Dynamics of noise-induced heating in atom traps

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A Fokker-Planck equation is derived for the energy distribution of atoms in a three-dimensional harmonic trap with fluctuations in the spring constant and the equilibrium position. Using this model, we predict trap lifetimes based on the measurable noise spectra of the fluctuations. The energy distributions evolve into a single eigenmode where the apparent temperature of the distribution remains constant while the population decays as a consequence of the energy input. The method of analysis and the corresponding results are applicable to any optical, magnetic, or ion trap that is approximately harmonic, and offer useful insights into both noise-induced and optical heating processes. [S1050-2947(98)05211-1]

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I. INTRODUCTION

Stable atom traps have diverse applications in quantum optics ranging from studies of Bose-Einstein condensates (BEC's) [1–3] in magnetic traps to quantum computing [4] in ion traps. Optical far-off-resonance traps have been demonstrated for confining arbitrary atomic ground states in nearly identical potentials [5,6] and for optical lattices [7]. Multiple spin condensates have been stored in a shallow far-off-resonance trap to study interactions in the quantum degenerate regime [8]. These traps also offer an attractive possibility for investigating weakly interacting atomic fermions where multiple atomic states are required for S-wave scattering [9]. In all of these applications, achievement of long storage times is of great importance.

For some time it has been appreciated that fluctuations in the trap parameters can cause atom heating, and that the resulting trap loss limits the maximum storage period. This problem has been circumvented in BEC experiments by using magnetic traps with relatively low trap resonance frequencies, so that fluctuations in the potential are only weakly transmitted to the atoms. Consequently, adequate mechanical and power supply stability are not too difficult to achieve. In contrast, far-off-resonance optical traps with high resonance frequencies can be very sensitive to laser intensity fluctuations and beam-pointing noise which cause trap fluctuations and subsequent heating. However, the conditions on the trap stability needed to achieve long storage times have not been carefully studied.

In a recent paper, we considered fluctuations in a far-off-resonance optical trap using a simple harmonic-oscillator model [10]. Heating rates were estimated in terms of the intensity and position noise power spectra measured for an argon ion laser. Intensity noise causes fluctuations in the spring constant and results in exponential heating, while beam-pointing noise causes fluctuations in the center of the trap and leads to heating at a constant rate. It was shown that achieving heating time constants well beyond 10 sec imposes stringent requirements on the trap stability. Since this model is applicable to any harmonic trap, the results apply equally well to magnetic, optical, and ion traps, and provide estimates of the expected heating time scales.

In this paper, we apply the fluctuating harmonic-oscillator model to noise-induced trap dynamics by deriving a Fokker-Planck equation for the energy distribution of atoms in a three-dimensional fluctuating trap. Numerical modeling for a trap with a finite depth yields estimates of the atom loss rate and the average energy for a number of loading conditions. We begin by reviewing the results of our previous paper for the heating rates, and then derive an approximate Fokker-Planck equation that describes the evolution of the atom energy distribution in the trap.

II. HEATING RATES

As a simple model, we first consider an atom in a one-dimensional harmonic-oscillator potential with a fluctuating spring constant and a fluctuating equilibrium position [10]. This serves as a general model that is valid for small oscillations in any atom trap.

A. Fluctuations in the spring constant

To determine the heating rate arising from fluctuations in the spring constant, we take the model Hamiltonian for a trapped atom of mass M to be

$$H=\frac{p^2}{2M}+\frac{1}{2}M\omega_x^2[1+\epsilon(t)]\,x^2. \tag{1}$$

Here $\omega_x^2=k_x/M$ is the mean-square trap oscillation frequency in the **x** direction, and k_x is the mean value of the corresponding spring constant. Fluctuations in any additive, spatially constant potential V_0 exert no force and do not cause heating. The spring constant exhibits a fractional fluctuation $\epsilon(t)$. For example, in a far-off-resonance optical trap, the spring constant is proportional to the laser intensity, and $\epsilon(t)$ is the fractional fluctuation in the laser intensity.

