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Dynamics of generalized Poisson and Nambu–Poisson brackets

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A unified setting for generalized Poisson and Nambu–Poisson brackets is discussed. It is proved that a Nambu–Poisson bracket of even order is a generalized Poisson bracket. Characterizations of Poisson morphisms and generalized infinitesimal automorphisms are obtained as coisotropic and Lagrangian submanifolds of product and tangent manifolds, respectively. © *1997 American Institute of Physics.* [S0022-2488(97)04305-3]

I. INTRODUCTION

In 1973 Nambu[1] proposed a generalization of Hamiltonian mechanics to the case of a three-dimensional phase space instead of the usual phase space of two canonical variables (q,p). He considered three dynamical variables (x_1,x_2,x_3) and two Hamiltonian functions h_1 and h_2 with the following motion equations:

$$\dot{f}=\{h_1,h_2,f\},$$

where the bracket $\{f_1,f_2,f_3\}$ of three arbitrary functions f_1,f_2,f_3 is given by

$$\{f_1,f_2,f_3\}=\frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)}.$$

Nambu mechanics was widely discussed by many authors, but was essentially forgotten for almost 20 years. A recent paper by Takhtajan[2] gave a new interest to this subject. In Takhtajan's paper, a geometrical setting for Nambu brackets is introduced. He also established the so-called fundamental identity, which is a generalization of the Jacobi identity. Let us recall that the Jacobi identity permits one to obtain new constants of motion from the old ones. So, a Nambu bracket is a natural generalization of Poisson brackets.

On the other hand, Azcarraga *et al.*[3] have considered an alternative way to generalize Poisson brackets. Since a bracket of n functions can be obtained from a skew-symmetric contravariant

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tensor of order n, they have deduced a different fundamental identity from an integrability condition on the tensor. It should be noted that this way implies that only brackets of even order are allowable.

Our purpose is to give a unified treatment for both kinds of brackets, in order to put in evidence their similarities as well as their differences. So, we consider general brackets of functions defined for skew-symmetric contravariant tensors. More precisely, if Λ is a skew-symmetric contravariant tensor of order n on a manifold M (an almost Poisson tensor, in our terminology), it defines an n-bracket $\{,...,\}$ on the algebra of C^∞ functions on M by the formula

$$\Lambda(df_1,\ldots,df_n)=\{f_1,\ldots,f_n\}.$$

Next, we investigate the integrability conditions on Λ in two directions. For $n=2$, we are in the presence of ordinary almost Poisson tensors, for which the integrability condition can be alternatively expressed in terms of the vanishing of the Schouten–Nijenhuis bracket $[\Lambda,\Lambda]$ or in terms of the Jacobi identity. The first direction gives the so-called generalized Poisson tensors discussed by Azcarraga *et al.*,[3] and the second way gives the Nambu–Poisson tensors discussed by Takhtajan.[2]

In this paper, we also discuss the dynamics of n-brackets. In fact, we extend the results of Tulczyjew,[4] who characterized a locally Hamiltonian vector field on a symplectic manifold (M,Ω) as a Lagrangian submanifold of the symplectic manifold (TM,Ω^c), where TM is the tangent bundle of M and Ω^c is the complete or tangent lift of Ω to TM. Recently, this result was extended by Grabowski and Urbánski[5] for Poisson manifolds (see also Ref. 6), and by us[7] for Jacobi manifolds. The equivalent notion of a locally Hamiltonian vector field is the so-called infinitesimal automorphism of the generalized almost Poisson tensor, that is, a vector field X such that $\mathscr{L}_X\Lambda=0$. With a suitable notion of Lagrangian submanifold, we prove that a vector field X on a generalized almost Poisson manifold (M,Λ) is an infinitesimal automorphism if and only if its image $X(M)$ is a Lagrangian submanifold of the induced structure (TM,Λ^c) (Theorem VI.2). It should be noted that no integrability condition is invoked in order to obtain our result. In fact, if (M,Λ) is a generalized Poisson manifold, so is (TM,Λ^c). However, if (M,Λ) is Nambu–Poisson, (TM,Λ^c) is not Nambu–Poisson except in the trivial case (Theorem VI.1). A Darboux theorem is also obtained, and the global structure of a Nambu–Poisson manifold is elucidated. If the order of Λ is greater or equal to 3, the distribution spanned by the Hamiltonian vector fields is completely integrable and, then, it defines a foliation whose leaves are either n-dimensional manifolds endowed with a Nambu–Poisson bracket coming from a volumen form, or points. From these results, it follows that if Λ is Nambu–Poisson, then $[\Lambda,\Lambda]=0$, which shows that every Nambu–Poisson manifold of even order is also a generalized Poisson manifold. The converse is not true, as Example II.5 shows. For $n=2$ (Poisson manifolds) we have, of course, the well-known symplectic foliation. This means that the basic Nambu–Poisson structures are given by symplectic or volume forms. Both are examples of multisymplectic structures (see Ref. 8), and it seems very interesting to look for structures which would give a more general kind of foliation by multisymplectic leaves.

In what concerns Poisson morphisms, we introduce the notion of Poisson morphism in this general setting, and prove that a morphism $\phi:(M_1,\Lambda_1)\to(M_2,\Lambda_2)$ between two generalized almost Poisson manifolds (M_1,Λ_1) and (M_2,Λ_2) is Poisson if and only its graph is a coisotropic submanifold of the generalized almost Poisson manifold $(M_1\times M_2,\Lambda_1-\Lambda_2)$ (Theorem V.2). This theorem extends the well-known ones for symplectic and Poisson manifolds.[9,5,6,10,11] As above, no integrability conditions are required.

