Dynamics of nearly parallel interacting vortex filaments

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The dynamics of interacting vortex filaments in an incompressible fluid, which are nearly parallel, have been approximated in the Klein-Majda-Damodaran model. The regime considers the deflections of each filament from a central axis; that is to say, the vortex filaments are assumed to be roughly parallel and centered along parallel lines. While this model has attracted a fair amount of mathematical interest in the recent literature, particularly concerning the existence of certain specific vortex filament structures, our aim is to generalise several known interesting filament solutions, found in the self-induced motion of a single vortex filament, to the case of pairwise interactions between multiple vortex filaments under the Klein-Majda-Damodaran model by means of asymptotic and numerical methods. In particular, we obtain asymptotic solutions for counter-rotating and co-rotating vortex filament pairs that are separated by a distance, so that the vortex filaments always remain sufficiently far apart, as well as intertwined vortex filaments which are in close proximity, exhibiting orverlapping orbits. For each scenario, we consider both co- and counter-rotating pairwise interactions, and the specific kinds of solutions obtained for each case consist of planar filaments, for which motion is purely rotational, as well as traveling wave and self-similar solutions, both of which change their form as they evolve in time. We choose traveling waves, planar filaments, and self-similar solutions for the initial filament configurations, as these are common vortex filament structures in the literature, and we use the dynamics under the Klein-Majda-Damodaran model to see how these structures are modified in time under pairwise interaction dynamics. Numerical simulations for each case demonstrate the validity of the asymptotic solutions. Furthermore, we develop equations to study a co-rotating hierarchy of many satellite vortices orbiting around a central filament. We numerically show that such configurations are unstable for plane wave solutions, which lead to the collapse of the hierarchy. We also consider more general traveling wave and self-similar solutions for co-rotating hierarchies, and these give what appears to be chaotic dynamics.

1. Introduction

Although vortices are conceptually simple objects in translational and turbulent flow, the interactions of vortex filaments in three-dimensional fluids exhibit complex dynamics and appear in remarkably diverse systems. For example, much research into the phenomenon was motivated by understanding vortices generated in the wake of aircraft wings and whether their subsequent interactions would prove hazardous for following vehicles (Breitsamter, 2011; Leweke *et al.*, 2016). During landing or take-off, co-rotating vortices are shed from the tip of the wings and the lowered flaps (Meunier *et al.*, 2005) which

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then merge into a pair of counter-rotating vortices (Parslew & Crowther, 2013) that exhibit long and short wave instabilities (Leweke & Williamson, 1998, 2011). Furthermore, of interest is the evolution of vortex dynamics produced in the wake of propellers, which produce a main hub vortex and opposite circulating double helical tip vortices that collapse and breakdown in finite time (Felli *et al.*, 2011).

In nature, fish similarly generate intricate wakes and patterns of vortices from the shape of their fins and bodies (Lauder *et al.*, 2002; Tytell, 2006). However, the state of such vortices after their creation is of biological importance because fish have been shown to detect these flows using the *lateral-line system*, a network of tubular organs covering their bodies (Bleckmann & Zelick, 2009; Chagnaud *et al.*, 2006), which allows schools of fish to swim together and predators to find their prey (Franosch *et al.*, 2009; Pohlmann *et al.*, 2001).

To gain understanding into the configurations that arise from interacting vortices, we study the simplified system of equations derived by Klein *et al.* (1995), who originally considered deformations of the straight vortex filaments which were small in amplitude and wavelength. We summarize their work as follows: Suppose there are N vortices in a three-dimensional incompressible fluid. We define the position of the j^{th} vortex filament at some time t and arc-length parameter s in Cartesian coordinates as $(x_j(t,s), y_j(t,s), s)$ and introduce the complex dependent variable $\psi_j(t,s) = x_j(t,s) + iy_j(t,s)$, so that the time-evolution of the j^{th} vortex filament reads

$$\mathrm{i}\frac{\partial \psi_j}{\partial t} + \alpha_j \Gamma_j \frac{\partial^2 \psi_j}{\partial s^2} + 2\sum_{k\neq j} \Gamma_k \frac{\psi_j - \psi_k}{|\psi_j - \psi_k|^2} = 0, \qquad j = \{1, 2, ..., N\}, \tag{1.1}$$

where α_j is related to the shape of the vortex core, as described in the work of Klein & Majda (1991) and Klein & Knio (1995), whilst Γ_j is the corresponding circulation of the vortex filament.

Under the framework of Klein *et al.* (1995), the shape of an individual vortex filament is influenced by self-induction (the second term of (1.1)) and the interactions of neighboring vortices (the third term of (1.1)), all of which are assumed to be dominated by local contributions; that is, $\psi_j(t,s)$ at a particular arc-length coordinate s is determined by the velocity contributions from all vortex filaments within this small region of s. For such filaments which deviate from being nearly parallel, the nonlinear Schrödinger-type equations in (1.1) break down for two reasons: First, the geometric ansatz describing the position $(x_j(t,s), y_j(t,s), s)$ is no longer valid as the z-axis coordinate is not adequately described by s. Second, the dynamics is now dominated by non-local contributions (Klein *et al.*, 1992). As the filament curls, the velocity contributions at a particular point on the vortex become increasingly determined by the vorticity generated further up or down the arc-length, not only from the same filament, but neighboring filaments as well.

Using (1.1), Klein *et al.* (1995) were able to predict the periodic long-wave instabilities that appear in oppositely-rotating vortex pairs due to the Crow mechanism, and neutral stability for configurations rotating in the same direction (Crow, 1970; Jimenez, 1975). However, more recent work on the model has considered the finite time collapse of counter-rotating vortex pairs and configurations with N co-rotating vortices with imposed polygonal symmetry; in particular, resulting from perturbing exactly straight filaments with self-similar solutions as the filaments are held parallel at $s \to \pm\infty$ (Banica & Miot, 2013; Banica *et al.*, 2014, 2016). The existence of traveling wave-type solutions (similar to those found by Hasimoto (1972) for a single vortex) as well as standing waves has been proven for vortex systems with the same polygonal symmetry (Banica & Miot, 2011; Craig *et al.*, 2016). Many of these results concern existence of solutions from very specific initial configurations and boundary data. In the present paper, we shall be more concerned with describing the temporal dynamics of a wide variety of interacting filament structures, motivated by an interest in generalising known isolated filament structures to account for pairwise interactions. Indeed, save for the original Klein *et al.* (1995) paper, asymptotic solutions for more complicated vortex filament structures under (1.1) are not present in the literature.

We shall study various families of solutions to (1.1), not only for co-rotating and counter-rotating pairs, but also a N co-rotating hierarchy around a central filament. We do not impose any assumptions on the symmetry of the system and consider more general initial and boundary conditions. Without these constraints, we are able to derive a partial differential equation for the counter-rotating vortex pair with parallels in the study of nonlinear beams and cables (Lazer & McKenna, 1990). We apply a similar treatment to the co-rotating vortex hierarchy, which produces compatibility relations describing how the vortices are distributed around the central filament for the case of plane wave solutions, and numerically study the case of two satellite vortices orbiting around a central filament for various forms of solutions.

For each of the physical configurations considered, we shall consider traveling wave, planar, and self-similar vortex filament structures. Not only are these natural mathematical choices for solutions, but these also correspond to well-studied regimes for the single vortex filament under the Biot-Savart and Local Induction Approximation (LIA) models governing its self-induced motion. Perhaps the most commonly studied traveling wave solution under the non-local Biot-Savart law is the helical filament (Widnall, 1972; Moore & Saffman, 1972; Ricca, 1994; Boersma & Wood, 1999). Analytical results are common under the LIA (Zhou, 1997), which is much simpler to solve in the helical case (Kida, 1981; Sonin, 2012). Hybrid results on the helical filament which use the LIA to regularise the core region (rather than using the cut-off method) whilst keeping the Biot-Savart integral "tails" were presented in Van Gorder (2015*a*).

The planar vortex filament is as studied (Da Rios, 1906; Ricca, 1996; Kida, 1981) and was related to problems in elastica (Hasimoto, 1971). The solution takes the form of a plane curve, with the motion being rotational orthogonal to the plane in which the curve lies. A direct derivation in Cartesian coordinates was given in Van Gorder (2012*b*). The Poincaré - Lindstedt method was used to study the spatial structure of a spatially periodic planar filament in the Cartesian frame by Van Gorder (2013*b*). Kida (1982) considered a numerical stability analysis for the planar filament, while Van Gorder (2013*a*) was able to obtain an analytical result for the orbital stability (spectral stability). Fukumoto (1997) studied the influence of background flows on planar filaments. There is an alternate formulation, given by Umeki (2010), which formulates the LIA in terms of the unknown tangent vector as a function of arclength and time, and a planar solution in this framework was given by Van Gorder (2012*a*). Aside from these LIA results, the planar vortex filament solution was also recently shown to exist for the non-local Biot-Savart dynamics by Van Gorder (2015*b*). Planar filaments correspond to stationary states consisting of rotating open space curves. We should also note that torus knot solutions are vortex filament solutions which may be described as stationary states which form a closed, braided space curve. Such torus knot solutions were presorted in Kida (1981) in terms of elliptic integrals and in Ricca (1993) as linear perturbations of circular solutions.

Self-similar solutions under the LIA have been used to understand quantized vortex filament motion in superfluid Helium (Bewley *et al.*, 2008; Lipniacki, 2000, 2003*a*,*b*; Van Gorder, 2013*c*). Similarity solutions can model sharp kinks along vortex filaments as well as the development of singularities (Gutiérrez *et al.*, 2003). Pelz (1997) numerically simulated vortex tangles which exhibit some self-similarity at very small times (although this may not persist at large times). Likewise, self-similar, singular-like structures localised in time have been observed in collapsing vortex rings just prior to core overlapping (Fernandez *et al.*, 1995). Self-similar dynamics are useful in the study of vortex collapse and turbulence (Das *et al.*, 2001; del Álamo *et al.*, 2006; Kimura, 1987, 2009, 2010; Yoshimoto & Goto, 2007). In addition to these results for LIA, self-similar solutions were very recently shown to exist for the non-local Biot-Savart dynamics by Van Gorder (2016).

The rotation and stability of N helical vortices of the same circulation, and whose motion is confined to a cylinder, has been studied (Okulov, 2004) and further extended to consider the changes brought on by an assigned axisymmetric vorticity field generated by the wake of a propeller's hub vortex (Okulov & Sørensen, 2007). In particular, the results are derived from previous work by the exact solution of Hardin (1982) involving explicit computations of Kapteyn series and are further compared to previous work by (Boersma & Wood, 1999).

The remainder of the paper is organised as follows. In Section 2, we shall consider asymptotic and numerical solutions for vortex filaments which are always separated from one another. Both co- and counter-rotating configurations are taken, and we obtain traveling wave and self-similar solutions for each. In Section 3, we consider asymptotic and numerical solutions for intertwined vortex filaments which share the same orbital envelope. Again, both co- and counter-rotating configurations are taken, and we are able to exhibit traveling wave, planar, and self-similar solutions for each configuration. In Section 4, we consider numerical solutions for the more complicated case of many satellite vortex filaments surrounding a central vortex, thereby constructing a co-rotating hierarchy of filaments. We demonstrate instability in a plane wave configuration, while more general traveling wave and self-similar solutions result in what appear to be chaotic dynamics. Finally, we give concluding remarks and summarise some of the interesting findings in Section 5.

2. Same sized vortex filaments held apart by a distance function

We consider the instance whereby two vortices of the same uniform core structure α are separated by a distance in the x-axis that depends on time t; explicitly, the filaments are some function $D(t) \in \mathbb{R}$ apart, which is taken to be very large. The intention is to gain physical intuition into the role that the vortex interaction plays in the system as the mean separation distance between filaments is varied. We make the transformation $\psi_2(t,s) \to \psi_2(t,s) + D(t)$ so that the position of the vortices is given by $(\mathrm{Re}(\psi_1), \mathrm{Im}(\psi_1), s)$ and $(\mathrm{Re}(\psi_2) + D(t), \mathrm{Im}(\psi_2), s)$ for filaments 1 and 2 respectively. The time evolution of $\psi_1(t,s)$ and $\psi_2(t,s)$ are then given by

$$\mathrm{i}\frac{\partial \psi_1}{\partial t} + \alpha\Gamma_1 \frac{\partial^2 \psi_1}{\partial s^2} + 2\Gamma_2 \frac{\psi_1 - \psi_2 - D}{|\psi_1 - \psi_2 - D|^2} = 0, \tag{2.1}$$

$$\mathrm{i}\frac{\partial \psi_2}{\partial t} + \alpha\Gamma_2 \frac{\partial^2 \psi_2}{\partial s^2} + 2\Gamma_1 \frac{\psi_2 - \psi_1 + D}{|\psi_2 - \psi_1 + D|^2} = 0, \tag{2.2}$$

where $\alpha_1 = \alpha_2 = \alpha$.

We simplify (2.1) and (2.2) by introducing new dependent variables $\nu(t,s) = \psi_2(t,s) + \psi_1(t,s)$ and $\mu(t,s) = \psi_2(t,s) - \psi_1(t,s)$, so that $\psi_1(t,s) = (\nu(t,s) - \mu(t,s))/2$ and $\psi_2 = (\nu(t,s) + \mu(t,s))/2$. One can interpret $\nu(t,s)$ as the position of the vortex pair's center whilst $\mu(t,s)$ describes the deviations from this center with the additional $D(t)$ offset (see Fig. 1). Furthermore, we rescale the independent variables $\hat{s} = s/\sqrt{\alpha}$ and

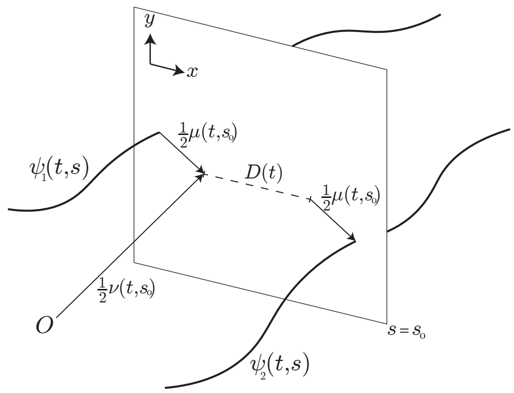

FIGURE 1. A pictorial representation of transformed dependent variables $\nu(t,s)$ and $\mu(t,s)$ and how they relate to the original variables $\psi_1(t,s)$ and $\psi_2(t,s)$ defined in (2.1) and (2.2). The variable $\nu(t,s)$ determines the center of the vortex pair whilst $\mu(t,s)$ measures the deviations from this center with an offset $D(t)$.

$\hat{t} = \Gamma_1 t$ to obtain (upon dropping the hat symbol):

$$\mathrm{i}\frac{\partial \nu}{\partial t} + \frac{1}{2}(1+\Pi)\frac{\partial^2 \nu}{\partial s^2} - \frac{1}{2}(1-\Pi)\frac{\partial^2 \mu}{\partial s^2} + 2(1-\Pi)\frac{\mu + D}{|\mu + D|^2} = 0, \tag{2.3}$$

$$\mathrm{i}\frac{\partial \mu}{\partial t} - \frac{1}{2}(1-\Pi)\frac{\partial^2 \nu}{\partial s^2} + \frac{1}{2}(1+\Pi)\frac{\partial^2 \mu}{\partial s^2} + 2(1+\Pi)\frac{\mu + D}{|\mu + D|^2} = 0, \tag{2.4}$$

where $\Pi = \Gamma_2/\Gamma_1$ is the ratio of vortex circulations. Note that in rescaling t, we have assumed that $\Gamma_1 > 0$, however the case of $\Gamma_1 < 0$ can readily be accounted for by transforming $t \to -t$. For the forthcoming analysis, we only consider the former case of $\Gamma_1 > 0$.

2.1. *Co-rotating vortex pair at large separations*

We suppose the two vortices have the same circulation (i.e. $\Pi = 1$). In this case, (2.3) and (2.4) decouple, meaning that an individual vortex filament's deviations from the center and the position of this center do not depend on each other. The evolution of the vortex pair is now described by:

$$\mathrm{i}\frac{\partial \nu}{\partial t} + \frac{\partial^2 \nu}{\partial s^2} = 0, \tag{2.5}$$

$$\mathrm{i}\frac{\partial \mu}{\partial t} + \frac{\partial^2 \mu}{\partial s^2} + 4\frac{\mu + D}{|\mu + D|^2} = 0. \tag{2.6}$$

By making the transformation to $\nu(t,s)$ and $\mu(t,s)$, we have removed the nonlinear interaction term, leaving us with the linear Schrödinger equation in (2.5). For this reason, we only focus on (2.6) and study the traveling wave and self-similar solutions that result.

2.1.1. *Traveling wave solutions*

We consider the case where the vortex pair are separated by a constant distance; that is, $D(t) = d \gg 1$, and study traveling wave solutions to (2.6). We do this by introducing the independent variable $\xi = s - vt$, where v is the velocity of the traveling wave (which is assumed to be constant and $O(1)$), and find that the transformed equation becomes:

$$\frac{\mathrm{d}^2 \mu}{\mathrm{d}\xi^2} - \mathrm{i}v\frac{\mathrm{d}\mu}{\mathrm{d}\xi} + 4\frac{\mu + d}{|\mu + d|^2} = 0. \tag{2.7}$$

We make use of the fact that we are studying large separation distances and expand $\mu(\xi)$ as a series in d^{-1}. In particular, $\mu = \sum_{m=0}^{\infty} (d)^{-m} \mu^{(m)}$, so that, the second order asymptotic solution is (see Section 6.1.1 for details):

$$\mu(t,s) = c_1 \exp(\mathrm{i}v(s-vt)) + c_2 + \frac{1}{d}\left(c_3 \exp(\mathrm{i}v(s-vt)) + c_4 - \frac{4\mathrm{i}}{v}(s-vt)\right)$$
$$+ \frac{1}{d^2}\left(-\frac{2c_1^*}{v^2}\exp(-\mathrm{i}v(s-vt)) + \frac{4\mathrm{i}c_2^*}{v}(s-vt) + c_5 \exp(\mathrm{i}v(s-vt)) + c_6\right) + O\left(\frac{1}{d^3}\right), \quad (2.8)$$

where $c_n \in \mathbb{C}$ are integration constants and $(\ )^*$ is the complex conjugate of the variable.

Equation (2.8) suggests that if the vortices are held at infinite separation, such that self-induction effects are only included, the resulting shape of the individual filaments is that of a helix with wavenumber v that oscillates at an angular frequency of v^2. As the pair is brought closer together, the vortex interaction manifests in two ways: First, there is repulsion in the perpendicular direction due to the $-4\mathrm{i}(s-vt)/v$ term, which is independent of any imposed conditions on $\mu(\xi)$; an expected result in light of point vortices. In a 2D flow, each co-rotating point vortex will feel a velocity field that is opposite in direction to the field felt by the other vortex, both of which are perpendicular to the direction of separation, thus driving the pair apart in the same manner as (2.8) predicts. Noting that $\Gamma_1 = \Gamma_2 > 0$, the direction of repulsion is such that the filament described by $\psi_1(t,s)$ will drift in the positive y direction, whilst the filament described by $\psi_2(t,s)$ will drift in the negative y direction. If we consider the case whereby $\Gamma_1 < 0$, such that $t \to -t$, the vortices are repelled in the opposite direction along the y-axis.

Second, an additional wave is generated in the configuration as a response to the zeroth order solution; one which has the form $-2c_1^* \exp(-\mathrm{i}v\xi)/v$. If we suppose that $\mathrm{Im}(c_1) = 0$, then for $\mathrm{Re}(c_1) > 0$, the secondary wave suppresses oscillations in the y direction, whilst vibrations in the x direction are similarly subdued when $\mathrm{Re}(c_1) < 0$. Physically, the helical shapes of the filaments flatten into ellipses due to the pairwise vortex interaction.

Furthermore, there is additional attraction or repulsion at $O(d^{-2})$ that depends on the initial off-set of the helical vortices; that is, depending on $4\mathrm{i}c_2^*\xi/v$. An offset which is in the x direction results in repulsion in the y direction, as seen in the first order correction. However, if the constant off-set is in the y direction, then the opposite holds true; namely, that the vortex pair translates in the x direction in either repulsion or attraction. For $\Gamma_1 > 0$, if $\mathrm{Im}(c_2) > 0$, then $4\mathrm{i}c_2^*\xi/v > 0$ and the pair attract each other, whilst if $\mathrm{Im}(c_2) < 0$, then $4\mathrm{i}c_2^*\xi/v < 0$ and the vortices repel. The opposite applies when $\Gamma_1 < 0$. This secondary repulsion or attraction is not predicted by the 2D theory and is a consequence of the vortex filaments existing in a 3D flow.

We numerically solve (2.7) for the parameters $v = 4$ and $d = 10$, with initial conditions $\mu(\xi = 0) = -\mathrm{i}$ and $\mu'(\xi = 0) = v$, and plot the vortex filament shapes that result when we set $\nu(t,s) = 0$ (shown in Fig. 2(a)-(d) for times $t = \{0, 40, 80, 120\}$ respectively). We observe that (2.8) breaks down for times past the initial repulsion of the vortex pair and does not capture the slowly oscillatory behavior of the numerical solution. This periodic solution on the slow wave-scale physically corresponds to the binary orbit of the vortex pair, whereby the filaments rotate around each other (see Fig. 2). This result is predicted if one considers the same problem from the perspective of 2D vortices. As the vortices separate vertically from their initial repulsion (Fig. 2(a)), the velocity field felt by each vortex becomes increasingly horizontal and decreasingly vertical, resulting in the pair becoming vertically displaced from each other (Fig. 2(b)). The velocity field for each vortex is now horizontal with the repulsion now leading to increasing vertical contributions and decreasing horizontal contributions in the velocity. The pair become horizontally separated again (Fig. 2(c)) and the cycle repeats.

Given that the pair are orbiting on a slow wave-scale and are separated by a constant d through all orientations, we instead look for a slowly varying solution of the form:

$$1+\frac{1}{d}\mu_{\text{slow}}(\xi)=\exp\left(\frac{\mathrm{i}\theta(\xi)}{d^{\beta}}\right), \tag{2.9}$$

with β being a positive constant and $\beta,\theta\in\mathbb{R}$. Ansatz (2.9) replaces the helical shape of each vortex in the pair, which individually rotate on a fast time scale, with straight filaments that no longer exhibit quick oscillations.

Substituting (2.9) into (2.7), we find the general solution (see Section 6.1.2 for details):

$$\theta(\xi)=dc_7-\frac{4\xi}{v}+c_8+\frac{c_9}{d}+O\left(\frac{1}{d^3}\right), \tag{2.10}$$

where $c_7,c_8,c_9\in\mathbb{R}$ are integration constants determining the initial orientation of the vortex pair. The odd powers of d^{-n} in (2.10) are corrections so that (2.9) satisfies the initial conditions imposed on $\mu_{\text{slow}}(\xi)$, whilst even powers give corrections to the frequency of the vortex pair's orbit.

We compare $\mu_{\text{slow}}(\xi)$ in (2.9), with $\theta(\xi)$ calculated from (2.10), to the numerical solution of (6.1) for the same initial conditions and value for v as seen previously. To impose $\mu_{\text{slow}}(\xi=0)=-\mathrm{i}$, we expand the exponential in (2.9) as a series and match powers of d^{-1} on both sides, giving $c_7=-1$, $c_8=0$, and $c_9=0$, which incurs an error of $O\left(d^{-2}\right)$ at $\xi=0$. We show these comparisons for $d=10$ in Fig. 3(a) and (b), and $d=5$ in Fig. 3(c) and (d), for the real and imaginary parts of $\mu(\xi)$. We observe that the slowly-varying approximation agrees very well with the numerical solutions showing the vortex pair's binary orbit, even for $d=5$. However, we note that (2.9) and (2.10) begin to break down as $\xi\approx 2000$, suggesting higher order frequency corrections are necessary to keep this solution valid for large ξ.

