# DYNAMICS OF HOMEOMORPHISMS ON MINIMAL SETS GENERATED BY TRIANGULAR MAPPINGS 

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#### Abstract

The main goal of the paper is the construction of a triangular mapping $F$ of the square with zero topological entropy, possessing a minimal set $M$ such that $\left.F\right|_{M}$ is a strongly chaotic homeomorphism, as well as other properties that are impossible for continuous maps on an interval.

To do this we define a parametric class of triangular maps on $Q \times I$, where $Q$ is an infinite minimal set on the interval, which are extendable to continuous triangular maps $F: I^{2} \rightarrow I^{2}$. This class can be used to create other examples.


## 1. Introduction

Let $I=[0,1]$ be the closed unit interval. Let $\mathcal{C}$ denote the class of continuous maps $f: I \rightarrow I$, and $\Delta$ the class of triangular maps $F: I^{2} \rightarrow I^{2}$, that is, the continuous functions defined by

$$
F(x, y)=(f(x), g(x, y))=\left(f(x), g_{x}(y)\right)
$$

The map $f \in \mathcal{C}$ is the base for $F$, and $g_{x}: I \rightarrow I$ is a family of continuous maps depending continuously on $x$. Note that $F$ transforms the layer $I_{x}:=\{x\} \times I$ into the layer $I_{f(x)}$.

Triangular maps have much simpler dynamics than continuous maps of the square in general [7]. This is because the projection $\pi_{1}:(x, y) \mapsto x$ semiconjugates any $F \in \Delta$ to its base $f$ via $f \circ \pi_{1}=\pi_{1} \circ F$. This implies, for example, that Sharkovsky's theorem on the coexistence of periodic orbits remains valid in $\Delta[6]$. Moreover, the projection $\pi_{1}$ maps the class $\operatorname{Per}(F)$ of periodic points of $F$ onto $\operatorname{Per}(f)$, or the class $U R(F)$ of uniformly recurrent points of $F$ onto $U R(f)$. However there are exceptions: homoclinic orbits [7] or isochronically recurrent points [4] of $F$ are not mapped by $\pi_{1}$ onto the corresponding classes of $f$.

[^0]A big difference between the dynamics of maps in $\mathcal{C}$ and in $\Delta$ already appears in the simplest cases in which every periodic point of $F$ is a fixed point and the base is linear, see [5, 8] (see also [7, Theorem 3]).

However, the class of maps in $\Delta$ of type $2^{\infty}$ (with respect to the Sharkovsky's ordering) is more interesting. There are, for example, maps in $\Delta$ of type $2^{\infty}$ with positive topological entropy [7] but with recurrent points which are not uniformly recurrent [5]. Such maps are impossible in $\mathcal{C}$. In both of the preceeding examples, the map $F$ has a base $f$ of type $2^{\infty}$ with an infinite minimal set $Q$ such that $F$ has "bad" behaviour on the set $\pi_{1}^{-1}(Q)=Q \times I$. (Recall that a set $M$ is a minimal set for a map if it is non-empty, closed and invariant and if no proper subset of $\dot{M}$ has the same properties.)

In the present paper we show that maps of type $2^{\infty}$ in $\Delta$, even homeomorphisms on minimal sets, may have very complicated dynamics. Note that if $M$ is a minimal set for $F$ in $\Delta$, then $\pi_{1}(M)$ is a minimal set for $f$ (this is true for any general semi-conjugacy, see [11]), hence $\pi_{1}(M)$ is either a periodic orbit or a solenoid, that is, a Cantor-type set [1]. The first case, however, implies that $M$ is essentially one-dimensional, so non trivial behaviour is possible only of $\pi_{1}(M)$ is infinite. We shall consider only this case.

In Section 2, starting from a Cantor-type set $Q$ and a map $f: Q \rightarrow Q$ of type $2^{\infty}$, we define a family $\mathcal{T}$ of functions $F$ of type $2^{\infty}$, non-decreasing on any layer and such that $F(Q \times I) \subset Q \times I$. It is always possible to extend each $F \in \mathcal{T}$ to a function $\widetilde{F} \in \Delta$ preserving its type $2^{\infty}$ and the monotonicity on each layer. All these functions have zero topological entropy. Then we define a parametric family $\mathcal{T}_{0} \subset \mathcal{T}$. This construction is based on an idea from [5] and can be further modified to get more general maps.

In Section 3, we construct a subclass $\mathcal{T}_{01}$ of $\mathcal{T}_{0}$ and prove that the maps in this class have a minimal set containing an interval. (The existence of such maps was already proved in [5].)

In Section 4 we show that there are maps in $\mathcal{T}_{01}$ which are distributionally chaotic, and hence, chaotic in the sense of Li and Yorke on a minimal set. Recall that no map in $\mathcal{C}$ having zero topological entropy can be chaotic on a minimal set [3].

In Section 5 we prove some results concerning functions in $\mathcal{T}_{01}$ and in other classes $\mathcal{T}_{02} \subset \mathcal{T}_{0}$ and $\mathcal{T}_{1} \subset \mathcal{T}$. These results show properties which are impossible in $\mathcal{C}$.

## 2. A Parametric Class of triangular maps

Let $\{0,1\}^{\mathbb{N}}$ be the space of all sequences of two symbols equipped with the following metric $\rho: \rho(\underline{\alpha}, \underline{\beta}):=\max \{1 / i: \alpha(i) \neq \beta(i)\}$ for any distinct $\underline{\alpha}=\{\alpha(i)\}_{i \geqslant 1}$ and $\underline{\beta}=$ $\{\beta(i)\}_{i \geqslant 1}$ in $\{0,1\}^{N}$. Since, as is well known, any Cantor-type set $Q$ is homeomorphic to $\{0,1\}^{\mathbf{N}}$, we may identify an element $x \in Q$ with the corresponding sequence $\underline{x}=$ $x(1) x(2) \cdots$.

Consider now the function $f: Q \rightarrow Q$ acting on $Q$ as an adding machine, that is, for $\underline{\alpha} \in\{0,1\}^{N}, f(\underline{\alpha})=\underline{\alpha}+1000 \cdots$ where the adding is in base 2 from the left to right; for example, $f(101100 \cdots)=011100 \cdots, f(11100 \cdots)=00010 \cdots$, and so on. Given a point $\underline{x} \in Q$, the point $f^{s}(\underline{x}) \in Q$ is represented by the sequence $\underline{x}_{s}$ obtained by adding (in base 2) the sequence $\underline{x}$ and the eventually zero sequence representing the number $s$ written in base 2 from left to right. It is easy to see that $\omega_{f}(\underline{x})=Q$ for any $\underline{x} \in Q$.

Denote by $\mathcal{T}$ the class of maps $F: Q \times I \rightarrow Q \times I$, where $Q$ is a Cantortype set and $F(\underline{x}, y)=(f(\underline{x}), g(\underline{x}, y))$ where $f: Q \rightarrow Q$ is the adding machine, and $g(\underline{x}, \cdot): I \rightarrow I$ is continuous and non-decreasing for any $\underline{x} \in Q$, and the family $g(\underline{x}, \cdot)$ depends continuously on $\underline{x}$ with respect to the uniform metric. Thus $F$ is continuous on $Q \times I$.

Note that each map $F \in \mathcal{T}$ (and obviously also its monotonic extension $\tilde{F} \in \Delta$ ) has topologically entropy $h(F)=0$. Indeed, we have (see [7]).

$$
\sup \left\{h\left(F, I_{\underline{x}}\right) ; x \in Q\right\}+h(f) \geqslant h(F),
$$

where $h\left(F, I_{\underline{x}}\right)$ denotes the topological entropy of the map $F: Q \times I \rightarrow Q \times I$ with respect to the compact subset $I_{\underline{x}}$, that is, the entropy $h\left(F, I_{\underline{x}}\right)$ is computed only for trajectories starting from $I_{\underline{x}}$. But since $F^{i}$ is monotonic on $I_{\underline{x}}$ for any $i$, we have clearly $h\left(F, I_{\underline{x}}\right)=0$, and of course $h(f)=0$ since $f$ is of type $2^{\infty}$. Thus, $h(F)=0$.