Equation (1) is well known, and has been studied extensively in classical treatments of parametric resonance [11]. When $x(t)=x_0\cos\omega_x t$ and $\epsilon(t)=\epsilon_0\sin 2\omega_x t$, it is easily shown that the energy increases exponentially with a rate constant $\epsilon_0\omega_x$ that is also the width of the parametric resonance. Further, it is possible to excite subharmonic reso-

nances that grow exponentially at a slower rate [12]. For sinusoidal modulation, the relative phase between $\epsilon(t)$ and $x(t)$ is important.

In the present case, we are interested in stochastic fluctuation rather than sinusoidal modulation of the spring constant. The dominant parametric heating rate arises from the component of the noise power spectrum of $\epsilon(t)$ at the second harmonic. In this case, the energy also increases exponentially, as it does for sinusoidal driving. However, the rate is reduced by the ratio of the linewidth of the parametric resonance $\epsilon_0\omega_x$ to the bandwidth $\Delta\omega$ of the fluctuations, where ϵ_0 is the rms fractional fluctuation in the spring constant. Hence the rate is of order $\omega_x^2 S$, where $S \simeq \epsilon_0^2/\Delta\omega$ is the noise spectral density in fraction squared per rad/sec. The heating rate can be calculated classically, as one would expect for a harmonic-oscillator potential [10]. Unlike the case of sinusoidal modulation, it is independent of the phase of the atom oscillation at the fundamental trap frequency.

Although the heating rates and the corresponding Fokker-Planck equation all can be derived classically, they are easily determined quantum mechanically using first-order time-dependent perturbation theory to calculate the average transition rates between quantum states of the trap. Taking Eq. (1) as a quantum-mechanical Hamiltonian, the perturbation of interest is given by

$$H'(t) = \tfrac{1}{2}\,\epsilon(t) M\omega_x^2 x^2. \tag{2}$$

For an atom in the state $|n\rangle$ at time $t=0$, the average rate to make a transition to state $|m\neq n\rangle$ in a time interval T is

$$R_{m\leftarrow n} \equiv \frac{1}{T}\left|\frac{-i}{\hbar}\int_0^T dt' H'_{mn}(t')\, e^{i\omega_{mn}t'}\right|^2 = \left(\frac{M\omega_x^2}{2\hbar}\right)^2 \int_{-\infty}^{\infty} d\tau\, e^{i\omega_{mn}\tau}\langle \epsilon(t)\epsilon(t+\tau)\rangle |\langle m|x^2|n\rangle|^2. \tag{3}$$

Here we have assumed that the averaging time T is short compared to the time scale over which the level populations vary, but large compared to the correlation time of the fluctuations so that the range of τ extends formally to $\pm\infty$. The correlation function for fractional fluctuations in the spring constant is defined as

$$\langle \epsilon(t)\epsilon(t+\tau)\rangle \equiv \frac{1}{T}\int_0^T dt\ \epsilon(t)\epsilon(t+\tau). \tag{4}$$

Using the transition matrix elements $(m\neq n)$ of x^2 and $\omega_{n\pm 2,n} = \pm 2\omega_x$ in Eq. (3), the transition rates are given by

$$R_{n\pm 2\leftarrow n} = \frac{\pi\omega_x^2}{16} S_k(2\omega_x)(n+1\pm 1)(n\pm 1). \tag{5}$$

In Eq. (5), $S_k(\omega)$ is the one-sided power spectrum of the fractional fluctuation in the spring constant,

$$S_k(\omega) \equiv \frac{2}{\pi}\int_0^\infty d\tau \cos\omega\tau \langle \epsilon(t)\epsilon(t+\tau)\rangle. \tag{6}$$

The one-sided power spectrum is defined so that

$$\int_0^\infty d\omega\ S_k(\omega) = \int_0^\infty d\nu\ S_k(\nu) = \langle \epsilon^2(t)\rangle \equiv \epsilon_0^2, \tag{7}$$

where ϵ_0 is the root-mean-square fractional fluctuation in the spring constant, and $\omega = 2\pi\nu$, with ν the frequency in Hz.