2.1.2. *Self-similar solutions*

We consider the case where the separation of the vortex pair grows in time; that is, $D(t)=d\sqrt{t}$. As in Section 2.1.1, $d\in\mathbb{R}$ is a constant which is very large such that $1/d=\epsilon\ll 1$. For a vortex separation of this form, (2.6) has self-similar solutions given by $\mu(t,s)=\sqrt{t}\chi(\eta)$, where $\eta=s/\sqrt{t}$. Transforming so that η is the independent variable, (2.6) becomes:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2}-\frac{\mathrm{i}\eta}{2}\frac{\mathrm{d}\chi}{\mathrm{d}\eta}+\frac{\mathrm{i}}{2}\chi+4\frac{\chi+d}{|\chi+d|^2}=0. \tag{2.11}$$

Taking $d^{-1}\ll 1$, we expand $\chi(\eta)$ as a power series in d^{-1}, so that $\chi(\eta)=\sum_{n=0}^{\infty}d^{-n}\chi^{(n)}$, to find the first-order asymptotic solution (see Section 6.2):

$$\begin{aligned}\mu(t,s)=&(-1)^{\frac{1}{4}}a_1s+a_2\left(\sqrt{t}\exp\left(\frac{\mathrm{i}s^2}{4t}\right)-\frac{(-1)^{\frac{1}{4}}\sqrt{\pi}}{2}\operatorname{serfi}\left(\frac{(-1)^{\frac{1}{4}}s}{2\sqrt{t}}\right)\right)\\&+\frac{1}{d}\left(8\mathrm{i}\sqrt{t}+(-1)^{\frac{1}{4}}a_3s+a_4\left(\sqrt{t}\exp\left(\frac{\mathrm{i}s^2}{4t}\right)+\frac{(-1)^{\frac{3}{4}}\sqrt{\pi}}{2}\operatorname{serf}\left(\frac{(-1)^{\frac{3}{4}}s}{2\sqrt{t}}\right)\right)\right)+O\left(\frac{1}{d^2}\right),\end{aligned} \tag{2.12}$$

where $a_n\in\mathbb{C}$ are integration constants and $\operatorname{erfi}(z)=\mathrm{i}\operatorname{erf}(\mathrm{i}z)$ is the complex error function.

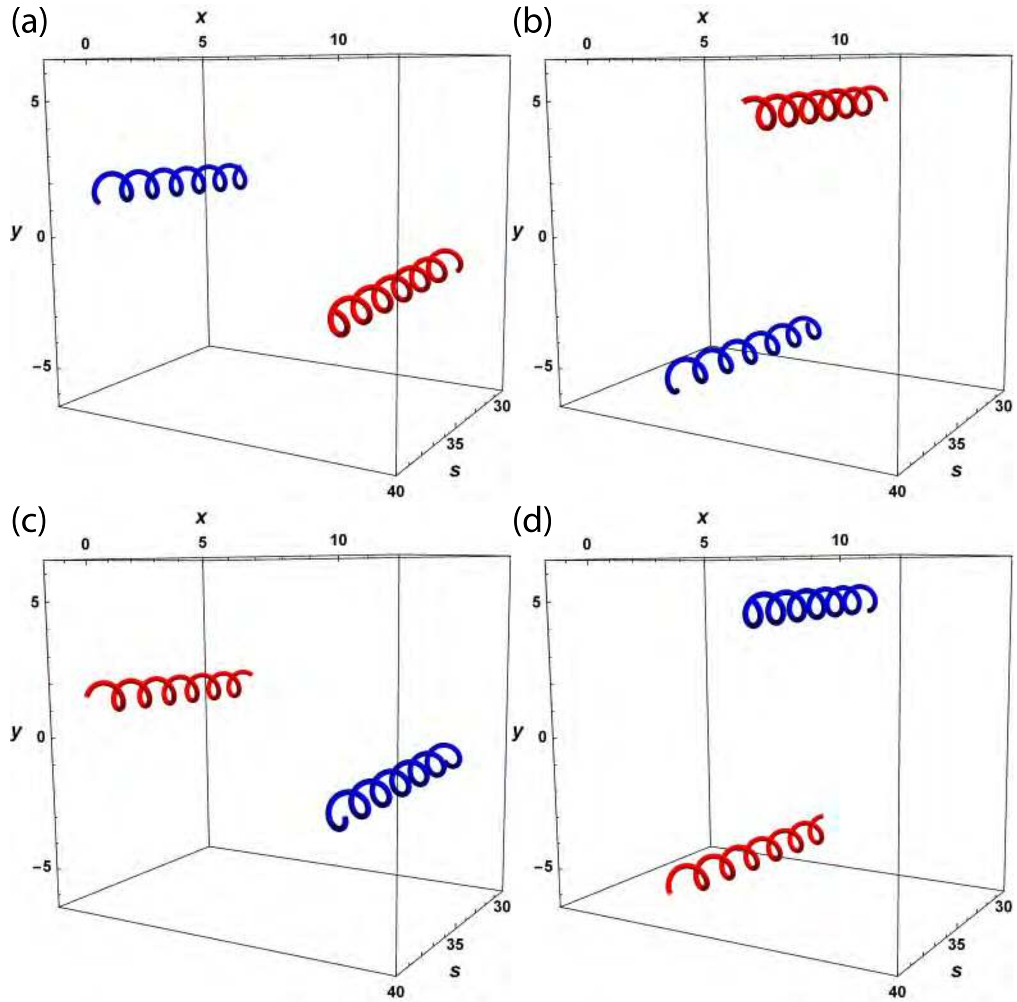

FIGURE 2. The time-evolution of traveling wave solutions in co-rotating vortices held at a constant distance, with $\psi_1(t,s)$ and $\psi_2(t,s)$ being denoted by blue and red respectively. In particular, $\nu(t,s)=0$ and the numerical solution to (2.7) have been plotted with initial conditions $\mu(\xi=0)=-\mathrm{i}$ and $\mu'(\xi=0)=v$, for $v=4$ and $d=10$, over $s\in[30,40]$. Times shown are: (a) $t=0$, (b) $t=40$, (c) $t=80$ and (d) $t=120$. (Color online)

We plot the vortex filament shapes obtained by numerically solving (2.11) for $\chi(\eta=100)=1+\mathrm{i}$, $\chi'(\eta=100)=1$, and $d=10$ with $\nu(t,s)=0$ for times $t=\{0.1,0.5,1,2\}$ in Fig. 4. These conditions correspond to $\mu(t,s)$ having an initial shape that satisfies the relations:

$$\left.\left(-\frac{2t^{\frac{3}{2}}}{s}\frac{\partial\mu}{\partial t}+t^{\frac{1}{2}}\frac{\partial\mu}{\partial s}\right)\right|_{\substack{t=t_0\\ s=s_0}}=1, \tag{2.13}$$

$$\mu(t_0,s_0)=\sqrt{t_0}\,(1+\mathrm{i}), \tag{2.14}$$

evaluated at $100=s_0/\sqrt{t_0}$.

We further compare the numerical solution to the asymptotic solution in (2.12) for the same boundary conditions in Fig. 5 and describe the form of solutions predicted for $t>0$ and $s>0$ as follows: At zeroth order, the initial shape of the pair is that of individual

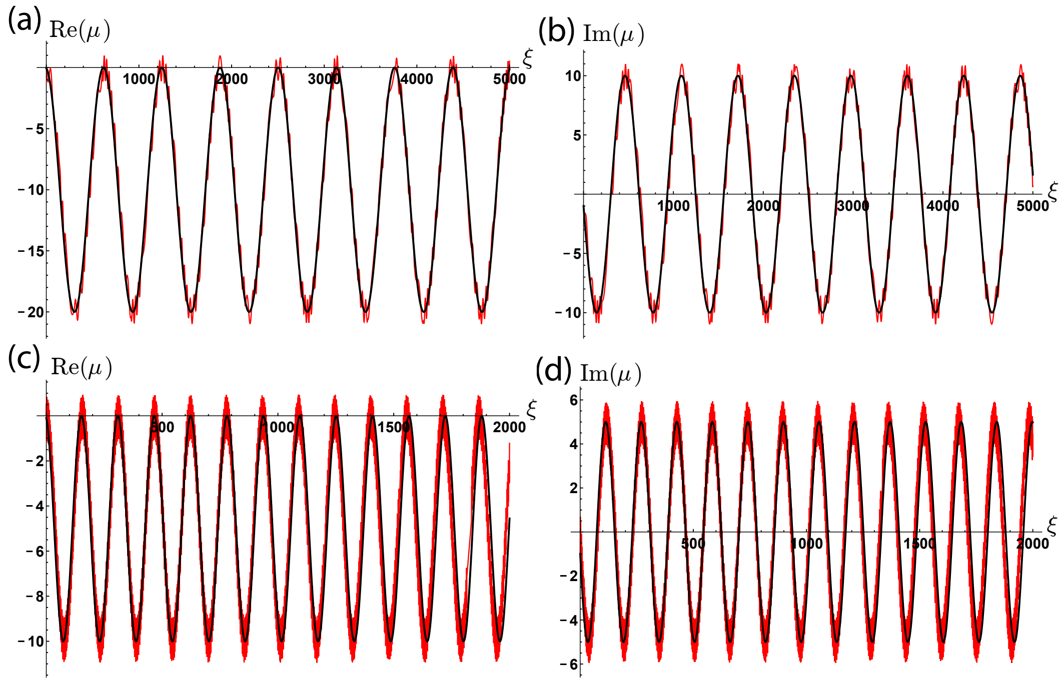

FIGURE 3. Real and imaginary components of $\mu(\xi)$ for the slowly-varying solution (2.9) and (2.10) (denoted by black) and numerical solution of the full problem (2.7) (denoted by red). The initial conditions imposed are $\mu(\xi = 0) = -\mathrm{i}$ and $\mu'(\xi = 0) = v$ for parameters $v = 4$ and ((a) and (b)) $d = 10$ and ((c) and (d)) $d = 5$. (Color online)

helical cones along an arbitrary orientation whose amplitude, wavelength, and separation grows with time. The conical nature of the helix arises from the complex error function whilst the increasing helix amplitude is due to the complex exponential in (2.12). Such conical-type solutions were previously seen for isolated vortex filaments (Van Gorder, 2016). The inclusion of vortex interaction at first order gives the same general shape as the zeroth order solution, with the exception of the monotonically increasing term in $\sqrt{t}$. As with Section 2.1.1, the vortex pair exhibit repulsion in opposite directions along the y-axis, however, in this instance, the asymptotic solution does not show significant disagreement with the numerical solution for $t \to \infty$. The reason is due to the separation function $D(t)$ used, which makes the binary orbits, as seen with traveling wave solutions, impossible due to the horizontal distance between the vortex pair increasing with time rather than staying constant. Lastly, we note that the rotation of the vortex helices is such that they rotate in the same direction with the orientation of one helix being a reflection of the other in both the x and y axis. In Fig. 4, both vortices have circulation $\Gamma > 0$, however the helices rotate in a clockwise direction, suggesting that the direction of a helix's rotation is opposite to an individual filament's circulation.

2.2. *Counter-rotating vortex pair at large separations*

For a vortex pair with opposite circulation, $\Pi = -1$ so that (2.3) and (2.4) now give:

$$\mathrm{i}\frac{\partial \nu}{\partial t} - \frac{\partial^2 \mu}{\partial s^2} + 4\frac{\mu + D}{|\mu + D|^2} = 0, \tag{2.15}$$

$$\mathrm{i}\frac{\partial \mu}{\partial t} - \frac{\partial^2 \nu}{\partial s^2} = 0. \tag{2.16}$$

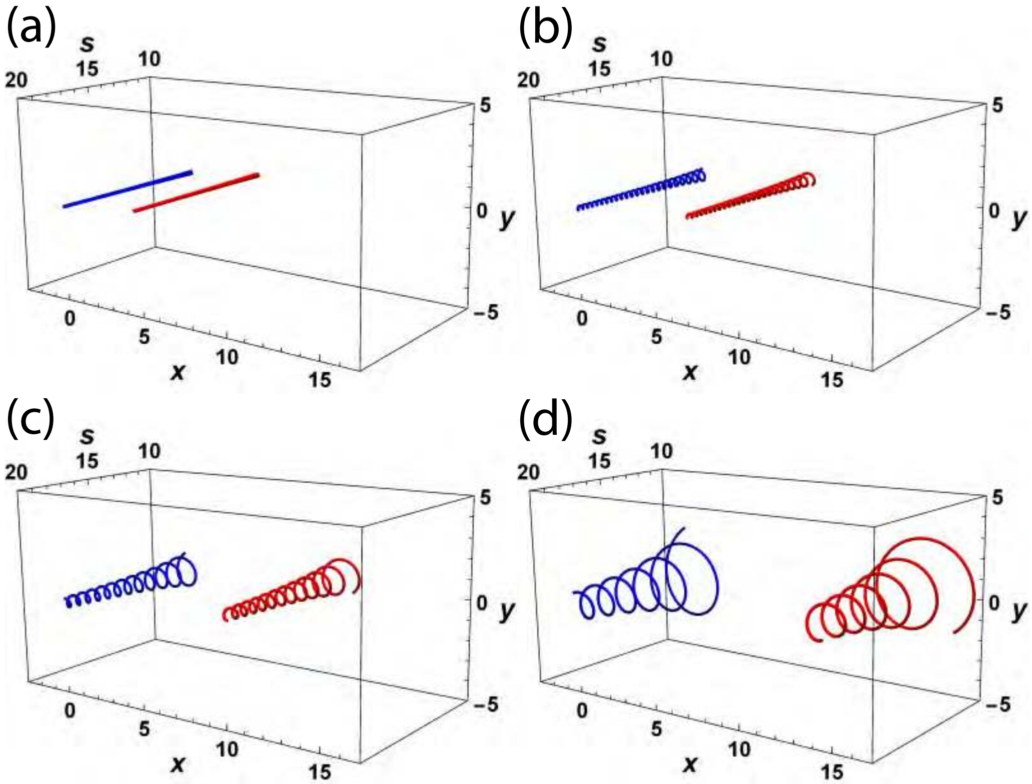

FIGURE 4. The time-evolution of self-similar solutions in co-rotating vortices held apart by a separation function $D(t) = \sqrt{t}$, with $\psi_1(t,s)$ and $\psi_2(t,s)$ being denoted by blue and red respectively. In particular, $\nu(t,s) = 0$ and (2.11), with conditions satisfying (2.13) and (2.14), have been plotted for $d = 10$ over $s \in [10, 20]$. Times shown are: (a) $t = 0.1$, (b) $t = 0.5$, (c) $t = 1$, and (d) $t = 2$. (Color online)

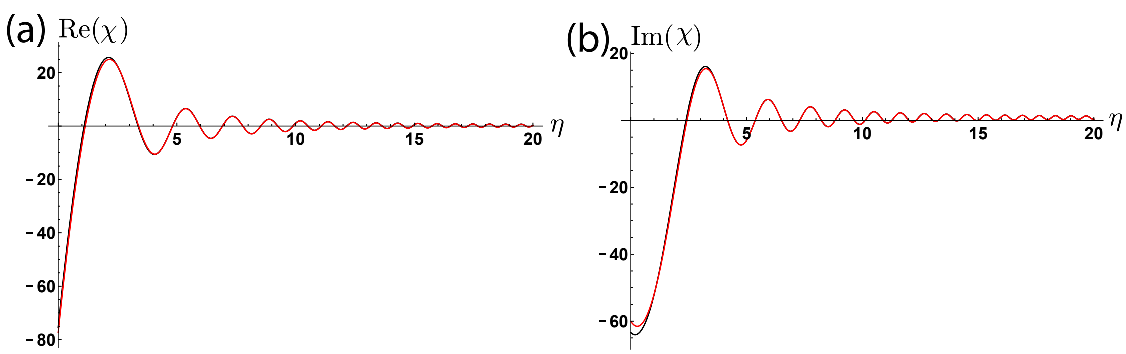

FIGURE 5. Solutions of $\chi(\eta)$ calculated from the first-order asymptotic solution (2.12) (shown in black) and the numerical solution of (2.11) (shown in red). In particular, the initial conditions $\chi(\eta = 100) = 1 + \mathrm{i}$ and $\chi'(\eta = 100) = 1$ were used with $d = 10$. (Color online)

However, supposing $\psi_1(t,s)$ and $\psi_2(t,s)$ are C^4 in s and C^2 in t, we differentiate (2.16) once with respect to t and differentiate (2.15) twice with respect to s to find a single time-evolution equation for $\mu(t,s)$:

$$\frac{\partial^2\mu}{\partial t^2}+\frac{\partial^4\mu}{\partial s^4}-4\frac{\partial^2}{\partial s^2}\left(\frac{\mu+D}{|\mu+D|^2}\right)=0, \tag{2.17}$$

which shares similarities to equations in the studies of beams and cables (Lazer & McKenna, 1990).

There are two noteworthy aspects of (2.17): First, $\mu(t,s)$ and $\nu(t,s)$ are not independent of each other and are still related by (2.16); that is, the position of the vortex pair's center and the deviations from this center are coupled, unlike the co-rotating vortices studied in Section 2.1. Second, the additional conditions necessary to solve (2.17) now come from those which are imposed on $\nu(t,s)$, along with its relation to $\mu(t,s)$, according to (2.15) and (2.16).

2.2.1. *Traveling wave solutions*

As in Section 2.1.1, we consider vortices that are separated by a constant distance $D(t)=d$ such that we have the small parameter $d^{-1}\ll 1$. Transforming (2.17) into wave coordinates $\xi=s-vt$, with $v=O(1)$, we find:

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\xi^2}+v^2\mu-4\frac{\mu+d}{|\mu+d|^2}=\delta, \tag{2.18}$$

where there is only one non-zero integration constant due to the relationship between $\mu(t,s)$ and $\nu(t,s)$, which imposes the constraints:

$$\frac{\mathrm{d}^3\mu}{\mathrm{d}\xi^3}+v^2\frac{\mathrm{d}\mu}{\mathrm{d}\xi}-4\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mu+d}{|\mu+d|^2}\right)=0, \tag{2.19}$$

$$v^2\mu-\mathrm{i}v\frac{\mathrm{d}\nu}{\mathrm{d}\xi}=\delta, \tag{2.20}$$

with the latter equation used to determine δ from conditions on $\mu(\xi)$ and $\nu'(\xi)$.

We note that (2.18) has solutions up to $O\left(d^{-2}\right)$ that remain finite for all time and along the entire arclength of the filament pair, provided the velocity of the traveling wave v is such that $v^2>4d^{-2}$. As a result, we are able to find solutions of the oscillatory Poincaré-Lindstedt form, which, to second order, are:

$$\begin{aligned}X(\Xi_1)=c_1\cos(v\Xi_1)+c_2\sin(v\Xi_1)+\frac{\delta_1}{v^2}+\frac{1}{d}\left(c_3\cos(v\Xi_1)+c_4\sin(v\Xi_1)+\frac{4}{v^2}\right)\\+\frac{1}{d^2}\left(c_5\cos(v\Xi_1)+c_6\sin(v\Xi_1)-\frac{4\delta_1}{v^4}\right),\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}Y(\Xi_2)=c_7\cos(v\Xi_2)+c_8\sin(v\Xi_2)+\frac{\delta_2}{v^2}+\frac{1}{d}\left(c_9\cos(v\Xi_2)+c_{10}\sin(v\Xi_2)\right)\\+\frac{1}{d^2}\left(c_{11}\cos(v\Xi_2)+c_{12}\sin(v\Xi_2)+\frac{4\delta_2}{v^4}\right),\end{aligned} \tag{2.22}$$

where $X,Y\in\mathbb{R}$ and are related to μ by $\mu=X+\mathrm{i}Y$, $\Xi_1=\left(1+2/v^2d^2\right)(s-vt)$, $\Xi_2=\left(1-2/v^2d^2\right)(s-vt)$, $\delta=\delta_1+\mathrm{i}\delta_2$, and all $c_n\in\mathbb{R}$ (see Section 6.3.1 for further details).

We comment on some important aspects of (2.21) and (2.22): For counter-rotating vortices, there is no secular vortex repulsion or attraction, unlike what was previously seen with the co-rotating pair in (2.8). Instead, the deviations of the vortex pair have a constant offset, which is determined by the speed of the traveling wave solution, as well as δ, which is dependent on the conditions imposed on $\mu(\xi)$ and $\nu'(\xi)$. However, we note that by integrating (2.20) that the constant offsets seen in $\mu(\xi)$ become secular in ξ for $\nu(\xi)$. Though the deviations of the vortex pair may not grow for all time by means of attraction or repulsion, the center of the vortex pair will move from its initial position; that is, the filaments will move together at a constant velocity through the 3D flow. However, the deviations in the y direction will tend to $-\infty$ and ∞ if $v^2 < 4\epsilon^2$, which suggests that the velocity of the traveling wave serves to stabilize the vortex pair so that they remain together. Note that (2.21) and (2.22) will not approximate such a regime accurately because of the assumption made at the beginning of Section 2.2.1; namely, that $v = O(1)$.

Lastly, the oscillations of the filaments is subject to dispersion that arises from the vortex interaction at $O\left(1/d^2\right)$. Vibrations in the x direction will increase in frequency by a factor of $2/v^2d^2$, whilst oscillations in y will decrease by $2/v^2d^2$. This orientation dependent oscillation causes a vortical helix to shear to a planar vibration and back to a helix again.

To illustrate some of the phenomena outlined, we solve (2.18) and (2.20) numerically for $d = 10$ and $v = 1$ with the initial conditions $\mu(\xi = 0) = 0$, $\mu'(\xi = 0) = v + \mathrm{i}v$, $\nu(\xi = 0) = 0$, and $\nu'(\xi = 0) = 0$. The corresponding vortex filament shapes are shown in Fig. 6(a)-(d) for times $t = \{25, 80, 135, 190\}$ respectively. In this case, the inclusion of $\nu(t, s)$ causes the vortex pair to travel together in the y direction as $t > 0$. This result is again predicted by considering the simpler problem of point vortices in a 2D flow. A pair that has opposite circulation will generate a velocity field that is the same direction for both vortices, thus pushing them together. For the case shown in Fig. 6, $\psi_1(t, s)$ has positive circulation whilst ψ_2 has negative circulation, leading to the respective vortices feeling a velocity field that is in the positive y direction.

However, there is an additional effect that is not predicted by the 2D theory; that is, the oscillations of $\nu(t, s)$. These vibrations cause the amplitude of the helices to periodically vary in a manner such that the individual helices are out of phase with each other. We observe that as time evolves, $\psi_1(t, s)$, which initially has a helical shape (Fig. 6(a)), has an amplitude that shrinks (Fig. 6(b)), before returning to its original shape (Fig. 6(c)). The vortex described by $\psi_2(t, s)$ is out of phase in that its amplitude shrinks when $\psi_1(t, s)$ grows (Fig. 6(d)).

We compare the asymptotic solutions given by (2.21) and (2.22) to the numerical solution of (2.18) for the same parameters and initial conditions on $\mu(\xi)$ and $\nu(\xi)$ (shown in Fig. 7). We note that the asymptotic solutions provide a very good approximation to the full vortex interaction with a notable exception: The solution with the untruncated vortex interaction predicts that $\nu(\xi)$ exhibits slow amplitude modulation when compared to the second-order asymptotic correction in Fig. 7(c). This result suggests that, in a similar manner as for the co-rotating vortices, we require a slow-scale analysis of the problem to capture this effect.