II. GENERALIZED ALMOST POISSON BRACKETS

Let M be a differentiable manifold of dimension m. Our purpose is to define brackets of functions on M which generalize the well-known Poisson bracket. Denote by $\mathscr{X}(M)$ the Lie algebra of the vector fields on M and by $C^\infty(M,\mathbb{R})$ the algebra of C^∞ real-valued functions on M.

Definition II.1: An almost Poisson bracket of order n on M is an n-linear mapping $\{,\ldots,\}:C^\infty(M,\mathbb{R})\times\cdots\times C^\infty(M,\mathbb{R})\to C^\infty(M,\mathbb{R})$ *satisfying the following properties:*
(1) (Skew-symmetry)

$$\{f_1,\ldots,f_n\}=(-1)^{\epsilon(\sigma)}\{f_{\sigma(1)},\ldots,f_{\sigma(n)}\},$$

for all $f_1,\ldots,f_n\in C^\infty(M,\mathbb{R})$ *and* $\sigma\in Symm(n)$, *where* $Symm(n)$ *is a symmetric group of n elements and* $\epsilon(\sigma)$ *is the parity of permutation* σ.
(2) (Leibniz rule)

$$\{f_1g_1,\ldots,f_n\}=f_1\{g_1,\ldots,f_n\}+g_1\{f_1,\ldots,f_n\},$$

for all $f_1,\ldots,f_n,\ g_1\in C^\infty(M,\mathbb{R})$.
An alternative way to define an n-bracket of functions is to consider the skew-symmetric tensor Λ of type $(n,0)$ given by

$$\Lambda_x(df_1(x),\ldots,df_n(x))=\{f_1,\ldots,f_n\}(x),$$

for all $f_1,\ldots,f_n\in C^\infty(M,\mathbb{R})$ and $x\in M$. Conversely, given a skew-symmetric contravariant n-tensor, the above formula defines an n-bracket of functions satisfying (*1*) and (*2*). Notice that (*1*) is equivalent to the skew-symmetric character of Λ, and (*2*) is a consequence of its tensorial character, that is, Λ is linear on functions. In fact, the Leibniz rule says that for any $x\in M$, $\{f_1,\ldots,f_n\}(x)$ only depends on the 1-jets of the functions $f_1,\ldots,f_n$ at x. Thus, we call Λ an *almost Poisson n-tensor*, and (M,Λ) a *generalized almost Poisson manifold.*

It should be noted that Λ induces a linear mapping

$$\#:\Omega^{n-1}(M)\to\mathscr{X}(M)$$

by defining

$$\langle\#(\alpha_1\wedge\cdots\wedge\alpha_{n-1}),\beta\rangle=\Lambda(\alpha_1,\ldots,\alpha_{n-1},\beta),$$

for all $\alpha_1,\ldots,\alpha_{n-1},\ \beta\in\Omega^1(M)$, where $\Omega^r(M)$ is the space of r-forms on the manifold M. Therefore, given $n-1$ functions $f_1,\ldots,f_{n-1}$, we define a vector field

$$X_{f_1,\ldots,f_{n-1}}=\#(df_1\wedge\cdots\wedge df_{n-1}),$$

which is called the *Hamiltonian vector field* associated with the Hamiltonian functions $f_1,\ldots,f_{n-1}$. The fact that $X_{f_1,\ldots,f_{n-1}}$ is a vector field is a consequence of the Leibniz rule. An interesting problem is to look for the integrability conditions of almost Poisson structures. We can proceed in two different directions. First of all, notice that for $n=2$ the integrability condition is given by the vanishing of the Schouten–Nijenhuis bracket of Λ with itself.[12,10] In fact, $[\Lambda,\Lambda]=0$ is equivalent to the so-called Jacobi identity,

$$\{\{f_1,f_2\},f_3\}+\{\{f_2,f_3\},f_1\}+\{\{f_3,f_1\},f_2\}=0. \tag{1}$$

In such a case, Λ is a Poisson tensor and $\{\,,\}$ is an ordinary Poisson bracket.
Next, we will extend this integrability condition. The skew-symmetry of the Schouten-Nijenhuis bracket $[\,,]$ implies that

$$[\Lambda,\Lambda]=(-1)^{n^2}[\Lambda,\Lambda]$$

for any n-tensor Λ. Thus, if n is odd, we do not obtain any integrability property, since $[\Lambda,\Lambda]$ identically vanishes. So, the vanishing of the Schouten–Nijenhuis bracket provides an integrability condition only in case n is even.

Definition II.2: Let Λ be an almost Poisson tensor of order $2k$ on a manifold M. Λ is said to be a generalized Poisson tensor if $[\Lambda,\Lambda]=0$.

Proposition II.3: Let Λ be an almost Poisson tensor of order $n=2k$. Then Λ is generalized Poisson if and only if the following generalized Jacobi identity (called the fundamental identity) holds

$$\mathrm{Alt}(X_{f_1,\ldots,f_{2k-1}}(\{g_1,\ldots,g_{2k}\}))=0, \tag{2}$$

for all functions $f_1,\ldots,f_{2k-1}$, $g_1,\ldots,g_{2k}$ on M.

Proof: The result follows by a direct computation (Ref. 13).

Remark II.4: Notice that the fundamental identity for generalized Poisson manifolds can also be expressed as follows:

$$\mathrm{Alt}(X_{f_1,\ldots,f_{2k-1}}(X_{g_1,\ldots,g_{2k-1}}(g_{2k})))=0, \tag{3}$$

for all functions $f_1,\ldots,f_{2k-1}$, $g_1,\ldots,g_{2k}$.