Now we describe the construction of the mappings of a special subclass $\mathcal{T}_{0}$ of $\mathcal{T}$.
First we take an increasing sequence of non-negative integers $\left\{k_{i}\right\}_{i=0}^{\infty}$ with $k_{0}=0$ and such that, for all $i \geqslant 1, k_{i}-k_{i-1}-1=: m_{i} \geqslant 1$. Thus $k_{n}=k_{n-1}+m_{n}+1=$ $m_{1}+\cdots+m_{n}+n$. For any $\underline{x} \in Q$, the digits $x\left(k_{1}\right), x\left(k_{2}\right), \ldots$ are called control digits of $\underline{x}$. If

$$
\underline{x}=\underbrace{x(1) \cdots x\left(k_{1}-1\right)}_{m_{1}} x\left(k_{1}\right) \cdots x\left(k_{n-1}\right) \underbrace{x\left(k_{n-1}+1\right) \cdots x\left(k_{n}-1\right)}_{m_{n}} x\left(k_{n}\right) \cdots
$$

we define, for every $n \geqslant 1$,
$\chi_{n}(\underline{x}):=\left(x\left(k_{n-1}+1\right), \ldots, x\left(k_{n}-1\right)\right) \in\{0,1\}^{m_{n}} \quad$ and $\quad\left|\chi_{n}(\underline{x})\right|:=\sum_{i=1}^{m_{n}} x\left(k_{n-1}+i\right) 2^{i-1}$.
Then we consider a family $\Gamma_{n}:=\left\{\varphi(n, j), j=0, \ldots, 2^{m_{n}}-1\right\}$ of functions from $I$ into $I$ satisfying the following properties:

$$
\begin{equation*}
\text { each function } \varphi(n, j) \in \Gamma_{n} \text { is continuous and non-decreasing : } \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
\varphi^{r}\left(n, 2^{m_{n}}-1\right) \circ \cdots \circ \varphi^{r}(n, 0)=I d \quad \text { for all } r \geqslant 1 \tag{2}
\end{equation*}
$$

where $I d$ denotes the identity map. We call any map of $\Gamma_{n}$ a map of rank $n$. Moreover we assume that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \max _{j}\{\|\varphi(n, j)-I d\|\}=0 \tag{3}
\end{equation*}
$$

where $\|\cdot\|$ is the uniform norm.
Finally we define a function $F: Q \times I \rightarrow Q \times I$ as follows. Take an arbitrary point $\underline{x} \in Q$. If the first zero control digit of $\underline{x}$ is $x\left(k_{n}\right)$, then we define

$$
F(\underline{x}, y)=\left(f(\underline{x}), \varphi\left(n,\left|\chi_{n}(\underline{x})\right|\right)(y)\right)
$$

Otherwise, if $\underline{x}$ has no zero control digits, we set

$$
F(\underline{x}, y)=(f(\underline{x}), y)
$$

Note that (1) and (3) guarantee the continuity of $F$ in $Q \times I$.
The class $\mathcal{T}_{0}$ consists of the functions constructed in this way for any possible choice of the parameters $k_{n}$ and of the families $\Gamma_{n}$.

Let $\pi_{2}:(x, y) \mapsto y$ be the projection on the second variable and put $t_{n}:=2^{k_{n}-1}$ for all $n \geqslant 0$. (Note that $t_{0}=1 / 2$.) Given $F \in \mathcal{T}_{0}$, for any $i \geqslant 0$ and any $y_{0} \in I$, define $y_{i}:=\pi_{2}\left[F^{i}\left(\underline{0}, y_{0}\right)\right]$. Then, for any integer $i \geqslant 0$ we have

$$
F^{i}\left(\underline{0}, y_{0}\right)=\left(f^{i}(\underline{0}), y_{i}\right)=\left(f^{i}(\underline{0}), \psi(i)\left(y_{0}\right)\right)
$$

where $\psi(0)=I d$ and, if $1 \leqslant i<t_{n}, \psi(i)$ is a composition of maps $\varphi$ of rank not greater than $n$.

For all $0 \leqslant j \leqslant 2^{m_{n+1}}-1$ and $0 \leqslant r<t_{n}$, we have the following relations

$$
\begin{align*}
\psi\left(2 j t_{n}+r\right) & =\psi(r) \circ \psi\left(2 j t_{n}\right)  \tag{4}\\
\psi\left((2 j+1) t_{n}+r\right) & =\psi_{j}^{*}(r) \circ \psi\left(2 j t_{n}\right) \tag{5}
\end{align*}
$$

where $\psi_{j}^{*}(r)$ is the function obtained from $\psi(r)$ by replacing all maps $\varphi$ of rank $n$ with $\varphi(n+1, j)$.

Indeed,

$$
f^{2 j t_{n}}(\underline{0})=\underbrace{0 \cdots 0}_{k_{n}} \xi(1) \cdots \xi\left(m_{n+1}\right) 0 \cdots, \quad f^{(2 j+1) t_{n}}(\underline{0})=\underbrace{0 \cdots 01}_{k_{n}} \xi(1) \cdots \xi\left(m_{n+1}\right) 0 \cdots,
$$

with $\left|\chi_{n+1}\left(f^{2 j t_{n}}(\underline{0})\right)\right|=\left|\chi_{n+1}\left(f^{(2 j+1) t_{n}}(\underline{0})\right)\right|=\left|\left(\xi(1), \ldots, \xi\left(m_{n+1}\right)\right)\right|=j$. This means that after $2 j t_{n}$ iterations, all the first $n$ control digits are zero and so, for the
next $r$ iterations, we apply the same functions $\varphi$ as when starting from $\underline{0}$. Conversely, after $(2 j+1) t_{n}$ iterations the $n$-th control digit is equal to one and so, during the next $r$ iterations we proceed as in the previous case, but instead of using the functions $\varphi$ of rank $n$, we apply the function $\varphi(n+1, j)$. This is exactly what is written in formulas (4) and (5). Obviously, if $r<t_{n-1}$, the function $\psi(r)$ does not contain any map of rank $n$ and so $\psi_{j}^{*}(r)=\psi(r)$.

Note that from (5) with $r=0$ we obtain

$$
\begin{equation*}
\psi\left(t_{n}\right)=I d, \quad \psi\left((2 j+1) t_{n}\right)=\psi\left(2 j t_{n}\right) \tag{6}
\end{equation*}
$$

Now we prove some identities concerning the functions in $\mathcal{T}_{0}$.
Lemma 1. Let $F \in \mathcal{T}_{0}$. For every $i \geqslant 1$ take $n \geqslant 1$ such that $t_{n-1} \leqslant i<t_{n}$ and consider the representation of $i$ in the form

$$
i=\left(2 \alpha_{n-1}+\beta_{n-1}\right) t_{n-1}+\cdots+\left(2 \alpha_{1}+\beta_{1}\right) t_{1}+2 \alpha_{0} t_{0}
$$

with $0 \leqslant \alpha_{s} \leqslant 2^{m_{s+1}}-1, \beta_{s} \in\{0,1\}$ for $0 \leqslant s \leqslant n-1$ and $\beta_{0}=0$. If

$$
\nu(i)=\max \left\{s \leqslant n-1: \beta_{s}=0\right\}
$$

we write

$$
i=\left(2 \alpha_{n-1}+1\right) t_{n-1}+\cdots+\left(2 \alpha_{\nu(i)+1}+1\right) t_{\nu(i)+1}+2 \alpha_{\nu(i)} t_{\nu(i)}+\theta(i) .
$$

Then we have

$$
\begin{equation*}
\psi(i)=\psi(\theta(i)) \circ \varphi^{\gamma(i)}\left(n, \alpha_{n-1}\right) \circ \varphi^{2^{k_{n-1}-(n-1)}}\left(n, \alpha_{n-1}-1\right) \circ \cdots \circ \varphi^{2^{k_{n-1}-(n-1)}}(n, 0) \tag{7}
\end{equation*}
$$

$$
\begin{equation*}
\psi_{u}^{*}(i)=\psi(\theta(i)) \circ \varphi^{\bar{\gamma}(i)}(n+1, u) \tag{8}
\end{equation*}
$$

where $\gamma(i)=\sum_{j=\nu(i)}^{n-2} \alpha_{j} 2^{k_{j}-j}\left(<2^{k_{n-1}-(n-1)}\right)$ and $\bar{\gamma}(i)=\gamma(i)+\alpha_{n-1} 2^{k_{n-1}-(n-1)}$.
In particular, for all $n \geqslant 2$ and all $j$ with $0 \leqslant j \leqslant 2^{m_{n}}-1$,

$$
\begin{equation*}
\psi\left(2 j t_{n-1}\right)=\varphi^{2^{k_{n-1}-(n-1)}}(n, j-1) \circ \cdots \circ \varphi^{2^{k_{n-1}-(n-1)}}(n, 0) \tag{9}
\end{equation*}
$$

Proof: First we prove (7) by induction on $n$. Let $n=1$, that is, $1 \leqslant i<t_{1}$; we have

$$
\psi(i)=\varphi(1, i-1) \circ \cdots \circ \varphi(1,0)
$$

In this case $i=2 \alpha_{0} t_{0}=\alpha_{0}, \nu(i)=0$ and $\theta(i)=\gamma(i)=0$. So (7) is satisfied.