Assuming that the trapped atoms occupy state $|n\rangle$ with probability $P(n,t)$ at time t, the average heating rate is just

$$\langle \dot{E}_x(t)\rangle = \sum_n P(n,t) 2\hbar\omega_x (R_{n+2\leftarrow n} - R_{n-2\leftarrow n}) = \frac{\pi}{2}\omega_x^2 S_k(2\omega_x)\langle E_x(t)\rangle, \tag{8}$$

where the average energy is $\langle E_x(t)\rangle = \Sigma_n P(n,t)(n+1/2)\hbar\omega_x$.

Equation (8) shows that the average energy increases exponentially,

$$\langle \dot{E}_x\rangle = \Gamma_x \langle E_x\rangle, \tag{9}$$

where the rate constant Γ_x is given by

$$\Gamma_x \equiv \frac{1}{T_x(\text{sec})} = \pi^2 \nu_x^2 S_k(2\nu_x). \tag{10}$$

Here ν_x is the trap oscillation frequency in Hz, and T_x is the energy e-folding time (time to increase the energy by a factor e) in sec.

The heating rate is proportional to the energy because the mean-square force fluctuations increase as the mean-square distance from the trap center. The dependence of the heating rate on the second harmonic of the trap frequency shows that it is a parametric heating process. According to Eq. (10), to achieve an energy e-folding time greater than 100 s in a trap with an oscillation frequency of 10 kHz requires $\sqrt{S_k(2\nu_x)} \leq 3\times 10^{-6}/\sqrt{\text{Hz}}$. Hence, if most of the noise were evenly distributed over a 40-kHz bandwidth, the rms fractional fluctuation in the spring constant must be less than $\epsilon_0 = 6\times 10^{-4}$.

B. Fluctuations in the trap center

Fluctuations in the trap equilibrium position also cause heating. In this case, the effective Hamiltonian is

$$H = \frac{p^2}{2M} + \frac{1}{2}M\omega_x^2[x-\epsilon_x(t)]^2, \tag{11}$$

where $\epsilon_x(t)$ is the fluctuation in the location of the trap center. In this case, analogous to the methods used to obtain Eq. (9), the transition rates calculated for Eq. (11) are

$$R_{n\pm 1\leftarrow n} = \frac{\pi}{2\hbar} M\omega_x^3 S_x(\omega_x)(n+1/2\pm 1/2). \tag{12}$$

The corresponding heating rate is

$$\langle \dot{E}_x(t)\rangle = \sum_n P(n,t)\hbar\omega_x[R_{n+1\leftarrow n} - R_{n-1\leftarrow n}]. \tag{13}$$

This yields

$$\dot{Q}_x \equiv \langle \dot{E}_x \rangle = \frac{\pi}{2} M \omega_x^4 S_x(\omega_x). \tag{14}$$

Shaking the trap causes heating that is independent of the trap energy. Here $S_x(\omega)$ is the one-sided power spectrum of the position fluctuations in the trap center along the **x** axis,

$$S_x(\omega) = \frac{2}{\pi} \int_0^\infty d\tau \cos \omega \tau \langle \epsilon_x(t+\tau) \epsilon_x(t) \rangle. \tag{15}$$

The one-sided power spectrum of the position fluctuation is normalized so that $\int_0^\infty d\omega \, S_x(\omega) = \epsilon_x^2$ is the mean-square variation in the trap center position, analogous to Eq. (7).

An energy doubling time T_x' can be defined as the time needed to increase the energy by the average energy at $t=0$: $\dot{Q}_x / \langle E_x(0) \rangle \equiv 1/T_x'$. Then, using $\langle E_x(0) \rangle = M \omega_x^2 \langle x^2 \rangle$, where $\langle x^2 \rangle$ is the mean-square position of an atom in the trap at $t=0$, one obtains

$$\frac{\dot{Q}_x}{\langle E_x(0) \rangle} \equiv \frac{1}{T_x'(\mathrm{sec})} = \pi^2 \nu_x^2 \frac{S_x(\nu_x)}{\langle x^2 \rangle}. \tag{16}$$

According to Eq. (16), if an atom is confined in a trap to a dimension of a 1 μm, with an oscillation frequency of 10 kHz, achievement of an energy doubling time of 100 sec requires a position stability of $\sqrt{S_x(\nu_x)} = 3 \times 10^{-6} \, \mu\mathrm{m}/\sqrt{\mathrm{Hz}}$. Note that to achieve a given time scale, the required fractional position stability in units of the spatial width of the trapped atoms is the same as that required for the fractional stability in the spring constant [Eq. (10)].