We look for a slow solution of the form:

$$\nu_{\mathrm{slow}}(\xi) = \epsilon \exp\left(\mathrm{i}\epsilon^{\beta}\theta(\xi)\right) - \frac{4\mathrm{i}\epsilon\xi}{v}, \tag{2.23}$$

where we have included the previously mentioned constant translation of the vortex pair's center from the $4/dv^2$ term in (2.21).

We substitute (2.23) into (2.20) to obtain $\mu_{\mathrm{slow}}(\xi)$, the use of which, gives the following asymptotic solution upon using the result in (2.18) (see Section 6.3.2 for mathematical

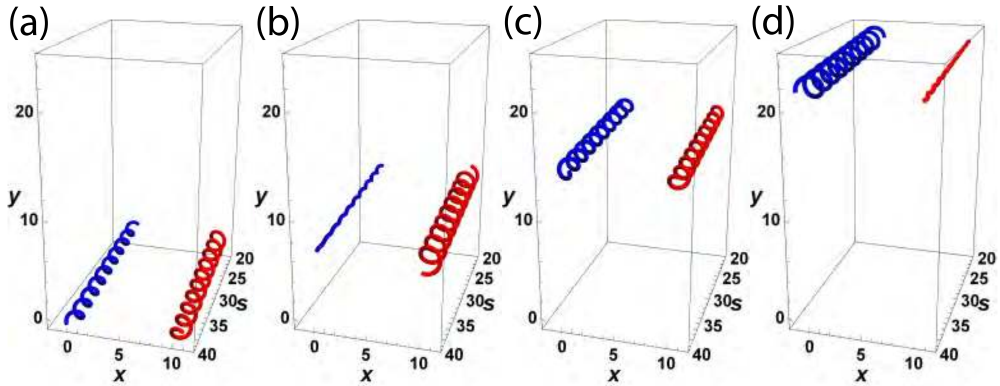

FIGURE 6. The time evolution of traveling wave solutions in counter-rotating vortices held apart by a constant distance d, with $\psi_1(t,s)$ and $\psi_2(t,s)$ being denoted by blue and red respectively. In particular, numerical solutions to (2.18) and (2.20) are plotted for $v=2$, $d=10$, and initial conditions $\mu(\xi=0)=0$, $\mu'(\xi=0)=v+\mathrm{i}v$, $\nu(\xi=0)=0$, and $\nu'(\xi=0)=0$ over $s\in[20,40]$. Times shown are (a) $t=25$, (b) $t=80$, (c) $t=135$, and (d) $t=190$. (Color online)

details):

$$\theta(\xi)=c_{13}-\frac{c_{14}}{v}\cos(v\xi)+\frac{c_{15}}{v}\sin(v\xi)+\frac{1}{d^2}\left(\frac{16\xi}{v^3}+c_{16}-\frac{c_{17}}{v}\cos(v\xi)+\frac{c_{18}}{v}\sin(v\xi)\right), \tag{2.24}$$

for $c_n\in\mathbb{R}$.

We compare (2.24) and (2.23) with the numerical solution of (2.18) and (2.20) for the same parameters and initial conditions on $\mu(\xi)$ and $\nu(\xi)$ as previously, but with $d=\{10,7.5\}$ in Fig. 8(a) and (b) respectively. In particular, the constants are $c_{13}=-\pi/2$, and all other $c_n=0$. Note that as $\xi\to\infty$, the monotically increasing term in ξ dominates in (2.24). Furthermore, we have vertically off-set the slowly-varying solution to show its fit to the slowly-varying envelope. We observe very good quantitative agreement for $d=10$, however, break down is apparent when $d=7.5$ for $\xi\approx 200$.

2.2.2. *Self-similar solutions*

We proceed in the same manner as Section 2.2.2 by considering a distance function $D(t)=d\sqrt{t}$ and self-similar solutions of the form $\mu=\sqrt{t}\chi(\eta)$, with the addition of $\nu=\sqrt{t}\rho(\eta)$, for $\eta=s/\sqrt{t}$. Transforming (2.15) and (2.16) gives:

$$\frac{\mathrm{i}}{2}\left(\rho-\eta\frac{\mathrm{d}\rho}{\mathrm{d}\eta}\right)-\frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2}+4\frac{\chi+d}{|\chi+d|^2}=0, \tag{2.25}$$

$$\frac{\mathrm{i}}{2}\left(\chi-\eta\frac{\mathrm{d}\chi}{\mathrm{d}\eta}\right)-\frac{\mathrm{d}^2\rho}{\mathrm{d}\eta^2}=0, \tag{2.26}$$

which can be combined by differentiating (2.25) once with respect to η, and substituting (2.26) into this result to obtain:

$$\frac{\mathrm{d}^3\chi}{\mathrm{d}\eta^3}+\frac{\eta^2}{4}\frac{\mathrm{d}\chi}{\mathrm{d}\eta}-\frac{\eta}{4}\chi-4\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\chi+d}{|\chi+d|^2}\right)=0, \tag{2.27}$$

where the additional initial condition on $\chi''(\eta)$ is calculated by the conditions imposed on $\chi(\eta)$, $\chi'(\eta)$, and $\rho'(\eta)$ in (2.25).

We expand the nonlinear interaction term in (2.27) and use the standard perturbation

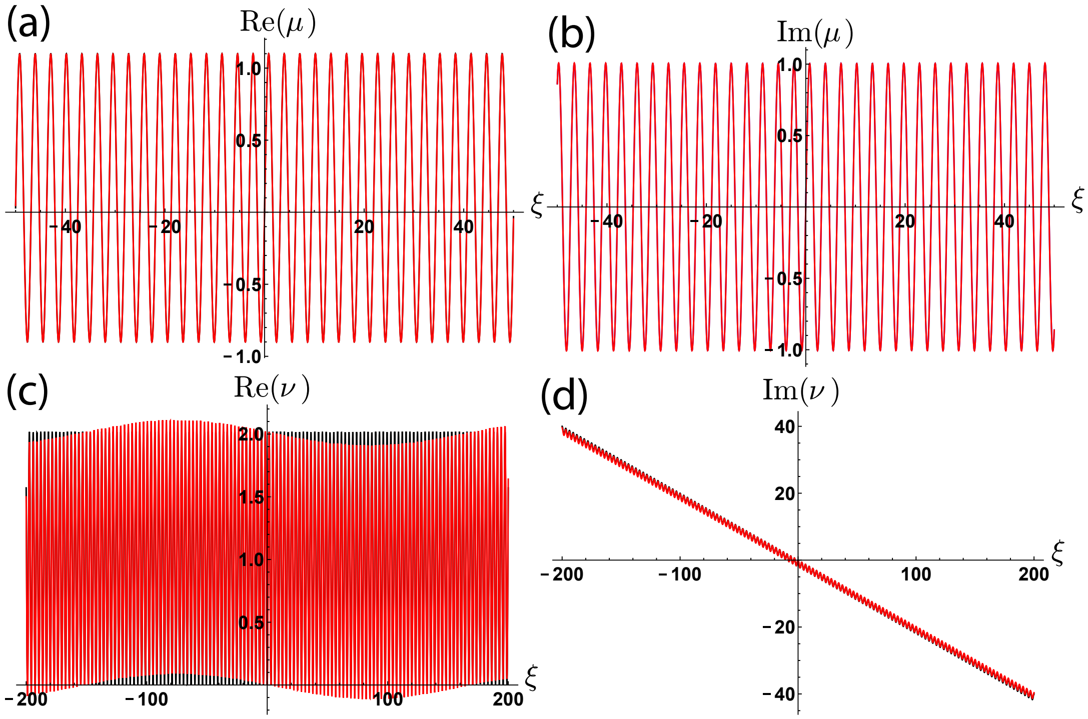

FIGURE 7. Real and imaginary components of $\mu(\xi)$ and $\nu(\xi)$ predicted by the second-order asymptotic correction (2.21) and (2.22) (shown in black) and numerical solutions of (2.18) and (2.20), with the parameters $v = 2$ and $d = 10$ and initial conditions $\mu(\xi = 0) = 0$, $\mu'(\xi = 0) = v + \mathrm{i}v$, $\nu(\xi = 0) = 0$, and $\nu'(\xi = 0) = 0$. (Color online)

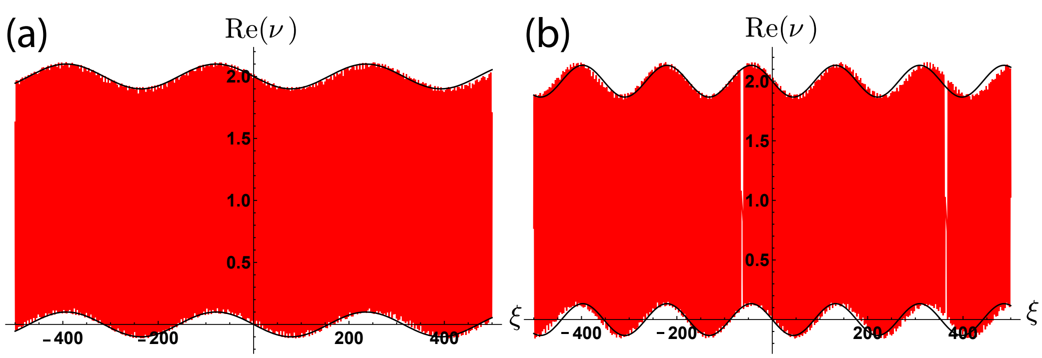

FIGURE 8. Real components of $\nu(\xi)$ predicted by the slowly varying solution (2.24) and (2.23) (shown in black) and the numerical solution of (2.18) and (2.20) (shown in red) for $v = 2$ and (a) $d = 10$ and (b) $d = 7.5$.

expansion for $\chi(\eta) = \sum_{q=0}^{\infty} d^{-q}\chi^{(q)}$ to find:

$$\frac{\mathrm{d}^3\chi^{(0)}}{\mathrm{d}\eta^3} + \frac{\eta^2}{4}\frac{\mathrm{d}\chi^{(0)}}{\mathrm{d}\eta} - \frac{\eta}{4}\frac{\mathrm{d}\chi^{(0)}}{\mathrm{d}\eta} + O\left(\frac{1}{d^2}\right) = 0, \tag{2.28}$$

which gives the first order solution:

$$\mu(t,s) = sc_1 - c_2\left[\sqrt{t}\exp\left(-\frac{\mathrm{i}s^2}{4t}\right) + \frac{(-1)^{\frac{1}{4}}\sqrt{\pi}}{2}\mathrm{serf}\left(\frac{(-1)^{\frac{1}{4}}s}{2\sqrt{t}}\right)\right]$$
$$+ c_3\left[\mathrm{i}\sqrt{t}\exp\left(\frac{\mathrm{i}s^2}{4t}\right) - \frac{(-1)^{\frac{1}{4}}\sqrt{\pi}}{2}\mathrm{serf}\left(\frac{(-1)^{\frac{3}{4}}s}{2\sqrt{t}}\right)\right] + O\left(\frac{1}{d^2}\right). \quad (2.29)$$

The absence of the $O\left(d^{-1}\right)$ vortex interaction term results in no repulsion of the vortex pair, as seen for the co-rotating pair. The interactions between the pair is $O\left(d^{-2}\right)$, so that the shape of the filaments in (2.29) is wholly determined by self-induction. The corresponding shape is similar to the co-rotating case in (2.12); that is, the vortices form conical spirals of increasing amplitude and wavelength. We show the vortex filament shapes obtained by numerically solving (2.25) and (2.26) for boundary conditions on $\mu(t,s)$ satisfying (2.13) and (2.14) with $\rho(\eta=100) = \rho'(\eta=100) = 0$ and $d = 20$ in Fig. 9 for times $t = \{0.05, 0.5, 1, 2\}$. We observe that, as with the co-rotating case, the helices rotate in the opposite direction to the vortex's respective circulation, with $\psi_1(t,s)$ and $\psi_2(t,s)$ having circulations $\Gamma_1 > 0$ and $\Gamma_2 < 0$. The helices are now reflections of each other across the y axis.

We compare the real and imaginary components of the asymptotic solution (2.29) with the numerical solutions for the same conditions in Fig. 10. In this case, the boundary condition necessary to determine the third condition on $\chi(\eta)$ is $\chi''(\eta=100) = 4d^{-1}\left(1+d^{-1}(1-\mathrm{i})\right)^{-1}$ by (2.25). Expanding the condition for $d^{-1} \ll 1$, gives $\chi''(\eta=100) = O\left(d^{-2}\right)$, which we impose to find the zeroth order solution $\chi^{(0)}(\eta)$ and subsequently find the corresponding $\rho^{(0)}(\eta)$ from (2.26). We note that the asymptotic solution shows good agreement to the numerical solutions, but breaks down as longer time scales are considered.

3. Intertwining Vortex Pairs

We now consider the case whereby the distance function $D(t)$ vanishes. The vortices now intertwine and revolve around each other, with $\nu(t,s)$ describing the center of revolution and $\mu(t,s)$ coinciding with the deviations from this center (see Fig. 11). The equations describing the time-evolution of the vortex pair are now given by:

$$\mathrm{i}\frac{\partial\nu}{\partial t} + \frac{1}{2}(1+\Pi)\frac{\partial^2\nu}{\partial s^2} - \frac{1}{2}(1-\Pi)\frac{\partial^2\mu}{\partial s^2} + 2(1-\Pi)\frac{\mu}{|\mu|^2} = 0, \quad (3.1)$$
$$\mathrm{i}\frac{\partial\mu}{\partial t} - \frac{1}{2}(1-\Pi)\frac{\partial^2\nu}{\partial s^2} + \frac{1}{2}(1+\Pi)\frac{\partial^2\mu}{\partial s^2} + 2(1+\Pi)\frac{\mu}{|\mu|^2} = 0. \quad (3.2)$$

3.1. *Co-rotating vortices*

We again study pairs whose vortices have the same circulation $\Pi = 1$. Equations (3.1) and (3.2) decouple in a similar manner as in Section 2.1:

$$\mathrm{i}\frac{\partial\nu}{\partial t} + \frac{\partial^2\nu}{\partial s^2} = 0, \quad (3.3)$$
$$\mathrm{i}\frac{\partial\mu}{\partial t} + \frac{\partial^2\mu}{\partial s^2} + \frac{4\mu}{|\mu|^2} = 0, \quad (3.4)$$

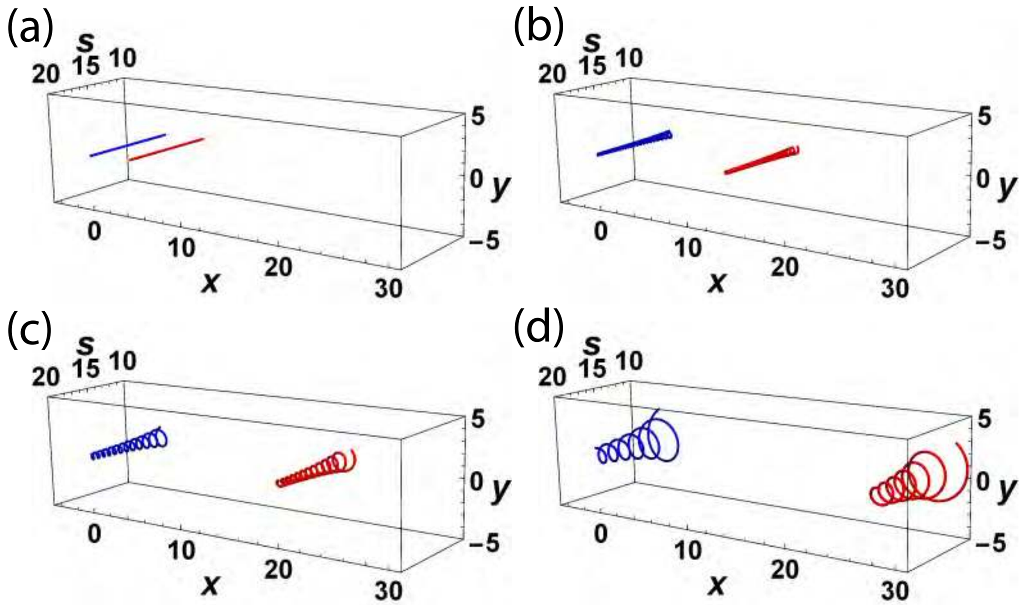

FIGURE 9. The time-evolution of self-similarity solutions in counter-rotating vortices held apart by a distance function $D(t) = d\sqrt{t}$, with $\psi_1(t,s)$ and $\psi_2(t,s)$ being denoted by blue and red respectively. In particular, numerical solutions of (2.25) and (2.26), with conditions satisfying (2.13), (2.14), and $\rho(\eta = 100) = \rho'(\eta = 100) = 0$ have been plotted for $d = 20$ over $s \in [10, 20]$. Times shown are: (a) $t = 0.05$, (b) $t = 0.5$, (c) $t = 1$, and (d) $t = 2$. (Color online)

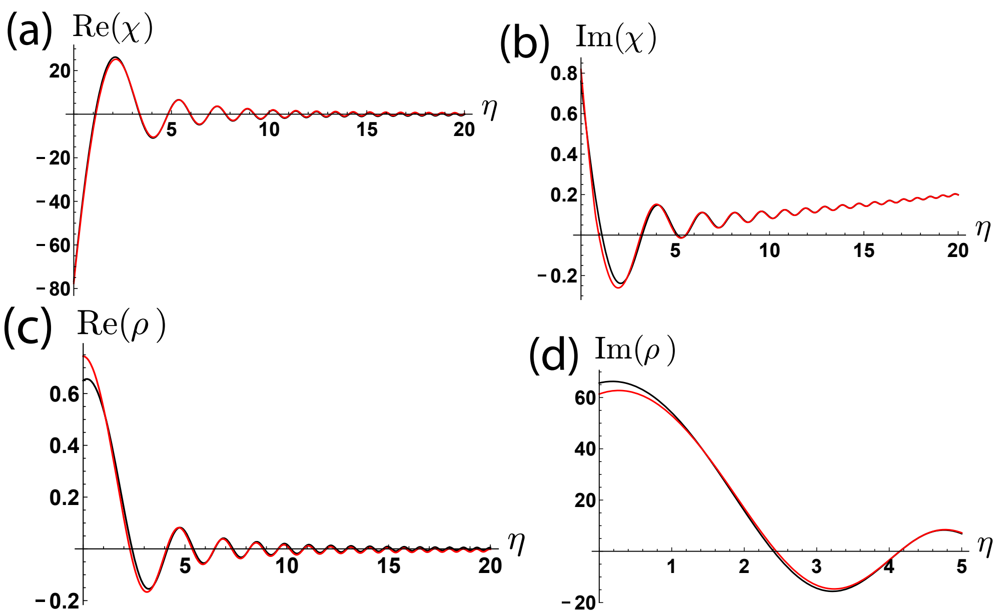

FIGURE 10. Real and imaginary components of $\chi(\eta)$ and $\rho(\eta)$ obtained by the asymptotic solution (2.26) and (2.27) (shown in black) and the numerical solution of (2.25) and (2.26) (shown in red). The initial conditions are $\chi(\eta = 100) = 1 + \mathrm{i}$, $\chi'(\eta = 100) = 1$, $\rho(\eta = 100) = 0$, and $\rho'(\eta = 100) = 0$, with $d = 20$. (Color online)

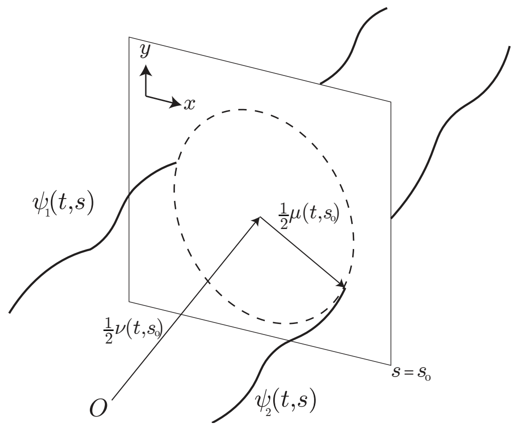

FIGURE 11. Intertwining vortex coordinates $\mu(t,s)$ and $\nu(t,s)$ and how they relate to filaments $\psi_1(t,s)$ and $\psi_2(t,s)$ in the case of no separation function (i.e. $D(t)=0$).

so that, for the following analysis, we will only consider the nonlinear Schrödinger-type equation given by (3.4).

We suppose that $\mu(t,s)$ has an amplitude and phase ansatz given by

$$\mu(t,s) = A(t,s)\exp(\mathrm{i}\theta(t,s)), \tag{3.5}$$

with $A(t,s) \in \mathbb{R}$ measuring the separation of the vortex pair and $\theta(t,s) \in \mathbb{R}$ measuring the orientation of separation from the positive x-axis.

Upon separating (3.4) into real and imaginary components using (3.5), we find:

$$-A\frac{\partial\theta}{\partial t} + \frac{\partial^2 A}{\partial s^2} - A\left(\frac{\partial\theta}{\partial s}\right)^2 + \frac{4}{A} = 0, \tag{3.6}$$

$$\frac{\partial A}{\partial t} + A\frac{\partial^2\theta}{\partial s^2} + 2\frac{\partial A}{\partial s}\frac{\partial\theta}{\partial s} = 0. \tag{3.7}$$

By multiplying (3.7) by $A(t,s)$, we note that it can be written as a conservation law

$$\frac{1}{2}\frac{\partial}{\partial t}\left(A^2\right) + \frac{\partial}{\partial s}\left(A^2\frac{\partial\theta}{\partial s}\right) = 0, \tag{3.8}$$

which is a statement on the evolution of the vortex pair's separation squared A^2 and how it is connected to the gradient of that same separation together with the pair's twist $\partial\theta/\partial s$, along the filament arclength s. For vortices which are straight under some orientation, $\partial\theta/\partial s = 0$ and the separation between vortices at a particular s will not change for all time. However, if the pair are highly coiled, such that $\partial\theta/\partial s \to \infty$, then slight deviations in $A(t,s)$ along the filaments will result in very quick changes in the pair's separation.

3.1.1. *Plane wave solutions*

We consider rotating helix solutions to (3.4) of the form:

$$\mu(t,s) = A_0\exp(\mathrm{i}(ks - \omega t)), \tag{3.9}$$

where A_0 is the constant amplitude of separation, k is the wavenumber related to the twist of the helix, ω is the rotational velocity, and $A_0, k, \omega \in \mathbb{R}$.

Substituting (3.9) into (3.4) gives the dispersion relation:

$$\omega = k^2 - \frac{4}{A_0}, \tag{3.10}$$

a result which has previously been found by Klein *et al.* (1995). However, (3.10) implies that the vortex pair can rotate clockwise, anti-clockwise, or not at all depending on the twist of the helical solution and its amplitude, regardless of the individual circulation of the vortices.

3.1.2. *Traveling wave solutions*

We introduce the same wave coordinate $\xi = s - vt$ seen in the aforementioned analysis, for v being a constant. Transforming relations (3.6) and (3.8), we find:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\xi^2} + vA\frac{\mathrm{d}\theta}{\mathrm{d}\xi} - A\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^2 + \frac{4}{A} = 0, \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(A^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) - \frac{v}{2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(A^2\right) = 0. \tag{3.12}$$

Solutions to these equations will effectively generalise the plane wave or helix solutions of Klein *et al.* (1995) mentioned above, as those solutions correspond to the case where A is constant.