Assume (7) true for $n$ and consider $n+1$. We have to find the representation of $\psi(i)$ for all $i$ with $t_{n} \leqslant i<t_{n+1}$. Let

$$
i=\left(2 \alpha_{n}+\beta_{n}\right) t_{n}+\cdots+\left(2 \alpha_{1}+\beta_{1}\right) t_{1}+2 \alpha_{0} t_{0}
$$

and assume first $\beta_{n}=0$. Then, $i=2 \alpha_{n} t_{n}+\theta(i)$ and, by (4),

$$
\psi(i)=\psi(\theta(i)) \circ \psi\left(2 \alpha_{n} t_{n}\right)
$$

Since in this case $\gamma(i)=0,(7)$ is proved if we show that

$$
\begin{equation*}
\psi\left(2 \alpha_{n} t_{n}\right)=\varphi^{2^{k_{n}-n}}\left(n+1, \alpha_{n}-1\right) \circ \cdots \circ \varphi^{2^{k_{n}-n}}(n+1,0) \tag{10}
\end{equation*}
$$

We prove (10) by induction on $\alpha_{n}$. By the induction hypothesis and the representation

$$
t_{n}-1=\left(2\left(2^{m_{n}}-1\right)+1\right) t_{n-1}+\cdots+2\left(2^{m_{i}}-1\right) t_{0}
$$

we have

$$
\begin{equation*}
\psi\left(t_{n}-1\right)=\varphi^{\gamma\left(t_{n}-1\right)}\left(n, 2^{m_{n}}-1\right) \circ \varphi^{2^{k_{n-1}-(n-1)}}\left(n, 2^{m_{n}}-2\right) \circ \cdots \circ \varphi^{2^{k_{n-1}-(n-1)}}(n, 0) \tag{11}
\end{equation*}
$$

where $\gamma\left(t_{n}-1\right)=\sum_{j=0}^{n-2}\left(2^{m_{j+1}}-1\right) 2^{k_{j}-j}=2^{k_{n-1}-(n-1)}-1$.
Now, by (5) and (11)

$$
\psi\left(t_{n}+\left(t_{n}-1\right)\right)=\psi_{0}^{*}\left(t_{n}-1\right) \circ \psi(0)=\psi_{0}^{*}\left(t_{n}-1\right)=\varphi^{2^{k_{n}-n}-1}(n+1,0)
$$

Since

$$
f^{2 t_{n}-1}(\underline{0})=\underbrace{1 \cdots 1}_{k_{n}} 0 \cdots
$$

at the next iteration we apply the map $\varphi(n+1,0)$, thus

$$
\varphi\left(2 t_{n}\right)=\varphi^{2^{k_{n}-n}}(n+1,0)
$$

hence (10) is proved for $\alpha_{n}=1$. Assume it is true for $\alpha_{n}=j$. By (6)

$$
\psi\left((2 j+1) t_{n}\right)=\psi\left(2 j t_{n}\right)
$$

and by (5) and the induction hypothesis we have

$$
\begin{aligned}
\psi\left((2 j+1) t_{n}\right. & \left.+t_{n}-1\right)=\psi_{j}^{*}\left(t_{n}-1\right) \circ \psi\left(2 j t_{n}\right) \\
& =\varphi^{2^{k_{n}-n}-1}(n+1, j) \circ \varphi^{2^{k_{n}-n}}(n+1, j-1) \circ \cdots \circ \varphi^{2^{k_{n}-n}}(n+1,0)
\end{aligned}
$$

Since

$$
f^{2 j t_{n}+2 t_{n}-1}(\underline{0})=\underbrace{1 \cdots 1}_{k_{n}} \xi(1) \cdots \xi\left(m_{n+1}\right) 0 \cdots
$$

with $\left|\chi_{n+1}\left(f^{2 j t_{n}+2 t_{n}-1}(\underline{0})\right)\right|=\left|\left(\xi(1), \ldots, \xi\left(m_{n+1}\right)\right)\right|=j$, at the next iteration we apply the map $\varphi(n+1, j)$, thus obtaining (10) for $\alpha_{n}=j+1$. Hence (10) is completely proved.

Assume now $\beta_{n}=1$, that is,

$$
i=\left(2 \alpha_{n}+1\right) t_{n}+\cdots+\theta(i)=\left(2 \alpha_{n}+1\right) t_{n}+r
$$

and observe that $\theta(i)=\theta(r)$.
By (5) and (10) we obtain

$$
\psi(i)=\psi_{\alpha_{n}}^{*}(r) \circ \psi\left(2 \alpha_{n} t_{n}\right)=\psi_{\alpha_{n}}^{*}(r) \circ \varphi^{2^{k_{n}-n}}\left(n+1, \alpha_{n}-1\right) \circ \cdots \circ \varphi^{2^{k_{n}-n}}(n+1,0)
$$

If $t_{n-1} \leqslant r<t_{n}$, then $\nu(i)=\nu(r)$ and

$$
\psi_{\alpha_{n}}^{*}(r)=\psi(\theta(r)) \circ \varphi^{\bar{\gamma}(r)}\left(n+1, \alpha_{n}\right)=\psi(\theta(i)) \circ \varphi^{\gamma(i)}\left(n+1, \alpha_{n}\right)
$$

since $\bar{\gamma}(r)=\sum_{j=\nu(i)}^{n-1} \alpha_{j} 2^{k_{j}-j}=\gamma(i)$.
If $r<t_{n-1}$, then $\nu(i)=n-1, \alpha_{n-1}=0$ and so $\bar{\gamma}(r)=\gamma(i)=0$; in this case

$$
\psi_{\alpha_{n}}^{*}(r)=\psi(\theta(r))=\psi(\theta(i))
$$

Thus (7) is proved for $n+1$.

## 3. Properties of minimal sets for maps in $\mathcal{T}$ and $\mathcal{T}_{0}$

Theorem 1. No $F \in \mathcal{T}$ can have a minimal set with non-empty interior in $Q \times I$.
Proof: Assume there is a function $F \in \mathcal{T}$ with a minimal set $M$ containing a nonempty open set $G$ of $Q \times I$. We may assume, without loss of generality, $G \subset Q \times(0,1)$.

Since $\pi_{1}(M)$ is minimal for the base map, $\pi_{1}(M)=Q$ and so, for any $\underline{x} \in Q$ the set $M \cap I_{\underline{x}}$ is non-empty. Let $\underline{x}_{0} \in Q$ and $M_{0}:=M \cap I_{x_{0}}$; define $y_{0}=\max \{y$ : $\left.\left(\underline{x}_{0}, y\right) \in M\right\}$. By the minimality of $M$ we have $\omega_{F}\left(\underline{x}_{0}, y_{0}\right)=M$, hence there is an integer $n$ such that $\left(\underline{x}_{n}, y_{n}\right):=F^{n}\left(\underline{x}_{0}, y_{0}\right) \in G$. Since $\underline{x}_{0}$ is the unique preimage of $\underline{x}_{n}$ with respect to $f^{n}$ and $F(M)=M$ (see [1]), we have $F^{n}\left(M_{0}\right)=M \cap I_{\underline{x}_{n}}$. But $\left.F^{n}\right|_{M_{0}}$ is non-decreasing and so $y_{n}=\max \left\{y:\left(\underline{x}_{n}, y\right) \in M\right\}$, contrary to the fact that $\sup \left\{y:\left(\underline{x}_{n}, y\right) \in M\right\}>y_{n}$.

THEOREM 2. Suppose that $F \in \mathcal{T}_{0}$ has a minimal set $M$ containing the layer $I_{\underline{0}}$. Then $\left.F\right|_{M}$ is a homeomorphism.

Proof: Since $M$ is a compact set and $F$ is continuous, $\left.F\right|_{M}$ is a homeomorphism if and only if it is one-to-one on any set $M_{\underline{x}}=M \cap I_{\underline{x}}, \underline{x} \in Q$. Consider first the case $\underline{x} \in \operatorname{Orb}(\underline{0})$, that is, $\underline{x}=f^{s}(\underline{0})$ for some $s \geqslant 0$ and let $t_{n}>s$. By $(6), \psi\left(t_{n}\right)=I d$ and this implies that at any step $j<t_{n}$ the function $\varphi$ to be applied to is injective on $\pi_{2}\left[F^{j}\left(I_{\underline{0}}\right)\right]$. Thus $F$ is injective on $F^{s}\left(I_{\underline{0}}\right)$, which, by the minimality of $M$, equals $M_{\underline{x}}$. Take now an aribtrary point $\underline{x} \in Q \backslash \operatorname{Orb}(\underline{0})$. If all control digits of $\underline{x}$ are equal to one, then the function to be applied to is the identity. Assume now that the first zero control digit of $\underline{x}$ is $x\left(k_{n}\right)$ and take the neighbourhood $U$ of $\underline{x}$ in $Q$ given by all $\underline{t} \in Q$ with the first $k_{n}$ digits of their representations equal to those of $\underline{x}$, that is, $\underline{t}(i)=\underline{x}(i)$, $1 \leqslant i \leqslant k_{n}$. Thus, for every $\underline{t} \in U$,

$$
F(\underline{t}, y)=\left(f(\underline{t}), \varphi\left(n,\left|\chi_{n}(\underline{t})\right|\right)(y)\right)=\left(f(\underline{t}), \varphi\left(n,\left|\chi_{n}(\underline{x})\right|\right)(y)\right),
$$

that is, the function $\varphi$ to be applied to is the same for all $\underline{t} \in U$.
Let $\underline{t}_{0}$ be the first point in $\operatorname{Orb}(\underline{0})$ belonging to $U$, hence

$$
\underline{t}_{0}=f^{r_{0}}(\underline{0})=x(1) \cdots x\left(k_{n}-1\right) 0 \cdots \in U \cap \operatorname{Orb}(\underline{0})
$$

with $\underline{t}_{0}(i)=0$ for $i \geqslant k_{n}$. Every $\underline{t} \in U \cap \operatorname{Orb}(\underline{0})$ is of the form $\underline{t}=f^{r}(\underline{0})$ with $r \geqslant r_{0}$ and so,

$$
f^{r-r_{0}}(\underline{0})=\underbrace{0 \cdots 0}_{k_{n}} \cdots .
$$

Hence, for the first $r_{0}$ iterations we apply the same maps either starting from $\underline{0}$ or from $f^{r-r_{0}}(\underline{0})$. This implies that, for every $y \in I$,

$$
\begin{equation*}
\pi_{2}\left[F^{r_{0}}(\underline{0}, y)\right]=\pi_{2}\left[F^{r_{0}}\left(f^{r-r_{0}}(\underline{0}), y\right)\right] . \tag{12}
\end{equation*}
$$

