

Dynamics of k -essence

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Abstract

There are a number of mathematical theorems in the literature on the dynamics of cosmological models with accelerated expansion driven by a positive cosmological constant Λ or a nonlinear scalar field with potential V (quintessence) which do not assume homogeneity and isotropy from the beginning. The aim of this paper is to generalize these results to the case of k -essence models which are defined by a Lagrangian having a nonlinear dependence on the kinetic energy. In particular, Lagrangians are included where late time acceleration is driven by the kinetic energy, an effect which is qualitatively different from anything seen in quintessence models. A general criterion for isotropization is derived and used to strengthen known results in the case of quintessence.

1 Introduction

In recent years many models have been introduced as possible explanations of the observed acceleration of the expansion of the universe. The simplest and best-known of these involve a positive cosmological constant Λ or a nonlinear scalar field with potential V . There is a huge literature on this subject - see for instance [24] for a review. In most papers the analysis is restricted to homogeneous and isotropic cosmological models since it is expected that these will give a good approximate description of more general spacetimes.

For some of the simplest cases with late time accelerated expansion this expectation has been confirmed by rigorous proofs. There are a number of papers where mathematical results on the dynamical properties of models of this kind are obtained in the homogeneous and inhomogeneous cases. Homogeneous models with $\Lambda > 0$ were studied in [27] and [17] and those results were extended to scalar fields with potentials belonging to various classes in [15], [18], [21] and [23]. In the inhomogeneous case theorems were proved for vacuum solutions with $\Lambda > 0$ in [9], [22] and [1], the case of an exponential potential in [13] and for some inhomogeneous cases with matter in [26] and [25]. Note also the results of [6] on curvature coupled scalar fields. The purpose of this paper is to extend some of these results to a larger class of models, the k -essence models.

The name k -essence denotes a nonlinear scalar field ϕ with Lagrangian $L(\phi, X)$, where $X = -\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi$, which is used to describe late-time cosmic acceleration. It was introduced by Armendariz-Picon, Mukhanov and Steinhardt [4] following the earlier application of similar models to the very early universe under the name k -inflation [2]. Fields of this kind arise in effective field theories coming from string theory when higher order corrections in the string tension or number of loops are included. Special cases of the k -essence Lagrangian are the ordinary minimally coupled nonlinear scalar field (associated with the name quintessence) where $L(\phi, X) = -V(\phi) + X$ and the tachyon, where $L = -V(\phi)\sqrt{1 - 2X}$ (see [8], [11]). Here the potential V is a smooth function, assumed in the following to be non-negative. The equation of motion is

$$\left(\frac{\partial L}{\partial X}g^{\alpha\beta} - \frac{\partial^2 L}{\partial X^2}\nabla^\alpha\phi\nabla^\beta\phi\right)\nabla_\alpha\nabla_\beta\phi + \frac{\partial^2 L}{\partial\phi\partial X}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi = -\frac{\partial L}{\partial\phi} \quad (1)$$

In the spatially homogeneous case this reduces to

$$\left(\frac{\partial L}{\partial X} + 2X\frac{\partial^2 L}{\partial X^2}\right)\ddot{\phi} + \frac{\partial L}{\partial X}(3H\dot{\phi}) + \frac{\partial^2 L}{\partial\phi\partial X}\dot{\phi}^2 - \frac{\partial L}{\partial\phi} = 0 \quad (2)$$

Note that in the spatially homogeneous case $X \geq 0$. To ensure the solvability of equation (2) for $\ddot{\phi}$ it should be assumed that $\partial L/\partial X + 2X\partial^2 L/\partial X^2 \neq 0$. For a given choice of L this is a restriction on the allowed values of ϕ and $\dot{\phi}$. In general L is not defined on the whole (ϕ, X) -plane but only on a subset of it. As an example, for the tachyon it is necessary that $X < 1/2$ in order that L be defined. Once this is ensured $\partial L/\partial X + 2X\partial^2 L/\partial X^2 > 0$ under the sole restriction that the potential be positive.

The energy-momentum tensor is given by

$$T^{\alpha\beta} = \frac{\partial L}{\partial X} \nabla^\alpha \phi \nabla^\beta \phi + L g^{\alpha\beta} \quad (3)$$

In the spatially homogeneous case it has two distinct non-vanishing components, ρ and p . In fact the pressure p is just equal to the Lagrangian. The energy density ρ is given by $2X\partial L/\partial X - L$. It follows that $\rho + p = 2X\partial L/\partial X$. The inequality $\rho + p \geq 0$ is a not very restrictive energy condition. In particular it follows from the dominant energy condition. If this energy condition is satisfied then it follows that $\partial L/\partial X \geq 0$ in the homogeneous case. In the following it will always be assumed that

$$\frac{\partial L}{\partial X} + 2X \frac{\partial^2 L}{\partial X^2} > 0, \quad \frac{\partial L}{\partial X} > 0 \quad (4)$$

As is discussed in Section 2 these conditions follow from the hyperbolicity of the equation of motion of ϕ and the energy condition mentioned above. The sign of $\rho + 3p = 2(X\partial L/\partial X + L)$ determines whether there is accelerated expansion in the homogeneous case. It follows from the equation of motion that

$$\frac{d}{dt} \left(2X \frac{\partial L}{\partial X} - L \right) = -6HX \frac{\partial L}{\partial X} \quad (5)$$