III. FOKKER-PLANCK EQUATION

An approximate Fokker-Planck equation for evolution of the energy distribution $n(E,t)$ in the atom trap is easily derived, and takes the form [13]

$$\frac{\partial n(E,t)}{\partial t} = -\frac{\partial}{\partial E}[M_1(E)\, n(E,t)] + \frac{\partial^2}{\partial E^2}[D(E)\, n(E,t)]. \tag{17}$$

Here the first moment $M_1(E) = \langle \Delta E \rangle / T$ is the mean heating rate, and the energy diffusion coefficient is $D(E) = M_2(E)/2 = \langle (\Delta E)^2 \rangle / (2T)$. M_1 and D are averaged over a time T that is long compared to the correlation time of the fluctuations, but short compared to the time scale over which the energy distribution evolves appreciably.

The moments M_1 and M_2 can be obtained classically, but they are readily determined from the transition rates found in Sec. II. First, we consider the one-dimensional case for motion in the **x** direction, and then generalize to three dimensions.

A. Fokker-Planck equation for a one-dimensional trap

For a one-dimensional trap with atoms of energy $E = E_x$, fluctuations in the spring constant cause transitions between states $n \to n \pm 2$ with energy changes $\Delta E_x = \pm 2 \hbar \omega_x$. The rates are given by Eq. (5). These rates can be written in terms of the heating rate constant Γ_x [Eq. (10)] for fluctuations in the spring constant k_x:

$$R_{n \pm 2 \leftarrow n} = \frac{\Gamma_x}{8}(n+1 \pm 1)(n \pm 1). \tag{18}$$

Similarly, position noise causes transitions between states $n \to n \pm 1$ with energy changes $\Delta E_x = \pm \hbar \omega_x$. The corresponding rates [Eq. (12)] can be written in terms of the heating rate $\dot{Q}_x$ [Eq. (14)] arising from position noise:

$$R_{n \pm 1 \leftarrow n} = \frac{\dot{Q}_x}{\hbar \omega_x}(n + 1/2 \pm 1/2). \tag{19}$$

From Eqs. (18) and (19), one obtains

$$M_1(E_x) = \sum_{m = \pm 1, \pm 2} m \hbar \omega_x R_{n+m \leftarrow n} = \dot{Q}_x + \Gamma_x E_x, \tag{20}$$

$$D(E_x) = \frac{1}{2} \sum_{m = \pm 1, \pm 2} (m \hbar \omega_x)^2 R_{n+m \leftarrow n} = \frac{\Gamma_x}{2} E_x^2 + \dot{Q}_x E_x. \tag{21}$$

Note that we neglect 3/4 compared to $(n+1/2)^2$ in the diffusion constant, and use $E_x = (n+1/2)\hbar \omega_x$.

For one-dimensional motion along the **x** axis, the Fokker-Planck equation for the energy distribution $n(E_x,t)$ then is given by Eq. (17) as

$$\frac{\partial n(E_x,t)}{\partial t} = \left(\frac{\Gamma_x}{2} E_x^2 + \dot{Q}_x E_x \right) \frac{\partial^2 n(E_x,t)}{\partial E_x^2} + (\Gamma_x E_x + \dot{Q}_x) \frac{\partial n(E_x,t)}{\partial E_x}. \tag{22}$$

This result also can be derived directly from the quantum rate equations for the state occupation numbers $n_0(E_x,t)$ by using the rates given in Eqs. (18) and (19) and Taylor expanding $n_0(E_x + m\hbar\omega_x, t)$ to second order in $\hbar \omega_x$. By taking the limit $\hbar \omega_x \ll E_x$ and using $n(E_x,t) = n_0(E_x,t)/(\hbar \omega_x)$, the same result is obtained.

