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Dynamics of Macrovariables for Nonuniform Systems

——*Scaling Theory*——

Hazime MORI

Department of Physics, Kyushu University, Fukuoka

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A scale transformation of the nonequilibrium macroscopic system to larger similar systems is introduced to find kinetic equations for the evolution and fluctuation of the macrovariables. In the scale transformation, we postulate that the probability distribution for the fluctuation of the macroscopic degrees of freedom and the quantities determined by the microscopic degrees of freedom per unit volume are invariant. The characteristic length of the macroscopic state l, the macroscopic state variables y_k and their fluctuation variables z_k are transformed by $l_L=Ll$, $(L\gg 1)$, $y_k{}^L=L^{-\alpha}y_k$ and $z_k{}^L=L^{-\beta}z_k$, respectively. The probability distribution then takes the form $P(\{z_k l^\beta\}, \{ql\}, \Omega/l^d, t/l^\theta)$, where q, Ω, d and t denote the wave vectors, volume, dimensionality and time, respectively. If $\alpha<\beta$, then the master equation is reduced to a linearly generalized Fokker-Planck equation with time-dependent coefficients and the probability distribution is normal around the mean evolution. If $\alpha\geqq\beta$, then the nonlinear drift terms are important and a renormalization of kinetic coefficients must be done to determine the mean evolution. For the isotropic Heisenberg ferromagnets near the Curie point, $\alpha=\beta=(d-2+\eta)/2$ and $\theta=(d+2-\eta)/2$, where η is the correlation critical exponent. For the isotropic homogeneous turbulence, $\alpha=1$, $\beta=-1/3$ and $\theta=2/3$, where Kolmogorov's spectrum is assumed. For example, this indicates that the turbulent viscosity has the form $q^{-4/3}V(\omega q^{-2/3})$, ω being the frequency.

§ 1. Introduction

Macroscopic systems have characteristic properties which do not appear in systems of small numbers of degrees of freedom. The central limit theorem and the phase transitions are outstanding examples. In a previous paper,[1] we have proposed a general type of kinetic equations from the statistical-mechanical point of view. In this paper we shall explore the most dominant features of macroscopic systems by introducing a new method of asymptotic evaluation for large systems and deriving the asymptotic form of the kinetic equations.

A similar attempt has been made by van Kampen[2] and by Kubo, Kitahara and Matsuo[3] for uniform systems by the use of the Kramers-Moyal expansion of the master equation. They extended the central limit theorem in the form of the system-size expansion of the master equation, and showed that, for large values of the system size Ω, the master equation is reduced to a linear Fokker-Planck equation and the probability distribution of the macrovariables is normal or Gaussian around their mean evolution. These works, however, are limited to the uniform disordered systems which are described by a small number of macrovariables.

The critical phenomena and the turbulence suggest that nonuniform systems can have quite different features and must be described by a large number of the macroscopic degrees of freedom which increase enormously as the system size Ω becomes large. For such systems the Ω expansion cannot be used. Instead a scale transformation of the nonequilibrium macroscopic state will be shown to be useful.

As the characteristic lengths, we have the linear range of the intermolecular force r_0 and the correlation length of fluctuation λ, where $\lambda > r_0$. In the case of critical phenomena near the equilibrium critical point, λ represents the correlation length of equilibrium fluctuation which becomes anomalously large near the critical point.[4] In the case of the fully-developed turbulence, λ would be the inverse of the lower limit of the inertial range of wave numbers, at which energy is fed into the turbulence.[5),6)]

It is convenient to distinguish the following three cases about the range of wave numbers of the macrovariables concerned: denoting the upper limit of wave numbers of the macrovariables by q^c,

$$(a)\ q^c\lambda \ll 1, \quad (b)\ q^c\lambda \gg 1, \quad (c)\ q^c\lambda \simeq 1, \tag{1}$$

where $q^c \ll 1/r_0$. The components with wave numbers $q > q^c$ will be called the eliminated degrees of freedom. In both (a) and (b), the Markov approximation is valid, whereas in (c) it is not valid since the separation of the time scale of the macroscopic and the eliminated degrees of freedom is not possible.[1] In (a), which will be called the normal case, the transport coefficients are determined by the eliminated degrees of freedom. In this case it will be shown except a few special systems that the master equation is reduced to a linearly generalized Fokker-Planck equation and the probability distribution is normal around the mean evolution. In (b), which will be called the extremely anomalous case, the transport coefficients are not determined by the eliminated degrees of freedom but by the nonlinear fluctuation of the macroscopic degrees of freedom.[7] In this case, therefore, a renormalization by eliminating the nonlinear mode coupling between the components in the wave number range $q^c > q > q^{c'}$, where $q^{c'}$ is a wave number satisfying $q^{c'}\lambda \ll 1$, must be done to obtain the transport coefficients.[1),8)] The turbulent viscosity[5] and the dynamic critical phenomena will be formulated from this point of view.

Although one of the main purposes is to develop a general theory, we shall keep our mind on two typical examples. One is the isotropic homogeneous turbulence in an incompressible, three-dimensional fluid whose velocity field $u_\alpha(\boldsymbol{r})$ obeys the Navier-Stokes equation[5]

$$\frac{\partial}{\partial t} u_\alpha(\boldsymbol{r}) = -\sum_\beta u_\beta(\boldsymbol{r}) \frac{\partial}{\partial r_\beta} u_\alpha(\boldsymbol{r}) + \nu \sum_\beta \left(\frac{\partial}{\partial r_\beta}\right)^2 u_\alpha(\boldsymbol{r}), \tag{2}$$

where the subscripts α and β indicate the x, y and z components and ν is the kinematic viscosity. In terms of the Fourier components

$$u_{\alpha q}=\frac{1}{\Omega}\int d\boldsymbol{r}\exp(i\boldsymbol{q}\cdot\boldsymbol{r})u_{\alpha}(\boldsymbol{r}), \tag{3}$$

Ω being the volume of the system, (2) takes the form

$$du_{\alpha q}/dt=i\sum_{p}{}'(\boldsymbol{p}\cdot\boldsymbol{u}_{q-p})u_{\alpha p}-q^2\nu u_{\alpha q}, \tag{4}$$

where $\sum_{p}{}'$ denotes the sum over the wave vectors whose magnitude is smaller than a cutoff q^c. The other is the dynamic critical phenomena in an isotropic, three-dimensional Heisenberg ferromagnet near the Curie point whose Fourier components of the spin density $s_q{}^\alpha$ ($\alpha=0, \pm$) obey[7)]

$$\frac{d}{dt}s_q{}^0=\frac{i}{\hbar\beta^e}\sum_{p}{}'\left[\frac{1}{\chi^e_{\perp p}}-\frac{1}{\chi^e_{\perp q-p}}\right]s_p{}^+s_{q-p}^- -q^2\nu_{\parallel}s_q{}^0, \tag{5}$$

$$\frac{d}{dt}s_q{}^\pm=\pm i\omega_q s_q{}^\pm\pm\frac{i}{\hbar\beta^e}\sum_{p}{}'\left[\frac{1}{\chi^e_{\parallel p}}-\frac{1}{\chi^e_{\perp q-p}}\right]s_p{}^0 s_{q-p}^\pm -q^2\nu_{\perp}s_q{}^\pm, \tag{6}$$

where $\beta^e=1/kT^e$, T^e being the equilibrium temperature,

$$\chi^e_{\parallel q}=\Omega\langle|s_q{}^0|^2\rangle^e,\quad \chi^e_{\perp q}=\Omega\langle|s_q{}^\pm|^2\rangle^e, \tag{7}$$

$\langle\cdots\rangle^e$ indicating the equilibrium average, ν_α are the spin diffusion coefficients and ω_q is the spin wave frequency. When $q^c\lambda\gtrsim 1$, ν_α depend on the wave number q.

In § 2, basic kinetic equations are summarized from the viewpoint of the theory of generalized Brownian motions, and it is pointed out that the Ω expansion method is not valid for nonuniform systems. In § 3, scaling exponents α, β, θ_S and θ_D are introduced to characterize the most dominant features of macroscopic systems and it is shown that the form of the master equation critically depends on the inequality relations between these scaling exponents. Examples of the normal case (a) and the anomalous case (b) are studied in §§ 4 and 5, respectively. Section 6 is devoted to a summary and remarks, where Table I summarizes the values of the scaling exponents in typical examples studied in this paper.