The paper is organized as follows. Section 2 deals with the question, for which choices of the Lagrangian L the equation of motion of k -essence is hyperbolic and under what conditions on L various energy conditions are satisfied. In Section 3 late time acceleration and isotropization are shown for homogeneous spacetimes of Bianchi type I-VIII with k -essence defined by a Lagrangian chosen from a wide class. The conditions defining this class are modelled on those previously used in analysing nonlinear scalar fields with a potential which have a positive lower bound. In particular tachyon models whose potential has a positive lower bound are treated. In these results the spacetime is allowed to contain normal matter in addition to the k -essence. The normal matter is only supposed to satisfy the dominant and strong energy conditions. Section 4 proceeds to consider homogeneous spacetimes where late time accelerated expansion is driven by the kinetic energy. In these spacetimes the kinetic energy does not go to zero at late times, in contrast to those studied in Section 3. In Section 5 a general criterion for isotropization is obtained and is applied in the special case of quintessence to generalize

the results of [23]. Appendix 1 contains some technical background required for Section 3 while Appendix 2 contains continuation criteria which are used to obtain global existence theorems in Section 3 and 4.

2 The general inhomogeneous case

The equations of a k -essence field ϕ coupled to the Einstein equations form a well-posed system. This means that if suitable initial data are prescribed there exists a unique local solution of the Einstein-matter equations inducing those initial data. This can be proved following the general scheme described in [10], section 5.4. Under the assumptions (4) the equation of motion of ϕ is a nonlinear wave equation. This follows from the fact that the tensor multiplying the second derivatives of ϕ in (1) is a Lorentz metric. The proof of this is given in Appendix 1. By reduction to first order this wave equation can be written as a symmetric hyperbolic system. The coupling of this system to the Einstein equations is such that the whole Einstein-matter system can be written in symmetric hyperbolic form. A local existence and uniqueness theorem follows. This argument also applies to the case where in addition to the coupling to the k -essence field the Einstein equations are coupled to other matter having a well-posed initial value problem, e.g. a perfect fluid with reasonable equation of state.

One cautionary remark is necessary. For the coupled system initial data may only be prescribed on a hypersurface which is spacelike with respect to both the spacetime metric and the metric in the principal part of the wave equation for ϕ . If $\partial^2 L/\partial X^2 \geq 0$ then it is enough to assume that the hypersurface is spacelike with respect to $g_{\alpha\beta}$. Otherwise it is not. In the latter case superluminal propagation of signals is possible. This is proved in Appendix 1.

Energy conditions can be investigated on a case by case basis as is done in Appendix 1 for the signature of the metric in the equations of motion. The results are as follows. The dominant energy condition is satisfied iff $\partial L/\partial X \geq 0$ and $X\partial L/\partial X - L \geq 0$. For $X > 0$ the weak energy condition is equivalent to $\partial L/\partial X \geq 0$ and $2X\partial L/\partial X - L \geq 0$. For $X < 0$ the weak energy condition is equivalent to $\partial L/\partial X \geq 0$ and $L \leq 0$. The strong energy condition is equivalent to $\partial L/\partial X \geq 0$ and $X\partial L/\partial X + L \geq 0$ for $X > 0$ and $\partial L/\partial X \geq 0$ and $-X\partial L/\partial X + L \geq 0$ for $X < 0$.

A potential difficulty with k -essence models is that the matter may form

singularities which have nothing to do with the familiar spacetime singularities of general relativity. (Cf. the discussion in [8].) It would be interesting to know for which choices of L this can be avoided. A simple place to start is to look at a k -essence field in Minkowski space and consider solutions evolving from data which are close to zero. A criterion for the corresponding smooth solution to exist globally in time is the null condition [16]. This is an algebraic condition on the nonlinearity. It turns out that it is satisfied for the k -essence field if and only if $\partial^k L / \partial \phi^k = 0$ at $(\phi, X) = (0, 0)$ for $k = 1, 2, 3$. In that case there are no singularities for small data. In particular this is true if L only depends on X .

3 Solutions with a positive lower bound for ρ

Consider a spatially homogeneous solution of Bianchi type I-VIII of the Einstein equations with matter. Suppose that the energy-momentum tensor is a sum of two parts. One of these $T_{\alpha\beta}^M$ represents ordinary matter and satisfies the dominant and strong energy conditions. The other represents dark energy and matter quantities derived from it will be given the subscript DE . It is assumed in the following that the dark energy satisfies the dominant energy condition. Some basic equations are as follows:

$$\frac{dH}{dt} = -H^2 - \frac{4\pi}{3}(\rho_{DE} + \text{tr}S_{DE}) - \frac{1}{3}\sigma_{ab}\sigma^{ab} - \frac{4\pi}{3}(\rho^M + \text{tr}S^M) \quad (6)$$

$$H^2 = \frac{8\pi}{3}\rho_{DE} + \frac{1}{6}(\sigma_{ab}\sigma^{ab} - R) + \frac{8\pi}{3}\rho^M \quad (7)$$

$$\frac{dH}{dt} = -4\pi(\rho_{DE} + \frac{1}{3}\text{tr}S_{DE}) - \frac{1}{2}\sigma_{ab}\sigma^{ab} + \frac{1}{6}R - 4\pi(\rho^M + \frac{1}{3}\text{tr}S^M) \quad (8)$$

These are the direct generalizations of equations (4), (6) and (7) of [21]. Suppose that H is zero at some time t_1 . Then by (7) ρ_{DE} is zero at that time. By (5) it vanishes at all later times. Since the k -essence field satisfies the dominant energy condition it follows that it makes no contribution to the energy-momentum tensor. Hence the geometry and normal matter satisfy the equations in the absence of k -essence. By uniqueness for the full system of equations this must be true at all times and not just after time t_1 . It may be said that the k -essence field is passive and these solutions are physically uninteresting. The results of [21] show that the normal matter vanishes and

that spacetime is flat. For this reason it will be assumed in the following that H is everywhere positive.