B. Fokker-Planck equation for a three-dimensional trap

Using the results for the one-dimensional trap, the Fokker-Planck equation for the energy distribution of atoms in a three-dimensional trap can be derived as a function of the total energy $E = E_x + E_y + E_z$ in an approximation of sufficient ergodicity: We assume that some mechanism exists for assuring that all states of equal total energy are equally probable. Possible mechanisms include reflections from trap imperfections or collisions that occur rapidly compared to the long heating time scales (seconds) of interest here. For a three-dimensional harmonic trap, the heating rates can be different for each direction of motion. Hence we can define three heating rate constants arising from fluctuations in the spring constants, Γ_x, Γ_y, and Γ_z, where the rates are given by Eq. (10). These rates can differ because the trap oscillation frequencies ν_x, ν_y, and ν_z generally are different for each direction or, more generally, the spring constants may fluctuate independently in some cases. Similarly, heating rates arising from fluctuations in the trap position will be different for each direction, and we define three heating rates

$\dot{Q}_x$, $\dot{Q}_y$, and $\dot{Q}_z$, each taking the form of Eq. (14). In this case, the Fokker-Plank equation again takes the form given by Eq. (17). However, the first moment and diffusion coefficient are the sums of the corresponding first moments and diffusion coefficients for the **x**, **y**, and **z** directions:

$$M_1(E)=\sum_{i=x,y,z}(\Gamma_i\langle E_i\rangle_E+\dot{Q}_i), \tag{23}$$

$$D(E)=\sum_{i=x,y,z}\left(\frac{\Gamma_i}{2}\langle E_i^2\rangle_E+\dot{Q}_i\langle E_i\rangle_E\right). \tag{24}$$

Here, $\langle\cdots\rangle_E$ denotes an average over all states of fixed total energy $E=E_x+E_y+E_z$. Assuming sufficient ergodicity, this average is carried out using the probability distribution

$$P(\vec{E})d^3\vec{E}=\frac{d^3\vec{E}}{\hbar\omega_x\hbar\omega_y\hbar\omega_z}\frac{\delta(E-E_x-E_y-E_z)dE}{g(E)dE}. \tag{25}$$

Here $\vec{E}\equiv E_x,\ E_y,\ E_z$, and $d^3\vec{E}\equiv dE_xdE_ydE_z$. The number of states within a fixed dE of a given total energy E is $g(E)dE$, where $g(E)$ is the density of states given by

$$g(E)=\int\int\int\frac{d^3\vec{E}}{\hbar\omega_x\hbar\omega_y\hbar\omega_z}\delta(E-E_x-E_y-E_z)=\frac{E^2}{2\hbar\omega_x\hbar\omega_y\hbar\omega_z}. \tag{26}$$

With Eq. (25), one obtains $\langle E_{x,y,z}\rangle_E=E/3$, as expected, and $\langle E_{x,y,z}^2\rangle_E=E^2/6$. Equations (23) and (24) then yield

$$M_1(E)=\Gamma E+3\dot{Q}, \tag{27}$$

$$D(E)=\frac{\Gamma}{4}E^2+\dot{Q}E. \tag{28}$$

Here we have defined the average rate constant Γ as

$$\Gamma\equiv\frac{\Gamma_x+\Gamma_y+\Gamma_z}{3}, \tag{29}$$

and an average heating rate $\dot{Q}$ as

$$\dot{Q}=\frac{\dot{Q}_x+\dot{Q}_y+\dot{Q}_z}{3}. \tag{30}$$

Using Eq. (17), the Fokker-Planck equation for the energy distribution in the three-dimensional trap is then given by

$$\frac{\partial n(E,t)}{\partial t}=\left(\frac{\Gamma}{4}E^2+\dot{Q}E\right)\frac{\partial^2 n(E,t)}{\partial E^2}-\dot{Q}\frac{\partial n(E,t)}{\partial E}-\frac{\Gamma}{2}n(E,t). \tag{31}$$