We note that (3.11) and (3.12) allow the exact implicit solution:

$$\xi - \xi_0 = \pm \int \frac{\mathrm{d}A}{\sqrt{2\left(E - V\left(A\right)\right)}}, \tag{3.13}$$

where $\xi_0, E \in \mathbb{R}$ are integration constants, and $V(A)$ is a Hamiltonian potential given by:

$$V(A) = \frac{v^2}{8}A^2 + 4\ln A + \frac{\gamma^2}{2A^2}, \tag{3.14}$$

for $\gamma \in \mathbb{R}$ being another integration constant that is a measure of coupling between the vortex pair's separation and changes in its orientation (see Section 7.1).

We comment on some important aspects of (3.13): The potential function implies that all traveling wave solutions will be attracted to $A \to 0^+$ as $\xi \to \pm\infty$ if $\gamma = 0$, resulting in the collapse of the vortex pair as $t \to \pm\infty$. Coupling between the amplitude of separation and the angular velocity of the pair is necessary to sustain traveling wave solutions for co-rotating pairs. When $\gamma \neq 0$, (3.8) predicts that A will have periodic trajectories such that $A > 0$ for all time. However, for $\gamma \ll 1$, there are three different asymptotic regimes in (3.11) and (3.12) (for full details of the other regimes see Section 7.1), the most noteworthy of which is $A(\xi) = O(\gamma^n)$ for $n > 1$. In this case, there exists an initial layer of width γ^{2n-1} so that $A'' \sim 1/A^3$ and the leading-order behavior is:

$$A\left(\frac{\xi}{\gamma^{2n-1}}\right) \sim \frac{\pm 1}{\sqrt{c_1}}\sqrt{1 + c_1^2 c_2^2 + 2c_1^2 c_2 \frac{\xi}{\gamma^{2n-1}} + c_1^2\frac{\xi^2}{\gamma^{4n-2}}}, \tag{3.15}$$

$$\theta\left(\frac{\xi}{\gamma^{2n-1}}\right) \sim \gamma \tan^{-1}\left(\frac{\xi}{\gamma^{2n-1}}c_1 + c_1 c_2\right) + \frac{v}{2}\frac{\xi}{\gamma^{2n-1}} + \theta_0, \tag{3.16}$$

for $c_1, c_2, \theta_0 \in \mathbb{R}$.

We illustrate the vortex pair by numerically solving (3.11) and (3.12) and compare these results with those from the leading-order behavior and the implicit solution (3.13). In particular, we consider the conditions $A(\xi = 0) = 1$, $A'(\xi = 0) = 0$, and $\theta(\xi = 0) = 0$,

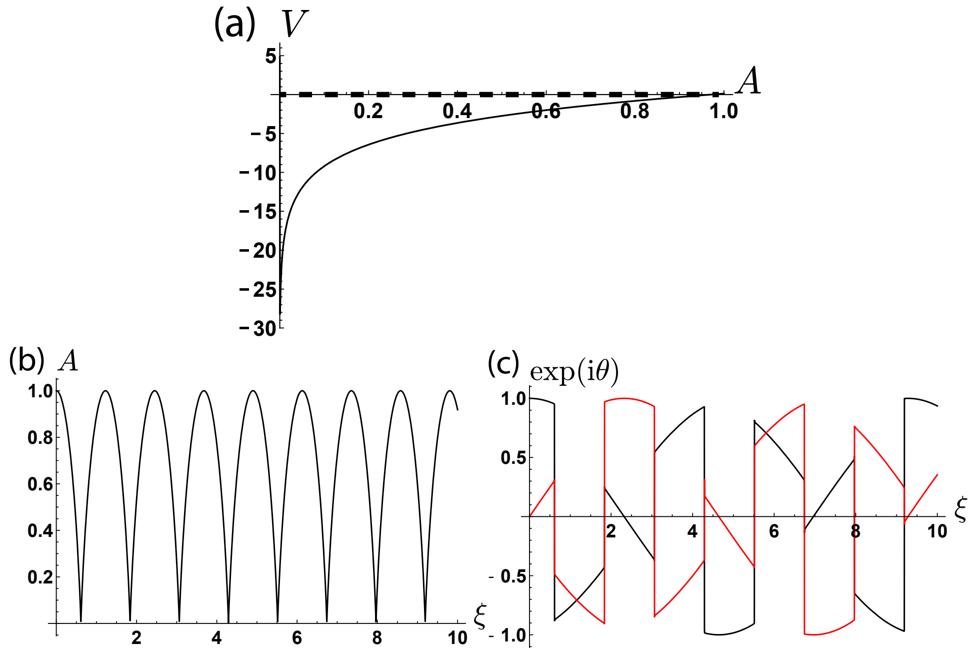

FIGURE 12. (a) Potential of the separation equation as defined in (3.14), and numerical solutions of (b) $A(\xi)$ and (c) the real and imaginary parts of the complex exponential $\exp(i\theta(\xi))$ (shown in black and red respectively). The numerical solutions were obtained by solving (7.1) and (7.2) for $v = 1$ and $\gamma = 0.001$, with initial conditions $A(\xi = 0) = 1$, $A'(\xi = 0) = 0$, and $\theta(\xi = 0) = 0$. The trajectories of $A(\xi)$ are restricted in the potential well such that they never exceed the potential $v^2/8 + \gamma^2/2$, shown by the dashed line, and oscillate with period $T \approx 1.226$ by (3.13). (Color online)

with $\gamma = 0.001$ and $v = 1$. For these parameters, the trajectories of A are such that they never exceed the potential given by $v^2/8 + \gamma^2/2$, according to (3.14), constraining the values of A to the range $A \in [1.17 \times 10^{-5}, 1]$ and, furthermore, the period of oscillation in $A(\xi)$ is $T \approx 1.226$ by solving (3.13). We plot the potential (3.14), corresponding $A(\xi)$, and the real and imaginary components of $\exp(\mathrm{i}\theta(\xi))$ in Fig. 12(a), (b), and (c) respectively as well as the shape of the vortex pair at $t = \{0, 2, 4, 6\}$ with $\nu(t, s) = 0$ (shown in Fig. 13(a)-(d)).

From Fig. 12(b), we observe that the period of oscillation is indeed $T \approx 1.226$, and furthermore note that the numerical solutions show similarities to (3.15) for small $A(\xi)$; in particular, the sharp cusp that was predicted in the regime $A = O(\gamma^n)$ for $n > 1$. As the vortices approach each other, their attraction occurs on a very fast time-scale, for $\gamma \ll 1$, until the pair reaches their minimum separation whereby they repel. The orientation of the vortices at this moment changes almost instantaneously, as $\gamma \tan^{-1}\left(c_1 c_2 + c_1 \xi/\gamma^{2n-1}\right)$ dominates in (3.16). As the vortices repel and $A \to O(1)$, the time-scale slows down and the orientation of the pair evolves according to the $v\xi/2$ term, which now plays a role in (7.6). The vortices reach their maximum separation before attracting each other again and repeating the cycle.

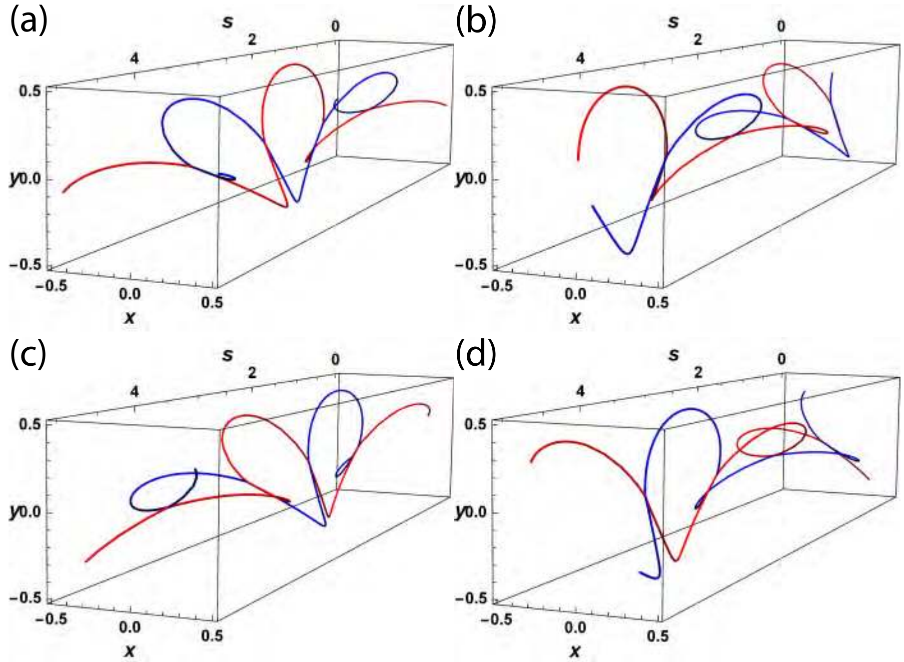

FIGURE 13. Time evolution of traveling wave solutions in intertwining co-rotating vortices, with $\psi_1(t,s)$ and $\psi_2(t,s)$ shown in blue and red respectively. Numerical solutions of (7.1) and (7.2), with $\nu(\xi)=0$, are plotted over $s\in[0,5]$. In particular, the initial conditions are $A(\xi=0)=1$, $A'(\xi=0)=0$, and $\theta(\xi=0)=0$, with parameters $v=1$ and $\gamma=0.001$. Times shown are: (a) $t=0$, (b) $t=2$, (c) $t=4$, and (d) $t=8$. (Color online)

3.1.3. *Purely rotating wave solutions*

We study solutions of the rotating-wave ansatz:

$$A(t,s)=A(s), \tag{3.17}$$

$$\theta(t,s)=-\omega t, \tag{3.18}$$

where ω is the constant angular velocity of the rotating wave. The resulting solutions are each effectively planar filaments, as they maintain their spatial structure while the only motion is pure rotation; see Van Gorder (2015*b*) and references therein.

Substituting this ansatz into (3.6) gives:

$$\frac{\mathrm{d}^2A}{\mathrm{d}s^2}+\omega A+\frac{4}{A}=0, \tag{3.19}$$

which, following the treatment of Section 7.1, has the exact implicit solution:

$$s-s_0=\int\frac{\mathrm{d}A}{\sqrt{2\,(E-V(A))}}, \tag{3.20}$$

where $s_0, E\in\mathbb{R}$ are integration constants and the Hamiltonian potential is given by:

$$V(A)=\frac{\omega A^2}{2}+4\ln A. \tag{3.21}$$

We note that (3.21) is not bounded from below, suggesting that, for all rotating wave solutions with $\omega>0$, $A\to 0^+$ as $s\to\pm\infty$; the vortex pair will collapse if their respective circulations are in the same direction as the pair rotates. If $\omega<0$, then another possibility

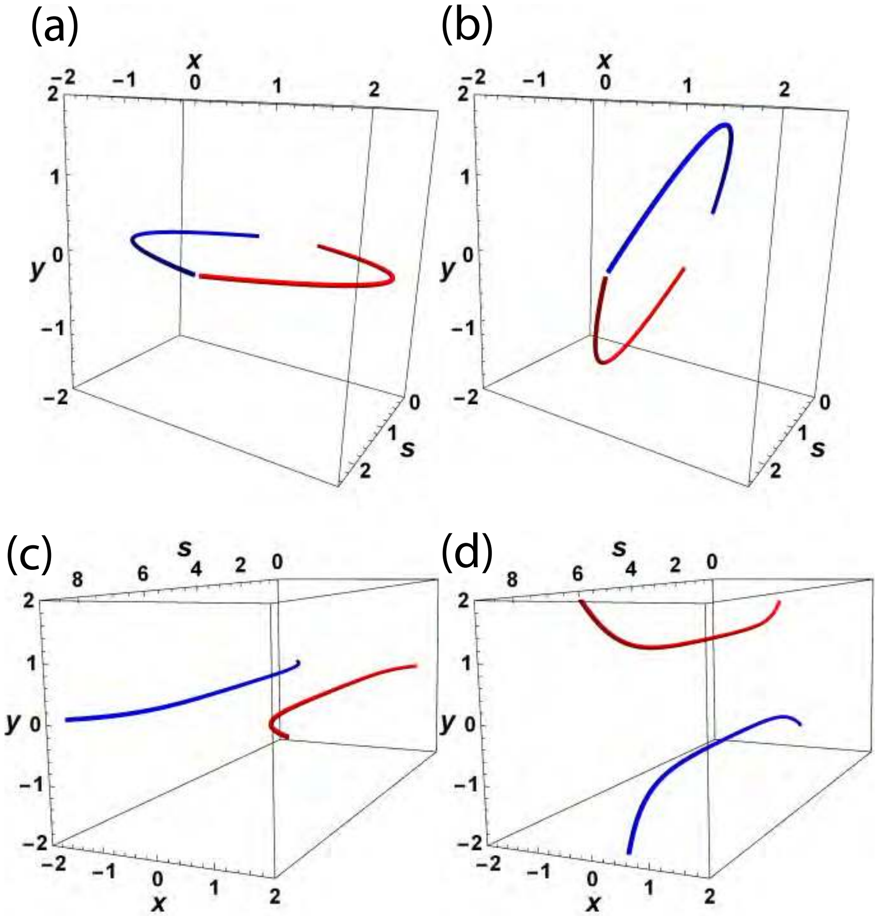

FIGURE 14. Time evolution of planar rotating filament solutions in intertwining co-rotating vortices, with $\psi_1(t,s)$ and $\psi_2(t,s)$ shown in blue and red respectively. Numerical solutions of (3.19), with $\nu(t,s) = 0$, are plotted over $s \in [0,9]$. In particular, the initial conditions and parameters are $A(\xi = 0) = 1$, $A'(\xi = 0) = 5$, and $\omega = 1$ ((a) and (b)) and $A(\xi = 0) = 1$, $A'(\xi = 0) = 1.8$, and $\omega = -1$ ((c) and (d)), both of which are plotted at times $t = \{0, 2\}$. (Color online)

arises, namely, $A \to \infty$ as $s \to \infty$, as the potential is not bound from above. If we consider a vortex pair with circulations in the opposite direction, then rotating pairs with $\omega < 0$ will collapse or, if $\omega > 0$, may also repel without bound.

To illustrate the aforementioned description of the vortex shapes, we plot the numerical solution of (3.19) with $\nu(t,s) = 0$ for the case of $\omega > 0$ and $\omega < 0$. In particular, we consider $\omega = 1$ for the initial conditions $A(s = 0) = 1$ and $A(s = 0) = 5$ (shown in Fig. 14(a) and (b)) and $\omega = -1$ for conditions $A(s = 0) = 1$ and $A(s = 0) = 1.8$, both at times $t = \{0, 2\}$. We note that, as predicted, the solutions for $\omega > 0$ exhibit the eventual collapse of the pair for increasing s, whilst solutions for $\omega < 0$ show the unbounded growth of the pair for increasing s.

3.1.4. *Self-similar solutions*

We transform (3.4) into the variables of $\mu(t,s) = s\chi(\eta)$, with $\eta = s/\sqrt{t}$:

$$\eta^2 \frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2} + \left(2\eta - \frac{\mathrm{i}\eta^3}{2}\right)\frac{\mathrm{d}\chi}{\mathrm{d}\eta} + \frac{4\chi}{|\chi|^2} = 0, \tag{3.22}$$

and use the amplitude-phase ansatz for $\chi(\eta)$ to simplify the vortex interaction term:

$$\chi(\eta) = A(\eta)\, e^{\mathrm{i}\theta(\eta)}, \tag{3.23}$$

which, substituting into (3.22) and separating real and imaginary components, gives:

$$\eta^2 \frac{\mathrm{d}^2 A}{\mathrm{d}\eta^2} - \eta^2 A\left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)^2 + 2\eta\frac{\mathrm{d}A}{\mathrm{d}\eta} + \frac{\eta^3}{2}A\frac{\mathrm{d}\theta}{\mathrm{d}\eta} + \frac{4}{A} = 0, \tag{3.24}$$

$$\eta^2 A\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} + 2\eta^2\frac{\mathrm{d}A}{\mathrm{d}\eta}\frac{\mathrm{d}\theta}{\mathrm{d}\eta} + 2\eta A\frac{\mathrm{d}\theta}{\mathrm{d}\eta} - \frac{\eta^3}{2}\frac{\mathrm{d}A}{\mathrm{d}\eta} = 0. \tag{3.25}$$

By multiplying (3.25) through by $A(\eta)$ and dividing by η^2, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(A^2\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right) + \frac{2}{\eta}A^2\frac{\mathrm{d}\theta}{\mathrm{d}\eta} - \frac{\eta}{4}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(A^2\right) = 0, \tag{3.26}$$

which is a linear equation in $A^2\theta'(\eta)$ that can be integrated to give:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\eta} = \frac{\gamma}{\eta^2 A^2} + \frac{1}{\eta^2 A^2}\int \frac{\eta^3}{4}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(A^2\right)\mathrm{d}\eta, \tag{3.27}$$

where $\gamma \in \mathbb{R}$ is an integration constant. Note that, compared to the traveling wave solutions in Section 3.1.2, γ does not quantify the degree of coupling between the changing orientation of the vortex separation and the amplitude of separation, because if $\gamma = 0$, there is still dependence of $\theta'(\eta)$ on $A(\eta)$.

We numerically solve (3.24) and (3.27) for the boundary conditions $A(\eta = 0.001) = 5$, $A'(\eta = 0.001) = 1$, $\theta(\eta = 0.001) = 0$, $\theta'(\eta = 0.001) = 0$ and plot $A(\eta)$ and $\exp(\mathrm{i}\theta(\eta))$ in Fig. 15(a) and (b) respectively, and the corresponding vortex filament shapes in Fig. 16 for $\nu(t,s) = 0$ at times $t = \{10, 50, 100, 500, 1000, 5000\}$.

We observe a number of interesting aspects: First, given the conditions on $A(\eta)$ and $\theta(\eta)$, we note that $\gamma \neq 0$, which implies that when $A(\eta) \to 0^+$, then $\theta'(\eta) \to \pm\infty$. In a similar manner as in Section 3.1.2 for the case of $\gamma \neq 0$, when the vortex pair come into close proximity, the orientation of separation changes rapidly, as we note in Fig. 15(a) and (b). Furthermore, as $\eta \to 0^+$, corresponding to time $t \to \infty$, the amplitude and the wavelength of the vortex separation becomes larger and the changes in orientation become slower. We note that these results follow directly from the self-similar nature of the vortex pair which exhibit an initial shape (Fig. 16(a)) that becomes increasingly zoomed in (Fig. 16(b)-(f)) for increasing times. As $t \to \infty$, the separation of the vortices themselves will grow to infinity, and their orientation will be planar along a single direction.

3.2. *Counter-rotating vortices*

We now study vortex pairs with opposite circulation, so that $\Pi = -1$. In this case, (3.1) and (3.2) give:

$$\mathrm{i}\frac{\partial\nu}{\partial t} - \frac{\partial^2\mu}{\partial s^2} + \frac{4\mu}{|\mu|^2} = 0, \tag{3.28}$$

$$\mathrm{i}\frac{\partial\mu}{\partial t} - \frac{\partial^2\nu}{\partial s^2} = 0. \tag{3.29}$$

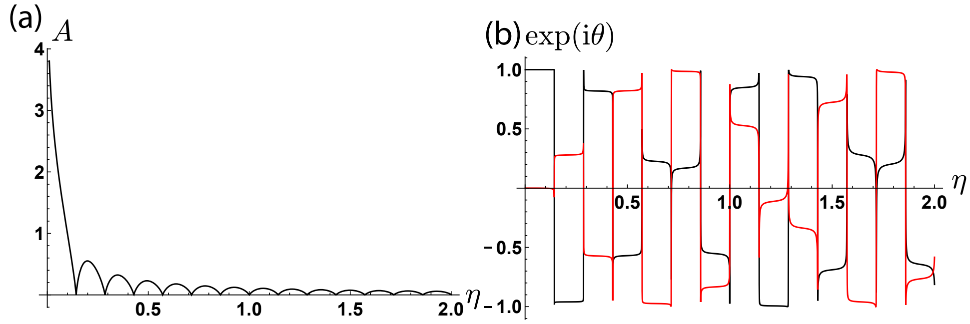

FIGURE 15. Numerical solutions of (a) $A(\eta)$ and (b) real and imaginary parts of the complex exponential $\exp(i\theta(\eta))$ (shown in black and red respectively), obtained by solving (3.24) and (3.27) with initial conditions $A(\eta = 0.001) = 5$, $A'(\eta = 0.001) = 1$, $\theta(\eta = 0.001) = 0$, $\theta'(\eta = 0.001) = 0$. (Color online)

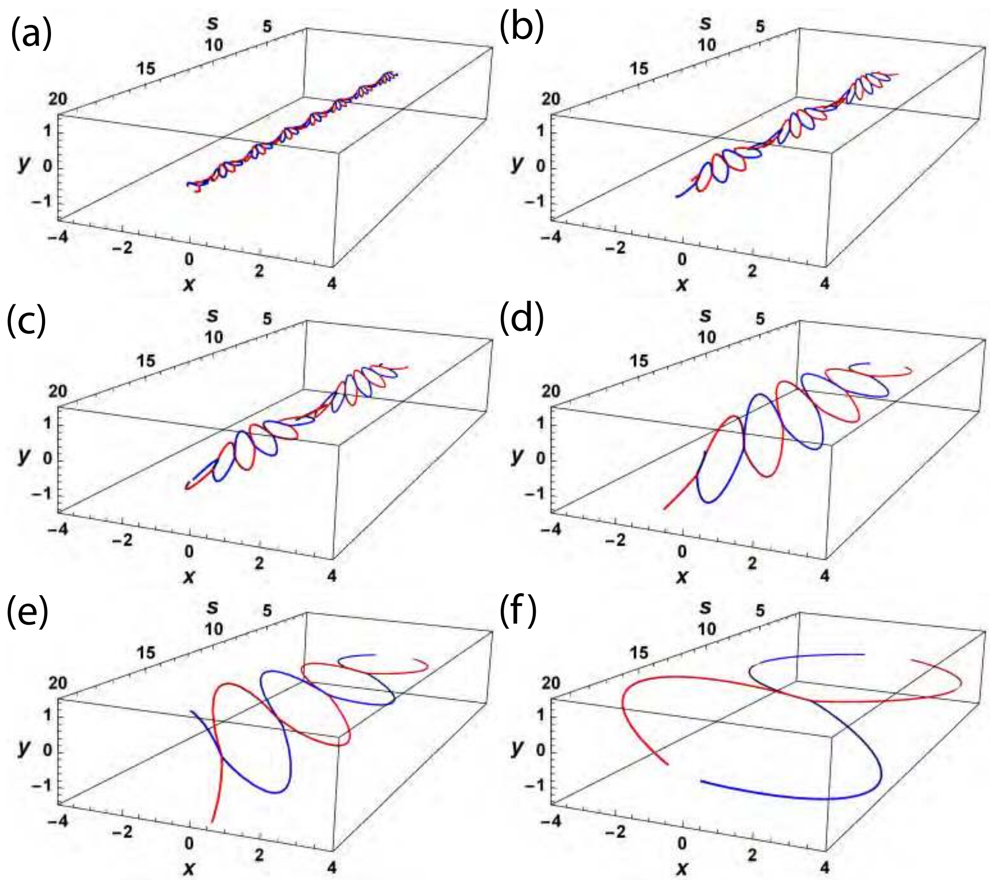

FIGURE 16. Time evolution of self–similar solutions in intertwining co-rotating vortices, with $\psi_1(t,s)$ and $\psi_2(t,s)$ shown in blue and red respectively. Numerical solutions of (3.24) and (3.27), with $\nu(t,s) = 0$, are plotted over $s \in [0.5, 20]$. In particular, the initial conditions are $A(\eta = 0.001) = 5$, $A'(\eta = 0.001) = 1$, $\theta(\eta = 0.001) = 0$, and $\theta'(\eta = 0.001) = 0$. Times shown are: (a) $t = 10$, (b) $t = 50$, (c) $t = 100$, (d) $t = 500$, (e) $t = 1000$, and (f) $t = 5000$. (Color online)

As in Section 2.2, we can rewrite (3.28) and (3.29) as a single partial differential equation for $\mu(t,s)$ by supposing the same differentiability constraints on $\psi_1(t,s)$ and $\psi_2(t,s)$:

$$\frac{\partial^2 \mu}{\partial t^2} + \frac{\partial^4 \mu}{\partial s^4} - 4\frac{\partial^2}{\partial s^2}\left(\frac{\mu}{|\mu|^2}\right) = 0, \tag{3.30}$$

and subsequently calculate $\nu(t,s)$ by solving (3.29).