Example II.5: Consider on $\mathbb{R}^5$ the 4-vector defined by

$$\Lambda=f\,\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}\wedge\frac{\partial}{\partial x^4}+g\,\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}\wedge\frac{\partial}{\partial x^4}\wedge\frac{\partial}{\partial x^5},$$

where (x^1,x^2,x^3,x^4,x^5) denote the standard coordinates in $\mathbb{R}^5$ and f and g are arbitrary functions on $\mathbb{R}^5$. Since Λ is of order 4, we deduce that $[\Lambda,\Lambda]=0$, which implies that Λ defines a Poisson tensor of order 4. Next, we will compute the Hamiltonian vector fields associated with the coordinate functions. If we denote $X_{ijk}=X_{x^ix^jx^k}$, a direct computation shows that

$$X_{123}=f\,\frac{\partial}{\partial x^4},\qquad X_{124}=-f\frac{\partial}{\partial x^3},$$

$$X_{125}=0,\qquad X_{134}=f\frac{\partial}{\partial x^2},$$

$$X_{135}=0,\qquad X_{145}=0,$$

$$X_{234}=-f\,\frac{\partial}{\partial x^1}+g\,\frac{\partial}{\partial x^5},\qquad X_{235}=-g\frac{\partial}{\partial x^4},$$

$$X_{245}=g\,\frac{\partial}{\partial x^3},\qquad X_{345}=-g\frac{\partial}{\partial x^2}.$$

Thus, the generalized distribution D generated by these Hamiltonian vector fields is not involutive. Take, for instance, $f=x^4$ and $g=1$. In this case, $[X_{123},X_{234}]$ does not belong to D.

The other direction is to look for a generalization of the Jacobi identity of a Poisson bracket (1). Notice that (1) can be equivalently written as follows:

$$X_{f_1}(\{f_2,f_3\})=\{X_{f_1}f_2,f_3\}+\{f_2,X_{f_1}f_3\},$$

that is, the Hamiltonian vector field X_{f_1} is a derivation of the algebra $(C^\infty(M,\mathbb{R}),\{\ ,\ \})$, for every function f_1.

We then introduce the following definition:

Definition II.6: A generalized almost Poisson tensor Λ *of order* n *on a manifold* M *is called Nambu–Poisson if the Hamiltonian vector field* $X_{f_1,\dots,f_{n-1}}$ *is a derivation of the algebra* $(C^\infty(M,\mathbb{R})\times\cdots\times C^\infty(M,\mathbb{R}),\{,\dots,\})$, *for all* $f_1,\dots,f_{n-1}$, *that is, the following fundamental identity holds:*

$$X_{f_1,\dots,f_{n-1}}\{g_1,\dots,g_n\}=\sum_{i=1}^{n}\{g_1,\dots,X_{f_1,\dots,f_{n-1}}g_i,\dots,g_n\},\tag{4}$$

for all functions $f_1,\dots,f_{n-1}$, $g_1,\dots,g_n$ *on* M.

Example II.7: Let M be an oriented n-dimensional manifold and choose a volume form ω. Given n functions $f_1,\dots,f_n$, on M, we define its bracket by the formula

$$df_1\wedge\cdots\wedge df_n=\{f_1,\dots,f_n\}\omega.$$

It is not hard to prove that it is a Nambu–Poisson bracket (see Ref. 14). If we take $M=\mathbb{R}^n$, and ω is the standard volume form $\omega=dx^1\wedge\cdots\wedge dx^n$, we recover the example discussed by Nambu.[1]

Denote by Λ_ω the Nambu–Poisson tensor of order n associated with a volume form ω on an oriented n-dimensional manifold M. Then, it is clear that $\Lambda_\omega\neq 0$ at every point of M. In fact, we have

Proposition II.8: Let (M,Λ) *be an* n*-dimensional Nambu–Poisson manifold of order* n *such that* $\Lambda\neq 0$ *at every point. Then, there exists a volume form* ω *on* M *with* $\Lambda=\Lambda_\omega$.

Proof: Since $\Lambda\neq 0$ at every point, we deduce that M is orientable. Thus, if $\overline{\omega}$ is a volume form on M we obtain that there exists $f\in C^\infty(M,\mathbb{R})$, $f\neq 0$ at every point, such that $\Lambda=f\Lambda_{\overline{\omega}}$. Now, we consider the volume form on M given by $\omega=(1/f)\overline{\omega}$. A direct computation proves that $\Lambda=\Lambda_\omega$.

Remark II.9: From the above definitions it follows that for $n=2$ Nambu–Poisson and generalized Poisson structures are the same geometrical object. However, for $n\geqslant 3$, both kinds of structures are in principle different. Of course, a trivial tensor, namely $\Lambda=0$ always defines a Nambu–Poisson as well as a generalized Poisson structure. Also, there exist Nambu–Poisson structures which are trivially generalized Poisson. For instance, if Λ is a Nambu–Poisson tensor with constant components, it trivially satisfies $[\Lambda,\Lambda]=0$. Also, if a Nambu–Poisson tensor Λ has order n, and dim $M<2n-1$, we have $[\Lambda,\Lambda]=0$. In Corollary III.8 we will give the relationship between generalized Poisson and Nambu–Poisson manifolds.

The following result will be very useful in the sequel.