In the next theorem no specific choice of a dark energy model is made. It will be specialized to the case of k -essence later.

Theorem 1 Consider a solution of the Einstein equations of Bianchi type I-VIII coupled to a matter model representing dark energy and other matter satisfying the strong and dominant energy conditions. Let the energy-momentum tensor $T_{DE}^{\alpha\beta}$ of the dark energy have the algebraic form of that of a perfect fluid and satisfy the dominant energy condition. Suppose that the energy density of dark matter ρ_{DE} has a positive lower bound $\rho_0 > 0$ and that the solution is initially expanding ($H > 0$). Then $H \geq H_0$ for some constant $H_0 > 0$ and if the solution exists globally in the future the quantities ρ^M , $\sigma_{ab}\sigma^{ab}$ and R decay exponentially.

Proof The first conclusion follows immediately from the Hamiltonian constraint. To prove the remainder of the theorem let $Z = 9H^2 - 24\pi\rho_{DE}$. As in the special case of quintessence considered in [21] the quantity Z satisfies the inequality $dZ/dt \leq -2HZ$. The only properties of the dark energy model which are used in deriving this are those included in the hypotheses of the theorem. From the inequality it follows that if the solution exists globally in the future the lower bound for H implies that Z decays exponentially as $t \rightarrow \infty$. The Hamiltonian constraint shows that

$$Z = \frac{3}{2}\sigma_{ab}\sigma^{ab} - \frac{3}{2}R + 24\pi\rho^M \quad (9)$$

and this concludes the proof.

The interpretation of this theorem is that global solutions isotropize and that the spatial curvature and the energy density of normal matter have very little effect on the dynamics at late times. In the case of k -essence the energy-momentum tensor of a homogeneous model always has perfect fluid form. To obtain the other conditions it suffices to supplement (4) by the inequality $2X\partial L/\partial X - L \geq C$ for a constant $C > 0$. The theorem also applies to fluid models for dark energy which satisfy the dominant energy condition and have a positive lower bound for the energy density, such as the Chaplygin gas [14]. Note that the argument of the theorem also applies to the situation where there is no global positive lower bound for ρ_{DE} but for a given solution which exists globally in the future ρ_{DE} tends to a positive limit as $t \rightarrow \infty$.

It follows from equation (5) and the lower bound for H that the quantity

$X\partial L/\partial X$ is integrable on the positive time axis for any global solution. Now

$$d/dt \left(X \frac{\partial L}{\partial X} \right) = \dot{\phi} \left[\ddot{\phi} \left(\frac{\partial L}{\partial X} + X \frac{\partial^2 L}{\partial X^2} \right) + X \frac{\partial^2 L}{\partial \phi \partial X} \right] \quad (10)$$

When the expression for $\ddot{\phi}$ is substituted into this the contribution containing H is non-positive. Discarding it gives an upper bound for the time derivative of $X\partial L/\partial X$ in terms of various other quantities.

Now a list of relevant assumptions on the Lagrangian will be collected. Given a constant $C_1 > 0$ there exists a constant $C_2 > 0$ such that:

1. if $\rho \leq C_1$ then $X \leq C_2$
2. if $\rho \leq C_1$ then $|\partial L/\partial X + X\partial^2 L/\partial X^2| \leq C_2$
3. if $X \leq C_1$ then $|\partial L/\partial X + 2X\partial^2 L/\partial X^2| \geq C_2^{-1}$
4. if $\rho \leq C_1$ then $|\partial^2 L/\partial \phi \partial X| \leq C_2$
5. if $\rho \leq C_1$ then $|\partial L/\partial \phi| \leq C_2$
6. if $X \leq C_1$ then $\partial L/\partial X \geq C_2^{-1}$
7. there are positive constants C_3 and C_4 such that $X \leq C_3$ implies $L(\phi, X) \leq -C_4$.

Theorem 2 Consider a solution of the Einstein equations of Bianchi type I-VIII coupled to a k -essence model and other matter satisfying the strong and dominant energy conditions. Let the energy-momentum tensor $T_{DE}^{\alpha\beta}$ of the k -essence satisfy the conditions (4) and the dominant energy condition and let there be a positive lower bound for the energy density. Suppose that the solution is initially expanding ($H > 0$). Then if the solution exists globally in the future and conditions 1.- 7. above are satisfied the expansion is accelerated at late times.

Proof Equation (5) shows that ρ_{DE} is bounded above. Then assumption 1. above implies that X is bounded. Assumptions 2.-5. imply that $X\partial L/\partial X \rightarrow 0$ as $t \rightarrow \infty$. By assumption 6. we can conclude that $X \rightarrow 0$ as $t \rightarrow \infty$. Finally assumption 7. implies that the expansion is accelerated at late times.