§ 2. Basic kinetic equations

The macroscopic state variables which we consider in this paper are the local densities $X_\mu(\boldsymbol{r})$, ($\mu=1, 2, \cdots$),*) which satisfy the conservation laws

$$\partial X_\mu(\boldsymbol{r})/\partial t=-\sum_{\beta}\partial J_{\mu\beta}(\boldsymbol{r})/\partial r_\beta, \tag{8}$$

where $\boldsymbol{J}_\mu(\boldsymbol{r})$ are the local fluxes. The local densities and fluxes are assumed to be coarse-grained in coordinate space so that they consist of the Fourier components with the wave vectors whose magnitude is smaller than a cutoff q^c.[9)]

*) The capital letters A, X, Z, Ξ, etc., indicate the phase functions, and the small letters a, x, z, ξ, etc., denote their values.

Following the theory of generalized Brownian motions, we write (8) in the form of the generalized Langevin equation

$$\partial X_\mu(\boldsymbol{r})/\partial t = -h_{\mu\boldsymbol{r}}(X) + R_{\mu\boldsymbol{r}}(t). \tag{9}$$

The first term represents the systematic part and $R_{\mu r}(t)$ is the fluctuating forces. The systematic part consists of a streaming term $v_{\mu r}$ and a dissipative term $C_{\mu r}$;[1)]

$$h_{\mu\boldsymbol{r}}(X) \equiv -[v_{\mu\boldsymbol{r}}(X) + C_{\mu\boldsymbol{r}}(X)]. \tag{10}$$

The dissipative term $C_{\mu r}$ is related to the fluctuating forces $R_{\mu r}(t)$ through the kinetic coefficients involved. In (2), $v_{\mu r}$ and $C_{\mu r}$ represent the first and the second term, respectively. In general, we may assume that the systematic part is a functional of the local densities $\{X_\nu(\boldsymbol{r}), \nu=1, 2, \cdots\}$ and their coordinate derivatives at $\boldsymbol{r}$;

$$h_{\mu\boldsymbol{r}}(X) = f_\mu\left[X_\nu(\boldsymbol{r}), \frac{\partial}{\partial r_\alpha} X_\nu(\boldsymbol{r}), \frac{\partial^2}{\partial r_\alpha \partial r_\beta} X_\nu(\boldsymbol{r}), \cdots\right]. \tag{11}$$

Let us suppose that the macroscopic state is described by the solution of the deterministic equation

$$\partial y_\mu(\boldsymbol{r}, t)/\partial t = -h_{\mu\boldsymbol{r}}(y) \tag{12}$$

with the boundary conditions imposed on the system. Then the fluctuation of $X_\mu(\boldsymbol{r})$ from $y_\mu(\boldsymbol{r})$,

$$Z_\mu(\boldsymbol{r}) \equiv X_\mu(\boldsymbol{r}) - y_\mu(\boldsymbol{r}, t), \tag{13}$$

is determined by

$$\partial Z_\mu(\boldsymbol{r})/\partial t = -\Delta h_{\mu\boldsymbol{r}}(Z, t) + R_{\mu\boldsymbol{r}}(t), \tag{14}$$

where

$$\Delta h_{\mu\boldsymbol{r}}(Z, t) = \left\{\exp\left[\sum_\nu Z_\nu(\boldsymbol{r}) \frac{\partial}{\partial y_\nu(\boldsymbol{r}, t)} + \sum_\nu \sum_\alpha \frac{\partial Z_\nu(\boldsymbol{r})}{\partial r_\alpha} \frac{\partial}{\partial(\partial y_\nu(\boldsymbol{r}, t)/\partial r_\alpha)} + \cdots\right] - 1\right\} h_{\mu\boldsymbol{r}}(y(t)). \tag{15}$$

When an external force is applied, it is added to (12). In describing the fluctuation, it is more convenient to use

$$\Xi_{\mu\boldsymbol{q}}(t) \equiv \frac{1}{\sqrt{\Omega}} \int d\boldsymbol{r} \exp(i\boldsymbol{q}\cdot\boldsymbol{r}) Z_\mu(\boldsymbol{r}, t), \tag{16}$$

since their two-time correlation functions

$$\Psi_{kl}(t, t_0) \equiv \langle [\Xi_k(t) - \langle \Xi_k(t) \rangle_0] [\Xi_l^*(t_0) - \langle \Xi_l^*(t_0) \rangle_0] \rangle_0 \tag{17}$$

can be assumed to have definite values for large Ω, where k represents the set

$(\mu, \boldsymbol{q})$ and $\langle\cdots\rangle_0$ indicates the average over the initial ensemble. Then (14) leads to

$$d\Xi_k(t)/dt = -\Delta\tilde{h}_k(\Xi, t) + \mathcal{R}_k(t) \tag{18}$$

with

$$\Delta\tilde{h}_k(\Xi, t) \equiv \frac{1}{\sqrt{\Omega}} \int d\boldsymbol{r} \exp(i\boldsymbol{q}\cdot\boldsymbol{r}) \Delta h_{\mu\boldsymbol{r}}(Z, t), \tag{19}$$

$$\mathcal{R}_k(t) \equiv \sum_\beta iq_\beta \frac{1}{\sqrt{\Omega}} \int d\boldsymbol{r} \exp(i\boldsymbol{q}\cdot\boldsymbol{r}) \mathcal{J}_{\mu\beta}(\boldsymbol{r}, t), \tag{20}$$

where $\mathcal{J}_\mu(\boldsymbol{r}, t)$ are the fluctuating local fluxes corresponding to $R_{\mu\boldsymbol{r}}(t)$.

The probability distribution function for $\Xi(t) \equiv \{\Xi_k(t)\}$ to take a set of values $\xi \equiv \{\xi_k\}$ is given by

$$\Pi(\xi, t) \equiv \langle \prod_k \delta(\Xi_k(t) - \xi_k) \rangle_0 . \tag{21}$$

If we may assume that the fluctuating forces $\mathcal{R}_k(t)$ are represented by a Gaussian white process, then[1] we obtain

$$\frac{\partial}{\partial t} \Pi(\xi, t) \doteqdot \sum_k \frac{\partial}{\partial \xi_k} \left\{ \Delta\tilde{h}_k(\xi, t) \Pi(\xi, t) + \sum_l \frac{\partial}{\partial \xi_l{}^*} [\hat{L}_{kl} \Pi(\xi, t)] \right\}, \tag{22}$$

where

$$\hat{L}_{kl} = q^2 \delta_{\boldsymbol{q},\boldsymbol{q}'} D_{\mu\nu} , \tag{23}$$

$$D_{\mu\nu} \equiv \int d\boldsymbol{r}' \int_0^\infty dt \langle \mathcal{J}_{\mu\beta}(\boldsymbol{r}, t) \mathcal{J}^*_{\nu\beta}(\boldsymbol{r}', 0) \rangle^e . \tag{24}$$

$D_{\mu\nu}$ are related to ν_α by $D_{\alpha\beta} = \delta_{\alpha,\beta} \nu_\alpha \chi^e_{\alpha q}$. The Gaussian assumption, however, is not valid at least when $q^c \lambda \gtrsim 1$. Therefore, we go back to a general type of the master equation proposed in a previous paper,[1] which will be referred to as I hereafter.

