, $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$.

(iii) $(\bar{x}, \bar{y})$ is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.

(iv) Unstable if it is not stable.

Theorem 2.1 ([11]) *Assume that $X(n+1) = F(X(n))$, $n = 0, 1, \ldots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system, i.e., $F(\overline{X}) = \overline{X}$. If all eigenvalues of the Jacobian matrix J_F, evaluated at $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

Theorem 2.2 ([12]) *Assume that $X(n+1) = F(X(n))$, $n = 0, 1, \ldots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system, the characteristic polynomial of this system about the equilibrium point $\overline{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$,* *with real coefficients and $a_0 > 0$. Then all roots of the polynomial $p(\lambda)$ lie inside the open unit disk $|\lambda| < 1$ if and only if*

$$\Delta_k > 0 \quad \text{for } k = 1, 2, \ldots, n, \tag{5}$$

where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

3 Main results

Consider the system (3), if $A < 1$, $B < 1$, system (3) has equilibrium $(0,0)$ and $(\sqrt[3]{1-A}, \sqrt[3]{1-B})$. In addition, if $A < 1$, $B = 1$, then system (3) has an equilibrium point $(\sqrt[3]{1-A}, 0)$, and if $A = 1$, $B < 1$, then system (3) has an equilibrium point $(0, \sqrt[3]{1-B})$. Finally, if $A > 1$ and $B > 1$, $(0,0)$ is the unique equilibrium point.

Theorem 3.1 *Let (x_n, y_n) be positive solution of system (3), then for all $k \geq 0$,*

$$\text{(i)}\quad 0 \leq x_n \leq \begin{cases} \frac{x_{-2}}{B^{k+1}}, & n = 3k+1; \\ \frac{x_{-1}}{B^{k+1}}, & n = 3k+2; \\ \frac{x_0}{B^{k+1}}, & n = 3k+3; \end{cases} \qquad \text{(ii)}\quad 0 \leq y_n \leq \begin{cases} \frac{y_{-2}}{A^{k+1}}, & n = 3k+1; \\ \frac{y_{-1}}{A^{k+1}}, & n = 3k+2; \\ \frac{y_0}{A^{k+1}}, & n = 3k+3. \end{cases} \tag{6}$$

Proof This assertion is true for $k = 0$. Assume that it is true for $k = m$, for $k = m+1$, we have

$$x_n = \begin{cases} x_{3(m+1)+1} \leq \frac{x_{3(m+1)-2}}{B} = \frac{x_{3m+1}}{B} \leq \frac{1}{B}\frac{x_{-2}}{B^{m+1}}, & n = 3(m+1)+1; \\ x_{3(m+1)+2} \leq \frac{x_{3(m+1)+1-2}}{B} = \frac{x_{3m+2}}{B} \leq \frac{1}{B}\frac{x_{-1}}{B^{m+1}}, & n = 3(m+1)+2; \\ x_{3(m+1)+3} \leq \frac{x_{3(m+1)+3-2}}{B} = \frac{x_{3m+4}}{B} \leq \frac{1}{B}\frac{x_0}{B^{m+1}}, & n = 3(m+1)+3; \end{cases}$$

$$y_n = \begin{cases} y_{3(m+1)+1} \leq \frac{y_{3(m+1)-2}}{A} = \frac{y_{3m+1}}{A} \leq \frac{1}{A}\frac{y_{-2}}{A^{m+1}}, & n = 3(m+1)+1; \\ y_{3(m+1)+2} \leq \frac{y_{3(m+1)+1-2}}{A} = \frac{y_{3m+2}}{A} \leq \frac{1}{A}\frac{y_{-1}}{A^{m+1}}, & n = 3(m+1)+2; \\ y_{3(m+1)+3} \leq \frac{y_{3(m+1)+3-2}}{A} = \frac{y_{3m+4}}{A} \leq \frac{1}{A}\frac{y_0}{A^{m+1}}, & n = 3(m+1)+3. \end{cases}$$

This completes our inductive proof. □

Corollary 3.1 *If $A > 1$, $B > 1$, then by Theorem 3.1 $\{(x_n, y_n)\}$ converges exponentially to the equilibrium point $(0,0)$.*

Theorem 3.2 *If*

$$A > 1, \qquad B > 1. \tag{7}$$

Then the equilibrium $(0,0)$ is locally asymptotically stable.

Proof We can easily obtain that the linearized system of (3) about the equilibrium $(0,0)$ is

$$\Phi_{n+1} = D\Phi_n, \tag{8}$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & \frac{1}{B} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{A} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of (8) is

$$f(\lambda) = \left(\lambda^3 + \frac{1}{A}\right)\left(\lambda^3 + \frac{1}{B}\right) = 0. \tag{9}$$

This shows that all the roots of the characteristic equation lie inside unit disk. So the unique equilibrium $(0,0)$ is locally asymptotically stable. □

Theorem 3.3 *If*

$$A < 1, \qquad B < 1. \tag{10}$$

Then

(i) *the equilibrium* $(0,0)$ *is locally unstable,*
(ii) *the positive equilibrium* $(\bar{x},\bar{y}) = (\sqrt[3]{1-A}, \sqrt[3]{1-B})$ *is locally unstable.*

Proof (i) From (9), we have that all the roots of characteristic equation lie outside unit disk. So the unique equilibrium $(0,0)$ is locally unstable.