3.2.1. *Plane wave solutions*

We consider solutions to (3.30) of the form given in (3.9). The vortex center $\nu(t,s)$ is not arbitrary in this case, and is explicitly given by solving (3.29):

$$\nu(t,s) = -\frac{\omega}{k^2} A_0 \exp\left(\mathrm{i}\left(ks - \omega t\right)\right) + c_1 s + c_2, \tag{3.31}$$

for $c_1, c_2 \in \mathbb{C}$ and where A_0, k, and ω have the same meaning as in Section 3.1.1. The center of the pair traces out a helix with the same wavenumber and rotates at the same angular velocity as the separation between the vortex pair, up to an arbitrary orientation determined by c_1 and translation defined by c_2.

Substituting (3.9) into (3.30) gives the dispersion relation:

$$\omega = \pm k\sqrt{k^2 + \frac{4}{A_0^2}}. \tag{3.32}$$

The angular velocity is again determined by the wavenumber k and the amplitude of vortex separation A_0. Similar to Section 3.1.1, ω can be positive, negative, or zero, however, for all nonzero values, the angular velocity can be either clockwise or anti-clockwise rotating, regardless of the parameters imposed on the system. To illustrate, we plot the filament shapes of the vortex pair in Fig. 17 for $A_0 = 1$ with (a) and (b) $k = 1$ with $\omega = -\sqrt{5}$, and (c) and (d) $k = 1$ with $\omega = \sqrt{5}$. The positive root of (3.32) was taken in both cases.

3.2.2. *Traveling wave solutions*

We transform (3.30) into wave coordinate $\xi = s - vt$ and integrate twice with respect to ξ to obtain:

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\xi^2} + v^2\mu - \frac{4\mu}{|\mu|^2} = \delta, \tag{3.33}$$

for $\delta \in \mathbb{C}$, which has constraints:

$$\frac{\mathrm{d}^3\mu}{\mathrm{d}\xi^3} + v^2\frac{\mathrm{d}\mu}{\mathrm{d}\xi} - \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{4\mu}{|\mu|^2}\right) = 0, \tag{3.34}$$

$$v^2\mu - \mathrm{i}v\frac{\mathrm{d}\nu}{\mathrm{d}\xi} = \delta. \tag{3.35}$$

As before, we determine δ by imposing conditions on $\mu(\xi)$ and $\nu'(\xi)$. However, for the forthcoming analysis, we only consider cases whereby $\delta = 0$.

We study solutions to (3.33) of the amplitude-phase ansatz (3.5) and find, upon sepa-

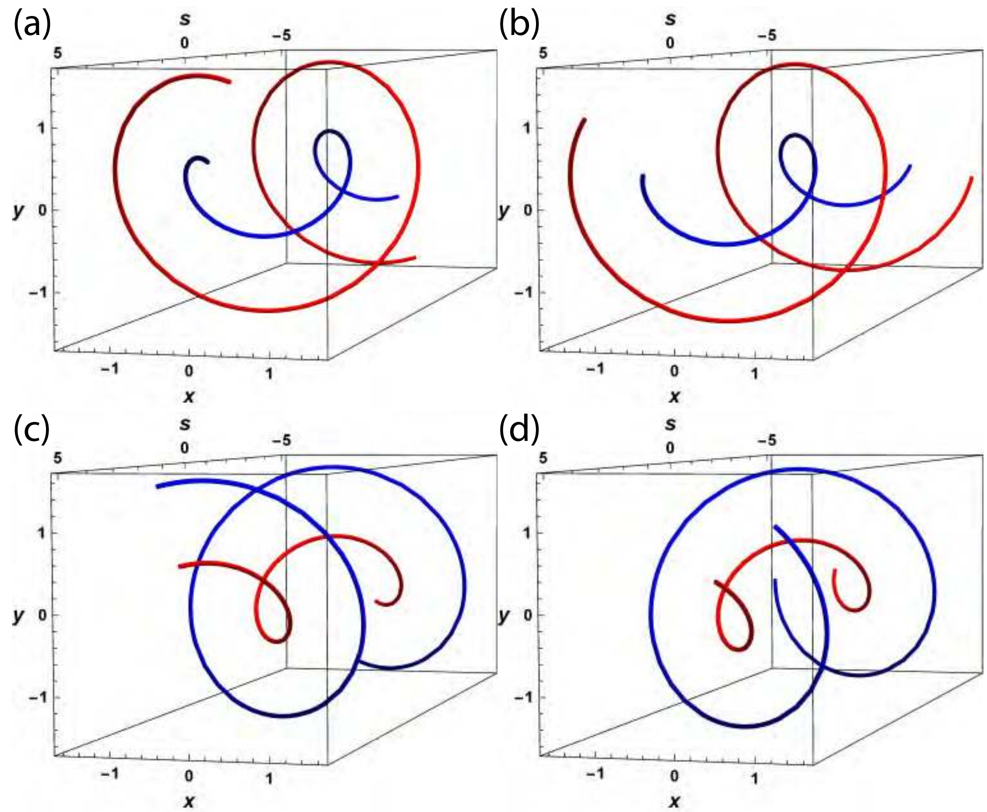

FIGURE 17. Time evolution of plane wave solutions in intertwining counter-rotating vortices, with $\psi_1(t,s)$ and $\psi_2(t,s)$ shown in blue and red respectively. In particular, (3.9) and (3.31), with the dispersion relation determined by (3.32), is plotted over $s \in [-5,5]$. The parameters are $A_0 = 1$ and: (a) and (b) $k=-1$ with $\omega = -\sqrt{5}$, and (c) and (d) $k=1$ with $\omega = \sqrt{5}$. Times shown are: (a) and (c) $t=0$ and (b) and (d) $t=0.5$. (Color online)

rating real and imaginary components:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\xi^2} + v^2 A - A\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)^2 - \frac{4}{A} = 0, \tag{3.36}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(A^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0. \tag{3.37}$$

Solutions to these equations naturally generalise the plane wave helix solutions of the previous subsection, as those solutions correspond to constant A.

We note that (3.36) and (3.37) have an exact solution given by (3.13) with a potential function given by:

$$V(A) = \frac{v^2 A^2}{2} - 4\ln(A) + \frac{\gamma^2}{2A^2}, \tag{3.38}$$

where $\gamma \in \mathbb{R}$ is an integration constant that has the same physical meaning as in Section 3.1.2, measuring the coupling between the vortex pair's separation and its change in orientation.

We note some important differences compared to the co-rotating case: First, if $\gamma = 0$, then the orientation of the counter-rotating pair does not change and the oscillation remains completely planar for all time. For example, vortices which only vibrate in the x

axis will do so for all time. Furthermore, the pair no longer collapses as $t \to \infty$, according to (3.38), as the lower bound of the potential is determined by $-4/A$ in (7.8). For any value of v or γ, the counter-rotating vortices will not meet, however, if the velocity of the traveling wave solution $v = 0$, the pair will repel as $A \to \infty$ for $\xi \to \pm\infty$. If $v \neq 0$, then periodic solutions in A are guaranteed for all values of γ. This aspect leads to the counter-rotating vortices having a different asymptotic structure that no longer depends on γ, unlike what was previously seen for the co-rotating vortices (see Section 7.3 for further details).

We illustrate the potential (3.38) and numerically solve (7.7) and (7.8) for $A(\xi)$ and $\theta(\xi)$ with the conditions $A(\xi = 0) = 2$, $A'(\xi = 0) = 0$, and $\theta(\xi = 0) = 0$, for parameters $v = 2$ and $\gamma = 1$ (shown in Fig. 18(a), (b), and (c) respectively). In this case, the trajectories of A are restricted such that they always remain within the range $A \in [0.494376, 2]$ for all t and s, with period given by $T \approx 1.925$. We note that both Fig. 18(b) and (c) show that the separation of the vortex pair is periodic, which, by (3.35), suggests that the position of the pair's center will periodically oscillate too.

We numerically solve (3.35) to obtain $\nu(\xi)$, imposing the condition $\nu(\xi = 0) = 0$, and plot the associated filament shapes at times $t = \{1, 1.5, 2, 2.5\}$ to obtain Fig. 19(a)-(d) respectively. Furthermore, we show the shape the vortex pair traces on the $x - y$ plane, for $s = 0$ during the time $t \in [0, 25]$, in Fig. 20. We observe that $\psi_2(t, s)$ oscillates in a trefoil knot-like shape whilst $\psi_1(t, s)$ traces out a helical shape that is deformed such that its curvature minima coincide with the maximal tips of the trefoil shape; a result that arises due to the coupling of $\mu(t, s)$ and $\nu(t, s)$, which causes the center of the vortex pair to form the periodically deformed helical shape for all time and filament arclength.

3.2.3. *Purely rotating wave solutions*

We study solutions of the rotating-wave ansatz of (3.5), (3.17), and (3.18) and apply it to (3.30) to obtain

$$\frac{\mathrm{d}^4 A}{\mathrm{d}s^4} - \omega^2 A - \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{4}{A}\right) = 0, \tag{3.39}$$

where the general solution to $\nu(t, s)$ is given by a combination (3.28) and (3.29) as

$$\nu(t, s) = \frac{1}{\omega}\left(\frac{\mathrm{d}^2 A}{\mathrm{d}s^2} - \frac{4}{A}\right)\exp(-\mathrm{i}\omega t) + c_1 s + c_2, \tag{3.40}$$

for $c_1, c_2 \in \mathbb{C}$. Again, the solutions we obtain will correspond to planar vortex filaments which maintain their spatial form as the only motion is purely rotational.

We observe that, with the characteristic coupling between the deviations and the center of the counter-rotating vortices, the pair not only rotate around each other, but also around a center point in the fluid, determined by the conditions on $\nu(t, s)$.

We numerically solve (3.39) and (3.40) with boundary conditions given by $A(s = 0) = 2$, $A(s = 10) = 2$, $A'(s = 0) = 0$, $A'(s = 10) = 0$, and $\nu(t, 0) = \partial\nu(t, 0)/\partial s = 0$, with $\omega = 0.5$ and $c_1 = c_2 = 0$. We plot $\mu(t, s)$ and $\nu(t, s)$ at $t = 0$ in Fig. 21(a) and (b), and the corresponding filament shapes over $s \in [0, 10]$ at times $t = \{2.5, 7.5\}$ in Fig. 21(c) and (d). We observe that, for the boundary conditions chosen, the separation of the vortex pair is oscillatory along the arc-length (Fig. 21(a)), a solution not possible for the co-rotating case seen in Section 3.1.3. By (3.40), $\nu(t, s)$ is also bounded and non-zero (Fig. 21(b)). The filament shapes are therefore asymmetrical when compared to the symmetrical co-rotating vortices (Fig. 21); a direct result of the coupling of $\mu(t, s)$ and $\nu(t, s)$ in counter-rotating vortices.

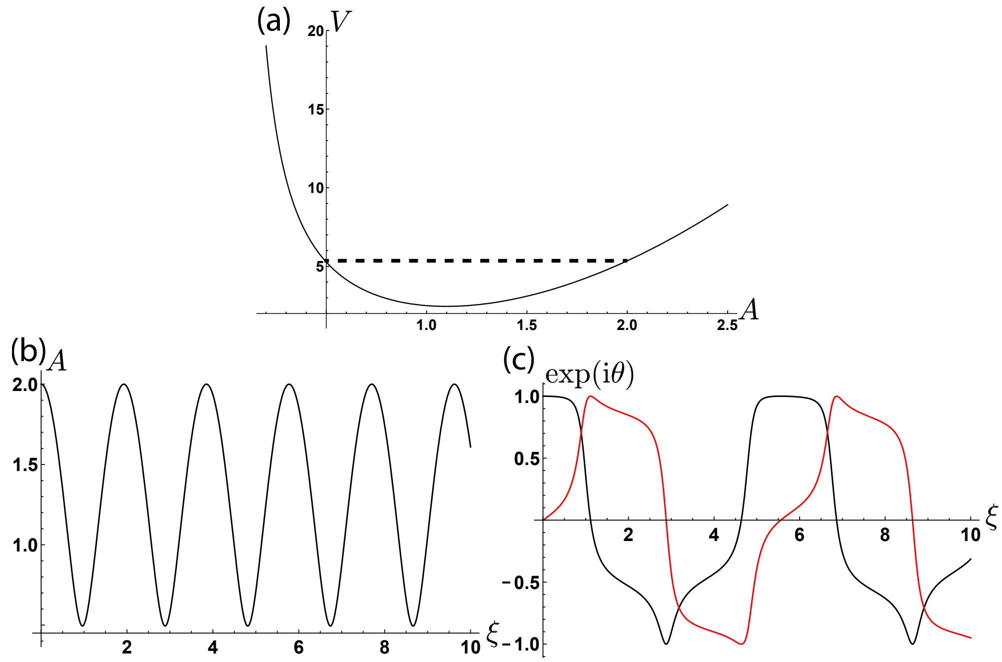

FIGURE 18. (a) Potential of the separation equation as defined in (3.38), and numerical solutions of (b) $A(\xi)$, (c) real and imaginary parts of the complex exponential $\exp(i\theta(\xi))$ (shown in black and red respectively) obtained by solving (3.37) and (3.36) for $v = 2$, $\gamma = 1$, and $\delta = 0$ with initial conditions $A(\xi = 0) = 2$, $A'(\xi = 0) = 0$, and $\theta(\xi = 0) = 0$. The trajectories of A are restricted in the potential well such that they never exceed the potential $2v^2/8 - 4\ln(2) + \gamma^2/8$, shown by the dashed line, and oscillate with period $T \approx 1.925$ according to (3.13). (Color online)

3.2.4. *Self-similar solutions*

We transform (3.28) and (3.29) into the variables $\mu(t, s) = s\chi(\eta)$ and $\nu(t, s) = s\rho(\eta)$, for $\eta = s/\sqrt{t}$, to find:

$$-\frac{\mathrm{i}\eta^3}{2}\frac{\mathrm{d}\rho}{\mathrm{d}\eta} - \eta^2\frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2} - 2\eta\frac{\mathrm{d}\chi}{\mathrm{d}\eta} + \frac{4\chi}{|\chi|^2} = 0, \tag{3.41}$$

$$-\frac{\mathrm{i}\eta^3}{2}\frac{\mathrm{d}\chi}{\mathrm{d}\eta} - \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2\frac{\mathrm{d}\rho}{\mathrm{d}\eta}\right) = 0, \tag{3.42}$$

and a single equation for $\chi(\eta)$ upon substituting (3.41) into (3.42):

$$\eta\frac{\mathrm{d}^3\chi}{\mathrm{d}\eta^3} + 3\frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2} + \frac{\eta^3}{4}\frac{\mathrm{d}\chi}{\mathrm{d}\eta} - \frac{4}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\chi}{|\chi|^2}\right) + \frac{4}{\eta^2}\left(\frac{\chi}{|\chi|^2}\right) = 0, \tag{3.43}$$

with the additional initial condition $\chi''(\eta)$ obtained from (3.41) using conditions on $\chi(\eta)$, $\chi'(\eta)$, and $\rho'(\eta)$.

We study solutions to $\chi(\eta)$ of the amplitude-phase ansatz in (3.23) and find, upon

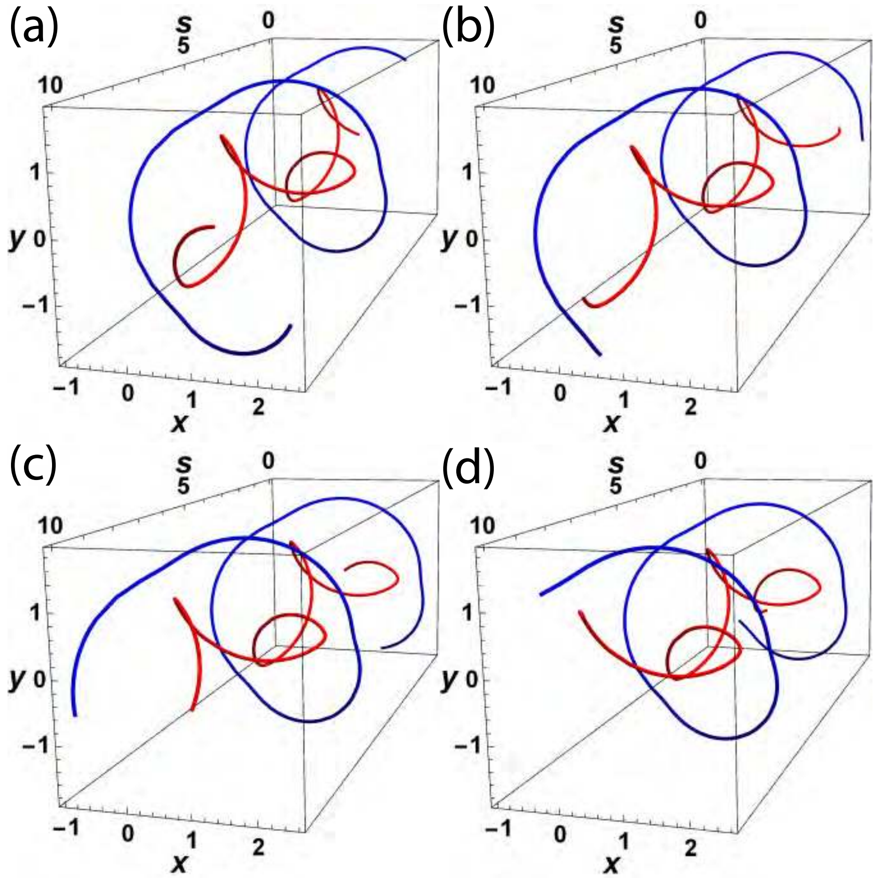

FIGURE 19. Time evolution of traveling wave solutions in intertwining counter-rotating vortices, with $\psi_1(t,s)$ and $\psi_2(t,s)$ shown in blue and red respectively. Numerical solutions of (3.35), (3.37), and (3.36) are plotted over $s \in [0,10]$. In particular, the initial conditions are $A(\xi=0)=2$, $A'(\xi=0)=0$, $\theta(\xi=0)=0$, and $\nu(\xi=0)=0$ with parameters $v=2$, $\gamma=1$, and $\delta=0$. Times shown are: (a) $t=0$, (b) $t=2$, (c) $t=4$, and (d) $t=8$. (Color online)

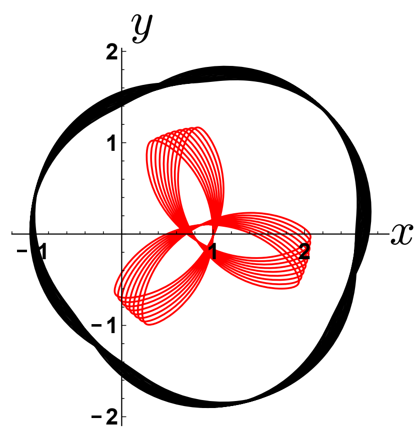

FIGURE 20. The shape that the filament pair in Fig. 19 traces out in the $x-y$ plane during the time $t \in [0,25]$ at $s=0$, with $\psi_1(t,s)$ and $\psi_2(t,s)$ corresponding to black and red lines respectively. (Color online)

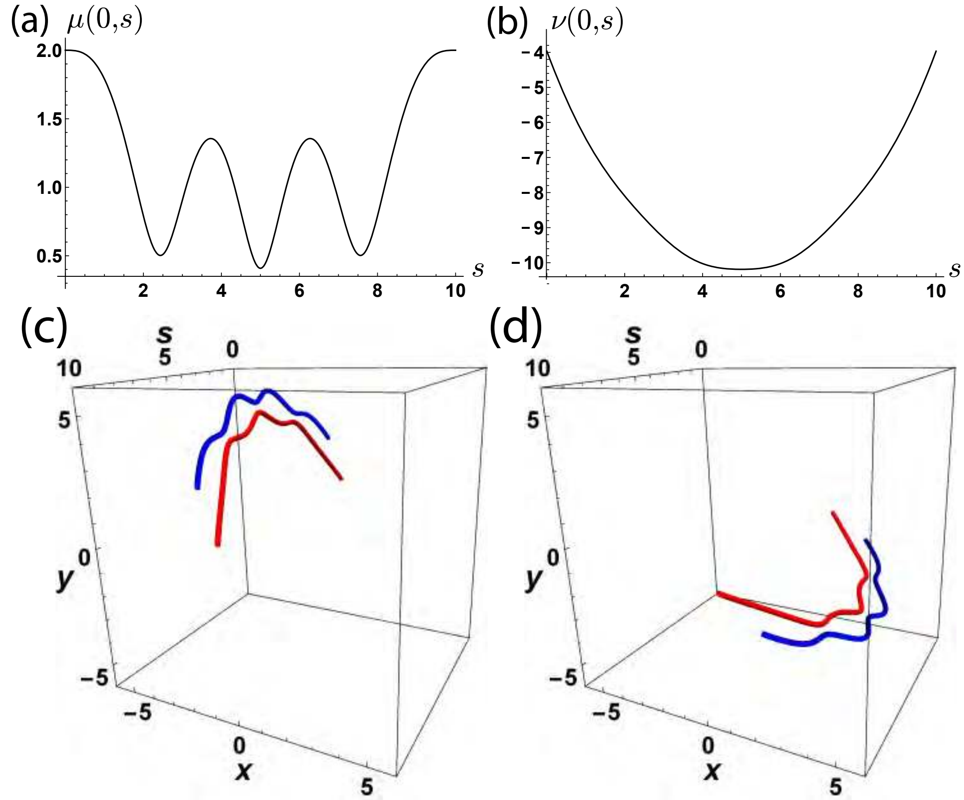

FIGURE 21. Numerical solutions of (3.39) and (3.40) at $t = 0$ (shown in (a) and (b)) for the conditions $A\,(s=0) = 2$, $A\,(s=10) = 2$, $A'\,(s=0) = 0$, $A'\,(s=10) = 0$, and $\nu\,(t,0) = \partial\nu\,(t,0)\,/\partial s = 0$, with $\omega = 0.5$ and $c_1 = c_2 = 0$. The corresponding filament shapes of $\psi_1\,(t,s)$ (shown in blue) and $\psi_2\,(t,s)$ (shown in red) are plotted over $s \in [0,10]$ for times (c) $t = 2.5$ and (d) $t = 7.5$. The solutions clearly take on a planar filament structure. (Color online)

substitution into (3.43) and separating real and imaginary parts:

$$\eta\frac{\mathrm{d}^3A}{\mathrm{d}\eta^3} - 3\eta A\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} - 3\eta\frac{\mathrm{d}A}{\mathrm{d}\eta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)^2 + 3\frac{\mathrm{d}^2A}{\mathrm{d}\eta^2} - 3A\left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)^2 + \frac{\eta^3}{4}\frac{\mathrm{d}A}{\mathrm{d}\eta} + \frac{4}{\eta A^2}\frac{\mathrm{d}A}{\mathrm{d}\eta} + \frac{4}{\eta^2 A} = 0, \tag{3.44}$$

$$\eta A\frac{\mathrm{d}^3\theta}{\mathrm{d}\eta^3} + 3\eta\frac{\mathrm{d}A}{\mathrm{d}\eta}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} + 3\eta\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\frac{\mathrm{d}^2A}{\mathrm{d}\eta^2} - \eta A\left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)^3 + 6\frac{\mathrm{d}A}{\mathrm{d}\eta}\frac{\mathrm{d}\theta}{\mathrm{d}\eta} + 3A\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} + \frac{\eta^3 A}{4}\frac{\mathrm{d}\theta}{\mathrm{d}\eta} - \frac{4\eta}{A}\frac{\mathrm{d}\theta}{\mathrm{d}\eta} = 0. \tag{3.45}$$