Let us introduce the "extensive" variables

$$A_{\mu\boldsymbol{q}}(t) \equiv \int d\boldsymbol{r} \exp(i\boldsymbol{q}\cdot\boldsymbol{r}) X_\mu(\boldsymbol{r}, t) \tag{25}$$

and denote the set of them by $A(t) \equiv \{A_k(t)\}$, where A_{k^*} is included to denote $A_k{}^*$. Then, as was shown in I, the probability distribution function for $A(t)$ to take a set of values $a \equiv \{a_k\}$ obeys the following master equation:

$$\frac{\partial}{\partial t} P(a, t) = \sum_k \frac{\partial}{\partial a_k} \{-[v_k(a) + C_k(a)] P\}$$
$$+ \sum_{n=2}^\infty \frac{(-1)^n}{n!} \sum_{k_1} \cdots \sum_{k_n} \frac{\partial}{\partial a_{k_1}} \cdots \frac{\partial}{\partial a_{k_n}} [\alpha_{k_1 \cdots k_n}(a) P], \tag{26}$$

where

$$v_k(a) = (1/\beta^e) \sum_l \left[\langle \{A_k, A_l^*\}; a\rangle^e \frac{\partial F(a)}{\partial a_l^*} - \frac{\partial \langle \{A_k, A_l^*\}; a\rangle^e}{\partial a_l^*} \right], \tag{27}$$

$$C_{k_1 \cdots k_n}(a) = -\sum_l \left[L_{k_1 \cdots k_n; l}(a) \frac{\partial F(a)}{\partial a_l^*} - \frac{\partial L_{k_1 \cdots k_n; l}(a)}{\partial a_l^*} \right], \tag{28}$$

$$\alpha_{k_1 \cdots k_n}(a)/n! = L_{k_1 \cdots k_{n-1}; k_n^*}(a) + C_{k_1 \cdots k_n}(a). \qquad (n \geqq 2) \tag{29}$$

In (27), the curly brackets denote the Poisson brackets and $\langle \cdots ; a\rangle^e$ indicates the conditional average over the canonical ensemble with a fixed value a of A. $F(a)$ is the dimensionless generalized free energy

$$F(a) = -\ln w(a), \quad [w(a) = \langle \delta(A(0) - a)\rangle^e]. \tag{30}$$

$L_{k_1 \cdots k_n; l}(a)$ are the generalized diffusion coefficients whose explicit expressions are given by (36) and (41) of I. Since the conservation laws (8) lead to

$$\dot{A}_k(t) = \sum_\beta i q_\beta \int d\boldsymbol{r} \exp(i\boldsymbol{q} \cdot \boldsymbol{r}) J_{\mu\beta}(\boldsymbol{r}, t), \tag{31}$$

we have

$$L_{k_1 \cdots k_n; l}(a) \sim \delta^{n+1}, \tag{32}$$

where $\delta \equiv q^c \lambda$. In most cases we may assume that

$$L_{k_0 \cdots k_{2n}; l}(a) \doteqdot \Omega(-1)^n \sum_\beta q_{0\beta} \cdots q_{2n\beta} q'_\beta \delta_{\boldsymbol{q}_0 + \cdots + \boldsymbol{q}_{2n}, \boldsymbol{q}'} D_{\mu_0 \cdots \mu_{2n}; \nu}, \tag{33}$$

where $D_{\mu_0 \cdots \mu_{2n}; \nu}$ are constants independent of Ω, the wave vector and the macroscopic state, and the diffusion tensors of odd rank vanish due to the local isotropy in the local equilibrium state.

Introducing the "densities" x_k by

$$a_k = \Omega x_k, \tag{34}$$

we denote the quantities per unit volume by $\widehat{G}(x)$;

$$\widehat{G}(x) \equiv G(a)/\Omega. \tag{35}$$

Then (26) can be written as

$$\frac{\partial}{\partial t} P(x, t) = \sum_k \frac{\partial}{\partial x_k} [h_k(x) P(x, t)]$$
$$+ \sum_{n=2}^\infty \frac{(-)^n}{n!} \frac{1}{\Omega^{n-1}} \sum_{k_1} \cdots \sum_{k_n} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_n}} [\widehat{\alpha}_{k_1 \cdots k_n}(x) P(x, t)], \tag{36}$$

where

$$h_k(x) \equiv -[\widehat{v}_k(x) + \widehat{C}_k(x)]. \tag{37}$$

Following van Kampen,[2] let us introduce the deterministic equation

$$dy_k(t)/dt = -h_k(y), \tag{38}$$

and define the fluctuation variables z_k and ξ_k by

$$x_k = y_k(t) + z_k, \quad z_k = \frac{1}{\sqrt{\Omega}} \xi_k . \tag{39}$$

Examples of (38) are provided by (4), (5) and (6). The probability distribution function for $z \equiv \{z_k\}$ is given by

$$P(z, t) = P(y(t) + z, t). \tag{40}$$

Since

$$\frac{\partial}{\partial t} P(z, t) = \frac{\partial P(x, t)}{\partial t} + \sum_k \dot{y}_k(t) \frac{\partial P(z, t)}{\partial z_k}, \tag{41}$$

(36) leads to

$$\frac{\partial}{\partial t} P(z, t) = \sum_k \frac{\partial}{\partial z_k} \{\Delta h_k(z, t) P(z, t)\} + \sum_{n=2}^{\infty} \frac{(-)^n}{n!} \frac{1}{\Omega^{n-1}} \sum_{k_1} \cdots \sum_{k_n} \frac{\partial}{\partial z_{k_1}} \cdots \frac{\partial}{\partial z_{k_n}} \{\hat{\alpha}_{k_1 \cdots k_n}(y(t) + z) P(z, t)\}, \tag{42}$$

where

$$\Delta h_k(z, t) \equiv h_k(y+z) - h_k(y) = \left[\exp\left(\sum_j z_j \frac{\partial}{\partial y_j}\right) - 1\right] h_k(y). \tag{43}$$

Rewritting in terms of $\xi_k = \sqrt{\Omega} z_k$, we thus obtain

$$\frac{\partial}{\partial t} \Pi(\xi, t) = \sum_k \frac{\partial}{\partial \xi_k} \{\Delta \tilde{h}_k(\xi, t) \Pi(\xi, t)\} + \sum_{n=2}^{\infty} \frac{(-)^n}{n!} \frac{1}{\Omega^{(n-2)/2}} \times \sum_{k_1} \cdots \sum_{k_n} \frac{\partial}{\partial \xi_{k_1}} \cdots \frac{\partial}{\partial \xi_{k_n}} \left\{\hat{\alpha}_{k_1 \cdots k_n}\left(y(t) + \frac{\xi}{\sqrt{\Omega}}\right) \Pi(\xi, t)\right\}, \tag{44}$$

where

$$\Delta \tilde{h}_k(\xi, t) \equiv \sqrt{\Omega} \Delta h_k(z, t) = \sqrt{\Omega} \left[\exp\left(\frac{1}{\sqrt{\Omega}} \sum_j \xi_j \frac{\partial}{\partial y_j}\right) - 1\right] h_k(y). \tag{45}$$

When $\Delta h_k(z)$ is a linear function of z, $\Delta \tilde{h}_k(\xi) = \Delta h_k(\xi)$. Equation (19) with (15) is the local representation of (45). Equation (44) is the most fundamental equation on which the present theory is developed. The corresponding Langevin equation is given by (18). If $\delta \equiv q^c \lambda \ll 1$, then the terms of higher derivatives with $n \geqq 3$ are of order δ^n and the second term of (29) is of higher order than the first term by δ. Then, to order δ^2, (44) is reduced to the generalized Fokker-Planck equation (22). When $\delta \gtrsim 1$, however, such an approximation is not valid.

In uniform disordered systems which can be described by a small number of macrovariables, the sums in (44) and (45) do not depend on Ω, and $\Pi(\xi, t)$ must be a definite function of ξ and t for large Ω. Then, in the thermodynamic limit $\Omega\to\infty$, (44) is reduced to

$$\frac{\partial}{\partial t}\Pi(\xi,t)=\sum_k\sum_j\left\{\lambda_{kj}(y)\frac{\partial}{\partial\xi_k}[\xi_j\Pi]+\frac{1}{2}\hat{\alpha}_{kj^*}(y)\frac{\partial^2\Pi}{\partial\xi_k\partial\xi_j{}^*}\right\}, \tag{46}$$

where

$$\lambda_{kj}(y)\equiv\partial h_k(y)/\partial y_j, \tag{47}$$

$$\hat{\alpha}_{kj^*}(y)/2=\hat{L}_{k;j}(y)-\sum_m\hat{L}_{kj^*;m}(y)\partial\hat{F}(y)/\partial y_m{}^*, \tag{48}$$

and (18) takes the form

$$d\Xi_k(t)/dt=-\sum_j\lambda_{kj}(y)\Xi_j(t)+\mathcal{R}_k(t). \tag{49}$$

Thus, in uniform systems, the fluctuation of the macrovariables is described by the linear Fokker-Planck and Langevin equations with time-dependent coefficients. In nonuniform systems, however, the sums in (44) and (45) are taken over all the macroscopic degrees of freedom whose number is proportional to Ω, and $\Pi(\xi,t)$ cannot be a definite function of ξ and t. Therefore, the thermodynamic limit $\Omega\to\infty$ cannot be taken in (44) and (45). This difficulty, however, will be removed by introducing a scale transformation of the nonequilibrium macroscopic state.