Proposition II.10: Let (M,Λ) *be a Nambu–Poisson manifold of order* n. *Then the bracket of two Hamiltonian vector fields is also a Hamiltonian vector field.*

Proof: In fact, we have

$$\begin{aligned}X_{f_1,\dots,f_{n-1}}(X_{g_1,\dots,g_{n-1}}g_n)&=X_{g_1,\dots,g_{n-1}}(X_{f_1,\dots,f_{n-1}}g_n)+\sum_{i=1}^{n-1}\{g_1,\dots,X_{f_1,\dots,f_{n-1}}g_i,\dots,g_n\}\\&=X_{g_1,\dots,g_{n-1}}(X_{f_1,\dots,f_{n-1}}g_n)+\sum_{i=1}^{n-1}X_{g_1,\dots,X_{f_1,\dots,f_{n-1}}g_i,\dots,g_{n-1}}(g_n),\end{aligned}$$

which implies that

$$[X_{f_1,\ldots,f_{n-1}},X_{g_1,\ldots,g_{n-1}}]=\sum_{i=1}^{n-1} X_{g_1,\ldots,X_{f_1,\ldots,f_{n-1}}g_i,\ldots,g_{n-1}}.$$

Given an almost Poisson tensor Λ of order n on M and fixing a function $f\in C^\infty(M,\mathbb{R})$, we define a generalized almost Poisson tensor Λ_f of order $n-1$ by setting

$$\Lambda_f=i_{df}\Lambda.$$

Proposition II.11: If Λ is Nambu–Poisson, then Λ_f is also Nambu–Poisson.

Proof: It follows from the fundamental identity (4).

Remark II.12: From Proposition II.11 we deduce that given r functions $f_1,\ldots,f_r$, a Nambu–Poisson tensor Λ of order n defines a Nambu–Poisson tensor $\Lambda_{f_1,\ldots,f_r}=i_{df_1\wedge\cdots\wedge df_r}\Lambda$ of order $n-r$. Thus, we obtain a family of subordinate Nambu–Poisson structures. For $r=n-2$, we obtain a family of Poisson structures, $\Lambda_{f_1,\ldots,f_{n-2}}$. Many of these subordinate Nambu–Poisson structures are compatible. For instance, if $n=3$, Λ_f and Λ_g are compatible for all functions f and g on M, since $\Lambda_f+\Lambda_g=\Lambda_{f+g}$. One can says that a Nambu–Poisson manifold is strongly bi-Hamiltonian.

III. MORPHISMS AND INFINITESIMAL AUTOMORPHISMS

Let (M_1,Λ_1) and (M_2,Λ_2) be two generalized almost Poisson manifolds of order n, and $\phi:M_1\to M_2$ a differentiable mapping.

Definition III.1: ϕ is said to be a Poisson morphism if

$$\{f_1\circ\phi,\ldots,f_n\circ\phi\}_1=\{f_1,\ldots,f_n\}_2\circ\phi,$$

for all functions $f_1,\ldots,f_n\in C^\infty(M_2,\mathbb{R})$, where $\{,\ldots,\}_i$ denotes the bracket associated with Λ_i, $i=1,2$.

The next result is an immediate consequence of the above definition.

Proposition III.2: The following statements are equivalent:

(1) ϕ is a Poisson morphism.

(2) Λ_1 and Λ_2 are ϕ-related, i.e.,

$$(\Lambda_1)(x)(\phi^*\alpha_1,\ldots,\phi^*\alpha_n)=(\Lambda_2)(\phi(x))(\alpha_1,\ldots,\alpha_n),$$

for all $\alpha_1,\ldots,\alpha_n\in T^*_{\phi(x)}M_2$, and $x\in M_1$.

(3) The Hamiltonian vector fields $X_{f_1,\ldots,f_{n-1}}$ and $X_{f_1\circ\phi,\ldots,f_{n-1}\circ\phi}$ are ϕ-related.

(4) The corresponding mappings $\#_1$ and $\#_2$ satisfy the formula

$$(\#_2)_{\phi(x)}=(\phi_*)_x\circ(\#_1)_x\circ(\phi^*)_{\phi(x)},$$

for every $x\in M_1$.

Definition III.3: Let (M,Λ) be a generalized almost Poisson manifold of order n. A vector field X is called an infinitesimal automorphism of Λ if $\mathscr{L}_X\Lambda=0$, where $\mathscr{L}$ denotes the Lie derivative.

Remark III.4: It should be noted that X is an infinitesimal automorphism of Λ if and only if its flow consists of Poisson morphisms.

Proposition III.5. A vector field X on M is an infinitesimal automorphism of Λ if and only if X is a derivation of the algebra $(C^\infty(M,\mathbb{R})\times\cdots\times C^\infty(M,\mathbb{R}),\{,\ldots,\})$.

Proof: In fact, we have

$$
\begin{aligned}
X(\{f_1,\ldots,f_n\}) &= \mathscr{L}_X(\{f_1,\ldots,f_n\}) \\
&= \mathscr{L}_X(\Lambda(df_1,\ldots,df_n)) \\
&= (\mathscr{L}_X\Lambda)(df_1,\ldots,df_n) + \sum_{i=1}^{n} \Lambda(df_1,\ldots,\mathscr{L}_X(df_i),\ldots,df_n) \\
&= (\mathscr{L}_X\Lambda)(df_1,\ldots,df_n) + \sum_{i=1}^{n} \{f_1,\ldots,Xf_i,\ldots,f_n\},
\end{aligned}
$$

where $\mathscr{L}_X$ denotes the Lie derivative with respect to X.

From Proposition III.5 and the definition of Nambu–Poisson manifold, we deduce the following.