To obtain physical intuition into the types of solutions predicted by (3.43), we numerically solve (3.44) and (3.45) with the conditions $A\,(\eta=3) = 1$, $A'\,(\eta=3) = 0$, $A''\,(\eta=3) = 0$, $\theta\,(\eta=3) = 0$, $\theta'\,(\eta=3) = 1$, and $\theta''\,(\eta=1) = -5$, and plot the real and imaginary parts of $\chi\,(\eta)$ and $\rho\,(\eta)$, using the condition $\rho\,(\eta=3) = 0$, in Fig. 22(a), (b), and (c), (d) respectively. Furthermore, we illustrate the corresponding filament shapes over the arclength $s \in [5,15]$ at times $t = \{0.1, 1, 2, 2.5\}$ in Fig. 23. We observe some parallels to self-similar solutions in other vortex configurations (i.e. Section 2.1.2, 2.2.2, and 3.1.4); namely, the progression of the vortex pair's shape from a straight filament in some orientation to oscillatory solutions with increasing amplitude and wavelength as $t \to \infty$. However, in contrast with the self-similar solutions in Section 3.1.4, the vortices

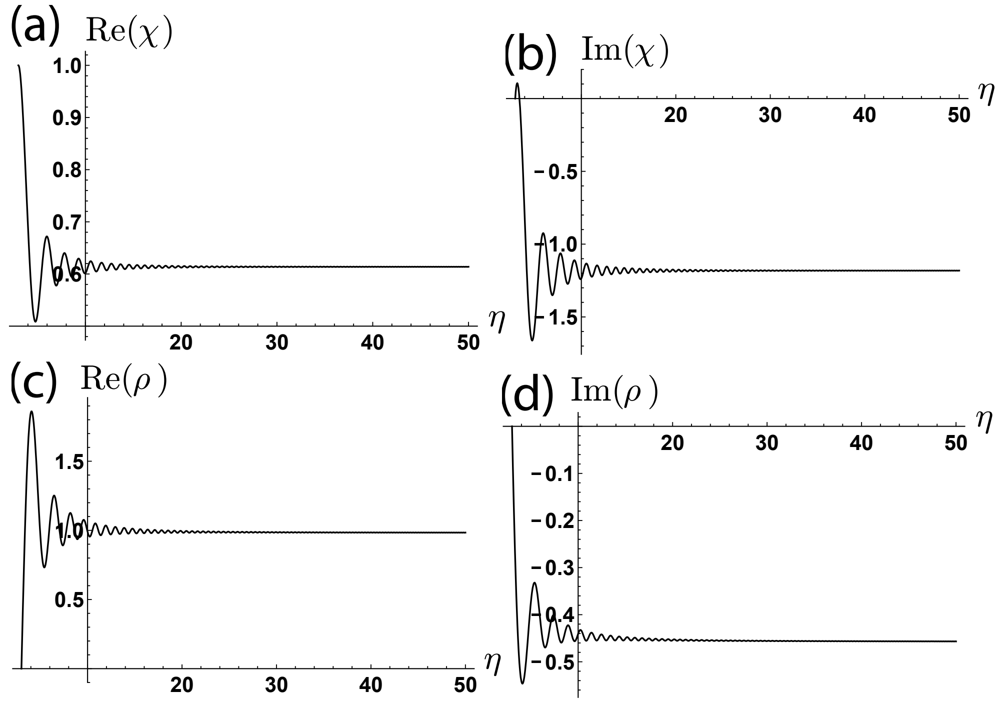

FIGURE 22. Real and imaginary components of $\chi(\eta)$ and $\rho(\eta)$ calculated by the numerical solution of (3.44), (3.45), and (3.42). The initial conditions are $A(\eta = 3) = 1$, $A'(\eta = 3) = 0$, $A''(\eta = 3) = 0$, $\theta(\eta = 3) = 0$, $\theta'(\eta = 3) = 1$, $\theta''(\eta = 3) = -5$, and $\rho(\eta = 3) = 0$.

are never in very close proximity, so that the orientation of the pair does not rapidly change according to the dependence of $1/A$ in (3.44). The vortex pair remains separated due to the constant offset in $\rho(\eta)$ in Fig. 22(c) and (d), but as $t \to \infty$, the center of the vortex pair moves such that the filaments begin to intertwine and rotate around each other (Fig. 23(d)).

4. Co-Rotating Vortex Hierarchy

We generalize the intertwining vortex coordinates introduced in Section 3 to study N co-rotating vortices around a central filament, all of which have the same size α and circulation Γ. We denote the central filament by $\psi_0(t, s) \in \mathbb{C}$ and the satellite vortices that revolve around this central filament as $\psi_n(t, s) \in \mathbb{C}$ for $n \in \{1, ..., N\}$. We transform $\psi_0(t, s)$ and $\psi_n(t, s)$ into coordinates describing the center of the vortex hierarchy and the deviations from this center by defining new variables $\nu(t, s)$ and $\mu_n(t, s)$, which satisfy:

$$\nu = \sum_{j=0}^{N} \psi_j, \tag{4.1}$$

$$\mu_n = \psi_0 - \psi_n, \tag{4.2}$$

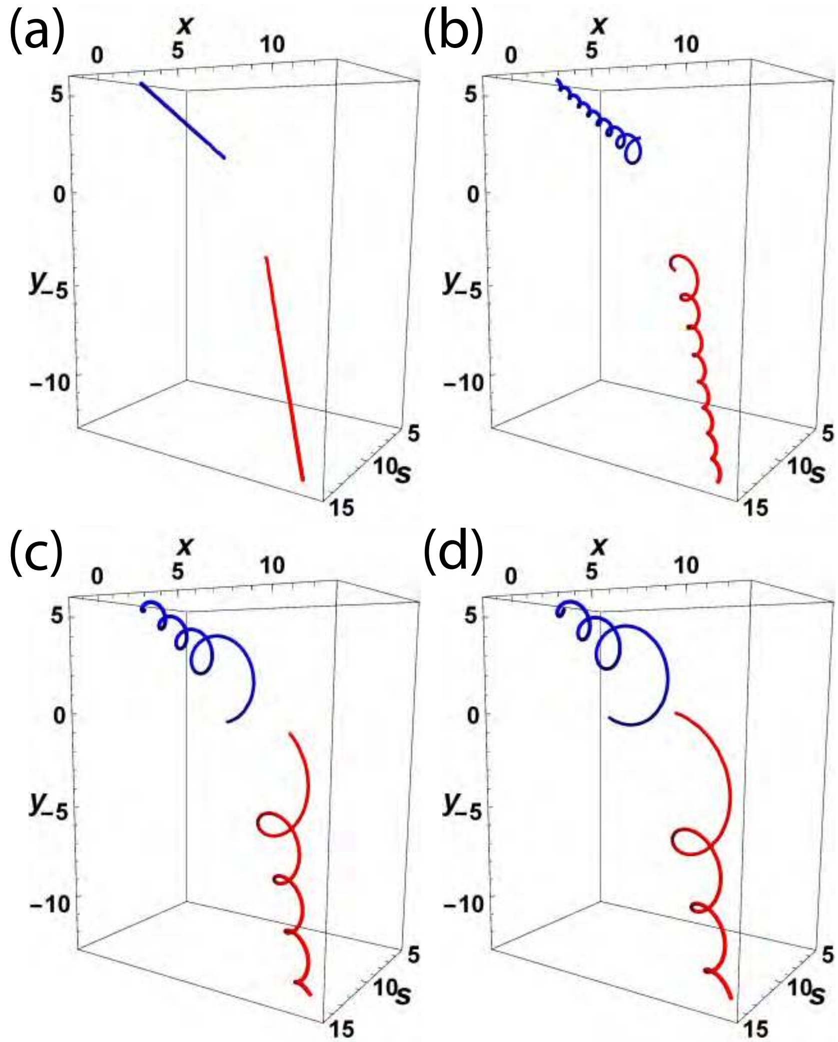

FIGURE 23. Time evolution of self-similar solutions in intertwining counter-rotating vortices, with $\psi_1(t,s)$ and $\psi_2(t,s)$ shown in blue and red respectively. Numerical solutions of (3.44), (3.45), and (3.42) are plotted over $s \in [5,15]$ with the initial conditions $A(\eta=3)=1$, $A'(\eta=3)=0$, $A''(\eta=3)=0$, $\theta(\eta=3)=0$, $\theta'(\eta=3)=1$, $\theta''(\eta=3)=-5$, and $\rho(\eta=3)=0$. Times shown are: (a) $t=0.1$, (b) $t=1$, (c) $t=2$, and (d) $t=2.5$. (Color online)

so that:

$$\psi_0 = \frac{1}{N+1}\left(\nu + \sum_{j=1}^{N}\mu_j\right), \tag{4.3}$$

$$\psi_n = \frac{1}{N+1}\left(\nu - N\mu_n + \sum_{j=1}^{N}\mu_j\right). \tag{4.4}$$

Using (4.3) and (4.4), we obtain the following system of $N+1$ equations from (1.1):

$$i\frac{\partial \nu}{\partial \tau}+\alpha\Gamma\frac{\partial^2 \nu}{\partial S^2}=0, \tag{4.5}$$

$$i\frac{\partial \mu_n}{\partial t}+\alpha\Gamma\frac{\partial^2 \mu_n}{\partial s^2}+\frac{4\Gamma\mu_n}{|\mu_n|^2}+2\Gamma\sum_{j\neq n}\left[\frac{\mu_j}{|\mu_j|^2}+\frac{\mu_n-\mu_j}{|\mu_n-\mu_j|^2}\right]=0. \tag{4.6}$$

The third term in (4.6) describes the effect of interactions between the satellite vortex $\psi_n(t,s)$ and the central filament $\psi_0(t,s)$. However there are additional vortex interactions which play a role in the shaping of an individual filament; namely, those occurring between the central filament and the other satellites (i.e. fourth term) as well as the interactions between the satellite vortices themselves (i.e. fifth term).

4.1. *Plane wave solutions*

To determine the role of these other interactions, we consider plane wave solutions to (4.6) of the form:

$$\mu_n = B_n \exp\left(i\left(k_n s-\omega_n t+\theta_n\right)\right), \tag{4.7}$$

where B_n is the amplitude of the separation between the central and satellite filaments, k_n is the wavenumber of oscillation, ω_n is its angular velocity, and θ_n is a constant phase off-set specifying the initial angular separation from the positive x axis, all of which are constant and real for $n\in\{1,...,N\}$.

For the case of $B_n=B$, $k_n=k$, $\omega_n=\omega$, and the vortices being evenly distributed around the center, we obtain the following dispersion relation (see Section 8):

$$\omega=\alpha\Gamma k^2-\frac{\Gamma}{B^2}\left(N+1\right), \tag{4.8}$$

for $N\geq 2$.

Setting $\alpha=\Gamma=1$, we note that the angular velocity of the vortex configuration depends on the wavenumber k, the distance of the satellite vortices from the central filament B, and the number of satellite vortices N. Even though we have not specified the direction in which individual vortices rotate (i.e. we have only set their circulations relative to one another and we have not rescaled time), the vortices can rotate clockwise, anti-clockwise, or not at all, depending only on the aforementioned parameters. The inclusion of more satellite vortices will lead to the system rotating in a clockwise direction with increasing angular velocity.

It is interesting to note that (4.8) has explicit dependence on the vortex core size, which is absent from the dispersion relations found in Okulov (2004) and Boersma & Wood (1999), who consider infinitesimally thin vortex filaments. However, their solutions would be valid for larger values of wavenumber k

We determine the effect of perturbations to the plane wave solutions for the simplest case of $N=2$. In particular, we use the ansatz:

$$\mu_1(t,s)=B\exp\left(\mathrm{i}\left(ks-\omega t\right)\right)+\epsilon\lambda_1(t,s), \tag{4.9}$$

$$\mu_2(t,s)=B\exp\left(\mathrm{i}\left(ks-\omega t+\pi\right)\right)+\epsilon\lambda_2(t,s), \tag{4.10}$$

for $\lambda_1,\lambda_2\in\mathbb{C}$ and $\epsilon\ll 1$.

Substituting (4.9) and (4.10) into (4.6), we find that the $O\left(\epsilon^0\right)$ terms give the dispersion relation (4.8) for $N=2$, whilst to $O\left(\epsilon^{1}\right)$:

$$\mathrm{i}\frac{\partial \lambda_1}{\partial t}+\alpha\Gamma\frac{\partial^2 \lambda_1}{\partial s_1}-\frac{\Gamma}{2B^2}\left(9\lambda_1^*+3\lambda_2^*\right)\exp\left(2\mathrm{i}\left(ks-\omega t\right)\right)=0, \tag{4.11}$$

$$\mathrm{i}\frac{\partial \lambda_2}{\partial t}+\alpha\Gamma\frac{\partial^2 \lambda_2}{\partial s_1}-\frac{\Gamma}{2B^2}\left(9\lambda_2^*+3\lambda_1^*\right)\exp\left(2\mathrm{i}\left(ks-\omega t\right)\right)=0. \tag{4.12}$$

We illustrate that such vortex hierarchies are unstable to perturbations. To do so, we numerically solve (4.6) for $\nu\left(t,s\right)=0$ over $s\in\left[-5,5\right]$ for two satellite vortices orbiting as plane-waves around a straight central filament. We impose initial conditions $\psi_0\left(t=0,s\right)=0$, $\psi_1\left(t=0,s\right)=\exp\left(\mathrm{i}\left(s/2+0.999\pi\right)\right)$, and $\psi_2\left(t=0,s\right)=\exp\left(\mathrm{i}\left(s/2+2\pi\right)\right)$, so that the satellite vortices are evenly distributed around the center, except for $\psi_1\left(t,s\right)$, which has a 0.001 perturbation in angular displacement. To define the boundary conditions, we introduce the position vector of the filament $\psi_j\left(t,s\right)$ in the $x-y$ plane as $\boldsymbol{r}_j=\left(x_j,y_j\right)$, and demand $\boldsymbol{r}_j\cdot\left(\partial\boldsymbol{r}_j/\partial s\right)=0$ and $\left|\partial\boldsymbol{r}_j/\partial s\right|^2=B^2k^2=1/4$, at $s=\{-5,5\}$ for $j\in\{1,2\}$. These conditions correspond to the satellite vortices being able to freely rotate on a circle, with the circle radius satisfying the second condition for all time. The central filament $\psi_0\left(t,s\right)$ is pinned so that $\psi_0\left(t,s=5\right)=\psi_0'\left(t,s=5\right)=0$. We plot the numerical solutions in Fig. 24 for times $t=\{0,1.7,1.9,2.1,2.3,2.4\}$.

We note that the central filament exhibits an instability, opposite to where it is pinned, at $t=1.7$ (Fig. 24(b)), which grows and spirals inwards through the hierarchy. In response, small wavelength perturbations develop in the satellite vortices (Fig. 24(d)) which grow without bound leading to the eventual collapse of the vortices. It is worth noting that such a hierarchy with two vortices is similarly unstable for this particular wavenumber k without a central vortex, as shown in Okulov (2004).

4.2. *Traveling wave solutions*

To generalise the plane wave solution dynamics to the case of non-constant amplitudes, we now consider traveling wave solutions that exist for $N=2$ satellite vortices around a central vortex. We consider wave coordinates $\xi=s-vt$, for constant v, and suppose $\mu_1\left(\xi\right)$ and $\mu_2\left(\xi\right)$ have the ansatz

$$\mu_j\left(\xi\right)=A_j\left(\xi\right)\exp\left(\mathrm{i}\theta_j\left(\xi\right)\right), \tag{4.13}$$

for $A_j,\theta_j\in\mathbb{R}$ and $j=\{1,2\}$.

We transform (4.6) into ξ coordinates and substitute (4.13) into the result to obtain, upon taking real and imaginary parts:

$$\begin{aligned}\alpha\Gamma\frac{\mathrm{d}^2A_1}{\mathrm{d}\xi^2}+vA_1\frac{\mathrm{d}\theta_1}{\mathrm{d}\xi}-\alpha\Gamma A_1\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}\xi}\right)^2+\frac{4\Gamma}{A_1}&\\+\frac{2\Gamma}{A_2}\cos\left(\theta_2-\theta_1\right)+\frac{2\Gamma\left(A_1-A_2\cos\left(\theta_2-\theta_1\right)\right)}{A_1^2-2A_1A_2\cos\left(\theta_2-\theta_1\right)+A_2^2}&=0,\end{aligned} \tag{4.14}$$

$$\begin{aligned}\alpha\Gamma A_1\frac{\mathrm{d}^2\theta_1}{\mathrm{d}\xi^2}+2\alpha\Gamma\frac{\mathrm{d}A_1}{\mathrm{d}\xi}\frac{\mathrm{d}\theta_1}{\mathrm{d}\xi}-v\frac{\mathrm{d}A_1}{\mathrm{d}\xi}&\\+\frac{2\Gamma}{A_2}\sin\left(\theta_2-\theta_1\right)+\frac{2\Gamma A_2\sin\left(\theta_2-\theta_1\right)}{A_1^2-2A_1A_2\cos\left(\theta_2-\theta_1\right)+A_2^2}&=0,\end{aligned} \tag{4.15}$$

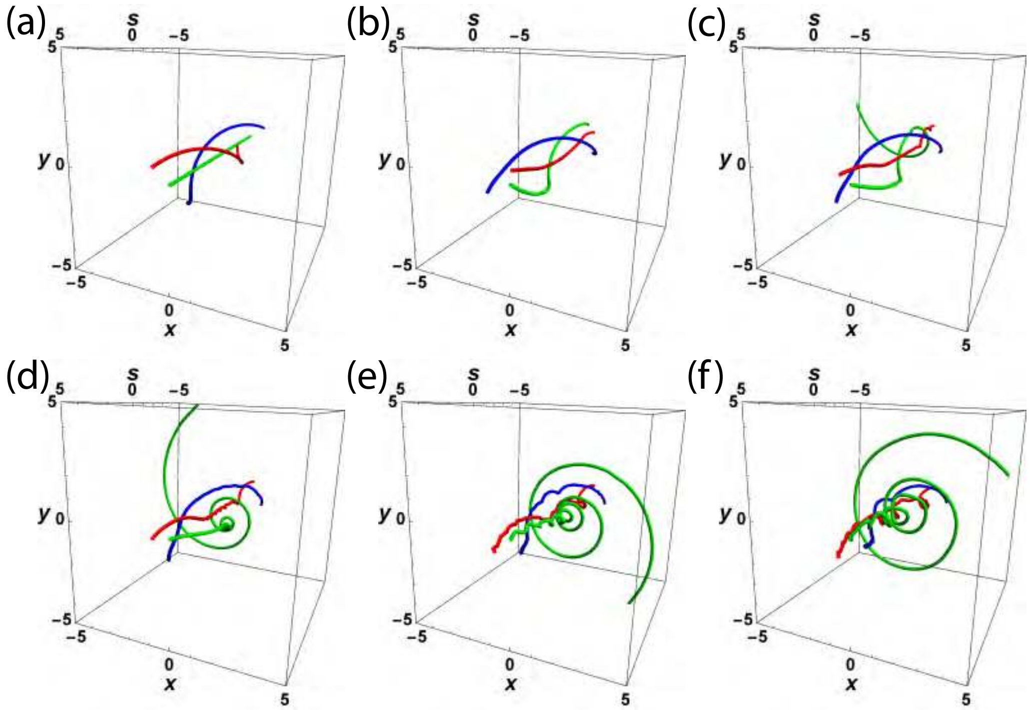

FIGURE 24. Time evolution of plane-wave solutions in a perturbed co-rotating vortex hierarchy featuring two satellite vortices orbiting around a central filament. Plots of $\psi_0\,(t,s)$, $\psi_1\,(t,s)$, and $\psi_2\,(t,s)$ (shown in green, blue, and red, respectively) were obtained by solving (4.6) with $\nu\,(t,s)=0$ for the initial conditions $\psi_0\,(t=0,s)=0$, $\psi_1\,(t=0,s)=\exp\left(\mathrm{i}\left(s/2+0.999\pi\right)\right)$, and $\psi_2\,(t=0,s)=\exp\left(\mathrm{i}\left(s/2+2\pi\right)\right)$. The boundary conditions $\boldsymbol{r}_j\cdot\left(\partial\boldsymbol{r}_j/\partial s\right)=0$ and $\left|\partial\boldsymbol{r}_j/\partial s\right|^2=B^2k^2=1/4$, at $s=\{-5,5\}$ are imposed for $j\in\{1,2\}$, with $\boldsymbol{r}_j=(x_j,y_j)$ being the position vector of the filament in the $x-y$ plane, whilst $\psi_0\,(t,s)$ is pinned so that $\psi_0\,(t,s=5)=\psi_0'\,(t,s=5)=0$. Times shown are: (a) $t=0$, (b) $t=1.7$, (c) $t=1.9$, (d) $t=2.1$, (e) $t=2.3$, and (f) $t=2.4$ with $\alpha=\Gamma=1$. (Color online)

$$\alpha\Gamma\frac{\mathrm{d}^2A_2}{\mathrm{d}\xi^2}+vA_2\frac{\mathrm{d}\theta_2}{\mathrm{d}\xi}-\alpha\Gamma A_2\left(\frac{\mathrm{d}\theta_2}{\mathrm{d}\xi}\right)^2+\frac{4\Gamma}{A_2}+\frac{2\Gamma}{A_1}\cos\left(\theta_1-\theta_2\right)-\frac{2\Gamma\left(A_1\cos\left(\theta_1-\theta_2\right)-A_2\right)}{A_1^2-2A_1A_2\cos\left(\theta_2-\theta_1\right)+A_2^2}=0, \quad (4.16)$$

$$\alpha\Gamma A_2\frac{\mathrm{d}^2\theta_2}{\mathrm{d}\xi^2}+2\alpha\Gamma\frac{\mathrm{d}A_2}{\mathrm{d}\xi}\frac{\mathrm{d}\theta_2}{\mathrm{d}\xi}-v\frac{\mathrm{d}A_2}{\mathrm{d}\xi}+\frac{2\Gamma}{A_1}\sin\left(\theta_1-\theta_2\right)-\frac{2\Gamma A_1\sin\left(\theta_1-\theta_2\right)}{A_1^2-2A_1A_2\cos\left(\theta_2-\theta_1\right)+A_2^2}=0. \quad (4.17)$$

We numerically solve (4.14)-(4.17) for the initial conditions $A_1\,(\xi=0)=0.5$, $A_1'\,(\xi=0)=0$, $\theta_1\,(\xi=0)=0$, $\theta_1'\,(\xi=0)=1$, $A_2\,(\xi=0)=0.5$, $A_2'\,(\xi=0)=0$, $\theta_2\,(\xi=0)=\pi$, and $\theta_2'\,(\xi=0)=1.1$ with $v=1$ in Fig. 25 and show the corresponding filament shapes in Fig. 26 for $\nu\,(t,s)=0$, $\alpha=\Gamma=1$, and times $t=\{0,2,4,6\}$. We observe that the result solutions are chaotic in nature, exhibiting aperiodicity and a hypersensitivity to the initial conditions imposed.