§ 3. Scale transformation

A scale transformation which we introduce here differs from the similarity laws in fluid mechanics which are obtained by changing all linear dimensions in the same ratio and by keeping all dimensionless quantities invariant.[5] In the following scale transformation, molecular quantities such as the intermolecular force and the mean molecular density are kept constants.

Denoting one of the characteristic lengths of the macroscopic state by l, we introduce a larger similar system whose characteristic length l_L and volume Ω_L are given by

$$l_L=Ll,\quad \Omega_L=L^d\Omega,\qquad (L\geqq 1) \tag{50}$$

where d is the dimensionality. The smallest distance of the spatial variation of the macroscopic state variables $b\equiv 1/q^c$ is one of the characteristic lengths. Therefore, the number of the macroscopic degrees of freedom is invariant under the scale transformation;

$$\sum_k=\sum_\mu\frac{\Omega}{(2\pi)^d}\int_{(<q^c)}d\boldsymbol{q}=\sum_\mu\frac{\Omega_L}{(2\pi)^d}\int_{(<q_L{}^c)}d\boldsymbol{q}_L, \tag{51}$$

where

$$\boldsymbol{q}_L=L^{-1}\boldsymbol{q},\quad q_L{}^c=L^{-1}q^c. \tag{52}$$

Equation (50) also leads to

$$\boldsymbol{r}_L = L\boldsymbol{r}\,, \quad \boldsymbol{q}_L \cdot \boldsymbol{r}_L = \boldsymbol{q} \cdot \boldsymbol{r}\,. \tag{53}$$

Since the force range r_0, the lattice constant and the mean molecular density are not changed, the total number of the degrees of the freedom of the system becomes L^d times, and thus the number of the microscopic degrees of freedom becomes L^d times. We postulate, however, that the quantities determined by the microscopic degrees of freedom per unit volume are invariant. This will be called the *Ansatz I.* For instance, $D_{\mu_1 \cdots \mu_n;\nu}$ in (33) are invariant, which leads to

$$\hat{L}^L_{k_1 \cdots k_{n-1}; k_n{}^*} = L^{-n} \hat{L}_{k_1 \cdots k_{n-1}; k_n{}^*}\,. \tag{54}$$

This ansatz is the most crucial point of our scale transformation. In the normal case (a) and the extremely-anomalous case (b), (54) is satisfied. It will be pointed out in § 6 that (54) can be extended in a more general form.

Let us define the scaling exponents for the time and the state variables by

$$t_L = L^\theta t\,, \quad y_k{}^L = L^{-\alpha} y_k\,, \tag{55}$$

$$z_k{}^L = L^{-\beta} z_k\,, \quad \xi_k{}^L = L^{(d/2)-\beta} \xi_k\,. \tag{56}$$

The time scaling exponent θ must be positive. In most cases β differs from α. The values of α and β depend on the variable. For instance, the mass density $\rho(\boldsymbol{r})$ and the momentum density $\boldsymbol{j}(\boldsymbol{r})$ of fluids can have different exponents which are related to each other by $\beta_\rho = \beta_j - \theta + 1$ from the conservation law. In the following, however, we assume that the relevant variables lead to a single β. We next postulate that the probability distribution for the fluctuation of the macroscopic state variables is invariant under the transformation $l \to l_L$ when L is large:

$$P^L(z^L, t_L) = P(z, t)\,, \quad \Pi^L(\xi^L, t_L) = \Pi(\xi, t)\,. \tag{57}$$

In other words, the macroscopic probability distribution has a certain functional form independent of L when L is large. This will be called the *Ansatz II.* In accordance with this ansatz, the macroscopic fluctuation part of the generalized free energy (30) must be invariant;

$$\Delta \hat{F}^L(\xi^L, t_L) = L^{-d} \Delta \hat{F}(\xi, t)\,, \tag{58}$$

where $\Delta \hat{F}(\xi, t) \equiv \hat{F}(y(t) + \xi/\sqrt{\Omega}) - \hat{F}(y(t))$. From (28), (29) and (54), therefore, we obtain

$$\hat{C}^L_{k_1 \cdots k_n}(z^L, t_L) = L^{-(n+1+d-\beta)} \hat{C}_{k_1 \cdots k_n}(z, t)\,, \tag{59}$$

$$\hat{\alpha}^L_{k_1 \cdots k_n}(z^L, t_L) = L^{-n} n!\, \hat{L}_{k_1 \cdots k_{n-1}; k_n{}^*}\,. \tag{60}$$

To obtain (60), we have assumed $\beta < d+1$, which is satisfied in all the examples we shall study in this paper.

The fluctuation drift coefficient $\Delta h_k(z, t)$ in (42) and (44) consists of the streaming and dissipative part. Let us define their scaling exponents for large L by

$$\Delta\hat{v}_k{}^L(z^L, t_L) = L^{-(\beta+\theta_S)}\Delta\hat{v}_k(z, t), \tag{61}$$

$$\Delta\hat{C}_k{}^L(z^L, t_L) = L^{-(\beta+\theta_D)}\Delta\hat{C}_k(z, t), \tag{62}$$

where (59) leads to

$$\theta_D = d+2-2\beta. \tag{63}$$

From (45) we have

$$\Delta\tilde{h}_k{}^L(\xi^L, t_L) = L^{(d/2)}\Delta h_k{}^L(z^L, t_L), \tag{64}$$

and similar equations for $\Delta\tilde{v}_k(\xi, t)$ and $\Delta\tilde{C}_k(\xi, t)$. The scaling exponents of $\Delta h_k(z, t)$ and $\Delta\tilde{h}_k(\xi, t)$ are thus determined by either of θ_S and θ_D which is smaller.

The scale transformation of the master equation (44) thus leads to

$$\frac{\partial}{\partial t}\Pi(\xi, t) = -\sum_k \frac{\partial}{\partial \xi_k}\{[L^{\theta-\theta_S}\Delta\tilde{v}_k(\xi, t) + L^{\theta-\theta_D}\Delta\tilde{C}_k(\xi, t)]\Pi(\xi, t)\}$$
$$+\sum_{n=2}^{\infty}\frac{(-)^n\Omega^{-(n-2)/2}}{L^{(n-1)\zeta_n}}\sum_{k_1}\cdots\sum_{k_n}\frac{\partial}{\partial\xi_{k_1}}\cdots\frac{\partial}{\partial\xi_{k_n}}\{\hat{L}_{k_1\cdots k_{n-1};k_n*}\Pi(\xi, t)\}, \tag{65}$$

where

$$\zeta_n = (\theta_D-\theta) + \frac{1}{2}\left(\frac{n-2}{n-1}\right)(d+2\theta-\theta_D), \tag{66}$$

θ_D being given by (63). Equation (18) leads to

$$d\Xi_k(t)/dt = L^{\theta-\theta_S}\Delta\tilde{v}_k(\Xi, t) + L^{\theta-\theta_D}\Delta\tilde{C}_k(\Xi, t) + L^{(\theta-\theta_D)/2}\mathcal{R}_k(t), \tag{67}$$

where it has been used that $\mathcal{R}_k{}^L(t_L) = L^{(d/2)-\beta-(\theta+\theta_D)/2}\mathcal{R}_k(t)$. In (65), (67) and the following, $\Delta\tilde{v}_k$ and $\Delta\tilde{C}_k$ represent their most dominant parts for large L. The scale transformation of (42) is similarly obtained and each term yields the same scaling exponent as the corresponding term of (65).

The time scaling exponent θ must be chosen to equal either of θ_S and θ_D which is smaller. Thus we have the following two cases:

[A] $\theta_S \geqq \theta_D$:

Then $\theta=\theta_D$. From (66), $\zeta_n=0$ if $n=2$, and $\zeta_n>0$ if $n\geqq 3$. Thus for large L,

$$\frac{\partial}{\partial t}\Pi(\xi, t) = \sum_k\frac{\partial}{\partial\xi_k}\left\{\Delta\tilde{h}_k(\xi, t)\Pi(\xi, t) + \sum_l\frac{\partial}{\partial\xi_l{}^*}\hat{L}_{k;l}\Pi(\xi, t)\right\}, \tag{68}$$

$$d\Xi_k(t)/dt = -\Delta\tilde{h}_k(\Xi, t) + \mathcal{R}_k(t), \tag{69}$$

where

$$\Delta\tilde{h}_k(\xi, t) = \begin{cases} -\Delta\tilde{C}_k(\xi, t), & \text{if} \quad \theta_S>\theta_D, \quad (70)\\ -[\Delta\tilde{v}_k(\xi, t)+\Delta\tilde{C}_k(\xi, t)] & \text{if} \quad \theta_S=\theta_D. \quad (71)\end{cases}$$

The case $\hat{v}_k(y)\equiv 0$ is included as a special case of (70). Equation (68) is a generalized Fokker-Planck equation, where $\Delta\tilde{h}_k$ can be a nonlinear function of ξ.