Corollary III.6: Let (M,Λ) be a Nambu–Poisson manifold. Then, every Hamiltonian vector field $X_{f_1,\ldots,f_{n-1}}$ is an infinitesimal automorphism of Λ.

Let (M,Λ) be a generalized almost Poisson manifold of order n and consider the space of infinitesimal automorphisms $\mathfrak{X}_\Lambda(M)$, that is,

$$\mathfrak{X}_\Lambda(M) = \{X \in \mathfrak{X}(M) \,|\, \mathscr{L}_X\Lambda = 0\}.$$

Since $[\mathscr{L}_X,\mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$, for two vector fields X and Y on M, we deduce that $\mathfrak{X}_\Lambda(M)$ defined an involutive distribution which is also invariant.

We can also consider the space of Hamiltonian vector fields. In fact, if D_x denotes the subspace of T_xM spanned by the Hamiltonian vector fields $X_{f_1,\ldots,f_{n-1}}$ evaluated at a point $x \in M$, we obtain a generalized distribution D on M, which will be called the *characteristic distribution.* For $n=2$, it is always involutive, and, in fact, it defines the symplectic foliation of the Poisson manifold.[15,10,11] The things are very different for $n \geqslant 3$.

Theorem III.7: *Let (M,Λ) be a generalized almost Poisson m-dimensional manifold of order n with $n \geqslant 3$.*

(1) If Λ is a generalized Poisson tensor, then D is not in general involutive.

(2) If Λ is Nambu–Poisson, D is completely integrable, and, therefore, it defines a foliation on M such that the restriction of Λ to each leaf defines an induced Nambu–Poisson structure. There are two kinds of leaves. If at a point $x \in M$, we have $\Lambda(x) \neq 0$, then the leaf passing through x has dimension n, and the induced Nambu–Poisson structure on it comes from a volume form. Moreover, there exist local coordinates $(x^1,\ldots,x^n,x^{n+1},\ldots,x^m)$ around x on M such that $\Lambda = (\partial/\partial x^1)\wedge\cdots\wedge(\partial/\partial x^n)$. If $\Lambda_x = 0$, the leaf passing through x reduces to the point x, and the induced Nambu–Poisson structure is trivial.

Proof:

(1) See Example II.5.

(2) That D is involutive is a consequence of Proposition II.10. On the other hand, let $\Phi:\mathbb{R}\times M \to M$ be the flow of the Hamiltonian vector field $X_{f_1,\ldots,f_{n-1}}$. Then, using Corollary III.6, we deduce that

$$(\Phi_t)_*(X_{g_1,\ldots,g_{n-1}}) = X_{h_1,\ldots,h_{n-1}},$$

for $t \in \mathbb{R}$ and $g_1,\ldots,g_{n-1} \in C^\infty(M,\mathbb{R})$, where $h_i \in C^\infty(M,\mathbb{R})$ is given by $h_i = g_i \circ \Phi_{-t}$, for $i \in \{1,\ldots,n-1\}$. From the above results, we conclude that the characteristic distribution is completely integrable (see Ref. 10, Theorem 2.6), and it defines a generalized foliation in the sense of Sussmann.[16]

Now, consider a leaf L of this foliation. If we take n functions $f_1,\ldots,f_n$ defined on L, a bracket $\{f_1,\ldots,f_n\}_{|L}$ can be defined as follows. We extend each f_j, $1 \leqslant j \leqslant n$ to a function $\tilde{f}_j$ on M and put

$$\{f_1,\ldots,f_n\}_{|L}(y)=(\{\tilde{f}_1,\ldots,\tilde{f}_n\})(y),$$

for every $y\in L$. Since the bracket is a derivation on each argument, we deduce that the result is independent of the chosen extensions. Of course, $\{,\ldots,\}_{|L}$ satisfies the fundamental identity.

Next, suppose that x is a point of M such that $\Lambda(x)\neq 0$. Using the results obtained by Gautheron,[14] we deduce that the n-vector $\Lambda(x)$ is decomposable. Thus, the dimension of the space $D(x)$ is n. Therefore, if L is the leaf of D passing through x we obtain that the dimension of L is n. Denote by Λ_L the induced Nambu–Poisson tensor of order n on L. If $y\in L$, then, since $\Lambda(y)\neq 0$ [notice that dim $D(y)=n$], we have that $(\Lambda_L)(y)\neq 0$. Consequently, from Proposition II.8, we conclude that there exists a volume form ω on L such that $\Lambda_L=\Lambda_\omega$.

Moreover, there exist local coordinates $(x^1,\ldots,x^n,x^{n+1},\ldots,x^m)$ around x such that

$$\Lambda=\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^n}.$$

In fact, if $\Lambda(x)\neq 0$, there exist functions $f_1,\ldots,f_n$ such that $X_{f_1,\ldots,f_{n-1}}(x)(f_n)=\{f_1,\ldots,f_n\}(x)\neq 0$. Thus, there is an open neighborhood V around x and a function g on V with

$$\{f_1,\ldots,f_{n-1},g\}=1$$

at every point of V. Consider the following set of vector fields:

$$\{X_{f_1,\ldots,f_{n-1}},X_{g,f_2,\ldots,f_{n-1}},X_{f_1,g,f_3,\ldots,f_{n-1}},\ldots,X_{f_1,\ldots,f_{n-2},g}\}.$$