Since the assumptions 1.-7. look quite abstract, let us see what they mean for quintessence and for the tachyon. Consider first quintessence with

a non-negative potential. Conditions 1.- 4. obviously hold. Condition 5. holds provided V' is bounded on any interval where V is. This assumption played an important role in [21]. Condition 6. is obvious. Condition 7. holds if V has a positive lower bound. This special case of Theorem 2 was proved in [21]

Next consider the tachyon. Since X is a priori bounded by one in this case condition 1. is obvious. In the case of the tachyon if ρ is bounded then so is V . If in addition V has a positive lower bound then X is bounded away from one. Then condition 2. follows. If V has a positive lower bound then condition 3. holds. For conditions 4. and 5. it should be assumed that V' is bounded whenever V is. With a positive lower bound for V condition 6. holds. Condition 7. is automatic for the tachyon. Thus Theorem 2. implies the following result

Theorem 3 Consider a solution of the Einstein equations of Bianchi type I-VIII coupled to a tachyon field and other matter satisfying the strong and dominant energy conditions. Suppose that the tachyon potential V has a positive lower bound, that V'/V is bounded and that the solution is initially expanding ($H > 0$). Then if the solution exists globally in the future the expansion is accelerated at late times.

For a perfect fluid or collisionless matter the global existence assumption in Theorem 3 holds automatically. For

$$d/dt(\log(1 - \dot{\phi}^2)) = (1 - \dot{\phi}^2)(3H\dot{\phi}^2 + V'/V\dot{\phi}) \quad (11)$$

and this prevents $|\dot{\phi}|$ approaching one in finite time. This in turn ensures that $(\phi, \dot{\phi})$ remains in a compact subset of the domain of definition of L . It follows that Lemma 1 and Lemma 2 of Appendix 2 can be applied.

If L depends only on X and is globally defined and if it satisfies condition 1. then conditions 2.- 6. are automatic. Moreover condition 7. can be replaced by the condition $L(0) < 0$. In [2] the case where L only depends on X plays an important role but L does not satisfy the positivity conditions (4) everywhere. An alternative interpretation is to keep those conditions and say that L is only defined locally. An interesting generalization is given by $L(\phi, X) = L_0(\phi) + L_1(\phi)X + L_2(\phi)X^2$. This is considered in [2] under the condition $L_0 = 0$ (no potential). Theorem 2 applies if $-L_0$ and L_1 are bounded below by a positive constant.

4 Solutions with a positive lower bound for the kinetic energy

The original aim of k -essence models was to produce a scenario where the accelerated expansion of the universe is driven by the kinetic energy of a scalar field. This is very different from the usual case of quintessence where the potential energy is the driving force. In the models studied in the previous section it was the latter mechanism which was at work. The present section is concerned with models closer to the original motivation for considering k -essence. They have the property that they have late time accelerated expansion where the kinetic energy does not tend to zero.

What can happen is illustrated by the following specific example. Let $L = X^2 - 2X$ on the interval $I = (1, \infty)$. The conditions (4) are satisfied on I , as is the dominant energy condition. The equation of motion of ϕ implies

$$\dot{X} = -6(3X - 1)^{-1}(X - 1)XH \quad (12)$$

In particular X is decreasing. Also $H \geq (8\pi/3)(3X^2 - 2X)$. It follows that if a solution of the Einstein-matter equations exists globally in the future then $X \rightarrow 1$ as $t \rightarrow \infty$. The continuation criteria of Appendix 2 show that when the normal matter is described by a perfect fluid or collisionless matter solutions exist globally. The mean curvature remains bounded away from zero at late times because ρ_{DE} does. The inequality $dZ/dt \leq 2HZ$ then implies as in the last section that at late times the influence of the spatial curvature and the energy density of ordinary matter becomes negligible and the geometry isotropizes. Since $XL' + L < 0$ for $X \in (1, 4/3)$ it follows that all solutions are accelerated at late times.

This example can easily be generalized. Consider a Lagrangian $L = L(X)$ defined on an interval (X_0, X_1) with $X_0 > 0$ and X_1 finite or infinite. Suppose further that (4) holds together with the dominant energy condition. Finally, suppose that $2XL' - L$ is never zero, that $\lim_{X \rightarrow X_0} (X - X_0)^{-1}L'$ exists and that $\lim_{X \rightarrow X_0} L$ exists and is negative. The equation of motion is

$$\dot{X} = -6(L' + XL'')^{-1}L'XH \quad (13)$$

Thus X is decreasing. From the Hamiltonian constraint it follows that $H^2 \geq (8\pi/3)(2XL' - L)$. Hence under the given conditions if a solution exists globally in the future then $X \rightarrow X_0$ as $t \rightarrow \infty$. Provided $L' + XL''$ is

bounded away from zero in a neighbourhood of X_0 global existence theorems follow from the continuation criteria of Appendix 2. In this case it can be concluded as before that the geometry isotropizes and that there is late time acceleration.

In the general case where the Lagrangian also depends on ϕ the equation of motion implies that

$$\dot{X} = - \left(\frac{\partial L}{\partial X} + 2X \frac{\partial^2 L}{\partial X^2} \right)^{-1} \left[6X \frac{\partial L}{\partial X} H + 2\sqrt{2}X^{3/2} \frac{\partial^2 L}{\partial \phi \partial X} - \sqrt{2}X^{1/2} \frac{\partial L}{\partial \phi} \right] \quad (14)$$

As a global simplifying assumption, suppose that the domain of definition of L is of the form $J \times I$ where J is an open interval and I is the interval (X_0, ∞) with $X_0 > 0$. Assume as usual (4) and the dominant energy condition. In the following a theorem is proved which concerns a situation where the dependence of L on ϕ does not disturb the behaviour seen for a Lagrangian only depending on X too much. Looking at (14) suggests that a smallness assumption should be made on $\partial L/\partial \phi$ and $\partial^2 L/\partial \phi \partial X$. It will be assumed that there is a positive constant α such that

$$|\phi \partial L/\partial \phi| + 2|\phi X \partial^2 L/\partial \phi \partial X| \leq \alpha(\partial L/\partial X) \quad (15)$$