[B] $\theta_S<\theta_D$:

Then $\theta=\theta_S$. From (66), $\zeta_n>0$. This is obvious if $d+2\theta_S-\theta_D\geqq 0$. If $d+2\theta_S-\theta_D<0$, then this comes from the inequality $\zeta_n>(\theta_D-\theta_S)+(d+2\theta_S-\theta_D)/2=(d/2)+(\theta_D/2)>0$. Thus for large L,

$$\frac{\partial}{\partial t}\Pi(\xi,t)=-\sum_k\frac{\partial}{\partial\xi_k}\{\Delta\tilde{v}_k(\xi,t)\Pi(\xi,t)\},\tag{72}$$

$$d\Xi_k(t)/dt=\Delta\tilde{v}_k(\Xi,t).\tag{73}$$

Equation (72) quite differs from (68). This form will appear in the extremely anomalous case (*b*).

When $\alpha<\beta$, we have a simple situation.

[A′] $\alpha<\beta$:

Let us consider (43). Since $z_j{}^L\partial/\partial y_j{}^L\sim L^{-(\beta-\alpha)}$ is small for large L,

$$\Delta\tilde{h}_k{}^L(\xi^L,t_L)=L^{(d/2)-(\beta-\alpha)-\delta}\Delta\tilde{h}_k(\xi,t),$$

where

$$\Delta\tilde{h}_k(\xi,t)=\sum_j\lambda_{kj}(y)\xi_j,\tag{74}$$

λ_{kj} being given by (47), and δ denotes the scaling exponent of $h_k(y)$. In this case, we have $\theta=\delta-\alpha=\theta_D$ from (62) and (63), and thus arrive at the case [A]. Insertion of (74) into (68) and (69) leads to the linearly generalized Fokker-Planck and the Langevin equation with time-dependent coefficients, respectively.

The probability distribution function $\Pi(\xi,t)$ depends on the macroscopic characteristic length l; for the l_L system,

$$\Pi^L(\xi^L,t_L)=F(\{\xi_k{}^L\},\{q_L\},\Omega_L,t_L,l_L).\tag{75}$$

The L dependence of the arguments are given by (50), (52), (55) and (56), but $\Pi^L(\xi^L,t_L)$ must have such a functional form that L cancels out the right-hand side of (75). This can only happen if $\Pi(\xi,t)$ is of the form

$$\Pi(\xi,t)=P(\{\xi_k l^{-(d/2)+\beta}\},\{ql\},\Omega l^{-d},tl^{-\theta}).\tag{76}$$

Equation (76) is the most general representation of the scaling of the macroscopic fluctuation, including the dynamic scaling. Similarly (58) leads to

$$\Delta\hat{F}(\xi,t)=l^{-d}f(\{\xi_k l^{-(d/2)+\beta}\},\{ql\},\Omega l^{-d},tl^{-\theta}).\tag{77}$$

It should be noted that we can replace $\xi_k l^{-(d/2)+\beta}$ in (76) and (77) by $z_k l^{\beta}$, but then their Ω dependence changes. The explicit functional form of (76) is not known to us until we solve the kinetic equation (68) or (72) explicitly. Nevertheless, (76) gives us useful information. As an example let us consider the mean evolution of ξ_k;

$$\bar{\xi}_k(t)=\langle\xi_k\rangle(t)\equiv\int d\xi\Pi(\xi,t)\xi_k\,. \tag{78}$$

Equation (76) leads to the form

$$\bar{\xi}_{\mu q}(t)=q^{-(d/2)+\beta}\Psi_\mu(ql,tq^\theta)\,. \tag{79}$$

The characteristic complex frequency of the time evolution of (78), therefore, must be of the form

$$\Omega_{\mu q}=q^\theta S_\mu(ql)\,. \tag{80}$$

Thus the exponent of the frequency spectrum is given by θ.

Since θ is a function of β, the unknown parameter is β only. To determine this we use the same-time correlation function

$$\chi_{\mu q}(t)\equiv\langle\xi'_{\mu q}\xi'^{*}_{\mu q}\rangle(t)=q^{-\gamma}C_\mu(ql,tq^\theta)\,, \tag{81}$$

where $\xi_k'\equiv\xi_k-\bar{\xi}_k(t)$ and γ must be non-negative. From (76),

$$\gamma=d-2\beta\geqq 0\,. \tag{82}$$

Let us take the correlation length of fluctuation λ as l. In the normal case where $q^c\lambda\ll 1, \gamma=0$ and $\beta=d/2$. In the critical regime of the equilibrium critical phenomena where $q^c\lambda\gg 1$, $\gamma=2-\eta$ and $\beta=(d-2+\eta)/2$, where η is the correlation critical exponent. In the isotropic homogeneous turbulence, $\gamma=11/3$ and $\beta=-1/3$ if Kolmogorov's spectrum is used.

Finally let us consider the scaling of the evolution equation (38) which describes the macroscopic state. The scale invariance of the macroscopic flow pattern leads to

$$y_{\mu q}(t)=q^\alpha Y_\mu(ql,tq^\tau)\,. \tag{83}$$

The scale transformation must keep the most dominant feature of the evolution equation (38) invariant. Thus α and τ are determined in such a way that

$$y_k{}^L=L^{-\alpha}y_k\,,\quad t_L=L^\tau t \tag{84}$$

satisfies

$$L^{\alpha+\tau}[\hat{v}_k(y)]^L=L^{\alpha+\tau}[\hat{C}_k(y)]^L=L^0, \tag{85}$$

where $[\cdots]^L$ denote the L factors of $\hat{v}_k{}^L$ and $\hat{C}_k{}^L$. For the equilibrium systems, (38) leads to the free energy minimum $\partial\hat{F}(y)/\partial y_k=0$, which determines α. In most cases the evolution scaling exponents α and τ differ from the fluctuation scaling exponents β and θ. For example, the Navier-Stokes equation (4) has $\alpha=1$ and $\tau=2$. If τ could be smaller than both of θ_S and θ_D, then θ had to be set to be τ and all the terms of (65) and (67) would vanish for large L. This, however, cannot occur. Although there are two kinds of time scales τ

and θ in the nonsteady states, therefore, $\tau \geqq \theta$ and we can single out the θ time scale by taking large L.

§ 4. Normal cases ($q^C\lambda \ll 1$)

Since $q^c\lambda \ll 1$, we obtain

$$\beta = d/2\,, \quad \gamma = 0\,, \quad \theta_D = 2\,. \tag{86}$$

To have exponents explicitly, we first consider three examples.

[1] *Navier-Stokes fluids*:

Equation (4) leads to

$$\alpha = 1\,, \quad \tau = 2\,, \tag{87}$$

$$\Delta \hat{v}_k(z, t) = i\sum_p{}' [(\boldsymbol{p}\cdot \boldsymbol{y}_{q-p}) z_{\alpha p} + y_{\alpha p}(\boldsymbol{p}\cdot \boldsymbol{z}_{q-p}) + (\boldsymbol{p}\cdot \boldsymbol{z}_{q-p}) z_{\alpha p}]. \tag{88}$$

If $d>2$, then $\alpha<\beta$, $\theta=\theta_S=2(=\theta_D)$ and

$$\Delta \tilde{v}_k(\xi, t) = i\sum_p{}' [(\boldsymbol{p}\cdot \boldsymbol{y}_{q-p}) \xi_{\alpha p} + y_{\alpha p}(\boldsymbol{p}\cdot \boldsymbol{\xi}_{q-p})]. \tag{89}$$

If $d=2$, then $\alpha=\beta$, $\theta=\theta_S=2(=\theta_D)$ and

$$\Delta \tilde{v}_k(\xi, t) = i\sum_p{}' \left[(\boldsymbol{p}\cdot \boldsymbol{y}_{q-p}) \xi_{\alpha p} + y_{\alpha p}(\boldsymbol{p}\cdot \boldsymbol{\xi}_{q-p}) + \frac{1}{\sqrt{\Omega}}(\boldsymbol{p}\cdot \boldsymbol{\xi}_{q-p}) \xi_{\alpha p}\right]. \tag{90}$$

If $d<2$, then $\alpha>\beta$, $\theta=\theta_S=(d+2)/2(<\theta_D)$ and

$$\Delta \tilde{v}_k(\xi, t) = i(1/\sqrt{\Omega})\sum_p{}' (\boldsymbol{p}\cdot \boldsymbol{\xi}_{q-p}) \xi_{\alpha p}\,. \tag{91}$$

In this case, since $\theta<\tau$, the scaling of the fluctuation is valid in the inertial range where the dissipative term can be neglected.