From Proposition II.10, it follows that

$$[X_{f_1,\ldots,f_{n-1}},X_{f_1,\ldots,f_{i-1},g,f_{i+1},\ldots,f_{n-1}}]=0,$$

$$[X_{f_1,\ldots,f_{i-1},g,f_{i+1},\ldots,f_{n-1}},X_{f_1,\ldots,f_{j-1},g,f_{j+1},\ldots,f_{n-1}}]=0,$$

where $i,j\in\{1,\ldots,n-1\}$. Since $X_{f_1,\ldots,f_{i-1},g,f_{i+1},\ldots,f_{n-1}}(g)=0$, and taking into account that the above set of vector fields on V are linearly independent, from the Frobenius theorem we obtain a set of independent functions $\{f_1,\ldots,f_{n-1},g,y^{n+1},\ldots,y^m\}$ on an open neighborhood $\bar{V}$ of x, with $\bar{V}\subset V$ such that

$$X_{f_1,\ldots,f_{n-1}}=\frac{\partial}{\partial g},\quad X_{g,f_2,\ldots,f_{n-1}}=-\frac{\partial}{\partial f_1},\ldots,X_{f_1,\ldots,f_{n-2},g}=(-1)^n\,\frac{\partial}{\partial f_{n-1}}.$$

Thus, if we rename the coordinates as $x^1=f_1,\ldots,x^{n-1}=f_{n-1}$, $x^n=g$, $x^{n+1}=y^{n+1},\ldots,x^m=y^m$, we obtain that

$$\Lambda=\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^n}+\Lambda_0,$$

where Λ_0 does not depend on the coordinates $x^1,\ldots,x^{n-1},x^n$. We omit the proof of the latter assertion, since it follows by applying the fundamental identity. Taking into account that Λ is decomposable on $\bar{V}$, we deduce that D has constant dimension n on $\bar{V}$. Thus, it immediately follows that $\Lambda_0=0$.

If $\Lambda(x)=0$, then all the Hamiltonian vector fields $X_{f_1,\ldots,f_{n-1}}$ vanish at x. Therefore, the leaf L through x reduces to the point x, namely $L=\{x\}$.

Corollary III.8: Every Nambu–Poisson manifold of even order is generalized Poisson, but the converse does not hold.

Proof: Let Λ be a Nambu–Poisson tensor of order $n=2k$. If $\Lambda(x)\neq 0$, we deduce that there exist local coordinates $(x^1,\ldots,x^{2k})$ such that

$$\Lambda=\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^{2k}},$$

which implies that $[\Lambda,\Lambda]=0$ at every point of the coordinate neighborhood. Since $\Lambda(x)=0$ implies $[\Lambda,\Lambda](x)=0$ (see Ref. 10), we deduce that $[\Lambda,\Lambda]$ identically vanishes.

Remark III.9: It should be noted that, since a Nambu–Poisson structure of order n has many subordinate Nambu–Poisson structures, we can apply Theorem III.7. If $\Lambda_{f_1,\ldots,f_{n-2}}$ is a subordinate Poisson structure, we deduce that the leaves of its characteristic foliation are oriented surfaces and/or points.

IV. LAGRANGIAN SUBMANIFOLDS

Let (M,Λ) be a generalized almost Poisson manifold of order n. Given a submanifold N of M, we define the jth annihilator of the tangent space T_xN of N at a point $x\in N$, $1\leqslant j\leqslant n-1$, as follows:

$$\mathrm{Ann}^j(T_xN)=\{\alpha\in\Lambda^{n-1}(T_x^*M)|i_{v_1\wedge\cdots\wedge v_j}\alpha=0,\forall v_1,\ldots,v_j\in T_xN\}.$$

We have

$$\mathrm{Ann}^1(T_xN)\subseteq\mathrm{Ann}^2(T_xN)\subseteq\cdots\subseteq\mathrm{Ann}^{n-1}(T_xN).$$

We introduce the following definitions.

Definition IV.1: (1) We say that N is j-coisotropic if

$$\#(\mathrm{Ann}^j(T_xN))\subseteq T_xN.$$

(2) We say that N is j-Lagrangian if

$$\#(\mathrm{Ann}^j(T_xN))=T_xN\cap\#(\Lambda^{n-1}(T_x^*M)).$$

If Λ is Nambu–Poisson, and Λ_f is a subordinate Nambu–Poisson tensor of order $n-1$ for a fixed function f on M, we deduce the following.

Proposition IV.2: If N is j-coisotropic in (M,Λ), and f weakly vanishes on N, then N is j-coisotropic in $(M,\ \Lambda_f)$.

Proof: If $f\approx 0$ on N, that is, f belongs to the ideal of functions which define N as a submanifold of M, then $v(f)=0$, for all $v\in TN$. The result follows taking into account the following facts:

(i) If we denote by $\#_f$ the linear mapping $\#_f:\Omega^{n-2}(M)\rightarrow\mathscr{X}(M)$ induced by Λ_f, we get $\#_f(\alpha)=\#(df\wedge\alpha)$.

(ii) If $\alpha\in\mathrm{Ann}^j(TN)$ (with respect to Λ_f,) then $df\wedge\alpha\in\mathrm{Ann}^j(TN)$ (with respect to Λ).

V. GRAPHS OF POISSON MORPHISMS

Let (M_1,Λ_1) and (M_2,Λ_2) be two generalized almost Poisson manifolds of order n, and $\phi:M_1\rightarrow M_2$ a differentiable mapping. Consider the product manifold $M=M_1\times M_2$ endowed with the following generalized almost Poisson tensor: $\Lambda=\Lambda_1-\Lambda_2$. We have

Proposition V.1: (1) *If (M_1,Λ_1) and (M_2,Λ_2) are generalized Poisson, then (M,Λ) is also generalized Poisson.*

(2) Assume that $n\geqslant 3$. If (M_1,Λ_1) and (M_2,Λ_2) are Nambu–Poisson, then (M,Λ) is not Nambu–Poisson, except in the trivial case when $\Lambda_1=0$ or $\Lambda_2=0$.