Equation (31) also can be obtained using the definition of the three-dimensional energy distribution $n(E,t)$ in the form

$$n(E,t)dE=dE\int\int\int\frac{d^3\vec{E}}{\hbar\omega_x\hbar\omega_y\hbar\omega_z}\delta(E-E_x-E_y-E_z)\times n_0(\vec{E},t), \tag{32}$$

where $n_0(\vec{E},t)$ is the occupation number of a single state with vibrational energies $E_x,\ E_y$, and E_z. $\dot{n}(E,t)$ is then an integral containing $\dot{n}_0(E_x,E_y,E_z,t)$ which is evaluated by adding the right-hand sides of one-dimensional Fokker-Planck equations of the form of Eq. (22) for each dimension. Sufficient ergodicity is imposed by assuming that the occupation number depends only on the total energy

$$n_0(\vec{E},t)=n_0(E=E_x+E_y+E_z,t). \tag{33}$$

Here $n(E,t)=g(E)\,n_0(E,t)$, where $g(E)$ is the density of states given by Eq. (26). Using Eq. (33) to eliminate $n_0(\vec{E},t)$ from the integral that appears in $\dot{n}(E,t)$, one obtains the same result as Eq. (31).

Equation (31) describes the evolution of the atom energy distribution for a harmonic well of infinite depth. It is easy to show for this case that $\dot{N}_{\rm TOT}=\int dE\,\dot{n}(E,t)=0$ and $\dot{E}_{\rm TOT}=\int dE\,E\,\dot{n}(E,t)=\Gamma E_{\rm TOT}+\dot{Q}N_{\rm TOT}$, as it should be.

C. Trap loss for finite well depth

To model the trap loss arising from noise-induced heating, we consider a harmonic well of finite depth. Trap loss occurs as the atoms, with an energy distribution evolving according to the Fokker-Planck equation, escape from this finite well. To model the finite well, we truncate the harmonic oscillator potential at the top, and assume that the heating rates are approximately unchanged from those of an infinite well. The maximum height of the well relative to the minimum is taken to be the well depth U_0. We assume that the atoms are lost when their energy is equal to U_0. In this way, we model loss as a transition to a virtual, unbound harmonic oscillator level at $E=U_0$. This implies the boundary condition $n(E=U_0,t)=0$. The second derivative with respect to energy in the Fokker-Planck equation necessitates a second boundary condition. Because the density of states for a three-dimensional harmonic oscillator varies as E^2, we assume that there are no atoms with zero energy, and we take this other boundary condition to be $n(E=0,t)=0$.

The Fokker-Planck equation (31) is easily converted into the equivalent finite-difference equation using standard methods. We can then compute the energy distribution of the trap at any future time given an initial distribution $n(E,0)$ consistent with the boundary conditions. The numerical method is validated by computing $\dot{N}_{\rm TOT}$ and $\dot{E}_{\rm TOT}$ for initial conditions with atoms that start low in the trap. The results display the infinite well behavior until the atoms begin to escape from the trap.

Initially, we examine solutions for the case of fluctuations in the spring constant, where $\Gamma\neq 0$ and $\dot{Q}=0$. The time is given in units of Γ^{-1} by defining a variable $\tau=\Gamma t$. Note that Γ is the average of the rate constants for the three dimensions, given by Eq. (29). The energy E is given in units of the well depth U_0, so that the energy of occupied states