[2] *Isotropic Heisenberg ferromagnets*:

Equations (5) and (6) lead to

$$\alpha = 0\,, \quad \tau = 2\,, \tag{92}$$

$$\Delta \hat{v}_{q}{}^0(z, t) = iC_\parallel \sum_p{}' (\boldsymbol{q}\cdot \boldsymbol{p})\, [y_p{}^+ z_{q-p}^- + y_{q-p}^- z_p{}^+ + z_p{}^+ z_{q-p}^-], \tag{93}$$

$$\Delta \hat{v}_{q}{}^\pm(z, t) = \pm iq^2 B z_q{}^\pm \pm iC_\perp \sum_p{}' (\boldsymbol{q}\cdot \boldsymbol{p})\, [y_p{}^0 z_{q-p}^\pm + y_{q-p}^\pm z_p{}^0 + z_p{}^0 z_{q-p}^\pm], \tag{94}$$

where B and C are constants. Since $\alpha<\beta$, $\theta=\theta_S=2(=\theta_D)$ and

$$\Delta \tilde{h}_{q}{}^0(\xi, t) = -iC_\parallel \sum_p{}' (\boldsymbol{q}\cdot \boldsymbol{p})\, [y_p{}^+ \xi_{q-p}^- + y_{q-p}^- \xi_p{}^+] + q^2 \nu_\parallel \xi_q{}^0, \tag{95}$$

$$\Delta \tilde{h}_{q}{}^\pm(\xi, t) = \mp iq^2 B \xi_q{}^\pm \mp iC_\perp \sum_p{}' (\boldsymbol{q}\cdot \boldsymbol{p})\, [y_p{}^0 \xi_{q-p}^\pm + y_{q-p}^\pm \xi_p{}^0] + q^2 \nu_\perp \xi_q{}^\pm. \tag{96}$$

Since $\theta=\tau$, and $\Delta \tilde{h}_k$ are linear functions of ξ, no singular phenomena are expected to occur.

[3] *Fluctuations in the equilibrium state*:

Since $q^c\lambda \ll 1$, the macroscopic densities $y_\mu(\boldsymbol{r}, t)$ and their Fourier transform

$y_k(t)$ do not depend on the system size Ω and are invariant under the scale transformation. This leads to $\alpha=0$.

Except for the case of the low-dimensional Navier-Stokes fluids ($d\leqq 2$), we have $\alpha<\beta$ and arrive at the case [A′]. Then (68), (69) and (74) lead to

$$d\Xi_k(t)/dt=-\sum_j \lambda_{kj}(y)\Xi_j(t)+\mathcal{R}_k(t), \tag{97}$$

$$\frac{\partial}{\partial t}\Pi(\xi,t)=\sum_k\sum_j\left\{\lambda_{kj}(y)\frac{\partial}{\partial \xi_k}[\xi_j\Pi]+\hat{L}_{k;j}\frac{\partial^2\Pi}{\partial\xi_k\partial\xi_j{}^*}\right\}. \tag{98}$$

Thus we obtain the linearly generalized Fokker-Planck equation of the same type as (46), even for nonuniform systems, though under the restricted conditions. Let us put

$$\Pi(\xi,t)=\exp[-\Phi(\xi,t)]. \tag{99}$$

Then (98) takes the form

$$\frac{\partial}{\partial t}\Phi(\xi,t)=-\sum_k\lambda_{kk}(y)+\sum_k\sum_j\lambda_{kj}(y)\left[\xi_j\frac{\partial\Phi}{\partial\xi_k}\right]$$
$$+\sum_k\sum_j\hat{L}_{k;j}\left[\frac{\partial^2\Phi}{\partial\xi_k\partial\xi_j{}^*}-\frac{\partial\Phi}{\partial\xi_k}\frac{\partial\Phi}{\partial\xi_j{}^*}\right]. \tag{100}$$

Noting that $\Phi(\xi,t)$ depends on ξ_k through $x_k=y_k(t)+(\xi_k/\sqrt{\Omega})$, we expand $\Phi(\xi,t)$ in the power series of ξ_k. Since $\Phi(\xi,t)=\Phi^L(\xi^L,t_L)$, we obtain

$$\Phi(\xi,t)=\Phi_0(t)+\sum_{n=1}^{\infty}\frac{L^{-(\beta-\alpha)n}}{n!\,\Omega^{n/2}}\sum_{k_1}\cdots\sum_{k_n}\xi_{k_1}\cdots\xi_{k_n}\frac{\partial^n\Phi(0,t)}{\partial y_{k_1}\cdots\partial y_{k_n}}. \tag{101}$$

Since $\alpha<\beta$, the terms with $n\geqq 3$ can be neglected for large L. Thus we can assume the quadratic form

$$\Phi(\xi,t)=\Phi_0(t)+\tfrac{1}{2}\sum_k\sum_l\phi_{kl}(t)[\xi_k-\bar{\xi}_k(t)][\xi_l{}^*-\bar{\xi}_l{}^*(t)], \tag{102}$$

where $\bar{\xi}_k(t)$ is the mean evolution (78), and $\phi_{kl}(t)$ is the inverse matrix of $\chi_{kl}(t)$,

$$\chi_{kl}(t)\equiv\langle[\xi_k-\bar{\xi}_k(t)][\xi_l{}^*-\bar{\xi}_l{}^*(t)]\rangle(t). \tag{103}$$

Both χ_{kl} and ϕ_{kl} are hermitian matrices;

$$\chi_{kl}=\chi_{l^*k^*}=\chi_{lk}^*,\quad \phi_{kl}=\phi_{l^*k^*}=\phi_{lk}^*. \tag{104}$$

Inserting (102) into (100), and equating the terms of the same order in $\xi_k'=\xi_k-\bar{\xi}_k(t)$, we find

$$d\bar{\xi}_k(t)/dt=-\sum_j\lambda_{kj}(y)\bar{\xi}_j(t), \tag{105}$$

$$d\chi_{kl}(t)/dt=-\sum_j[\chi_{kj}\lambda_{lj}(y)+\lambda_{kj}^*(y)\chi_{jl}]+2\hat{L}_{lk}^s, \tag{106}$$

where $\hat{L}_{kl}^s\equiv[\hat{L}_{k;l}+\hat{L}_{l^*;k^*}]/2=\hat{L}_{lk}^{s*}$. Equation (105) represents the average of (97). Since $\hat{L}_{kl}^s$ are real constants, $\hat{L}_{kl}^s$ form a real symmetric matrix. The $\lambda_{kj}(y)$, $\hat{L}_{kl}$, $\bar{\xi}_k(t)$, $\chi_{kl}(t)$, $\phi_{kl}(t)$ and $\Phi(\xi,t)/\Omega$ are definite functions in the thermodynamic limit $\Omega\to\infty$. In the homogeneous systems, the square matrices are diagonal with respect to the wave vector. Inserting (102) into (99), we obtain

$$\Pi(\xi,t)=N(t)\exp\{-\tfrac{1}{2}\sum_k\sum_l\phi_{kl}(t)[\xi_k-\bar{\xi}_k(t)][\xi_l^*-\bar{\xi}_l^*(t)]\}. \tag{107}$$

Thus it turns out that when $\alpha<\beta$, $\Pi(\xi,t)$ is given by the normal distribution.

§ 5. Extremely anomalous cases ($q^C\lambda\gg 1$)

As typical examples, we consider the fully-developed turbulence in the Navier-Stokes fluids and the dynamic critical phenomena in the isotropic Heisenberg ferromagnets near the Curie point.