This is motivated by the consideration of Lagrangians of the form $L(\phi, X) = \tilde{L}(X)\phi^{-2}$ as introduced in [4]. The time derivative of ϕ has constant sign and only solutions for which $\dot{\phi} > 0$ will be considered. Since a solution which exists globally in the future satisfies $\dot{\phi} > \sqrt{2X_0}$ it follows that ϕ grows at least linearly as $t \rightarrow \infty$ so that (15) implies decay of $(\partial L/\partial \phi)/(\partial L/\partial X)$. The following conditions are required in the following results. Given a constant $C_1 > 0$ there exists a constant $C_2 > 0$ such that

1. if $X \geq X_0 + C_1$ then $\partial L/\partial X \geq C_2^{-1}$
2. $(X - X_0)^{-1} \partial L/\partial X$ converges to a limiting function as $X \rightarrow X_0$
3. L converges to a negative limiting function L_0 as $X \rightarrow X_0$
4. if $X \leq X_0 + C_1$ then $(2X \partial L/\partial X - L) \geq C_2^{-1}$
5. $(\partial L/\partial X + X \partial^2 L/\partial X^2) \geq C_2^{-1}$ for all X
6. $\partial L/\partial X \leq C_2 X^{1/2}$ for all X

Theorem 4 Consider a solution of the Einstein equations of Bianchi type I-VIII coupled to a k -essence model and other matter satisfying the strong and dominant energy conditions. Let the energy-momentum tensor $T_{DE}^{\alpha\beta}$ of the k -essence satisfy the conditions (4) and the dominant energy condition. Suppose that the inequality (15) and conditions 1.-4. above are satisfied. If the solution is initially expanding and exists globally in the future then the expansion is accelerated at late times and the geometry isotropizes.

Proof Under the assumptions of the theorem the following estimate is obtained:

$$6X \frac{\partial L}{\partial X} H + 2\sqrt{2}X^{3/2} \frac{\partial^2 L}{\partial \phi \partial X} - \sqrt{2}X^{1/2} \frac{\partial L}{\partial \phi} \geq X^{1/2} \frac{\partial L}{\partial X} (6X^{1/2}H - \sqrt{2}\phi^{-1}\alpha) \quad (16)$$

The function ϕ^{-1} tends to zero as $t \rightarrow \infty$. Hence the second term in the bracket on the right hand side tends to zero. To get information about the first term, note that due to (4) the energy density ρ_{DE} is an increasing function of X and that $H \geq \sqrt{8\pi\rho_{DE}/3}$. Together with assumption 4. this implies a global positive lower bound for ρ_{DE} and for H . Hence $\dot{X} < 0$ for t sufficiently large. It follows using assumption 1. that $X \rightarrow X_0$ as $t \rightarrow \infty$. The remaining conclusions of the theorem follow using assumptions 2. and 3.

It is of interest to ask when global existence holds so that the theorem can be applied. First a lower bound for X will be obtained. At a given time either $X < 2X_0$ or $X \geq 2X_0$. If the former condition holds let t_2 be the last time before t_1 at which $X \geq 2X_0$ if such a time exists and let t_2 be the initial time otherwise. On this interval X and ϕ^{-1} are bounded. It follows by condition 5. that

$$|\dot{X}| \leq C\partial L/\partial X \quad (17)$$

for a constant C . Since $\partial L/\partial X$ is $O(X - X_0)$ as $X \rightarrow X_0$ it can be concluded that X remains bounded away from zero on the interval $[t_2, t_1]$ with a bound depending only on the initial data. It follows that for any solution X remains bounded away from X_0 on any finite time interval. Using assumption 6 above and Gronwall's inequality it can be concluded that X is also bounded on any finite time interval. It follows that the solution cannot approach the boundary of the domain of definition of L in finite time. The results of Appendix 2 can be applied to obtain global existence statements.

The conditions 1.-6. do not apply to the case $L(\phi, X) = \phi^{-2}\tilde{L}(X)$ occurring in [4]. Next some properties of solutions for Lagrangians of this form will be examined. The equation of motion takes the form:

$$(\tilde{L}' + 2X\tilde{L}'')\dot{X} + \tilde{L}'(6XH - 4\sqrt{2}\phi^{-1}X^{3/2}) + 2\sqrt{2}X^{1/2}\phi^{-1}\tilde{L} = 0 \quad (18)$$

In this case it is possible for \dot{X} to become zero, the condition for this being that $2X\tilde{L}' - \tilde{L} = (3/\sqrt{2})\phi X^{1/2}H\tilde{L}'$. In a FLRW model without ordinary matter this reduces to $\tilde{L} = 2X(-6\pi(\tilde{L}')^2 + \tilde{L}')$. Any solution of this equation leads to a solution of the Einstein equations coupled to k -essence alone in which X is constant and the scale factor has power-law behaviour. Solutions of this type were considered in section 5 of [2] and correspond to the k -attractors of [4]. If $X\tilde{L}' + \tilde{L} < 0$ the expansion is accelerated. Eliminating \tilde{L} gives the inequality $\tilde{L}' > 1/4\pi$. The dominant energy condition is equivalent to $X\tilde{L}' - \tilde{L} \geq 0$ which gives the inequality $\tilde{L}' > 1/12\pi$.