Define $J:=\pi_{2}\left[M_{\underline{t}_{0}}\right]=\pi_{2}\left[F^{r_{0}}\left(I_{\underline{0}}\right)\right]$; it follows that

$$
\begin{aligned}
\pi_{2}\left[M_{\underline{t}}\right]=\pi_{2}\left[F^{r_{0}}\left(F^{r-r_{0}}\left(I_{\underline{0}}\right)\right)\right] & =\pi_{2}\left[F^{r_{0}}\left(f^{r-r_{0}}(\underline{0}), \psi\left(r-r_{0}\right)(I)\right)\right] \\
& \subset \pi_{2}\left[F^{r_{0}}\left(f^{r-r_{0}}(\underline{0}), I\right)\right]=\pi_{2}\left[F^{r_{0}}\left(I_{\underline{0}}\right)\right]=J .
\end{aligned}
$$

By the previous argument concernig the points of the orbit of $\underline{0}$, the map $\varphi\left(n,\left|\chi_{n}(\underline{x})\right|\right)$ is injective on $J$. So it is sufficient to show that $\pi_{2}\left[M_{x}\right] \subset J$. Since, by the hypothesis, $I_{\underline{0}} \subset M$, the minimality of $M$ implies

$$
M=\overline{\bigcup_{i=0}^{\infty} F^{i}\left(I_{\underline{0}}\right)}
$$

whence

$$
\begin{equation*}
M_{\underline{x}} \subset \overline{\bigcup\left\{M_{\underline{t}}: \underline{t} \in \operatorname{Orb}(\underline{0}) \cap U\right\}} . \tag{13}
\end{equation*}
$$

Since $\pi_{2}\left[M_{t}\right] \subset J$ for every $\underline{t} \in \operatorname{Orb}(\underline{0}) \cap U$, by (13) the same holds for the set $\pi_{2}\left[M_{x}\right]$. $]$

Remark. We conjecture that Theorem 2 is still valid for functions $F \in \mathcal{T}$.
Let us denote by $\sigma_{\delta}$ and $\tau_{\delta}$ the following functions depending on the parameter $\delta \in(0,1)$ :

$$
\begin{equation*}
\sigma_{\delta}(t)=(1-\delta) t, \quad \tau_{\delta}(t)=(1-\delta) t+\delta \tag{14}
\end{equation*}
$$

Now we introduce the subclass $\mathcal{T}_{01}$ of $\mathcal{T}_{0}$ consisting of those functions $F \in \mathcal{T}_{0}$ that satisfy the following additional conditions:
$\forall n \geqslant 1 \quad \exists j_{n}, 0 \leqslant j_{n} \leqslant 2^{m_{n+1}}-2$, such that:

$$
\begin{gather*}
\varphi\left(2 n-1, j_{2 n-1}\right)(t)=\sigma_{\delta_{2 n-1}}, \quad \varphi\left(2 n, j_{2 n}\right)(t)=\tau_{\delta_{2 n}} ;  \tag{15}\\
\text { if } j_{n}>0 \text { then, for all } r \geqslant 1, \quad \varphi^{r}\left(n, j_{n}-1\right) \circ \cdots \circ \varphi^{r}(n, 0)=I d . \tag{16}
\end{gather*}
$$

$$
\begin{equation*}
\left\{\left(1-\delta_{2 n+1}\right)^{2^{k_{2 n}-(2 n)}}\right\}, \quad\left\{\left(1-\delta_{2 n}\right)^{2^{k_{2 n-1}-(2 n-1)}}\right\} \text { are sequences dense in }[0,1] \tag{17}
\end{equation*}
$$

(Of course, by (3), we must have also $\lim _{n \rightarrow \infty} \delta_{n}=0$ ).
Remark. Note that given the sequence $\left\{k_{n}\right\}$, it is always possible to construct a (decreasing) sequence $\left\{\delta_{n}\right\}$ converging to 0 and satisfying (17).

Now we prove the following
Theorem 3. Every $F \in \mathcal{T}_{01}$ has a minimal set $M \supset I_{\underline{0}}$.
Proof: Take a point $\left(\underline{0}, y_{0}\right) \in I_{\underline{0}}$. By (9) and (14)-(17) we have

$$
\begin{align*}
y_{2\left(j_{2 n+1}+1\right) t_{2 n}} & =\varphi^{2^{k_{2 n}-2 n}}\left(2 n+1, j_{2 n+1}\right) \circ \cdots \circ \varphi^{2^{k_{2 n}-2 n}}(2 n+1,0)\left(y_{0}\right)  \tag{18}\\
& =\varphi^{2^{k_{2 n}-2 n}}\left(2 n+1, j_{2 n+1}\right)\left(y_{0}\right)=y_{0}\left(1-\delta_{2 n+1}\right)^{2^{k_{2 n}-2 n}}
\end{align*}
$$

and similarly

$$
\begin{equation*}
y_{2\left(j_{2 n}+1\right) t_{2 n-1}}=\varphi^{2^{k_{2 n-1}-(2 n-1)}}\left(2 n, j_{2 n}\right)\left(y_{0}\right)=1+\left(y_{0}-1\right)\left(1-\delta_{2 n}\right)^{2^{k_{2 n-1}-(2 n-1)}} \tag{19}
\end{equation*}
$$

By the hypotheses on the sequence $\left\{\delta_{n}\right\}$, we have

$$
\begin{equation*}
\omega_{F}\left(\underline{\mathbf{0}}, y_{0}\right) \supset I_{\underline{\mathbf{0}}} . \tag{20}
\end{equation*}
$$

Set $M=\omega_{F}(\underline{0}, 0)$ and let $w=(\underline{u}, v) \in M$. Since $F^{i}(w)$ visits any neighbourhood of $I_{\underline{0}}, \omega_{F}(w)$ contains a point from $I_{\underline{0}}$ and consequently, by $(20),(\underline{0}, 0) \in \omega_{F}(w)$. This implies $\omega_{F}(w) \supset M$, that is, $M$ is a minimal set for $F$, containing $I_{\underline{0}}$.

## 4. Distributional Chaos

We start this section by defining the notion of distributional chaos.
Let $g$ be a map from a metric space $(S, d)$ into itself. For any pair $(x, y)$ of points of $S$ and any positive integer $n$, we define a distribution function $\Phi_{x y}^{n}: \mathbb{R} \rightarrow[0,1]$ by

$$
\Phi_{x y}^{n}(t)=\frac{1}{n} \#\left\{i: 0 \leqslant i<n \text { and } d\left(g^{i}(x), g^{i}(y)\right)<t\right\}
$$

Obviously $\Phi_{x y}^{n}$ is a left-continuous non-decreasing function, $\Phi_{x y}^{n}(0)=0$ and $\Phi_{x y}^{n}(t)=1$ for all $t$ greater than the maximum of the numbers $d\left(g^{i}(x), g^{i}(y)\right), 0 \leqslant i \leqslant n-1$. Note that for the definition of each $\Phi_{x y}^{n}$ we need only to know the first $n$ iterates of $g$.

Having the whole sequence $\left\{\Phi_{x y}^{n}(t)\right\}_{n \geqslant 1}$ we set

$$
\Phi_{x y}(t)=\liminf _{n \rightarrow \infty} \Phi_{x y}^{n}(t), \quad \Phi_{x y}^{*}(t)=\limsup _{n \rightarrow \infty} \Phi_{x y}^{n}(t)
$$

We shall refer to $\Phi_{x y}$ as the lower and $\Phi_{x y}^{*}$ as the upper functions of $x$ and $y$.
If there is a pair $(x, y)$ of points of $S$ such that $\Phi_{u v}(t)<\Phi_{u v}^{*}(t)$ for all $t$ in some non degenerate interval, then we say that $g$ is distributionally chaotic (see $[\mathbf{9}, \mathbf{1 0}]$ ).