(ii) We can easily obtain that the linearized system of (3) about the equilibrium $(0,0)$ is

$$\Phi_{n+1} = G\Phi_n, \tag{11}$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 0 & \frac{1}{B} & \alpha & \alpha & \alpha \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \beta & \beta & \beta & 0 & 0 & \frac{1}{A} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

in which $\alpha = -\sqrt[3]{(1-A)(1-B)^2}$, $\beta = -\sqrt[3]{(1-A)^2(1-B)}$. The characteristic equation of (11) is

$$P(\lambda) = \lambda^6 - \alpha\beta\lambda^4 - \left(2\alpha\beta + \frac{1}{A} - \frac{1}{B}\right)\lambda^3 - 3\alpha\beta\lambda^2 - 2\alpha\beta\lambda - \alpha\beta + \frac{1}{AB}. \tag{12}$$

From (12), we have

$$\Delta_6 = \begin{bmatrix} 0 & -(2\alpha\beta + \frac{1}{A} - \frac{1}{B}) & -2\alpha\beta & 0 & 0 & 0 \\ 1 & -\alpha\beta & -3\alpha\beta & -\alpha\beta + \frac{1}{AB} & 0 & 0 \\ 0 & 0 & -(2\alpha\beta + \frac{1}{A} - \frac{1}{B}) & -2\alpha\beta & 0 & 0 \\ 0 & 0 & -\alpha\beta & -3\alpha\beta & -\alpha\beta + \frac{1}{AB} & 0 \\ 0 & 0 & 0 & -(2\alpha\beta + \frac{1}{A} - \frac{1}{B}) & -2\alpha\beta & 0 \\ 0 & 0 & 0 & 0 & -3\alpha\beta & -\alpha\beta + \frac{1}{AB} \end{bmatrix}.$$

It is clear that not all of $\Delta_k > 0$, $k = 1, 2, \ldots, 6$. Therefore, by Theorem 2.2, the positive equilibrium $(\bar{x}, \bar{y}) = (\sqrt[3]{1-A}, \sqrt[3]{1-B})$ is locally unstable. □

Theorem 3.4 *Consider system (3), and suppose that (10) holds. Then the following statements are true, for* $i = -2, -1, 0$,

(i) $(x_i, y_i) \in (0, \sqrt[3]{1-A}) \times (\sqrt[3]{1-B}, +\infty) \Rightarrow (x_n, y_n) \in (0, \sqrt[3]{1-A}) \times (\sqrt[3]{1-B}, +\infty)$;

(ii) $(x_i, y_i) \in (\sqrt[3]{1-A}, +\infty) \times (0, \sqrt[3]{1-B}) \Rightarrow (x_n, y_n) \in (\sqrt[3]{1-A}, +\infty) \times (0, \sqrt[3]{1-B})$.

Proof (i) Let $(x_i, y_i) \in (0, \sqrt[3]{1-A}) \times (\sqrt[3]{1-B}, +\infty)$ $(i = -2, -1, 0)$, from system (3), we have

$$x_1 = \frac{x_{-2}}{B + y_{-2}y_{-1}y_0} < \frac{\bar{x}}{B + \bar{y}^3} = \bar{x}, \qquad y_1 = \frac{y_{-2}}{A + x_{-2}x_{-1}x_0} < \frac{\bar{y}}{A + \bar{x}^3} = \bar{y}. \tag{13}$$

We prove by induction that

$$(x_n, y_n) \in \left(0, \sqrt[3]{1-A}\right) \times \left(\sqrt[3]{1-B}, +\infty\right). \tag{14}$$

Suppose that (14) is true for $n = k > 1$. Then from (3), we have

$$x_{k+1} = \frac{x_{k-2}}{B + y_{k-2}y_{k-1}y_k} < \frac{\bar{x}}{B + \bar{y}^3} = \bar{x}, \qquad y_{k+1} = \frac{y_{k-2}}{A + x_{k-2}x_{k-1}x_k} < \frac{\bar{y}}{A + \bar{x}^3} = \bar{y}. \tag{15}$$

Therefore, (14) is true. This completes the proof of (i). Similarly, we can obtain the proof of (ii). Hence, it is omitted. □

4 Conclusion and future work

Since the system of the difference equation (3) is the extension of the third-order equation in [4] in the six-dimensional space. In this paper, we investigated the local behavior of solutions of the system of difference equation (3) using linearization. But as we saw linearization do not say anything about the global behavior and fails when the eigenvalues have modulus one. Some powerful tools such as semiconjugacy and weak contraction in [4] cannot be used to analyze global behavior of system (3). The global behavior of the system (3) will be next our aim to study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors indicated in parentheses made substantial contributions to the following tasks of research: drafting the manuscript (QH Zhang, LH Yang); participating in the design of the study (JZ Liu); writing and revision of the paper (QH Zhang, LH Yang).

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