It is clear that there are many k -essence Lagrangians together with solutions of the type just discussed where X is constant. Note however that it is not guaranteed that they will exist for any given Lagrangian. In the case of the tachyon solutions of this kind are known explicitly [7].

By introducing a new time coordinate we can consider (18) as an ordinary differential equation for X alone which does not involve ϕ . This makes it very easy to do a local stability analysis of the stationary solution in the class of spatially flat Friedmann models without normal matter. It turns out that the condition for stability is $\tilde{L}''(X\tilde{L}' - \tilde{L}) > 3\pi(\tilde{L}')^2$. Assuming the dominant energy condition this can only be satisfied if $\tilde{L}'' > 0$ which means that there is no superluminal propagation in inhomogeneous models for this Lagrangian. It is more difficult to prove something about the stability of these solutions within a class of anisotropic solutions. One reason this is difficult is because the accelerated expansion is only of power-law type and not faster. In [23] isotropization was proved for models whose expansion increases in time faster than any power of t corresponding to a potential which decays slower than any exponential. For general potentials which decay at a rate comparable to an exponential no result of this kind was obtained. The only case for which isotropization has been proved is that of an exact exponential [18]. In the next section some partial results on isotropization are obtained which, in particular, considerably extend what is known in the case of quintessence.

5 A criterion for isotropization

Consider a solution of Bianchi type I-VIII of the Einstein equations coupled to dark energy (satisfying the dominant energy condition) and ordinary matter (satisfying the dominant and strong energy conditions). A computation using the evolution equation for H and the Hamiltonian constraint shows that

$$\frac{dH}{dt} \geq -3H^2 \left(1 - \frac{4\pi}{3} \left(\rho_{DE} - \frac{1}{3} \text{tr} S_{DE} \right) / H^2 \right) \quad (19)$$

Suppose that $\liminf_{t \rightarrow \infty} (\rho_{DE} - \frac{1}{3} \text{tr} S_{DE}) / H^2 > \frac{1}{2\pi}$. Then it follows that for t sufficiently large $dH/dt \geq -\beta H^2$ for a constant $\beta < 1$. Hence $H(t) \geq \beta^{-1} t^{-1} + O(t^{-2})$. It follows that for t large $H(t) \geq \gamma t^{-1}$ for any γ less than β^{-1} . The constant γ may be chosen larger than one. Putting this inequality into the relation $dZ/dt \leq -2HZ$ shows that $Z = O(t^{-2\gamma})$ and that $Z/H^2 \rightarrow 0$ as $t \rightarrow \infty$. Thus a criterion for isotropization has been obtained.

What does the criterion look like in the case of quintessence? There

$$\left(\rho_{DE} - \frac{1}{3} \text{tr} S_{DE} \right) / H^2 = \frac{2V}{H^2} \quad (20)$$

so that we get the inequality $\liminf(8\pi V/3H^2) > 2/3$. It will now be shown that if $\alpha = \limsup_{\phi \rightarrow \infty} (-V'/V) < 4\sqrt{\pi/3}$ then the criterion is satisfied. The starting point of the proof is provided by equation (11) and the inequality (13) of [23]. There is a number ϕ_1 such that $-V'/V \leq \alpha/2 + \sqrt{4\pi/3}$ for all $\phi \geq \phi_1$. Now distinguish between the cases $\dot{\phi}/H \geq 1/\sqrt{12\pi}$ and $\dot{\phi}/H \leq 1/\sqrt{12\pi}$. In the first case equation (11) of [23] gives an upper bound for $d/d\phi(3H^2/8\pi V)$ while in the second case equation (11) and the inequality (13) of that reference imply the bound (14) for $d/d\phi(3H^2/8\pi V)$. Combining these two bounds shows that the following inequality holds in general:

$$\frac{d}{d\phi} \left(\frac{3H^2}{8\pi V} \right) \leq - \min \left\{ \left[\sqrt{48\pi} \left(1 - \frac{8\pi V}{3H^2} \right) + \frac{V'}{V} \right], \sqrt{\frac{4\pi}{3}} - \frac{\alpha}{2} \right\} \left(\frac{3H^2}{8\pi V} \right) \quad (21)$$

Suppose that

$$\frac{3H^2}{8\pi V} \geq \left(1 - \frac{\alpha}{\sqrt{48\pi}} \right)^{-1} + C_1 \quad (22)$$

on some interval for a positive constant C_1 . Then

$$1 - \frac{8\pi V}{3H^2} \geq \left[C_1 \left(1 - \frac{\alpha}{\sqrt{48\pi}} \right) + \frac{\alpha}{\sqrt{48\pi}} \right] \left[1 + C_1 \left(1 - \frac{\alpha}{\sqrt{48\pi}} \right) \right]^{-1} \quad (23)$$

The expression on the right hand side of this inequality is strictly greater than $\alpha/\sqrt{48\pi}$. Increase ϕ_1 if necessary so that

$$\left| \frac{V'}{V} \right| \leq \frac{1}{2} \sqrt{48\pi} \left[C_1 \left(1 - \frac{\alpha}{\sqrt{48\pi}} \right) + \frac{\alpha}{\sqrt{48\pi}} \right] \left[1 + C_1 \left(1 - \frac{\alpha}{\sqrt{48\pi}} \right) \right]^{-1} \quad (24)$$

for $\phi > \phi_1$. This is possible due to the restriction on α which has been assumed. It follows that the criterion for isotropization holds.