Proof: (1) In fact, we have

$$[\Lambda,\Lambda]=[\Lambda_1-\Lambda_2,\Lambda_1-\Lambda_2]=0,$$

since $[\Lambda_1,\Lambda_1]=0$, $[\Lambda_2,\Lambda_2]=0$ and $[\Lambda_1,\Lambda_2]=0$. (2) If Λ would be Nambu–Poisson, and there exists a point $(x_0,y_0)\in M$ such that $\Lambda_1(x_0)\neq 0$ and $\Lambda_2(y_0)\neq 0$, then the characteristic distribution of (M,Λ) at the point (x_0,y_0) would have dimension $2n$. But this is not possible, since Λ is of order n.

Theorem V.2: *Let (M_1,Λ_1) and (M_2,Λ_2) be two generalized almost Poisson manifolds of order n, and $\phi: M_1\to M_2$ a differentiable mapping. Then, ϕ is a Poisson morphism if and only if the graph of ϕ is a $(n-1)$-coisotropic submanifold of $(M=M_1\times M_2,\Lambda=\Lambda_1-\Lambda_2)$.*

Proof: Let us recall that

$$\text{Graph }\phi=\{(x,\phi(x))\,|\,x\in M_1\}\subset M_1\times M_2.$$

Thus, we have

$$T_{(x,\phi(x))}\text{Graph }\phi=\{(v_x,\phi_*(v_x))\,|\,v_x\in T_xM_1\},$$

for $x\in M_1$. Therefore, a direct computation shows that

$$\text{Ann}^{n-1}(T_{(x,\phi(x))}\text{Graph }\phi)=\{(-\phi^*\lambda_2,\lambda_2)\,|\,\lambda_2\in\Lambda^{n-1}(T^*_{\phi(x)}M_2)\}.$$

Consequently, we deduce that

$$\#_{(x,\phi(x))}(\text{Ann}^{n-1}(T_{(x,\phi(x))}\text{Graph }\phi))\subset T_{(x,\phi(x))}\text{Graph }\phi$$

if and only if

$$(\#_2)_{\phi(x)}(\lambda_2)=((\phi_*)(x)\circ(\#_1)_x\circ(\phi^*)(\phi(x)))(\lambda_2),\quad \forall\lambda_2\in\Lambda^{n-1}(T^*_{\phi(x)}M_2),$$

or, in other words, ϕ is a Poisson morphism (see Proposition III.2).

Remark V.3: Notice that Theorem V.2 is the generalization of the well-known result for Poisson manifolds.[6,10,9]

VI. TANGENT LIFTS OF BRACKETS

Let (M,Λ) be a generalized almost Poisson manifold, and denote by TM its tangent bundle. The canonical projection is denoted by $\tau_M: TM\to M$. If (x^i), $i=1,\ldots,m$ are local coordinates in M, we will use the notation $(x^i,\dot{x}^i)$ for the induced coordinates in TM.

Let Λ^c be the complete lift of Λ to TM (see Ref. 17). We recall that Λ^c is a skew-symmetric $(n,0)$-tensor field on TM characterized by the following formulas:

$$\Lambda^c(\alpha_1^c,\alpha_2^c,\ldots,\alpha_n^c)=(\Lambda(\alpha_1,\alpha_2,\ldots,\alpha_n))^c,$$

$$\Lambda^c(\alpha_1^v,\alpha_2^c,\ldots,\alpha_n^c)=(\Lambda(\alpha_1,\alpha_2,\ldots,\alpha_n))^v,$$

$$\Lambda^c(\alpha_1^v,\alpha_2^v,\ldots,\alpha_n^c)=0,$$

$$\vdots$$

$$\Lambda^c(\alpha_1^v,\alpha_2^v,\ldots,\alpha_n^v)=0,$$

for all 1-forms $\alpha_1,\ldots,\alpha_n$ on M, where α_i^v and α_i^c are, respectively, the vertical and complete lifts of the 1-form α_i to TM, and f^v, f^c are, respectively, the vertical and complete lifts of a function f on M to TM.

Therefore, for the corresponding bracket $\{,\ldots,\}^T$ on TM, we have

$$\{f_1^c,f_2^c,\ldots,f_n^c\}^T=\{f_1,f_2,\ldots,f_n\}^c,$$

$$\{f_1^v,f_2^c,\ldots,f_n^c\}^T=\{f_1,f_2,\ldots,f_n\}^v,$$

$$\{f_1^v,f_2^v,\ldots,f_n^c\}^T=0,$$

$$\vdots$$

$$\{f_1^v,f_2^v,\ldots,f_n^v\}^T=0.$$

If $\Lambda^{i_1,\ldots,i_n}$ are the local components of Λ, the local components of Λ^c are given by the following formulas:

$$(\Lambda^c)^{\overline{i}_1\overline{i}_2,\ldots,\overline{i}_n}=\dot{x}^l\,\frac{\partial\Lambda^{i_1i_2,\ldots,i_n}}{\partial x^l},$$

$$(\Lambda^c)^{i_1\overline{i}_2,\ldots,\overline{i}_n}=\Lambda^{i_1i_2,\ldots,i_n},$$

$$(\Lambda^c)^{i_1i_2,\ldots,\overline{i}_n}=0,$$

$$\vdots$$

$$(\Lambda^c)^{i_1i_2,\ldots,i_n}=0,$$

where $\overline{i}$ means that we are evaluating Λ^c on $d\dot{x}^i$.