The main result of this section is the following.
Theorem 4. For every $\varepsilon, 0<\varepsilon<1$, there exists a function $F_{\varepsilon} \in \mathcal{T}_{01}$ such that for $u=(\underline{0}, 0)$ and $v=(\underline{0}, 1)$,

$$
\Phi_{u v}^{*}(t)=1,0<t<1 \quad \text { and } \quad \Phi_{u v}(t) \leqslant \varepsilon, 0<t \leqslant 1-\varepsilon
$$

Proof: Fix $\varepsilon \in(0,1)$. We construct the function $F_{\varepsilon}$ by choosing $j_{n}=0$ for all $n$ and the functions $\varphi(n, j) \in \Gamma_{n}$, depending on integer parameters $a_{n}, b_{n}$ and $m_{n}$, as follows:

$$
\begin{aligned}
\varphi(2 n-1, j) & = \begin{cases}\sigma_{\delta_{2 n-1}}, & 0 \leqslant j<a_{2 n-1} \\
I d, & a_{2 n-1} \leqslant j<b_{2 n-1} \\
\sigma_{\delta_{2 n-1}}^{*}, & b_{2 n-1} \leqslant j<a_{2 n-1}+b_{2 n-1} \\
I d, & a_{2 n-1}+b_{2 n-1} \leqslant j<2^{m 2 n-1}\end{cases} \\
\varphi(2 n, j) & = \begin{cases}\tau_{\delta_{2 n}}, & 0 \leqslant j<a_{2 n} \\
I d, & a_{2 n} \leqslant j<b_{2 n} \\
\tau_{\delta_{2 n}}^{*}, & b_{2 n} \leqslant j<a_{2 n}+b_{2 n} \\
I d, & a_{2 n}+b_{2 n} \leqslant j<2^{m}\end{cases}
\end{aligned}
$$

where $\sigma_{\delta_{2 n-1}}$ and $\tau_{\delta_{2 n}}$ are the functions defined in (14), $\sigma_{\delta_{2 n-1}}^{*}$ and $\tau_{\delta_{2 n}}^{*}$ are their left-inverses given by
$\sigma_{\delta_{2 n-1}}^{*}(t)=\min \left\{1, t /\left(1-\delta_{2 n-1}\right)\right\}, \quad \tau_{\delta_{2 n}}^{*}(t)=\max \left\{0,\left(t-\delta_{2 n}\right) /\left(1-\delta_{2 n}\right)\right\} \quad t \in[0,1]$,
and $\left\{\delta_{n}\right\}$ is a sequence satifying (17) with $\lim _{n \rightarrow \infty} \delta_{n}=0$.
In this way we get a function $F_{\varepsilon} \in \mathcal{T}_{01}$. Thus what remains to be chosen are the parameters $a_{n}, b_{n}$ and $m_{n}$.

Before starting with the choice of these parameters we need some properties of $F_{\varepsilon}$. Let $J_{i}:=\pi_{2}\left[F_{\varepsilon}^{i}\left(I_{\underline{0}}\right)\right]$ and $\lambda_{i}:=\left|J_{i}\right|=|\psi(i)(I)|$.

By applying the functions $\sigma_{\delta}$ or $\tau_{\delta}$ to an interval $J \subset I$, we get

$$
\left|\sigma_{\delta}(J)\right|<|J|, \quad\left|\tau_{\delta}(J)\right|<|J| .
$$

Moreover, for every $j \leqslant 2^{m_{n}}-1$ and $s \leqslant r$,

$$
\left|\varphi^{s}(n, j) \circ \varphi^{r}(n, j-1) \circ \cdots \circ \varphi^{r}(n, 0)(J)\right| \leqslant|J|
$$

since first we apply the functions $\sigma_{\delta}$ or $\tau_{\delta}$ and then their left-inverses for a smaller number of times. Thus, by (7) it is easy to prove by induction that for any interval $J \subset I$

$$
\begin{equation*}
|\psi(i)(J)| \leqslant|J| . \tag{21}
\end{equation*}
$$

For any $j$ with $a_{n+1} \leqslant j \leqslant b_{n+1}-1$ we have $\varphi(n+1, j)=I d$; thus, by (9) we obtain $\psi\left(2 j t_{n}\right)=\psi\left(2 a_{n+1} t_{n}\right)$. Then, if we set

$$
\bar{J}:=J_{2 a_{n+1} t_{n}}=\varphi^{2^{k_{n}-n}}\left(n+1, a_{n+1}-1\right) \circ \cdots \circ \varphi^{2^{k_{n}-n}}(n+1,0)(I)
$$

by the structure of the family $\Gamma_{n+1}$ we have

$$
\begin{equation*}
\bar{J}=\sigma_{\delta_{n+1}}^{a_{n+1} 2^{k_{n}-n}}(I) \quad \text { or } \quad \bar{J}=\tau_{\delta_{n+1}}^{a_{n+1} 2^{k_{n}-n}}(I) \tag{22}
\end{equation*}
$$

according to whether $n$ is even or odd. By (4), (5), (8) and (21) we have

$$
\begin{aligned}
\lambda_{i}=|\psi(i)(I)| & =\left|\psi(r) \circ \psi\left(2 j t_{n}\right)(I)\right|=\left|\psi(r) \circ \psi\left(2 a_{n+1} t_{n}\right)(I)\right| \\
& =|\psi(r)(\bar{J})| \leqslant|\bar{J}|, \quad i=2 j t_{n}+r, 0 \leqslant r<t_{n} \\
\lambda_{i}=|\psi(i)(I)| & =\left|\psi_{j}^{*}(r) \circ \psi\left(2 j t_{n}\right)(I)\right|=\left|\psi_{j}^{*}(r) \circ \psi\left(2 a_{n+1} t_{n}\right)(I)\right| \\
& =\left|\psi_{j}^{*}(r)(\bar{J})\right|=|\psi(\theta(r))(\bar{J})| \leqslant|\bar{J}|, \quad i=(2 j+1) t_{n}+r, 0 \leqslant r<t_{n} .
\end{aligned}
$$

This implies that, if $\lambda_{2 a_{n+1} t_{n}}=|\bar{J}|<1 /(n+1)$, then

$$
\begin{equation*}
\lambda_{i} \leqslant \lambda_{2 a_{n+1} t_{n}}<\frac{1}{n+1}, \quad 2 a_{n+1} t_{n} \leqslant i<2 b_{n+1} t_{n} \tag{23}
\end{equation*}
$$

Now we have to choose the parameters $a_{n}, b_{n}$ and $m_{n}$. The choice will be made iteratively in order to assure that, for $n \geqslant 1$,

$$
\begin{equation*}
\Phi_{u v}^{2 b_{n} t_{n-1}}(1 / n) \geqslant 1-1 / n \quad \text { and } \quad \Phi_{u v}^{t_{n}}(1-\varepsilon) \leqslant \varepsilon \tag{24}
\end{equation*}
$$

The relation $\Phi_{u v}^{2 b_{n} t_{n-1}}(1 / n) \geqslant 1-1 / n$ means that the number of $i$ 's less than $2 b_{n} t_{n-1}$ for which $\lambda_{i}<1 / n$ is "almost the same" as $2 b_{n} t_{n-1}$ while $\Phi_{u v}^{t_{n}}(1-\varepsilon) \leqslant \varepsilon$ means that the number of $i$ 's less than $t_{n}$ for which $\lambda_{i}<1-\varepsilon$ is "small" in respect to $t_{n}$.

Let $n=1$. Take $a_{1}=b_{1}=1<2^{m_{1}}$. Then we have $\lambda_{0}=1, \lambda_{1}=1-\delta_{1}$ and $\lambda_{i}=1$ for all $2 \leqslant i \leqslant 2^{m_{1}}=t_{1}$. So, the first inequality of (24) is trivially satisfied and

$$
\Phi_{u v}^{t_{1}}(1-\varepsilon)=\frac{1}{2^{m_{1}}} \#\left\{i: 0 \leqslant i<2^{m_{1}} \text { and } \lambda_{i}<1-\varepsilon\right\} \leqslant \frac{1}{2^{m_{1}}} .
$$

If we choose $m_{1}$ such that $1 / 2^{m_{1}} \leqslant \varepsilon$, then the second inequality of (24) is satisfied.
Assuming we have determined $a_{r}, b_{r}$ and $m_{r}$ for all $r \leqslant n$, now we choose the parameters $a_{n+1}, b_{n+1}$ and $m_{n+1}$. By (22),

$$
\lambda_{2 a_{n+1} t_{n}}=\left(1-\delta_{n+1}\right)^{a_{n+1} 2^{k_{n}-n}}
$$

so we take $a_{n+1}$ so that $\lambda_{2 a_{n+1} t_{n}}<1 /(n+1)$. Now, by (23),

$$
\#\left\{i: 0 \leqslant i<2 b_{n+1} t_{n} \text { and } \lambda_{i}<1 /(n+1)\right\} \geqslant 2\left(b_{n+1}-a_{n+1}\right) t_{n}
$$

and so we can take $b_{n+1}$ so that

$$
\Phi_{u v}^{2 b_{n+1} t_{n}}\left(\frac{1}{n+1}\right) \geqslant \frac{2\left(b_{n+1}-a_{n+1}\right) t_{n}}{2 b_{n+1} t_{n}} \geqslant 1-\frac{1}{n+1}
$$

that is, the first inequality of (24) is satisfied for $n+1$. Assume $m_{n+1}$ has been chosen with $a_{n+1}+b_{n+1}<2^{m_{n+1}}$ and take $a_{n+1}+b_{n+1} \leqslant j<2^{m_{n+1}}$. Then $\varphi(n+1, j)=I d$. If $i=2 j t_{n}+r$ with $0 \leqslant r<t_{n}$ then, by (4), (9) and the structure of $\Gamma_{n+1}$ we have $\psi(i)=\psi(r) \circ \psi\left(2 j t_{n}\right)=\psi(r)$ and so $\lambda_{i}=\lambda_{r}$. The second inequality in (24) implies (25)
$\#\left\{i: i=2 j t_{n}+r\right.$ and $\left.0 \leqslant r<t_{n}: \lambda_{i}<1-\varepsilon\right\}=\#\left\{r: 0 \leqslant r<t_{n}\right.$ and $\left.\lambda_{r}<1-\varepsilon\right\} \leqslant \varepsilon t_{n}$.
Let now $i=(2 j+1) t_{n}+r$ with $0 \leqslant r<t_{n}$. Again, by (5), (8) (9) and the structure of $\Gamma_{n+1}$ we obtain

$$
\psi(i)=\psi_{j}^{*}(r) \circ \psi\left(2 j t_{n}\right)=\psi_{j}^{*}(r)=\psi(\theta(r))
$$

and so $\lambda_{i}=\lambda_{\theta(r)}$. Note that $\theta(r)$ may assume all values from 0 to $t_{\nu(r)}-1$ where $0 \leqslant \nu(r) \leqslant n-1$. (See the notation of Lemma 1.)