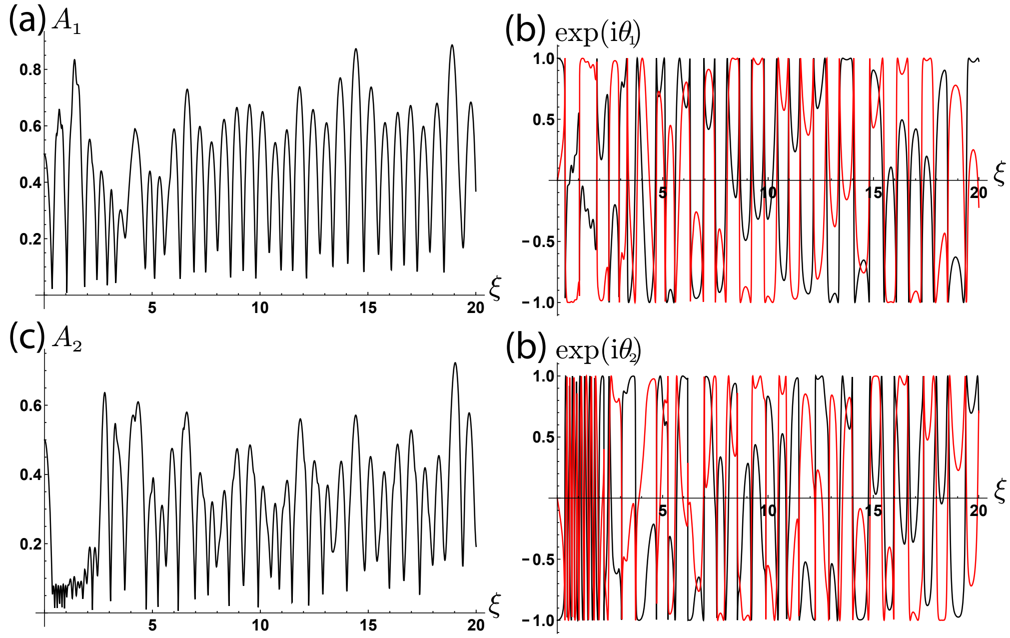

FIGURE 25. $A_j(\xi)$ and real and imaginary components of $\exp(\mathrm{i}\theta_j(\xi))$ (shown in black and red respectively) calculated by the numerical solution of (4.14)-(4.17) for $j = \{1, 2\}$. The initial conditions are $A_1(\xi = 0) = 0.5$, $A_1'(\xi = 0) = 0$, $\theta_1(\xi = 0) = 0$, $\theta_1'(\xi = 0) = 1$, $A_2(\xi = 0) = 0.5$, $A_2'(\xi = 0) = 0$, $\theta_2(\xi = 0) = \pi$, $\theta_2'(\xi = 0) = 1.1$, with $v = 1$. (Color online)

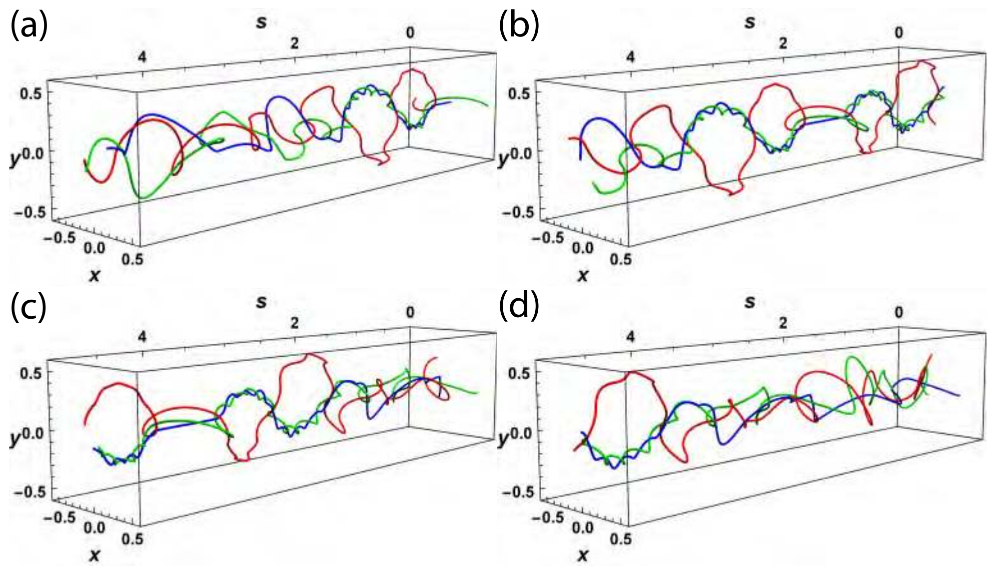

FIGURE 26. Time evolution of traveling wave solutions in a co-rotating vortex hierarchy featuring two satellite vortices orbiting around a central filament. Plots of $\psi_0(t, s)$, $\psi_1(t, s)$, and $\psi_2(t, s)$ (shown in blue, red, and green, respectively) were obtained by solving (4.14)-(4.17) with $\nu(t, s) = 0$ for the initial conditions $A_1(\xi = 0) = 0.5$, $A_1'(\xi = 0) = 0$, $\theta_1(\xi = 0) = 0$, $\theta_1'(\xi = 0) = 1$, $A_2(\xi = 0) = 0.5$, $A_2'(\xi = 0) = 0$, $\theta_2(\xi = 0) = \pi$, $\theta_2'(\xi = 0) = 1.1$, with $v = \alpha = \Gamma = 1$. Times (a) $t = 0$, (b) $t = 2$, (c) $t = 4$, and (d) $t = 6$ are shown over the arclength $s \in [0, 5]$. (Color online)

4.3. *Self-similar solutions*

We study solutions for $N = 2$ satellite vortices around a central filament of the form $\mu_j(t,s) = s\chi_j(\eta)$, for $\eta = s/\sqrt{t}$ and $j = \{1,2\}$. By using the amplitude-phase ansatz

$$\chi_j(\eta) = A_j(\eta)\exp(\mathrm{i}\theta_j(\eta)), \tag{4.18}$$

with $A_j, \theta_j \in \mathbb{R}$, we transform (4.6) and take real and imaginary parts to obtain:

$$\alpha\Gamma\eta^2\frac{\mathrm{d}^2A_1}{\mathrm{d}\eta^2} - \alpha\Gamma\eta^2A_1\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}\eta}\right)^2 + 2\eta\frac{\mathrm{d}A_1}{\mathrm{d}\eta} + \frac{\eta^3}{2}A_1\frac{\mathrm{d}\theta_1}{\mathrm{d}\eta} + \frac{4\Gamma}{A_1} + \frac{2\Gamma}{A_2}\cos(\theta_2-\theta_1) + \frac{2\Gamma(A_1 - A_2\cos(\theta_2-\theta_1))}{A_1^2 - 2A_1A_2\cos(\theta_2-\theta_1) + A_2^2} = 0, \tag{4.19}$$

$$\alpha\Gamma\eta^2A_1\frac{\mathrm{d}^2\theta_1}{\mathrm{d}\eta^2} + 2\alpha\eta^2\Gamma\frac{\mathrm{d}A_1}{\mathrm{d}\eta}\frac{\mathrm{d}\theta_1}{\mathrm{d}\eta} + 2\eta A_1\frac{\mathrm{d}\theta_1}{\mathrm{d}\eta} - \frac{\eta^3}{2}\frac{\mathrm{d}A_1}{\mathrm{d}\eta} + \frac{2\Gamma}{A_2}\sin(\theta_2-\theta_1) + \frac{2\Gamma A_2\sin(\theta_2-\theta_1)}{A_1^2 - 2A_1A_2\cos(\theta_2-\theta_1) + A_2^2} = 0, \tag{4.20}$$

$$\alpha\Gamma\eta^2\frac{\mathrm{d}^2A_2}{\mathrm{d}\eta^2} - \alpha\Gamma\eta^2A_2\left(\frac{\mathrm{d}\theta_2}{\mathrm{d}\eta}\right)^2 + 2\eta\frac{\mathrm{d}A_2}{\mathrm{d}\eta} + \frac{\eta^3}{2}A_2\frac{\mathrm{d}\theta_2}{\mathrm{d}\eta} + \frac{4\Gamma}{A_2} + \frac{2\Gamma}{A_1}\cos(\theta_1-\theta_2) - \frac{2\Gamma(A_1\cos(\theta_1-\theta_2) - A_2)}{A_1^2 - 2A_1A_2\cos(\theta_2-\theta_1) + A_2^2} = 0, \tag{4.21}$$

$$\alpha\Gamma\eta^2A_2\frac{\mathrm{d}^2\theta_2}{\mathrm{d}\eta^2} + 2\alpha\eta^2\Gamma\frac{\mathrm{d}A_2}{\mathrm{d}\eta}\frac{\mathrm{d}\theta_2}{\mathrm{d}\eta} + 2\eta A_2\frac{\mathrm{d}\theta_2}{\mathrm{d}\eta} - \frac{\eta^3}{2}\frac{\mathrm{d}A_2}{\mathrm{d}\eta} + \frac{2\Gamma}{A_1}\sin(\theta_1-\theta_2) - \frac{2\Gamma A_1\sin(\theta_1-\theta_2)}{A_1^2 - 2A_1A_2\cos(\theta_2-\theta_1) + A_2^2} = 0. \tag{4.22}$$

We numerically solve (4.19)-(4.22) for the initial conditions $A_1(\eta = 0.01) = 1$, $A_1'(\eta = 0.01) = 0$, $\theta_1(\eta = 0.01) = 0$, $\theta_1'(\eta = 0.01) = 0.5$, $A_2(\eta = 0.01) = 1$, $A_2'(\eta = 0.01) = 0$, $\theta_2(\eta = 0.01) = \pi$, and $\theta_2'(\eta = 0.01) = 0.6$ and plot the results in Fig. 27. Furthermore, we illustrate the corresponding vortex filament shapes with $\nu(t,s) = 0$ and $\alpha = \Gamma = 1$ in Fig. 28 for times $t = \{0.05, 0.5, 5, 50, 500, 5000\}$. Again, the vortex filament interactions appear to yield chaotic dynamics. Additionally, the filament pair with the larger initial twist $\theta'(\xi)$, in the present case $\psi_0(t,s)$ and $\psi_2(t,s)$, tightly intertwine and form a macrosopic filament that interacts with the other satellite vortex.

5. Discussion

When studying the co-rotating vortices with large separation between the filaments, we found self-similar solutions which exhibited a conical-helix like structure (conical-type solutions were previously seen for isolated vortex filaments (Van Gorder, 2016)) and also traveling wave solutions. For this case, the vortex pair repelled in opposite directions. In the self-similar case, the rotation of the vortex helices was such that they rotated in the same direction, with the orientation of one helix being a reflection of the other. In Fig. 4, both vortices have circulation $\Gamma > 0$, however the helices rotated in a clockwise direction, with the direction of a helix's rotation being opposite to an individual filament's circulation.

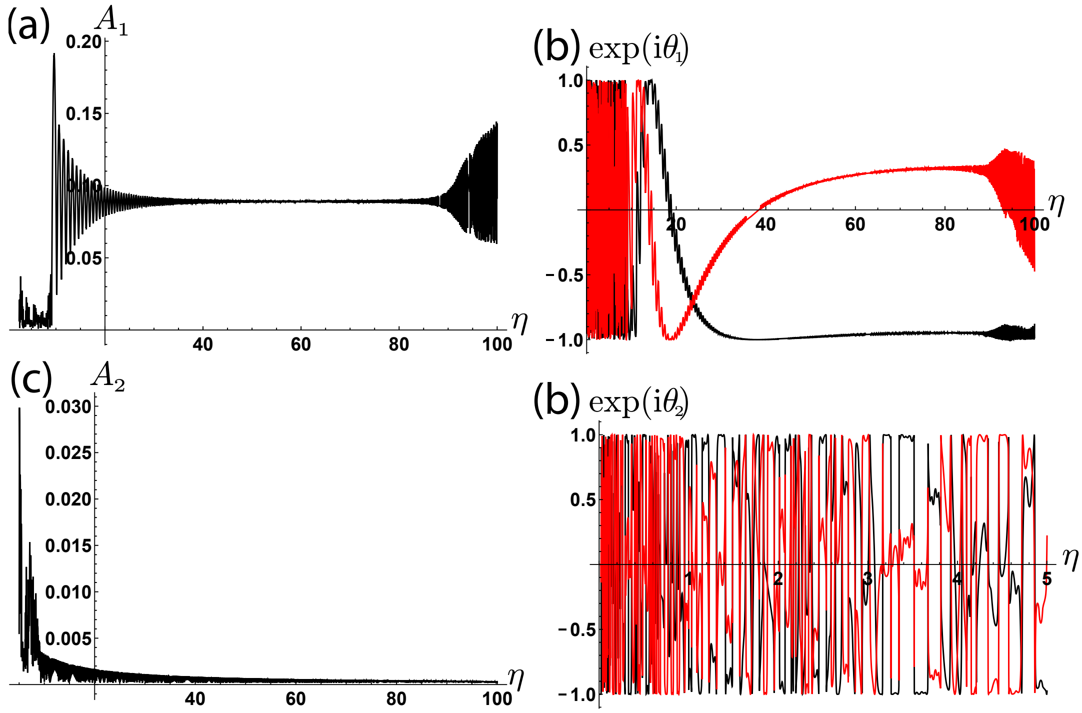

FIGURE 27. $A_j(\eta)$ and real and imaginary components of $\exp(\mathrm{i}\theta_j(\eta))$ (shown in black and red respectively) calculated by the numerical solution of (4.19)-(4.22) for $j = \{1, 2\}$. The initial conditions are $A_1(\eta = 0.01) = 1$, $A_1'(\eta = 0.01) = 0$, $\theta_1(\eta = 0.01) = 0$, $\theta_1'(\eta = 0.01) = 0.5$, $A_2(\eta = 0.01) = 1$, $A_2'(\eta = 0.01) = 0$, $\theta_2(\eta = 0.01) = \pi$, and $\theta_2'(\eta = 0.01) = 0.6$. (Color online)

In the case of counter-rotating filaments with large separation between the filaments, we found that the position of the vortex pair's center and the deviations from this center were strongly coupled, unlike what was shown for the co-rotating vortex filaments, where these two terms decoupled. This somewhat complicates the solution procedure. For the counter-rotating traveling wave solutions with large separation between the filaments, there was no secular vortex repulsion or attraction, contrary to what was previously seen with the co-rotating pair; rather, deviations of the vortex pair had a constant offset, which was determined by the speed of the traveling wave. The center of the vortex pair will move from its initial position over time; that is to say, the filaments will move together at a constant velocity through the 3D flow. Furthermore, we found that the wave speed of the traveling wave could stabilize the vortex pair so that they remain together. There was an additional effect that was not predicted by the 2D theory, namely oscillations in the deviations of the two filaments from the center. These vibrations caused the amplitude of the helices to periodically vary in a manner such that the individual helices were out of phase with each other. We observed that as time evolved, the filament initially had a helical shape (Fig. 6(a)), whose amplitude shrunk (Fig. 6(b)), before returning to its original shape (Fig. 6(c)). The other helix grew in amplitude as the first filament decreased. Meanwhile, for the counter-rotating self-similar solutions with large separation between the filaments, we found that the helices rotated in the opposite direction to the vortex's respective circulation, as was also seen in the co-rotating case.

Regarding traveling wave solutions for intertwined vortex filaments in the co-rotating case, as the filaments approached each other, their attraction occurred on a very fast

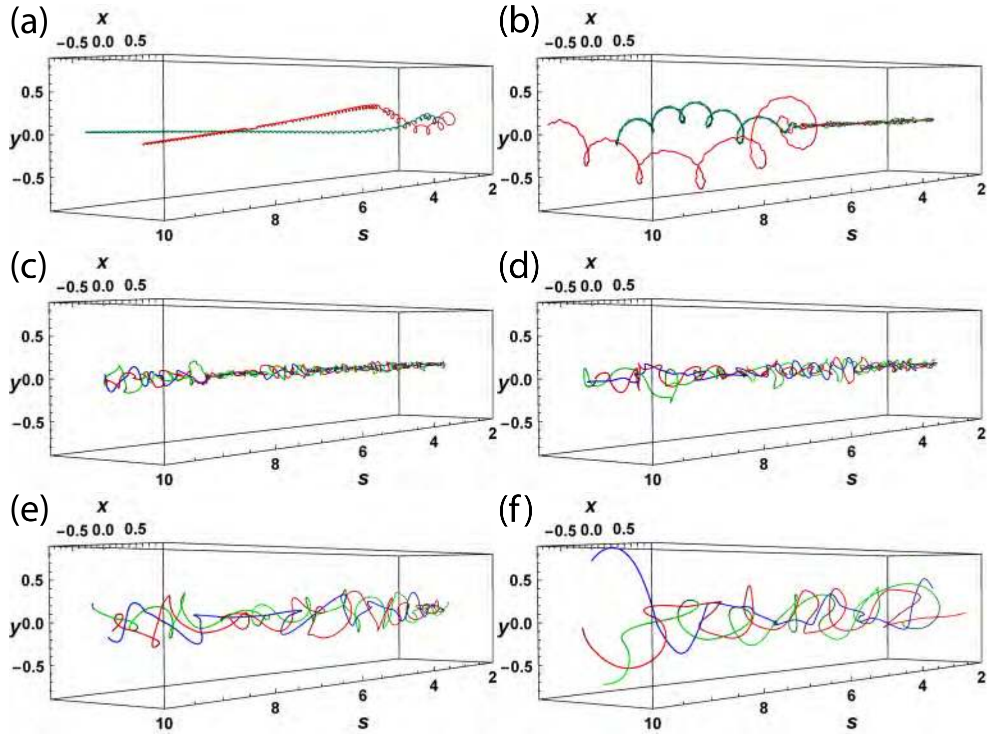

FIGURE 28. Time evolution of self-similar solutions in a co-rotating vortex hierarchy featuring two satellite vortices orbiting around a central filament. Plots of $\psi_0(t,s)$, $\psi_1(t,s)$, and $\psi_2(t,s)$ (shown in blue, red, and green, respectively) were obtained by solving (4.19)-(4.22) with $\nu(t,s)=0$ for the initial conditions $A_1(\eta=0.01)=1$, $A_1'(\eta=0.01)=0$, $\theta_1(\eta=0.01)=0$, $\theta_1'(\eta=0.01)=0.5$, $A_2(\eta=0.01)=1$, $A_2'(\eta=0.01)=0$, $\theta_2(\eta=0.01)=\pi$, and $\theta_2'(\eta=0.01)=0.6$, with $\alpha=\Gamma=1$. Times (a) $t=0.05$, (b) $t=0.5$, (c) $t=5$, (d) $t=50$, (e) $t=500$, and (f) $t=5000$ are shown over the arclength $s\in[2,10]$. (Color online)

time-scale, for $\gamma \ll 1$, until the pair reached their minimum separation, at which point they repelled one another. The orientation of the vortices at this moment changed almost instantaneously, as $\gamma\tan^{-1}\left(c_1c_2+c_1\xi/\gamma^{2n-1}\right)$ dominated in the expression (3.16). As the filaments separated, with their amplitudes $A\to O(1)$, the time-scale slowed down and the orientation of the pair evolved according to the $v\xi/2$ term, which now dominated in (7.6). The filaments then reached their maximum separation before attracting each other again and repeating the cycle.

For the intertwined rotating planar filaments in the co-rotating case, we observed a couple of behaviors, depending on the parameter value ω. When $\omega>0$, $A\to 0^+$ as $s\to\pm\infty$, and the vortex pair was found to collapse if the filaments' respective circulations were in the same direction as the pair rotated. On the other hand, if $\omega<0$, then $A\to\infty$ as $s\to\infty$. Considering a vortex pair with circulations in the opposite direction, rotating pairs with $\omega<0$ would also collapse or, if $\omega>0$, may also repel without bound. The strong dependence of solutions on the sign of the spectral parameter ω was previously discussed for single, isolated planar vortex filaments under the LIA (Van Gorder, 2013*c*) and Biot-Savart (Van Gorder, 2015*b*) dynamics.

When intertwined self-similar filament structures in the co-rotating case came into close proximity, the orientation of separation changed rapidly. In the large time limit $t\to\infty$, the amplitude and wavelength of the filament separation function became larger and variations in the orientation became slower. As $t \to \infty$, the separation of the filaments themselves grew to infinity, and their orientation became planar along a single direction. This smoothing out of an originally rapidly varying filament was exactly seen even in models of isolated self-similar filaments (see Van Gorder (2016) and references therein).

For traveling wave solutions to the counter-rotating intertwined filaments, some differences existed compared to the co-rotating case. If $\gamma = 0$, then the orientation of the counter-rotating pair did not change and the oscillation remained completely planar for all time, with the pair no longer collapsing as $t \to \infty$. For any value of v (the wave speed) or γ, the counter-rotating vortices would not meet. If the velocity of the traveling waves was $v = 0$, corresponding to standing or stationary waves, the filament pair repelled. If the wave speed v was non-zero, then periodic solutions in the wave amplitude were found for all values of γ. As such, the counter-rotating vortex filaments had a different asymptotic structure that no longer depended on γ, in contrast to what was found for the corresponding co-rotating traveling wave solutions.

For the counter-rotating intertwined planar vortex filaments, we imposed boundary conditions so that the quantity denoting the separation of the vortex filament pair was oscillatory along the arc-length parameter. The filament shapes were therefore asymmetric when compared to the symmetrical co-rotating filaments; a direct result of the coupling of the position of the vortex pair's center and the deviations from this center. In contrast with the self-similar solutions in earlier sections, the self-similar intertwined counter-rotating filaments were never in very close proximity, so that the orientation of the pair did not rapidly change according to the dependence of $1/A$ in (3.44). The vortex pair remained separated, but as $t \to \infty$, the center of the vortex pair moved such that the filaments began to intertwine and rotate around each other.

All of the above conclusions correspond to two mutually interacting vortex filaments. To extend such results, we considered the co-rotating vortex hierarchy with satellite vortex filaments surrounding a central vortex. In the case in which all vortex filaments had plane wave structure, the angular velocity of the vortex configuration depended on the wave number, k, the distance of the satellite filaments from the central filament, B, and the number of satellite filaments, N. In our numerical simulations, the central filament exhibited an instability, which grew and spiralled inwards through the hierarchy. In response, small wavelength perturbations developed in the satellite filaments, and these then grew without bound, leading to the eventual collapse of the structure.