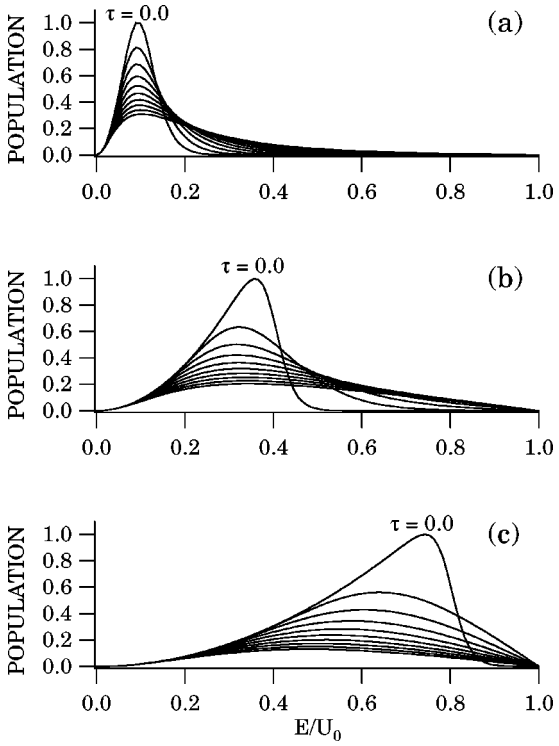

FIG. 1. Evolution of the energy distribution $n(E,t)$ as a function of $\tau \equiv \Gamma t = 0.0$–$1.0$ in steps of $\Delta\tau = 0.1$ for fluctuations in the spring constant ($\dot{Q} = 0$). The curves have been scaled so that the initial distribution has a peak value of 1.0. Three initial conditions are shown: (a) $E_c = 0.1\,U_0$, (b) $E_c = 0.4\,U_0$, and (c) $E_c = 0.8\,U_0$.

varies from 0 to 1. The initial condition is taken to be

$$n(E,0) = \frac{E^2}{e^{(E-E_c)/\Delta E}+1}. \tag{34}$$

Here we assume that the initial distribution is a product of the harmonic-oscillator density of states $\propto E^2$ and an occupation number that smoothly cuts off for $E > E_c$ over a range ΔE. This distribution avoids an abrupt cutoff of $n(E,0)$ that occurs for $E = U_0$ when the initial occupation number is Maxwellian and the thermal energy is comparable to the well depth. Typically, we vary the cutoff energy E_c from $0.1\,U_0$ to $0.8\,U_0$, and take $\Delta E = 0.1\,U_0$.

Figure 1 shows the evolution of the energy distributions, $n(E,t)$, for initial distributions with $E_c/U_o = 0.1$, 0.4, and 0.8. The distributions always evolve into an eigenmode that decays with a single time constant.

The fraction of the total number of atoms remaining, $N_{\mathrm{TOT}}(\tau)/N_{\mathrm{TOT}}(0)$, is plotted as a function of $\tau = \Gamma t$ in Fig. 2. When the atoms are loaded with an initial distribution that has a mean energy near the bottom of the trap, the evolution is not exponential until τ is substantially larger than 1. When the trap is loaded with an initial occupation number that is relatively flat ($E_c = 0.8$), the quadratic dependence of the density of states favors loading of high-lying states, and the total number decays by a factor of about 0.4 in a time $0.5\,\Gamma^{-1}$. Hence traps that are loaded at temperatures T_0 such that $k_B T_0$ is comparable to the well depth will decay with a time constant almost twice as fast as Γ^{-1}. From our previous measurements for an argon-ion laser [10], we find that Γ^{-1}

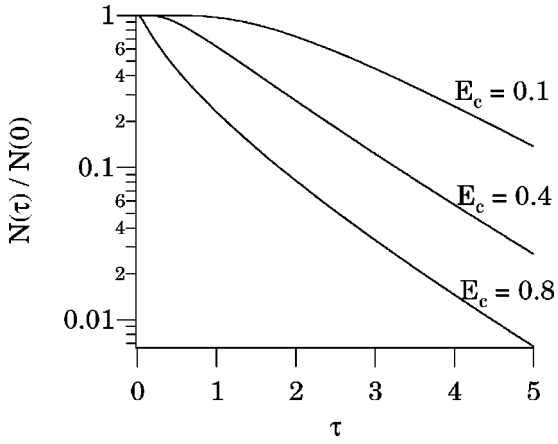

FIG. 2. Decay of the total trapped population vs $\tau = \Gamma t$ for fluctuations in the spring constant ($\dot{Q} = 0$).

$\leqslant 10$ sec for trap resonance frequencies above 5 kHz. Storage times well above 10 sec are unlikely in optical traps that employ argon-ion pump lasers, unless the atoms are initially loaded near the bottom of the trap.

Figure 3 shows the mean energy per atom $E_{\mathrm{TOT}}(\tau)/N_{\mathrm{TOT}}(\tau)$ in units of U_0 as a function of time for various loading conditions. For loading near the trap bottom, $E_c = 0.1\,U_0$, the mean energy first rises as the atoms heat and then asymptotically approaches a constant value $\simeq 0.36\,U_0$. For loading at high temperatures, $E_c = 0.8$, the mean energy initially drops as hot atoms are rapidly expelled from the trap and then approaches $0.36\,U_0$. This behavior is a consequence of the evolution of the energy distribution into a single eigenmode. This result is confirmed by the asymptotic forms of the analytic solutions for the total energy and number, each decaying exponentially at large times with a rate $5\Gamma/9$, and reaching a steady state ratio of 9/25 $= 0.36$. Hence, after a couple of time constants, the apparent temperature of the distribution does not change, although the population decays as a consequence of the energy input.