Now we intend counting the number of indices $r$ with $0 \leqslant r<t_{n}$ for which $\nu(r)=l$. This means counting the indices having in their binary representation 0 at the place $k_{l}, 1$ in the places $k_{p}, l<p<n$ and 0 in all places greater or equal to $k_{n}$. Thus the required number is $t_{n} / 2^{n-1}$. These indices can be collected in $t_{n} / t_{l} 2^{n-l}$ blocks of type $l$ containing the numbers having in their binary representation the same digits in the places greater or equal to $k_{l}$, that is, in blocks of indices of the form

$$
r=\left(2 \alpha_{n-1}+1\right) t_{n-1}+\cdots+\left(2 \alpha_{l+1}+1\right) t_{l+1}+2 \alpha_{l} t_{l}+s \quad \text { with } 0 \leqslant s<t_{l}
$$

with the same $a_{q}, l \leqslant q \leqslant n-1$.
For each block $B$ of type $l$ and any index $i=(2 j+1) t_{n}+r, r \in B$, by (7) and the structure of $\Gamma_{n+1}$, we have $\lambda_{i}=|\psi(i)(I)|=|\psi(s)(I)|=\lambda_{s}$ and, by the second inequality in (24), we get

$$
\#\left\{i: i \in B \text { and } \lambda_{i}<1-\varepsilon\right\}=\#\left\{s: 0 \leqslant s<t_{l} \text { and } \lambda_{s}<1-\varepsilon\right\} \leqslant \varepsilon t_{l} .
$$

Since the number of blocks of type $l$ is $t_{n} / t_{l} 2^{n-l}$ and $0 \leqslant l \leqslant n-1$ we obtain

$$
\#\left\{i: i=(2 j+1) t_{n}+r, 0 \leqslant r<t_{n} \text { and } \lambda_{i}<1-\varepsilon\right\} \leqslant \sum_{l=0}^{n-1} \frac{t_{n}}{t_{l} 2^{n-l}} \varepsilon t_{l}=\varepsilon t_{n} \sum_{l=0}^{n-1} \frac{1}{2^{n-l}}
$$

From (25) and this inequality we get

$$
\#\left\{i: i=2 j t_{n}+r, 0 \leqslant r<2 t_{n} \text { and } \lambda_{i}<1-\varepsilon\right\} \leqslant \varepsilon t_{n} \sum_{p=0}^{n} \frac{1}{2^{p}}=2 t_{n}\left(1-2^{-(n+1)}\right) \varepsilon
$$

Since $t_{n+1}=2^{m_{n+1}}\left(2 t_{n}\right)$, we conclude that

$$
\begin{aligned}
\Phi_{u v}^{t_{n+1}}(1-\varepsilon) & =\frac{1}{2^{m_{n+1}}\left(2 t_{n}\right)} \#\left\{i: 0 \leqslant i<2^{m_{n+1}}\left(2 t_{n}\right) \text { and } \lambda_{i}<1-\varepsilon\right\} \\
& \leqslant \frac{2\left(a_{n+1}+b_{n+1}\right) t_{n}+2 \varepsilon t_{n}\left(1-2^{-(n+1)}\right)\left(2^{m_{n+1}}-\left(a_{n+1}+b_{n+1}\right)\right)}{2^{m_{n+1}\left(2 t_{n}\right)}} \\
& =\frac{\left(a_{n+1}+b_{n+1}\right)\left(1-\varepsilon\left(1-2^{-(n+1)}\right)\right)}{2^{m_{n+1}}}+\varepsilon\left(1-2^{-(n+1)}\right)
\end{aligned}
$$

and we choose $m_{n+1}$ so that $\Phi_{u v}^{t_{n+1}}(1-\varepsilon) \leqslant \varepsilon$.
Summarising, by Theorems 2, 3 and 4 we have the following
Corollary 5. For every $\varepsilon, 0<\varepsilon<1$, there exists a function $F_{\varepsilon} \in \mathcal{T}_{01}$ satisfying the following properties:
(i) $F_{\varepsilon}$ has a minimal set $M \subset Q \times I$ such that $\left.F_{\varepsilon}\right|_{M}$ is a homeomorphism;
(ii) $M$ contains points $u$ and $v$ such that

$$
\Phi_{u v}^{*}(t)=1,0<t<1 \quad \text { and } \quad \Phi_{u v}(t) \leqslant \varepsilon, 0<t \leqslant 1-\varepsilon .
$$

Note that the behaviour described in Corollary 5 is impossible in dimension one. Indeed any $f \in \mathcal{C}$ with $h(f)=0$ is not chaotic (in the sense of Li and Yorke) on any minimal set [3].

## 5. Other results

In this section of the paper we present some other results about the functions in the class $\mathcal{T}_{01}$. Moreover we define other subclasses of $\mathcal{T}$ and we prove some properties of them.

ThEOREM 6. For every function $F$ in $\mathcal{T}_{01}$ no point of the layer $I_{\underline{0}}$ is isochronically recurrent.

Proof: We have to prove that for every $y_{0} \in I$ there exists a neighbourhood $U=U_{1} \times U_{2}$ of $\left(\underline{0}, y_{0}\right)$ such that for every positive integer $\nu$, there is an integer $r$ for which $F^{r \nu}\left(\underline{0}, y_{0}\right) \notin U$. The proof is analogous to that in [4].

We distinguish two cases: $y_{0}<1$ and $y_{0}=1$.
Let $y_{0}<1$. We choose $U=I \times U_{2}$ with $\sup U_{2}<1$, and take an integer $p$ such that

$$
1-\frac{1}{p}>\sup U_{2}
$$

Consider the set

$$
A:=\left\{n: n \geqslant 1 \text { and } \pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}(\underline{0}, 0)\right]>1-\frac{1}{p}\right\} .
$$

By (19), the set $A$ contains infinitely many elements and by the monotonicity of the functions $\varphi(n,$.$) , for every t \in I$ we have

$$
\pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}(\underline{0}, t)\right] \geqslant \pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}(\underline{0}, 0)\right]>1-\frac{1}{p}
$$

for any $n \in A$.
Take an integer $\nu$ and let $2^{q}+\sum_{i \geqslant q+1} c_{i} 2^{i}, c_{i} \in\{0,1\}, q \geqslant 0$, be its binary representation. Fix any $n \in A$; since

$$
2^{k_{n}-2^{m_{n+1}} q} \nu^{2^{m_{n+1}}}=\overbrace{\underbrace{\overbrace{\cdots 0}}_{k_{n}} 1 \underbrace{0 \cdots 0}_{m_{n+1}}}^{k_{n+1}} \cdots
$$

for $r=2\left(j_{n+1}+1\right) t_{n} 2^{k_{n}-2^{m_{n+1}} q} \nu^{2^{m_{n+1}}}-1$ we have

$$
r \nu=\underbrace{0 \cdots 0}_{k_{n}} \xi(1) \cdots \xi\left(m_{n+1}\right) 0 \cdots
$$

with $j_{n+1}+1=\left|\left(\xi(1), \ldots, \xi\left(m_{n+1}\right)\right)\right|$. Since $\mu=r \nu-2\left(j_{n+1}+1\right) t_{n}=\underbrace{0 \cdots 0}_{k_{n+1}} \cdots$, we have $f^{\mu}(\underline{0})=\underbrace{0 \cdots 0}_{k_{n+1}} \cdots$ and $F^{r \nu}\left(\underline{0}, y_{0}\right)=F^{2\left(j_{n+1}+1\right) t_{n}}\left(f^{\mu}(\underline{0}), y_{\mu}\right)$.

Since $\pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}\left(f^{\mu}(\underline{0}), t\right)\right]=\pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}(\underline{0}, t)\right]$ for every $t \in I$, by the definition of $A$ we get

$$
\begin{aligned}
\pi_{2}\left[F^{r \nu}\left(\underline{0}, y_{0}\right)\right] & =\pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}\left(f^{\mu}(\underline{0}), y_{\mu}\right)\right]=\pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}\left(\underline{0}, y_{\mu}\right)\right] \\
& \geqslant \pi_{2}\left[F^{2\left(j_{n+1}+1\right) t_{n}}(\underline{0}, 0)\right]>1-\frac{1}{p} .
\end{aligned}
$$

In the case $y_{0}=1$ we proceed in a similar way starting from a neighbourhood $U=I \times U_{2}$ with $\inf \left(U_{2}\right)>0$ and using formula (18).