[1] *Isotropic homogeneous turbulence in three dimensions*:

Assuming Kolmogorov's spectrum $\gamma=2+(5/3)$, we obtain

$$\beta=-1/3,\quad \gamma=11/3,\quad \theta_D=17/3. \tag{108}$$

The α and τ are given by (87). Since $\alpha>\beta$, (88) leads to $\theta_S=1+\beta=(d+2-\gamma)/2=2/3<\theta_D$, and

$$\Delta\tilde{v}_{\alpha q}(\xi)=i(1/\sqrt{\Omega})\sum_p{}'(p\cdot\xi_{q-p})\xi_{\alpha p}, \tag{109}$$

which is identical with (91). Thus we arrive at the case [B] with $\theta=2/3$. Since $\theta<\tau(=2)$, the range of wave numbers must be divided into two subranges; the range I, $q<q_d$, where the time scale is determined by the characteristic frequency $\Omega_q=q^\theta S(q\lambda)$, and the range II, $q_d\lesssim q<q^c$, where it is given by $\Omega_q=q^\tau S'(q\lambda)$. The range I is the inertial range where Kolmogorov's spectrum is expected to be valid[5),6)] and thus (108) and (109) are obtained. In the range II, the dissipative term is also important even for determining the fluctuation.

Since $\alpha>\beta$, the probability distribution of fluctuation must drastically differ from the normal distribution. It is, however, not easy to determine it since (73) is a highly-nonlinear equation. Recently Nelkin has discussed this problem, assuming a kinetic equation similar to (72).[6)] The turbulent viscosity $\nu_{\rm turb}$ may be regarded as an effective viscosity on a single mode $\xi_{\alpha Q}(t)$ due to the mode-mode coupling with other modes. Since $Q^2\nu_{\rm turb}=Q^\theta S(Q\lambda)$, its spectrum must be of the form

$$\nu_{\rm turb}(Q,\omega)=Q^{-4/3}V(Q\lambda,\omega Q^{-2/3}), \tag{110}$$

where ω is the frequency.

The turbulent viscosity can be formulated in the following way. Other modes exert a fluctuating force on the mode $\xi_{\alpha Q}(t)$ due to the nonlinear mode coupling (109). In order to express this effect in terms of a viscosity, we eliminate other modes by the projection-operator method.[10)] Using the master operator $M(\xi)$ and its adjoint $\Lambda(\xi)$

$$M(\xi) = -\sum_k \frac{\partial}{\partial \xi_k} \Delta\tilde{v}_k(\xi), \quad \Lambda(\xi) = \sum_k \Delta\tilde{v}_k(\xi) \frac{\partial}{\partial \xi_k}, \tag{111}$$

we first introduce

$$\xi_k(t) \equiv \exp[t\Lambda(\xi)]\xi_k . \tag{112}$$

Since (72) leads to $\Pi(\xi, t) = \exp[tM(\xi)]\Pi(\xi, 0)$, we can then write the mean evolution (78) as

$$\bar{\xi}_k(t) = \int d\xi \Pi(\xi, 0)\xi_k(t). \tag{113}$$

If we set $\Pi(\xi, 0) = [1 + B_l \delta\xi_l^*]\Pi^s(\xi)$ with a steady-state distribution $\Pi^s(\xi)$, B_l being a constant, then this takes the form

$$\delta\bar{\xi}_k(t) = \Psi_{kl}(t) B_l, \quad \Psi_{kl}(t) \equiv \langle \delta\xi_k(t) \delta\xi_l^*(0) \rangle^s, \tag{114}$$

where $\delta\xi_k(t) \equiv \xi_k(t) - \langle \xi_k \rangle^s$, and $\langle \cdots \rangle^s$ indicates the average over the steady-state distribution $\Pi^s(\xi)$. Equation (114) shows that the two-time correlation functions $\Psi_{kl}(t)$ describe the linear response of the steady state to a weak disturbance. In previous papers,[1),10)] the renormalization for the equilibrium fluctuation has been shown. This method, however, can be used also for the steady state by replacing the equilibrium average by the steady-state average. Namely, setting $\langle \delta\xi_k \delta\xi_j^* \rangle^s = \delta_{k,j} \langle |\delta\xi_j|^2 \rangle^s$, we introduce the projection operator

$$Pf(\xi) = \sum_j \langle f(\xi) \delta\xi_j^* \rangle^s \delta\xi_j / \langle |\delta\xi_j|^2 \rangle^s. \tag{115}$$

Then the equation of motion for (112) can be written as[10)]

$$d\xi_k(t)/dt = \sum_j i\omega_{kj} \delta\xi_j - \sum_j \int_0^t \psi_{kj}(s) \delta\xi_j(t-s) ds + q_k(t), \tag{116}$$

where

$$i\omega_{kj} \equiv \langle \delta\xi_j^* \Lambda(\xi) \delta\xi_k \rangle^s / \langle |\delta\xi_j|^2 \rangle^s, \tag{117}$$

$$\psi_{kj}(t) \equiv -\langle \delta\xi_j^* \Lambda(\xi) q_k(t) \rangle^s / \langle |\delta\xi_j|^2 \rangle^s, \tag{118}$$

$$q_k(t) \equiv \exp[t(1-P)\Lambda(\xi)](1-P)\Lambda(\xi)\delta\xi_k . \tag{119}$$

The $q_k(t)$ represents the fluctuating force exerted by other modes due to the nonlinear mode coupling. Since $\langle q_k(t) \delta\xi_l^*(0) \rangle^s = 0$, the mean evolution $\delta\bar{\xi}_k(t)$ is obtained from (116) through (114). The mean evolution has a quite different feature from in the normal case (105). That is the memory effect coming out from the memory kernel $\psi_{kj}(s)$ in (116). The turbulent viscosity is represented by this memory kernel. In the isotropic homogeneous turbulence, all the correlation functions are diagonal. Then (116) leads to

$$d\bar{\xi}_{\alpha Q}(t)/dt = i\omega_Q \delta\bar{\xi}_{\alpha Q} - Q^2 \int_0^t \nu_{\text{turb}}(Q, s) \delta\bar{\xi}_{\alpha Q}(t-s) ds, \tag{120}$$

where $i\omega_Q = \langle \delta\xi_{\alpha Q}^* \Delta\tilde{v}_{\alpha Q}(\xi) \rangle^s / \langle |\delta\xi_{\alpha Q}|^2 \rangle^s$, and (118) leads to

$$\nu_{\text{turb}}(Q,\omega)=\frac{Q^{\gamma-2}}{C_\alpha(Q\lambda)}\int_0^\infty\langle q^*_{\alpha Q}\exp[t(1-P)\Lambda(\xi)]q_{\alpha Q}\rangle^s e^{-i\omega t}dt\,, \tag{121}$$

where $q_{\alpha Q}=(1-P)\varDelta\tilde{v}_{\alpha Q}(\xi)$. Since $[q_{\alpha Q}]^L=L^{7/6}$ and $[t]^L=L^{2/3}$, (121) satisfies the dynamic scaling (110). To calculate (121) explicitly, we may employ a self-consistent equation approach developed for the equilibrium critical phenomena.[7),11)]

[2] *Magnetic critical phenomena*:

The equilibrium state y is determined by

$$\partial\hat{F}(y)/\partial y_k=0 \tag{122}$$

with the positive hermitian matrix $\hat{F}_{kj}\equiv\partial^2\hat{F}(y)/\partial y_k\partial y_j^*$. Since the second terms of (27) and (28) can be neglected for large Ω,

$$h_k(x)=\sum_j[\hat{M}_{kj}(x)+\hat{L}_{k;j}(x)]\partial\hat{F}(x)/\partial x_j^*, \tag{123}$$

where $\hat{M}_{kj}(x)\equiv-\langle\{A_k,A_j^*\};x\rangle^e/\beta^e\Omega$. Therefore, (122) gives a stable steady solution of the evolution equation (38). The correlation length of critical fluctuation is a function of temperature T;[4)] $\lambda\sim\varepsilon^{-\nu}$, $(\varepsilon\equiv|T-T_C|/T_C)$, where T_C is the Curie temperature and $\nu\simeq2/3$ for the three-dimensional isotropic Heisenberg ferromagnets. Now we take $q^c\lambda\gg1$. Then λ is one of the characteristic lengths of the macroscopic state, and the scale transformation (50) means T approaching to T_C according to $\varepsilon_L=L^{-(1/\nu)}\varepsilon$. This agrees with Kadanoff's scaling.[4)] According to the scaling theory of static critical phenomena, the thermodynamic free energy $\hat{F}(y)$ obeys the same scaling as the fluctuation free energy $\varDelta\hat{F}(z)$. If a magnetic field h is applied, then it must be transformed by $h_L=L^{-x}h$, where $x=d-\alpha$. Therefore,

$$\alpha=\beta=d-x=(d-2+\eta)/2\,, \tag{124}$$

and $\gamma=2-\eta$, $\theta_D=4-\eta$, where $\eta\simeq0.075$ for the three-dimensional isotropic Heisenberg ferromagnets. Since $\hat{M}_{kj}(x)\sim x_i$, (123) leads to $[\tilde{v}_k(x)]^L=[\hat{F}(x)]^L=L^{-d}$, giving

$$\theta_S=d-\beta=(d+\gamma)/2=(d+2-\eta)/2\,. \tag{125}$$

Since $\omega_q=\sigma/\hbar\beta^e\chi^e_{\perp q}$, σ being the spontaneous polarization, (5) and (6) lead to the same result for θ_S. Thus we have the following two cases.

If $d<6-\eta$, then $\theta=\theta_S(<\theta_D)$ and the dynamics of the critical fluctuation is described by (72) and (73). The streaming velocity $\varDelta\tilde{v}_k$ may be approximately given by

$$\varDelta\tilde{v}_k(\xi)\simeq\begin{cases}\dfrac{i}{\hbar\beta^e}\dfrac{1}{\sqrt{\Omega}}\sum_p{}'\left[\dfrac{1}{\chi^e_{\perp p}}-\dfrac{1}{\chi^e_{\perp q-p}}\right]\xi_p{}^+\xi^-_{q-p}\,, & (126)\\[2ex] \pm i\omega_q\xi_q{}^\pm\pm\dfrac{i}{\hbar\beta^e}\dfrac{1}{\sqrt{\Omega}}\sum_p{}'\left[\dfrac{1}{\chi^e_{\parallel p}}-\dfrac{1}{\chi^e_{\perp q-p}}\right]\xi_p{}^0\xi^\pm_{q-p}\,, & (127)\end{cases}$$

where, since $\alpha=\beta>0$, only the terms of lowest order in y_k and z_k have been retained. Then we meet a situation similar to the turbulence. Since $\alpha=\beta$, the probability distribution of fluctuation $\Pi(\xi, t)$ must deviate from the normal one. The mean evolution of fluctuation $\delta\bar{\xi}_k(t)$, however, can be determined by (116) and written in terms of the renormalized kinetic coefficients $\psi_{kj}(i\omega)$. The linear transport coefficients, observed by neutron scattering for example, are thus given by

$$\Gamma_{\alpha Q}(i\omega) = \frac{Q^2}{\chi^e_{\alpha Q}}\left[D_\alpha + \frac{1}{Q^2}\int_0^\infty \langle q^*_{\alpha Q} \exp[t(1-P)\Lambda(\xi)]q_{\alpha Q}\rangle^e e^{-i\omega t}dt\right], \tag{128}$$

where D_α is given by (24). From (80) the second term in the square brackets must have the scaling form $\lambda^{3/2}D_\alpha(Q\lambda, \omega\lambda^{5/2})$, where we have assumed $d=3$ and $\eta=0$. This form agrees with the Halperin-Hohenberg non-classical dynamic scaling.[12]

If $d\geqq 6-\eta$, then $\theta=\theta_D=4-\eta(\leqq\theta_S)$ and the kinetic equations are provided by (68) and (69) with (126), (127) and $\Delta\tilde{C}_k(\xi)=-q^2[D_\alpha/\chi^e_{\alpha q}]\xi_q{}^\alpha$. For $d>6-\eta$, therefore, (70) leads to a simple relaxation form for the mean evolution; $\bar{\xi}_q{}^\alpha(t) \sim \exp[-tq^2D_\alpha/\chi^e_{\alpha q}]$. Thus it turns out that the conventional theory[13] is valid only for $d>6-\eta$, and the non-classical dynamic scaling[12] holds for $d\leqq 6-\eta$.

§ 6. Summary and remarks

It has been shown that the scaling exponents α, β, θ_S and θ_D defined by (55), (56), (61) and (62), are the parameters characterizing the most dominant stochastic features of the macroscopic systems and provide us with a useful method for finding asymptotic kinetic equations for large systems. Table I summarizes

Table I. Scaling exponents in typical examples.

	1	2	3	4
	Equilibrium Fluctuations		Navier-Stokes Fluids ($d=3$)	
	$q^e\lambda\ll 1$	$q^e\lambda\gg 1$ [b]	$q^e\lambda\ll 1$	$q^e\lambda\gg 1$
region	hydrodynamic	critical	laminar	turbulent
α β	0 $d/2$	$(d-2+\eta)/2$ $(d-2+\eta)/2$	1 3/2	1 $-1/3$
	$\alpha<\beta$	$\alpha=\beta$	$\alpha<\beta$	$\alpha>\beta$
θ_D θ_S	ϕ [a]	$2-\eta+\phi$ $(d+2-\eta)/2$	2 2	17/3 2/3
	$\theta=\theta_D=\phi$	$\theta_S\lessgtr\theta_D$ if $d\lessgtr 2-\eta+2\phi$.	$\theta=\theta_D=2$	$\theta=\theta_S<\theta_D$

a) $\phi\simeq 2$ for the conserved densities and $\phi\simeq 0$ for the bulk-contact systems.
b) The ordered state $(T<T_C)$ is included in the case 2 irrespectively of the value of q^e since $\lambda\to\infty$.

their values in the typical examples studied in this paper. This table indicates that our scaling would be useful for constructing a framework of the statistical mechanics of fluctuations and nonequilibrium states, including the nonlinear steady states far from equilibrium. In fact, starting from a general type of the master equation, proposed in a previous paper from the statistical-mechanical viewpoint,[1] and applying our scaling, we have derived asymptotic kinetic equations for several examples.

In the uniform systems[2),8)] and in the generalized time-dependent Ginzburg-Landau model,[14)] the fluctuation of the macroscopic state variables also arises from the interaction with the contact system. Imagine, for example, the spin system in contact with the phonon system. To include such systems, we extend (54) in the following form:

$$[\hat{L}_{k_1\cdots k_{n-1};k_n*}]^L = L^{-n\phi/2}. \qquad (\phi \geqq 0) \tag{129}$$

Then, instead of (63) and (60), we obtain

$$\theta_D = d + \phi - 2\beta = \phi + \gamma\,, \tag{130}$$

$$[\hat{\alpha}_{k_1\cdots k_n}]^L = L^{-n\phi/2}, \tag{131}$$

where we have used $\beta < d+1$. These equations lead to the same expression for ζ_n as (66) but with (130) for θ_D. Thus (65), (67) and the subsequent equations are also valid for the bulk-contact systems with (130) for θ_D. It turns out that the Ω expansion method is valid only if $\alpha=0, \beta=d/2$, $(\gamma=0)$, $\phi=0$ and thus $\theta=\theta_D=0$. It is worth noting here that the Ω expansion method is not valid for the ordered or coherent state, since then α is not zero as suggested by Table I. In the kinetic Ising model near the Curie point,[14)] if $q^c\lambda \gg 1$, then $\alpha=\beta=(d-2+\eta)/2$, $(\gamma=2-\eta)$, and $\theta=\theta_D=2-\eta+\phi$, where $\phi\simeq 0$.[14),15)] These cases are also included in Table I. Thus our scaling theory gives us a generalization of the central limit theorem, the Ω expansion methods[2),8)] and the scaling theories of thermodynamic critical phenomena[4),12),14),16)] in a unifying manner.

The second merit of our scaling theory is that it can also be applied to nonlinear steady states far from equilibrium, such as the turbulence, as discussed in § 5 [1].

The third and probably the most important merit would be that the inequality relations between the scaling exponents give us a useful method for finding kinetic equations from the statistical-mechanical point of view. This has been formulated in [A], [A′] and [B] of § 3, and examples have been studied in §§ 4 and 5.

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