Theorem VI.1: *Let (M,Λ) be a generalized almost Poisson manifold of order n and (TM,Λ^c) its tangent lift. Then,*
(1) (M,Λ) is a generalized Poisson manifold if and only if (TM,Λ^c) is so also.
(2) If (M,Λ) is a Nambu–Poisson manifold and $n\geqslant 3$, (TM,Λ^c) is not Nambu–Poisson, except in the trivial case $\Lambda=0$.

Proof: (1) The result follows taking into account the formula

$$[\Lambda^c,\Lambda^c]=[\Lambda,\Lambda]^c,$$

(see Ref. 5) and the fact that for a contravariant n-tensor Γ on M, Γ vanishes if and only if Γ^c does so (see Ref. 17).
(2) Let Λ be a Nambu–Poisson tensor of order $n\geqslant 3$. Assume that Λ^c is Nambu–Poisson. If $\Lambda(x)\neq 0$, then $\Lambda^c(y)\neq 0$, for any tangent vector y at x. Since the characteristic distribution is generated by the vertical and complete lifts of the Hamiltonian vector fields defined by Λ on M, we deduce that it is the tangent lift of the characteristic distribution on M. Thus, it has dimension $2n$ at the points $y\in T_xM$ where $\Lambda(x)\neq 0$. Consequently, Λ^c is not Nambu-Poisson, except if Λ identically vanishes.

Theorem VI.2: *Let (M,Λ) be a generalized almost Poisson m-dimensional manifold and X a vector field on M. Then the image $X(M)$ of X is a 1-Lagrangian submanifold of (TM,Λ^c) if and only if X is an infinitesimal automorphism of Λ,* i.e., $\mathscr{L}_X\Lambda=0$.

Proof: Assume that the local expression of X is $X=X^i\partial/\partial x^i$ in some local coordinate system (x^i) on M. Therefore, the submanifold $X(M)$ is locally defined as follows:

$$x^i = x^i, \quad \dot{x}^i = X^i,$$

in the induced coordinates $(x^i, \dot{x}^i)$ in TM.
Define the local frame $\{B_i, C_i;\ i = 1,\ldots,m\}$ of TM along $X(M)$ by

$$B_i = X_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial \dot{x}^j}, \quad C_i = \frac{\partial}{\partial \dot{x}^i}.$$

[It should be noted that the local vector fields $\{B_i;\ i=1,\ldots,m\}$ are tangent to the submanifold $X(M)$]. Its dual local coframe $\{B_i^*, C_i^*\}$ along $X(M)$ is

$$B_i^* = dx^i, \quad C_i^* = d\dot{x}^i - \frac{\partial X^i}{\partial x^j}\, dx^j.$$

Next, we will compute Λ^c along $X(M)$,

$$\Lambda^c(C^*_{i_1},\ldots,C^*_{i_{n-1}},C^*_{i_n}) = \frac{\partial \Lambda^{i_1\cdots i_n}}{\partial x^l} X^l - \sum_{s=1}^{n} \frac{\partial X^{i_s}}{\partial x^j} \Lambda^{i_1,\ldots,j,\ldots,i_n},$$

$$\Lambda^c(C^*_{i_1},\ldots,C^*_{i_{n-1}},B^*_{i_n}) = \Lambda^{i_1,\ldots,i_n},$$

$$\Lambda^c(C^*_{i_1},\ldots,B^*_{i_{n-1}},B^*_{i_n}) = 0,$$

$$\vdots$$

$$\Lambda^c(B^*_{i_1},\ldots,B^*_{i_{n-1}},B^*_{i_n}) = 0.$$

From the above formulas, we get

$$\#^T(C^*_{i_1}\wedge\cdots\wedge C^*_{i_{n-1}}) = \Lambda^{i_1,\ldots,i_n} B_{i_n} + \left(\frac{\partial \Lambda^{i_1,\ldots,i_n}}{\partial x^l} X^l - \sum_{s=1}^{n} \frac{\partial X^{i_s}}{\partial x^j} \Lambda^{i_1,\ldots,j,\ldots,i_n}\right) C_{i_n},$$

where $\#^T$ denotes the induced mapping from Λ^c.
Since for every $y \in X(M)$, $\mathrm{Ann}^1(T_y(X(M)))$ is locally generated by the exterior powers of the 1-forms $\{C_i^*;\ i=1,\ldots,m\}$, we deduce that $X(M)$ is 1-Lagrangian if and only if

$$\frac{\partial \Lambda^{i_1,\ldots,i_n}}{\partial x^l} X^l - \sum_{s=1}^{n} \frac{\partial X^{i_s}}{\partial x^j} \Lambda^{i_1,\ldots,j,\ldots,i_n} = 0. \tag{5}$$

On the other hand, if we write

$$\Lambda = \frac{1}{k!} \Lambda^{i_1,\ldots,i_n} \frac{\partial}{\partial x^{i_1}} \wedge\cdots\wedge \frac{\partial}{\partial x^{i_n}},$$

we deduce that $\mathscr{L}_X\Lambda$ has components

$$(\mathscr{L}_X\Lambda)^{i_1,\ldots,i_n} = \sum_{s=1}^{n} \Lambda^{i_1,\ldots,j,\ldots,i_n} \frac{\partial X^{i_s}}{\partial x^j} - X^j \frac{\partial \Lambda^{i_1,\ldots,i_n}}{\partial x^j}. \tag{6}$$

From (5) and (6) the result follows.

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