Remark. The previous result shows that for each function in $\mathcal{T}_{01}$ the point $\underline{0}$ is isochronically recurrent for the base map while it is not the projection of any isochronically recurrent point of the triangular map $F$.

Hence, from Theorems 2, 4 and 6 we get the following
Corollary 7. For every $\varepsilon, 0<\varepsilon<1$, there exists a function $F_{\varepsilon} \in \mathcal{T}_{01}$ such that:
(i) $F_{\varepsilon}$ has a minimal set $M \supset I_{\underline{0}}$;
(ii) $\left.F_{\varepsilon}\right|_{M}$ is a homeomorphism;
(iii) no point of $I_{\underline{0}}$ is isochronically recurrent;
(iv) $\Phi_{u v}^{*}(t)=1$ for $0<t<1$ and $\Phi_{u v}(t) \leqslant \varepsilon$ for $0<t \leqslant 1-\varepsilon$, where $u=(\underline{0}, 0)$ and $v=(\underline{0}, 1)$.

Now we define another subclass $\mathcal{T}_{02}$ of $\mathcal{T}_{0}$ as follows: for every $n \geqslant 1$

$$
\begin{aligned}
& \varphi(n, 0)=\varphi(n, 1)=I d \\
& \varphi^{r}\left(n, 2^{p}-1\right) \circ \cdots \circ \varphi^{r}(n, 0)=I d, 1 \leqslant p \leqslant m_{n}, \text { for all } r \geqslant 1 .
\end{aligned}
$$

Theorem 8. For every $F \in \mathcal{T}_{02}$ we have

$$
\lim _{s \rightarrow \infty} F^{2^{s}}\left(\underline{0}, y_{0}\right)=\left(\underline{0}, y_{0}\right)
$$

for every $y_{0} \in I$.
Proof: Formulas (6) and (9) and the definition of $\mathcal{T}_{02}$ imply $y_{2^{s}}=y_{0}$ for every integer $s$.

Corollary 9. There exists $F \in \mathcal{T}_{02}$ such that:
(i) for every $y_{0} \in I, \lim _{s \rightarrow \infty} F^{2^{s}}\left(\underline{0}, y_{0}\right)=\left(\underline{0}, y_{0}\right)$;
(ii) $F$ has a minimal set $M \supset I_{\underline{0}}$;
(iii) $\left.F\right|_{M}$ is a homeomorphism;
(iv) no point of $I_{\underline{0}}$ is isochronically recurrent.

Proof: By Theorems 2 and 6, the only thing to be proved is that $\mathcal{T}_{01} \cap \mathcal{T}_{02} \neq \emptyset$. It is enough to take in the definition of $\mathcal{T}_{01}, j_{n}=2$ for all $n, \varphi(n, 0)=\varphi(n, 1)=I d$ and choose $\varphi\left(n, j_{n}+1\right)=\varphi(n, 3)$ as the left inverse of $\varphi(n, 2)$. All other functions can be chosen equal to the identity.

Remark. A one-dimensional map $f$ has zero topological entropy if and only if the set $\left\{x \in I: \lim _{s \rightarrow \infty} f^{2^{s}}(x)=x\right\}$ coincides with the set of the isochronically recurrent points [7, Table 1]. We recall (see Section 2) that our maps have zero topological entropy and so properties (i) and (iv) of Corollary 9 show a completely different behaviour with respect to the one-dimensional case.

To present the last results of the paper we introduce another subclass $\mathcal{T}_{1}$ of $\mathcal{T}$. Let $\left\{k_{i}\right\}_{i=1}^{\infty}$ be an increasing sequence if positive integers with $k_{i}-i \rightarrow+\infty$, and $\left\{\varphi_{i}\right\}_{i=1}^{\infty}$ a sequence of mappings from $I$ into $I$ of the form

$$
\varphi_{i}(t)=t^{s_{i}}, \quad \text { with } s_{i}>0, \lim _{i \rightarrow \infty} s_{i}=1
$$

As in the definition of the class $\mathcal{T}_{0}$, the digits $x\left(k_{1}\right), x\left(k_{2}\right), \ldots$ are called control digits of $\underline{x} \in Q$. We define a function $f: Q \times I \rightarrow Q \times I$ as follows:

If the first zero control digit of $\underline{x}$ is $x\left(k_{n}\right)$,

$$
F(\underline{x}, y)=\left(f(\underline{x}), \varphi_{n}(y)\right)
$$

otherwise $F(\underline{x}, y)=(f(\underline{x}), y)$. The condition $\lim _{i \rightarrow \infty} s_{i}=1$ assures the continuity of $F$. Moreover, it is easy to recognise that $F$ is a homeomorphism of $Q \times I$ onto itself.

Theorem 10. There exists a function $F \in \mathcal{T}_{1}$ with the following properties:
(i) for any $w \in\{\underline{0}\} \times(0,1)$ we have $\omega_{F}(w)=Q \times I$;
(ii) $F$ has two minimal sets, namely $Q \times\{0\}$ and $Q \times\{1\}$;
(iii) $\{\underline{0}\} \times(0,1) \subset \operatorname{Rec}(F) \backslash U R(F)$;
(iv) for any $u \in\{\underline{0}\} \times(0,1)$ and $v=(\underline{0}, 0)$ or $v=(\underline{0}, 1)$,

$$
\begin{equation*}
\Phi_{u v}^{*}(t)=1, \quad \Phi_{u v}(t)=0, \quad t \in(0,1) \tag{26}
\end{equation*}
$$

hence $F$ is distributionally chaotic.
Proof: (i) Since the functions $\varphi_{i}$ commute, the value $F^{m}\left(\underline{0}, y_{0}\right)=\left(f^{m}(\underline{0}), y_{m}\right)$ depends only on the number of times any function $\varphi_{i}$ is applied.

Given a positive number $r$, take $n$ so that $k_{n} \leqslant r<k_{n+1}$. Then the points $f^{i}(\underline{0}), 0 \leqslant i<2^{r}$ are represented by all the $2^{r}$ sequences

$$
a_{1}, \cdots a_{r} 0 \cdots, \quad a_{i} \in\{0,1\}, 1 \leqslant i \leqslant r
$$

which have the $(n+1)$-th control digit equal zero and so the only functions that may enter in the expression of $y_{i}, 1 \leqslant i \leqslant 2^{r}$, are $\varphi_{1}, \ldots, \varphi_{n+1}$. The number of times the function $\varphi_{i}, 1 \leqslant i \leqslant n+1$, enters the expression of $y_{2^{r}}$ equals the number of sequences $a_{1} \cdots a_{r} 0 \cdots$ having $a_{k_{i}}=0$ and $a_{k_{s}}=1$ for all $1 \leqslant s<i$. This number is $2^{r-i}$ for $1 \leqslant i \leqslant n$, and $2^{r-n}$ for $i=n+1$. So, we have

$$
y_{2^{k_{n}}}=\varphi_{1}^{2^{k_{n}}-1} \circ \varphi_{2}^{2^{k_{n}-2}} \circ \cdots \circ \varphi_{n-1}^{2^{k_{n}-(n-1)}} \circ \varphi_{n}^{2^{k_{n}-n}} \circ \varphi_{n+1}^{k_{n}-n}\left(y_{0}\right)
$$

Since

$$
f^{2^{k_{n}}}(\underline{0})=\underbrace{0 \cdots 0}_{k_{n}} 10 \cdots,
$$

for the next $2^{k_{n}}$ iterations we use exactly the same functions as starting from $\underline{0}$. We may proceed in this way until the $k_{n+1}$ digit is zero. Thus, for all $m$ with $2 \leqslant m \leqslant$ $2^{k_{n+1}-k_{n}-1}$ we get

$$
\begin{equation*}
y_{m 2^{k_{n}}}=\varphi_{1}^{m 2^{k_{n}-1}} \circ \varphi_{2}^{m 2^{k_{n}-2}} \circ \cdots \circ \varphi_{n-1}^{m 2^{k_{n}-(n-1)}} \circ \varphi_{n}^{m 2^{k_{n}-n}} \circ \varphi_{n+1}^{m 2^{k_{n}-n}}\left(y_{0}\right) \tag{27}
\end{equation*}
$$

In order to construct the function $F$ we start by imposing on the sequence $\left\{s_{i}\right\}$ the additional condition

$$
\begin{equation*}
s_{2 i-1}^{2} s_{2 i}=1, \quad i \geqslant 1 \tag{28}
\end{equation*}
$$