The result just proved extends the proof of isotropization and decay of spatial curvature and energy density of ordinary matter significantly in comparison with what is proved in [23]. It applies for example to the potentials $V(\phi) = V_0(\log \phi)^p \phi^n \exp(-\lambda\phi)$, the special case of equation (18) in [23] with $m = 1$ provided $\lambda < 4\sqrt{\pi}/3$ and to a number of other potentials listed in [23].

6 Conclusions and outlook

In this paper results are obtained on the dynamics of solutions of the Einstein equations with k -essence and normal matter. They include anisotropic spacetimes of Bianchi type I-VIII and not only FLRW models. In this setting late time isotropization is proved. They allow the normal matter to be anything which satisfies the strong and dominant energy conditions. Thus the results are not confined to fluids or multifluids and apply, for instance, to collisionless matter. All the arguments are rigorous and not dependent on heuristics.

Before coming to the global results for homogeneous solutions, basic information is presented concerning the initial value problem for the Einstein-matter equations with k -essence in general, i.e. without symmetry assumptions. A link is made with a concept in the theory of nonlinear hyperbolic equations, the null condition, and it would be desirable to establish further connections in the future. The paper [8] could be a good starting point.

It is shown that in general if ρ_{DE} satisfies a suitable lower bound homogeneous solutions of Bianchi type I-VIII which exist globally in the future become homogeneous with dimensionless measures of the spatial curvature and density of normal matter tending to zero exponentially. This works if a global lower bound is available for the given matter model or if it is available for a particular solution. It is shown further in Section 3 that if certain

bounds are assumed on the Lagrangian it is possible to prove late-time accelerated expansion. In this class of models $X \rightarrow 0$ as $t \rightarrow \infty$. They very much resemble quintessence models with a lower bound for the potential. It would be interesting to prove a similar result for a class generalizing quintessence models with a potentials tending to zero at infinity as analysed in [23].

In Section 4 results are obtained on late time behaviour of solutions whose kinetic energy does not tend to zero as $t \rightarrow \infty$. They achieve the aim of proving late time accelerated expansion and isotropization for models where these phenomena occur. The questions of behaviour at intermediate times and late time behaviour without acceleration are not addressed since they are not amenable to the techniques used here. If the expansion is not accelerated at late times then the behaviour is strongly dependent on Bianchi type and a unified treatment is difficult or impossible. As far as the behaviour at intermediate times is concerned it is difficult to even formulate interesting and precisely defined statements susceptible to rigorous proof. This is connected to the fact that the interesting statements often involve essentially quantitative elements while rigorous results tend to be qualitative in nature. These issues deserve a separate discussion.

The first case considered in Section 4 is that where L is independent of ϕ . A more general theorem is then proved which shows that certain kinds of dependence on ϕ do not change the qualitative behaviour in comparison to the case where L only depends on X . Late time acceleration and isotropization are proved. Next the case $L(\phi, X) = \phi^{-2}\tilde{L}(X)$ is considered. This allows solutions with $\dot{X} = 0$, the k -attractors. The stability of these solutions within the class of isotropic spatially flat models without ordinary matter is determined. Isotropization is not proved. The fact that the expansion of the candidate attractor is only of power-law type makes this difficult, as it does for quintessence models with this type of asymptotics. To make further progress a criterion for isotropization in dark energy models is developed in Section 5. It is shown that this criterion leads to a significant improvement in the results which can be obtained in the case of quintessence. Similar conclusions have not yet been obtained for more general k -essence models.

This paper contains new results on the mathematical properties of solutions of the Einstein equations coupled to dark energy and other matter and concentrates on establishing a basis for the mathematical study of k -essence models. The k -essence Lagrangians give rise to a considerable variety of behaviour and the task of obtaining an overview of the possibilities is a challenge for the future.

Appendix 1. Analysis of characteristics

Results similar to those of this appendix were obtained in [3]. The treatment here is intended to make the proofs more transparent by avoiding the use of series expansions.

Consider the expression $\frac{\partial L}{\partial X}g^{\alpha\beta} - \frac{\partial^2 L}{\partial X^2}\nabla^\alpha\phi\nabla^\beta\phi$ which occurs in the principal part of the k -essence equations of motion. Evidently this is degenerate if $\partial L/\partial X = 0$. We now assume that $\partial L/\partial X \neq 0$ and define

$$\tilde{h}^{\alpha\beta} = (\partial L/\partial X)^{-1}\left(\frac{\partial L}{\partial X}g^{\alpha\beta} - \frac{\partial^2 L}{\partial X^2}\nabla^\alpha\phi\nabla^\beta\phi\right) = g^{\alpha\beta} - K\nabla^\alpha\phi\nabla^\beta\phi \quad (25)$$

where $K = \frac{\partial^2 L/\partial X^2}{\partial L/\partial X}$. If $1 - K\nabla_\sigma\phi\nabla^\sigma\phi$ is non-zero let $h_{\alpha\beta} = g_{\alpha\beta} + \frac{K\nabla_\alpha\phi\nabla_\beta\phi}{1 - K\nabla_\sigma\phi\nabla^\sigma\phi}$. Then by a direct computation $h_{\alpha\beta}$ is the inverse of $\tilde{h}^{\alpha\beta}$. The condition $1 - K\nabla_\sigma\phi\nabla^\sigma\phi = 0$ is equivalent to $\frac{\partial^2 L}{\partial X^2}\nabla_\sigma\phi\nabla^\sigma\phi = \frac{\partial L}{\partial X}$. If $\frac{\partial^2 L}{\partial X^2} = 0$ then this never happens. Otherwise the assumption that it does not happen puts a restriction on $\nabla_\sigma\phi\nabla^\sigma\phi$.