For the more general traveling wave case, numerical simulations for the co-rotating vortex hierarchy allowed us to find solutions which appeared to be chaotic in nature, exhibiting aperiodicity and a strong sensitivity to the initial conditions imposed. In the case of self-similar vortex filament structures arranged in this hierarchy, we again observed what appeared to be chaotic dynamics as the system evolved in time away from the initial configuration and, furthermore, the filament pair with the larger initial twist tightly intertwined to form a macroscopic filament that interacted with the other vortex. Such results are interesting, as they show that chaotic dynamics are possible in configurations with relatively small numbers of vortex filaments. Chaos, either deterministic (Nemirovskii & Baltsevich, 2001) or stochastic (Nemirovskii, 2008), has previously been discussed in relation to quantum turbulence. Chaotic dynamics from Kelvin waves along quantised vortex filaments (corresponding to plane wave solutions, or generalisations such as traveling wave solutions we have considered) carrying energy to small scales and leading to a cascade has been suggested (Nemirovskii, 2013) as one route for the transition to turbulence in superfluid Helium. This direction is particularly promising in light of the fact that Kelvin waves have recently been observed experimentally in superfluid Helium (Fonda *et al.*, 2014).

6. Appendix: Asymptotic solutions of vortex pair configurations

6.1. *Asymptotic solutions for Section 2.1.1*

6.1.1. *Fast wave-scale dynamics*

We define a small parameter $\epsilon = 1/d \ll 1$ and expand the nonlinear interaction terms in (2.7) to give:

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\xi^2} - \mathrm{i}v\frac{\mathrm{d}\mu}{\mathrm{d}\xi} + 4\sum_{n=0}^{\infty}(-1)^n\epsilon^{n+1}(\mu^*)^n = 0. \tag{6.1}$$

where $(\ )^*$ is the complex conjugate of the variable.

We make use of the fact that we are studying large separation distances and expand $\mu(\xi)$ as a series in ϵ. In particular, $\mu = \sum_{m=0}^{\infty}\epsilon^m\mu^{(m)}$, so that, for increasing orders of ϵ:

$$O\left(\epsilon^0\right): \qquad 0 = \frac{\mathrm{d}^2\mu^{(0)}}{\mathrm{d}\xi^2} - \mathrm{i}v\frac{\mathrm{d}\mu^{(0)}}{\mathrm{d}\xi}, \tag{6.2}$$

$$O\left(\epsilon^1\right): \qquad 0 = \frac{\mathrm{d}^2\mu^{(1)}}{\mathrm{d}\xi^2} - \mathrm{i}v\frac{\mathrm{d}\mu^{(1)}}{\mathrm{d}\xi} + 4, \tag{6.3}$$

$$O\left(\epsilon^2\right): \qquad 0 = \frac{\mathrm{d}^2\mu^{(2)}}{\mathrm{d}\xi^2} - \mathrm{i}v\frac{\mathrm{d}\mu^{(2)}}{\mathrm{d}\xi} - 4\left(\mu^{(0)}\right)^*, \tag{6.4}$$

the exact solutions of which give the second order asymptotic solution (2.8).

6.1.2. *Slow wave-scale dynamics*

Let $\epsilon = 1/d \ll 1$. We note that $\beta = 2$ in (2.9) from balancing the vortex interaction term in (6.1). However, in order for $\theta(\xi)$ to satisfy initial conditions on $\mu(\xi)$, the perturbation expansion must begin at $O\left(\epsilon^{-1}\right)$. Therefore, we set $\theta(\xi) = \sum_{n=0}^{\infty}\epsilon^{n-1}\theta^{(n)}(\xi)$, which gives, upon extracting orders of ϵ:

$$O\left(\epsilon^0\right): \qquad 0 = \mathrm{i}\frac{\mathrm{d}^2\theta^{(0)}}{\mathrm{d}\xi^2} + v\frac{\mathrm{d}\theta^{(0)}}{\mathrm{d}\xi}, \tag{6.5}$$

$$O\left(\epsilon^1\right): \qquad 0 = \mathrm{i}\frac{\mathrm{d}^2\theta^{(1)}}{\mathrm{d}\xi^2} - \left(\frac{\mathrm{d}\theta^{(0)}}{\mathrm{d}\xi}\right)^2 + v\frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}\xi} + 4, \tag{6.6}$$

$$O\left(\epsilon^2\right): \qquad 0 = \mathrm{i}\frac{\mathrm{d}^2\theta^{(2)}}{\mathrm{d}\xi^2} - 2\frac{\mathrm{d}\theta^{(0)}}{\mathrm{d}\xi}\frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}\xi} + v\frac{\mathrm{d}\theta^{(2)}}{\mathrm{d}\xi}, \tag{6.7}$$

the solution of which results in the slow wave-scale solution (2.10).

6.2. *Asymptotic solutions for Section 2.1.2*

Making use of the fact that $\epsilon = d^{-1} \ll 1$, we expand the nonlinear interaction term in (2.11) as a power series to obtain:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2} - \frac{\mathrm{i}\eta}{2}\frac{\mathrm{d}\chi}{\mathrm{d}\eta} + \frac{\mathrm{i}}{2}\chi + 4\sum_{n=0}^{\infty}(-1)^n\epsilon^{n+1}(\chi^*)^n = 0, \tag{6.8}$$

and further expand $\chi(\eta)$ as a power series in ϵ, so that $\chi(\eta) = \sum_{n=0}^{\infty}\epsilon^n\chi^{(n)}$, to find:

$$O\left(\epsilon^0\right): \qquad 0 = \frac{\mathrm{d}^2\chi^{(0)}}{\mathrm{d}\eta^2} - \frac{\mathrm{i}\eta}{2}\frac{\mathrm{d}\chi^{(0)}}{\mathrm{d}\eta} + \frac{\mathrm{i}}{2}\chi^{(0)} \tag{6.9}$$

$$O\left(\epsilon^1\right): \qquad 0 = \frac{\mathrm{d}^2\chi^{(1)}}{\mathrm{d}\eta^2} - \frac{\mathrm{i}\eta}{2}\frac{\mathrm{d}\chi^{(1)}}{\mathrm{d}\eta} + \frac{\mathrm{i}}{2}\chi^{(1)} + 4, \tag{6.10}$$

the solutions of which gives the first order asymptotic expansion in (2.12).

6.3. *Section 2.2.1*

6.3.1. *Fast wave-scale dynamics*

We expand the interaction term in (2.18) as a series in $\epsilon = 1/d \ll 1$ to obtain:

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\xi^2} + v^2\mu - 4\sum_{n=0}^{\infty} (-1)^n \epsilon^{n+1} (\mu^*)^n = \delta, \tag{6.11}$$

and split $\mu(t,s)$ into real and imaginary components, so that $\mu(\xi) = X(\xi) + \mathrm{i}Y(\xi)$, where $X, Y \in \mathbb{R}$ are the components of the vortex pair's deviation in the x and y direction respectively.

Separating the real and imaginary components of (6.11), we find:

$$\frac{\mathrm{d}^2X}{\mathrm{d}\xi^2} + v^2X - 4\epsilon + 4\epsilon^2X + O\left(\epsilon^3\right) = \delta_1, \tag{6.12}$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\xi^2} + v^2Y - 4\epsilon^2Y + O\left(\epsilon^3\right) = \delta_2, \tag{6.13}$$

where $\delta = \delta_1 + \mathrm{i}\delta_2$ for $\delta_1, \delta_2 \in \mathbb{R}$.

Given that neither (6.12) nor (6.13) have dependence on the first derivative, both equations comprise of conservative systems for which the potential functions $V_1(X)$ and $V_2(Y)$, that constrains the trajectories of $X(\xi)$ and $Y(\xi)$, can be written:

$$V_1(X) = \frac{X^2}{2}\left(v^2 + 4\epsilon^2\right) - X(\delta_1 + 4\epsilon) + b_1 + O\left(\epsilon^3\right), \tag{6.14}$$

$$V_2(Y) = \frac{Y^2}{2}\left(v^2 - 4\epsilon^2\right) - \delta_2 Y + b_2 + O\left(\epsilon^3\right), \tag{6.15}$$

where $b_1, b_2 \in \mathbb{R}$.

Compared to the co-rotating case, (6.14) and (6.15) suggest that the traveling wave solutions of the counter-rotating vortex pair remain finite for all time and along the entire arclength to $O\left(\epsilon^2\right)$, provided the velocity of the traveling wave v is such that $v^2 > 4\epsilon^2$.

As a result, we look for asymptotic solutions to (6.12) and (6.13) of the Poincaré-Lindstedt form to suppress secular terms that would exist using the regular perturbation expansion. In particular, we introduce two separate rescaled wave coordinates $\Xi_1 = \omega_1\xi$ and $\Xi_2 = \omega_2\xi$, where $\omega_1 = 1+\epsilon^2\omega_x^{(2)}$ and $\omega_2 = 1+\epsilon^2\omega_y^{(2)}$ for $\omega_x, \omega_y \in \mathbb{R}$, and perturbation expansions for $X(\Xi_1) = \sum_{p=0}^{\infty} \epsilon^p X^{(p)}$ and $Y(\Xi_2) = \sum_{q=0}^{\infty} \epsilon^q Y^{(q)}$, to find the first-order general solutions:

$$X(\Xi_1) = c_1\cos(v\Xi_1) + c_2\sin(v\Xi_1) + \frac{\delta_1}{v^2} + \frac{1}{d}\left(c_3\cos(v\Xi_1) + c_4\sin(v\Xi_1) + \frac{4}{v^2}\right) + O\left(\frac{1}{d^2}\right), \tag{6.16}$$

$$Y(\Xi_2) = c_7\cos(v\Xi_2) + c_8\sin(v\Xi_2) + \frac{\delta_2}{v^2} + \frac{1}{d}\left(c_9\cos(v\Xi_2) + c_{10}\sin(v\Xi_2)\right) + O\left(\frac{1}{d^2}\right), \tag{6.17}$$

where all $c_n \in \mathbb{R}$.

At $O\left(\epsilon^2\right)$, we find:

$$\frac{\mathrm{d}^2X^{(2)}}{\mathrm{d}\Xi_1^2} + v^2X^{(2)} = -2\left(\omega_x^{(2)}\frac{\mathrm{d}^2X^{(0)}}{\mathrm{d}\Xi_1^2} + 2X^{(0)}\right), \tag{6.18}$$

$$\frac{\mathrm{d}^2Y^{(2)}}{\mathrm{d}\Xi_2^2} + v^2Y^{(2)} = -2\left(\omega_y^{(2)}\frac{\mathrm{d}^2Y^{(0)}}{\mathrm{d}\Xi_2^2} - 2Y^{(0)}\right), \tag{6.19}$$

which implies $\omega_x^{(2)} = 2/v^2$ and $\omega_y^{(2)} = -2/v^2$ to remove the secular terms. The same result follows if we apply a multiple-scales analysis of the problem.

Solving (6.18) and (6.19) and combining the result with (6.16) and (6.17), we find the second order general solutions (6.18) and (6.19).

6.3.2. *Slow wave-scale dynamics*

Substituting (2.23) into (2.20), we obtain the slow solution of the vortex pair's deviation from this center:

$$\mu_{\text{slow}}(\xi) = -\frac{\epsilon^{\beta+1}}{v}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\exp\left(\mathrm{i}\epsilon^{\beta}\theta(\xi)\right) + \frac{4\epsilon}{v^2}, \tag{6.20}$$

which, upon substitution into (6.11), cancels with the $O(\epsilon)$ interaction term, giving:

$$-\frac{\epsilon^{\beta+1}}{v}\exp\left(\mathrm{i}\epsilon^{\beta}\theta\right)\frac{\mathrm{d}^3\theta}{\mathrm{d}\xi^3} - v\epsilon^{\beta+1}\exp\left(\mathrm{i}\epsilon^{\beta}\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + \frac{16\epsilon^3}{v^2} + O\left(\epsilon^{2\beta+1}\right) = 0, \tag{6.21}$$

implying that $\beta = 2$ by balancing. We note that the oscillation of $\mu_{\text{slow}}(\xi)$ would be $O\left(\epsilon^3\right)$, which explains why there was no slowly varying solution evident in the Fig. 7(a).

We suppose that $\theta(\xi) = \sum_{q=0}\epsilon^{2q-2}\theta^{(q)}(\xi)$, with the $O\left(\epsilon^{-2}\right)$ term required to impose the initial condition on $\nu_{\text{slow}}(\xi)$, and find upon separating powers of ϵ in (6.21):

$$O\left(\epsilon^1\right): \qquad 0 = -\frac{1}{v}\frac{\mathrm{d}^3\theta^{(0)}}{\mathrm{d}\xi^3} - v\frac{\mathrm{d}\theta^{(0)}}{\mathrm{d}\xi}, \tag{6.22}$$

$$O\left(\epsilon^3\right): \qquad 0 = -\frac{1}{v}\frac{\mathrm{d}^3\theta^{(1)}}{\mathrm{d}\xi^3} - v\frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}\xi} + \frac{16}{v^2}, \tag{6.23}$$

the solution of which is (2.24).

7. Appendix B: Exact implicit solutions and asymptotic regimes

7.1. *Asymptotic results for Section 3.1.2*

Equation (3.12) can be integrated once to give:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = \frac{\gamma}{A^2} + \frac{v}{2}, \tag{7.1}$$

where $\gamma \in \mathbb{R}$ is an integration constant that quantifies the coupling between the separation of the vortex and changes to its orientation.

Using this result, we decouple (3.11) and write it in terms of the vortex separation:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\xi^2} + \frac{v^2}{4}A + \frac{4}{A} - \frac{\gamma^2}{A^3} = 0. \tag{7.2}$$

Given that (7.2) has no first derivatives, we multiply it through by $A'(\xi)$ and integrate once with respect to ξ to find:

$$\frac{1}{2}\left(\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)^2 + V(A) = E, \tag{7.3}$$

where $E \in \mathbb{R}$ is an integration constant analogous to the total mechanical energy of the system and $V(A)$ is the potential function, given by:

$$V(A) = \frac{v^2}{8}A^2 + 4\ln A + \frac{\gamma^2}{2A^2}. \tag{7.4}$$

Furthermore, for $\gamma \ll 1$, (7.2) has three asymptotic regimes, the first of which is detailed in Section 3.1.2), whilst the other two are given by:

• $A(\xi) = O(\gamma^p)$ for $p \in (0,1)$. The initial layer has width γ^p and $A'' \sim -4/A$ so that the dominant terms are:

$$A\left(\frac{\xi}{\gamma^p}\right) \sim \exp\left(\frac{1}{8}\left[c_3 - 8\left(\operatorname{erf}^{-1}\left[\pm\sqrt{\frac{8}{\pi}\left(\frac{\xi}{\gamma^p}+c_4\right)^2 \exp\left(-\frac{c_3}{4}\right)}\right]\right)^2\right]\right), \tag{7.5}$$

$$\theta\left(\frac{\xi}{\gamma^p}\right) \sim \pm\frac{\gamma\sqrt{\pi}}{2\sqrt{2}\left(\frac{\xi}{\gamma^p}+c_4\right)}\sqrt{\frac{8}{\pi}\left(\frac{\xi}{\gamma^p}+c_4\right)^2 \exp\left(-\frac{c_3}{4}\right)} \times \operatorname{erfi}\left(\operatorname{erf}^{-1}\left[\pm\sqrt{\frac{8}{\pi}\left(\frac{\xi}{\gamma^p}+c_4\right)^2 \exp\left(-\frac{c_3}{4}\right)}\right]\right) + \frac{v}{2}\frac{\xi}{\gamma^p} + \theta_1, \tag{7.6}$$

for $c_3, c_4, \theta_1 \in \mathbb{R}$.

• $A(\xi) = O(\gamma^q)$ for $q = 0$. The region has width of $O(1)$ so that $A'' \sim -4/A - v^2A/4$.

7.2. *Asymptotic results for Section 3.1.3*

For non-zero ω, there are two noteworthy asymptotic regimes in (3.19):

• $A(s) = O(\omega^n)$ for $n \neq -1/2$. In this case, there exists an initial layer of width ω^n so that $A'' \sim -4/A$ and the leading-order behavior is given by (7.5) with the scaling $s \to s/\omega^n$.

• $A(s) = O(\omega^n)$ for $n = -1/2$. The width of the region is ω^n giving the balance $A'' \sim -A - 4/A$.

7.3. *Asymptotic results for Section 2.2.1*

Equation (3.37) can be integrated once with respect to ξ to give:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} - \frac{\gamma}{A^2} = 0, \tag{7.7}$$

where $\gamma \in \mathbb{R}$ is an integration constant that has the same physical meaning as in Section 3.1.2, measuring the coupling between the vortex pair's separation and its change in orientation, the result of which we use to decouple (3.36):

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\xi^2} + v^2 A - \frac{4}{A} - \frac{\gamma^2}{A^3} = 0. \tag{7.8}$$

Following the treatment of Section 3.1.2, (7.8) can be integrated to obtain (7.3) with

$$V(A) = \frac{v^2A^2}{2} - 4\ln(A) + \frac{\gamma^2}{2A^2}, \tag{7.9}$$

and an exact implicit solution given by (3.13).

We consider the asymptotic structure of (7.8) as being dependent on v, given that periodic solutions in A are guaranteed for all values of γ. For $A(\xi) = O(v^n)$ with any nonzero n, there are three initial layers:

• One with width v^n, which corresponds to the balance $A'' \sim 4/A$, giving the leading-order behavior:

$$A\left(\frac{\xi}{v^n}\right) \sim \exp\left(\frac{1}{8}\left[-c_1 - 8\left(\operatorname{erf}^{-1}\left[\pm \mathrm{i}\sqrt{\frac{8}{\pi}\left(\frac{\xi}{v^n}+c_2\right)^2 \exp\left(-\frac{c_1}{4}\right)}\right]\right)^2\right]\right), \quad (7.10)$$

$$\theta\left(\frac{\xi}{v^n}\right) \sim \mp \frac{\mathrm{i}\gamma\sqrt{\pi}}{2\sqrt{2}\left(\frac{\xi}{v^n}+c_2\right)}\sqrt{\frac{8}{\pi}\left(\frac{\xi}{v^n}+c_2\right)^2 \exp\left(-\frac{c_1}{4}\right)}$$
$$\times \operatorname{erfi}\left(\operatorname{erf}^{-1}\left[\pm \mathrm{i}\sqrt{\frac{8}{\pi}\left(\frac{\xi}{v^n}+c_2\right)^2 \exp\left(-\frac{c_1}{4}\right)}\right]\right) + \theta_0, \quad (7.11)$$

for $c_1, c_2, \theta_0 \in \mathbb{R}$.

- Another with width $v^{n/2}$, which gives the balance $A'' \sim \gamma^2/A^3$, leading to the dominant terms:

$$A\left(\frac{\xi}{v^{n/2}}\right) \sim \frac{\pm 1}{\sqrt{c_3}}\sqrt{\gamma^2 + c_3^2 c_4^2 + 2c_3^2 c_4 \frac{\xi}{v^{n/2}} + c_3^2 \frac{\xi^2}{v^n}}, \quad (7.12)$$

$$\theta\left(\frac{\xi}{v^{n/2}}\right) \sim \tan^{-1}\left(\frac{c_3}{\gamma}\left(\frac{\xi}{v^{n/2}} + c_4\right)\right) + \theta_1, \quad (7.13)$$

for $c_3, c_4, \theta_1 \in \mathbb{R}$.

- The last with width v^{-1}, so that $A'' \sim -A$, giving the leading-order behavior:

$$A(v\xi) \sim c_5 \cos(v\xi) + c_6 \sin(v\xi), \quad (7.14)$$

$$\theta(v\xi) \sim \frac{2\gamma \sin(v\xi)}{vc_5\left(c_5\cos(v\xi) + c_6\sin(v\xi)\right)} + \theta_2, \quad (7.15)$$

for $c_5, c_6, \theta_2 \in \mathbb{R}$.

7.4. *Asymptotic results for Section 3.2.3*

Equation (3.39) has the following asymptotic structure in ω for $A = O(\omega^n)$ for any n:

- An initial layer of width ω^n, for $n \neq 1/2$ leading to the balance $A'''' \sim (4/A)''$. For the particular case of $\nu(0,s) = 0$ and having boundary conditions such that $c_1 = c_2 = 0$, we have $A''(s=0) - 4/A(s=0) = \left(A''(s=0) - 4/A(s=0)\right)' = 0$, so that the balance can be solved exactly:

$$A\left(\frac{s}{\omega^n}\right) \sim \exp\left(\frac{1}{8}\left[c_3 - 8\left(\operatorname{erf}^{-1}\left[\pm\sqrt{\frac{8}{\pi}\left(\frac{s}{\omega^n}+c_4\right)^2 \exp\left(-\frac{c_3}{4}\right)}\right]\right)^2\right]\right), \quad (7.16)$$

for $c_3, c_4 \in \mathbb{R}$.

- An initial layer of width ω^{-n-1}, for $n \neq 1/2$ which gives the balance $(4/A)'' \sim -A$, leading to the dominant term:

$$A\left(\omega^{n+1}s\right) \sim \exp\left(-2c_5 + \left(\operatorname{erf}^{-1}\left[\pm\frac{1}{\sqrt{2\pi}}\left(\omega^{n+1}s + c_6\right)^2 \exp\left(-2c_5\right)\right]\right)^2\right), \quad (7.17)$$

for $c_5, c_6 \in \mathbb{R}$.

- The last initial layer has width $\omega^{-1/2}$, so that $A'''' \sim A + (4/A)''$.

8. Appendix C: Dispersion relation for co-rotating hierarchy

The substitution of (4.7) into (4.6) produces a system of N coupled equations, the form of which is most generally given by:

$$\omega_n - \alpha\Gamma k_n^2 + \frac{4\Gamma}{B_n^2} + 2\Gamma \sum_{j\neq n} \left[\frac{\exp(i\phi_{jn})}{B_n B_j} + \frac{B_n - B_j \exp(i\phi_{jn})}{B_n^3 - 2B_n^2 B_j \cos(\phi_{jn}) + B_n B_j^2} \right] = 0, \quad (8.1)$$

where $\phi_{jn} = (k_j - k_n)\, s - (\omega_j - \omega_n)\, t + (\theta_j - \theta_n)$ is the difference between the phases of μ_j and μ_n. However, given that k_n and ω_n are assumed to be constant, this implies that $k_n = k$ and $\omega_n = \omega$ for every μ_n in order for ϕ_{jn} to be independent of s and t. In this case, the only degrees of freedom are the plane wave amplitudes B_n and phase offsets θ_n, so that ϕ_{jn} simplifies to $\theta_{jn} = \theta_j - \theta_n$.

Taking the real and imaginary parts of (8.1), we find:

$$\omega - \alpha\Gamma k^2 + \frac{4\Gamma}{B_n^2} + 2\Gamma \sum_{j\neq n} \left[\frac{B_j + B_n \cos(\theta_{jn}) - 2B_j \cos^2(\theta_{jn})}{B_j \left(B_n^2 - 2B_n B_j \cos(\theta_{jn}) + B_j^2\right)} \right] = 0, \quad (8.2)$$

$$\sum_{j\neq n} \sin(\theta_{jn}) \left[\frac{1}{B_n B_j} + \frac{B_j}{B_n^3 - 2B_n^2 B_j \cos(\theta_{jn}) + B_n B_j^2} \right] = 0. \quad (8.3)$$

By the constraint that $\omega_n = \omega$ and (8.3) must hold true for all $n \in \{1, 2, ..., N\}$, we obtain $2N-1$ equations for $2N$ unknowns. We have a single free parameter that specifies either the initial separation B or orientation θ of a single satellite vortex filament. All other B_n and θ_n are solved using (8.2) and (8.3).

One such solution which satisfies these constraints is $B_n = B$ and θ_n satisfying:

$$\theta_n = \frac{2\pi(n-1)}{N}, \quad n \in \{2, ..., N+1\}, \quad (8.4)$$

which physically corresponds to the vortices being evenly distributed around the center, so that the vortex configuration exhibits polygonal symmetry.

Substituting this result into (8.2), we obtain the dispersion relation given in (4.8).

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