This implies

$$
\varphi_{2 i-1}^{2} \circ \varphi_{2 i}=I d, \quad i \geqslant 1 .
$$

Hence, by (27) and (28) for all $n \geqslant 1$ we obtain

$$
\begin{cases}y_{m 2^{k} 2 n}=\varphi_{2 n+1}^{m 2^{k_{2 n}-2 n}}\left(y_{0}\right)=y_{0}^{s_{2 n+1}^{\left(m 2^{k_{2 n}-2 n}\right)}}, & 1 \leqslant m \leqslant 2^{k_{2 n+1}-k_{2 n}-1}  \tag{29}\\ y_{m 2^{k_{2 n-1}}}=\varphi_{2 n}^{m 2^{k_{2 n-1}-2 n}}\left(y_{0}\right)=y_{0}^{s_{2 n}}{ }^{\left(m 2^{k_{2 n-1}-2 n}\right)}, & 1 \leqslant m \leqslant 2^{k_{2 n}-k_{2 n-1}-1}\end{cases}
$$

We want to show that it is possible to choose the sequence of parameters $\left\{s_{n}\right.$ ) in order to assure that

$$
\forall y_{0} \in(0,1), \quad \omega_{F}\left(\left(\underline{0}, y_{0}\right)\right) \supset I_{\underline{\mathbf{0}}} .
$$

Since the $\omega$-limit sets are strongly $F$-invariant, this implies $\omega_{F}\left(\left(\underline{0}, y_{0}\right)\right)=Q \times I$.
To this aim it is enough to assure that the values given by (29) with $m=1$ are dense in $I$ and this is equivalent to requiring that

$$
\begin{equation*}
2^{k_{2 n}-2 n} \log \left(s_{2 n+1}\right) \quad \text { is dense in }(-\infty,+\infty) \tag{30}
\end{equation*}
$$

If $\left\{s_{2 n+1}\right\}$ is a sequence satisfying (30) and

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} s_{2 n+1}=1 \tag{31}
\end{equation*}
$$

then the whole sequence $\left\{s_{n}\right\}$ constructed by using (28) satisfies the required property $\lim _{n} s_{n}=1$. To satisfy (30) and (31) we define the sequence $\left\{s_{2 n+1}\right\}_{n=1}^{\infty}$ by

$$
\begin{equation*}
\log \left(s_{2 n+1}\right)=\frac{\sigma_{n}}{2^{k_{2 n}-2 n}} \tag{32}
\end{equation*}
$$

where $\left\{\sigma_{n}\right\}_{n=1}^{\infty}$ is a sequence dense in $(-\infty,+\infty)$ satisfying

$$
\begin{equation*}
\left|\frac{\sigma_{n}}{2^{k_{2 n}-2 n}}\right|<\frac{1}{n} . \tag{33}
\end{equation*}
$$

So (i) is proved. In the following, given the sequence $\left\{\sigma_{n}\right\}$, we show how to construct the sequence $\left\{k_{n}\right\}$ in order to satisfy (33).

Property (ii) is obvious. By (i), every point $w \in\{\underline{0}\} \times(0,1)$ is recurrent and, by (ii), $\omega_{F}(w)$ is not a minimal set. So $w$ is not uniformly recurrent (see [1]). Now we prove (iv). In order to assure (26) we take sequence $\left\{\sigma_{n}\right\}$, dense in $(-\infty,+\infty)$ and such that $\sigma_{2 n-1}<0$ and $\sigma_{2 n}>0$.

Now we recursively define the sequences $\left\{k_{n}\right\}$ and $\left\{s_{n}\right\}$.
We start the recursive process by taking $k_{1}$ arbitrarily, $s_{1}=s_{2}=1$ and $k_{2}>k_{1}$ satisfying (33). Assume now we have constructed $k_{i}, s_{i}$ for $i \leqslant 2 n$ so that (33) is satisfied. By (28) and (32) we immediately get $s_{2 n+1}$ and $s_{2 n+2}$.

Suppose now $n$ even [ $n$ odd]. We take $0<\rho_{n}<1 / 2 n$ such that, for $y_{0} \leqslant \rho_{n}$ [ $\left.y_{0} \geqslant 1-\rho_{n}\right]$ and for all $j$ with $0 \leqslant j<2^{k_{2 n}}$, we have

$$
\begin{equation*}
y_{j}<\frac{1}{2 n} \quad\left[y_{j}>1-\frac{1}{2 n}\right] . \tag{34}
\end{equation*}
$$

This is possible since only a finite number of continuous functions enter in the expression of $y_{j}$ and for them both points 0 and 1 are fixed.

Then we find an integer $p$ so that, for $y_{0}=1-(1 / 2 n)\left[y_{0}=(1 / 2 n)\right]$,

$$
\begin{equation*}
y_{p 2^{k_{2 n}}}=\varphi_{2 n+1}^{p 2^{\left(k_{2 n}-2 n\right)}}\left(y_{0}\right)<\rho_{n} \quad\left[y_{p 2^{k_{2 n}}}=\varphi_{2 n+1}^{p 2^{\left(k_{2 n}-2 n\right)}}\left(y_{0}\right)>1-\rho_{n}\right] \tag{35}
\end{equation*}
$$

Now, we choose $k_{2 n+1}$ so that

$$
\begin{equation*}
\frac{p 2^{k_{2 n}}}{2^{k_{2 n+1}-1}}<\frac{1}{2 n} \tag{36}
\end{equation*}
$$

and $k_{2 n+2}>k_{2 n+1}$ satisfying (33).

Now we show that the function $F$ constructed in this way satisfies (26). Take $y_{0} \in(0,1)$ and $n_{0}$ such that $y_{0} \in\left[\left(1 / 2 n_{0}\right), 1-\left(1 / 2 n_{0}\right)\right]$. Choose $n$ even and greater than $n_{0}$. For every $r, p 2^{k_{2 n}} \leqslant r \leqslant 2^{k_{2 n+1}-1}$ we can write $r=m 2^{k_{2 n}}+j$ with $p \leqslant m \leqslant 2^{k_{2 n+1}-k_{2 n}-1}$ and $0 \leqslant j<2^{k_{2 n}}$. So, by (29), (34) and (35)

$$
y_{m 2^{k}{ }^{k}} \leqslant y_{p 2^{k_{2 n}}}<\rho_{n} \quad \text { and } \quad y_{r}<\frac{1}{2 n} .
$$

Thus

$$
\#\left\{i: 0 \leqslant i<2^{k_{2 n+1}-1} \text { and } y_{i}<\frac{1}{2 n}\right\} \geqslant 2^{k_{2 n+1}-1}-p 2^{k_{2 n}}
$$

and so, by (36)

$$
\Phi_{u v}^{2^{k_{2 n+1}-1}}\left(\frac{1}{2 n}\right) \geqslant 1-\frac{p 2^{k_{2 n}}}{2^{k_{2 n+1}-1}}>1-\frac{1}{2 n}
$$

Hence we conclude that $\Phi_{u v}^{*}(t)=1$ for $t \in(0,1)$.
Similarly, if we take $n$ odd, we get

$$
\Phi_{u v}^{2^{k_{2 n+1}-1}}\left(1-\frac{1}{2 n}\right)<\frac{1}{2 n}
$$

and so $\Phi_{u v}(t)=0$ for $t \in(0,1)$.
Remark. Again the properties proved in Theorem 10 are impossible in $\mathcal{C}$. Indeed for a one-dimensional map $f$ with $h(f)=0$ we have $\operatorname{Rec}(f)=U R(f)$ and each $\omega$-limit set contains only one minimal set. The next theorem shows something more: the existence of a triangular map $F$ with $h(F)=0$ having an $\omega$-limit set containing infinitely many minimal sets.

Theorem 11. There exists a triangular map $F$ of type $2^{\infty}$, strictly increasing on any layer $I_{x}$, having an $\omega$-limit set containing uncountably many minimal sets.

Proof: By [2, Theorems 6.2,6.5] there exists a function $f \in \mathcal{C}$ of type $2^{\infty}$ having an infinite $\omega$-limit set $\widetilde{Q} \supset Q$ containing isolated points and such that $\widetilde{Q} \backslash Q$ is a single orbit disjoint from $Q$. Moreover, this function acts as the adding machine on $Q$ and for every $x \in \widetilde{Q} \backslash Q$ we have $\omega_{f}(x)=Q$. We take such a function as base of the triangular map we are constructing. We choose $p_{0} \in \widetilde{Q} \backslash Q$ with $\operatorname{Orb}\left(p_{0}\right)=\widetilde{Q} \backslash Q$ and associate to it the zero sequence. Then we code $\operatorname{Orb}\left(p_{0}\right)$ by associating to each point $p_{n}=f^{n}\left(p_{0}\right)$ the corresponding sequence $f^{n}(\underline{0})$. Now we define $\widetilde{F}(x, y)=\left(f(x), g_{x}(y)\right)$ on $\widetilde{Q} \times I$ as follows: for $x \in Q, g_{x}=I d$ and for $x \in \widetilde{Q} \backslash Q, g_{x}$ as in the construction of the class $\mathcal{T}_{1}$ on the corresponding points of $\operatorname{Orb}(\underline{0})$. Arguing as in the proof of Theorem 9 we get $\omega_{\widetilde{F}}(z)=Q \times I$ for any $z \in(\tilde{Q} \backslash Q) \times(0.1)$. Clearly, any set $Q \times\{a\}$ is a minimal set for $\widetilde{F}$ contained in $\omega_{\widetilde{F}}(z)$. It is easy to see that it is possible to extend $\widetilde{F}$ continuously to a triangular map $F: I^{2} \rightarrow I^{2}$ increasing on any layer.

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