Next the signature of the metric $\tilde{h}^{\alpha\beta}$ will be investigated. Suppose first that the gradient of ϕ is timelike. Then $\nabla^\alpha\phi = AT^\alpha$ for a unit timelike vector T^α . Then $\tilde{h}^{\alpha\beta}T_\alpha T_\beta = -1 - KA^2$ while $\tilde{h}^{\alpha\beta}S_\alpha S_\beta = 1$ for any unit spacelike vector S^α orthogonal to T^α . Moreover $\tilde{h}^{\alpha\beta}S_\alpha T_\beta = 0$. Thus the signature of \tilde{h} is Lorentzian for $KA^2 > -1$. For $KA^2 < -1$ it is positive definite. The equation becomes elliptic. If the gradient of ϕ is spacelike then it follows that $\nabla^\alpha\phi = AS^\alpha$ for a unit spacelike vector S^α . A calculation similar to the one just done shows that the signature is Lorentzian for $KA^2 < 1$ and $(-, -, +, +)$ for $KA^2 > 1$. If $\nabla^\alpha\phi$ is null the signature is always Lorentzian. Hence in all cases an equivalent condition to the metric being Lorentzian is $K\nabla_\alpha\phi\nabla^\alpha\phi < 1$. This can be re-expressed as $2X\frac{\partial^2 L}{\partial X^2} + \frac{\partial L}{\partial X} > 0$ when $\partial L/\partial X > 0$.

Let v^α be a vector which is null with respect to the metric $h_{\alpha\beta}$. Then $g_{\alpha\beta}v^\alpha v^\beta = -K(\nabla_\alpha\phi v^\alpha)^2/(1 - K\nabla_\alpha\phi\nabla^\alpha\phi)^2$. It follows that if K is positive the null cone of $h_{\alpha\beta}$ is inside (or on) that of $g_{\alpha\beta}$ while if K is negative it is outside (or on) that of $g_{\alpha\beta}$.

Appendix 2. Continuation criteria

In this appendix basic local existence theorems and continuation criteria will be proved for homogeneous spacetimes containing k -essence and normal matter. The choices of normal matter discussed are a large class of perfect fluids with positive pressure, a mixture of several non-interacting perfect

fluids and collisionless matter described by the Vlasov equation. Let G be the open subset of \mathbf{R}^2 where the Lagrangian L of the k -essence field is defined. Assume that the equation of state $p = f(\rho)$ of any perfect fluid occurring is such that $f : [0, \infty) \rightarrow [0, \infty)$ is continuous with $f(0) = 0$ and that for $\rho > 0$ the function f is C^1 with $0 \leq f'(\rho) \leq 1$

Lemma 1 Consider a solution of Bianchi type I-VIII of the Einstein equations coupled to a perfect fluid and a k -essence field ϕ on a time interval $[t_0, t_1)$ with $t_1 < \infty$. Suppose that the k -essence field satisfies the dominant energy condition. If $H(t)$ is bounded and if $(\phi, \dot{\phi}^2/2)$ remains in a compact subset of G then the solution can be extended to a longer time interval.

Proof The proof follows the discussion in section 4 of [20]. The equations of motion of the fluid and the k -essence field can be written in the form

$$A(z, g_{ij}, k_{ij})dz/dt = F(z, g_{ij}, k_{ij}) \quad (26)$$

where $z = (\rho, u^i, \phi, \dot{\phi})$ and tensors are expressed in components with respect to a left invariant frame of the Lie algebra defining a given Bianchi type. The matrix A is invertible. The standard local existence theorem for ordinary differential equations [12] applies to this system. It gives local existence of solutions and shows that the solution can be extended to a longer time interval provided (z, g_{ij}, k_{ij}) remains within a compact subset of the domain of definition of the coefficients. It will be shown that this follows from the assumptions of the Lemma. A first step is to show that under the given assumptions g_{ij} , $(\det g)^{-1}$ and k_{ij} remain bounded. The proof adapts an argument used in the proof of Lemma 2.1 of [19]. There the boundedness of the integral of $k_{ij}k^{ij}$ was deduced from certain energy conditions. In fact the dominant energy condition alone is enough. Equation (8) can be used to bound the integral of $\sigma_{ij}\sigma^{ij}$ where σ_{ij} is the tracefree part of k_{ij} . Together with the assumed boundedness of H this gives the desired conclusion. It remains to show that ρ , ρ^{-1} and u^i are bounded.. This can be done just as in [20].

As in [20], this result can immediately be extended to the case of several non-interacting fluids, e.g. a mixture of dust and radiation. It can also be adapted to the case of collisionless matter. For this purpose the existence theorem of section 2 of [19] has to be extended but this is straightforward. The quantities ϕ and $\dot{\phi}$ have to be added to those for which an iteration is carried out. The result is:

Lemma 2 Consider a solution of Bianchi type I-VIII of the Einstein equations coupled to a collisionless gas described by the Vlasov equation and a k -essence field ϕ on a time interval $[t_0, t_1)$ with $t_1 < \infty$. Suppose that the k -essence field satisfies the dominant energy condition. If $H(t)$ is bounded and if $(\phi, \dot{\phi}^2/2)$ remains in a compact subset of G then the solution can be extended to a longer time interval.

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