The evolution of the energy distribution for fluctuations in the center of the trap, $\dot{Q} \neq 0$ and $\Gamma = 0$, is shown in Fig. 4 for different values of E_c. Here, the unit of time is taken as $U_0/\dot{Q}$, the time needed to increase the energy per atom by

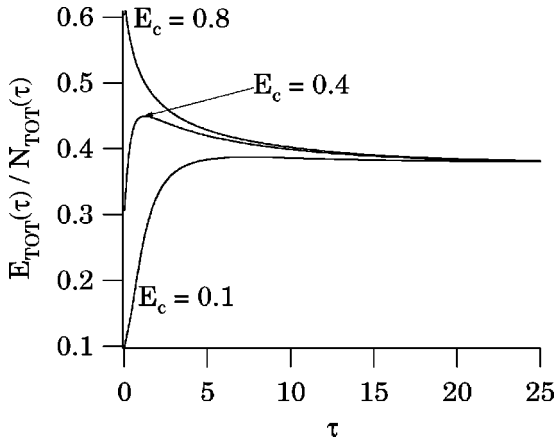

FIG. 3. Mean energy of the trapped atoms vs $\tau = \Gamma t$ for fluctuations in the spring constant ($\dot{Q} = 0$). The energy is in units of the well depth U_0.

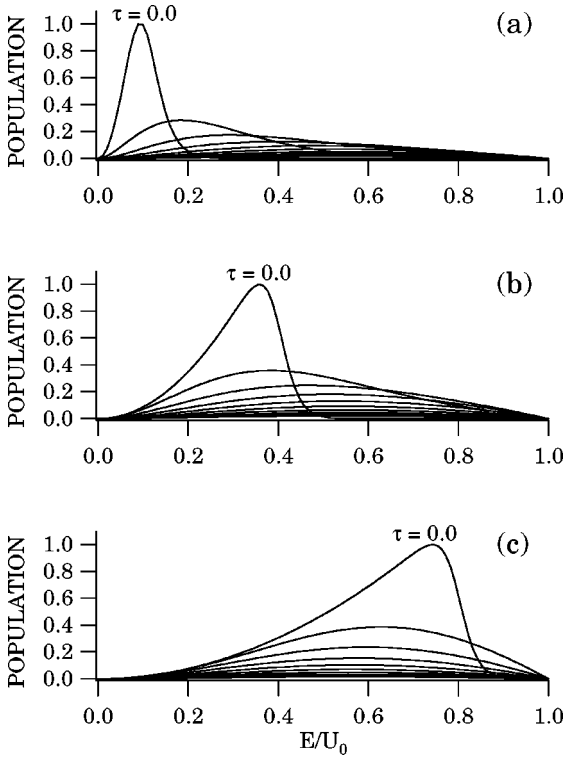

FIG. 4. Evolution of the energy distribution $n(E,t)$ as a function of $\tau=\dot{Q}t/U_0=0.0$–0.5 in steps of $\Delta\tau=0.05$ for fluctuations in the trap center ($\Gamma=0$). The curves have been scaled so that the initial distribution has a peak value of 1.0. Three initial conditions are shown: (a) $E_c=0.1\,U_0$, (b) $E_c=0.4\,U_0$, and (c) $E_c=0.8\,U_0$.

the well depth, and $\tau=\dot{Q}t/U_0$ is the time in dimensionless units. Note that $\dot{Q}$ is the average of the heating rates for the three dimensions given by Eq. (30). The corresponding decay of the total number $N_{\rm TOT}(\tau)/N_{\rm TOT}(0)$ is shown in Fig. 5. Since all atoms will have escaped the trap when the added energy equals the well depth and the energy distribution broadens by energy diffusion, the number decays exponentially after just a fraction of a time constant, $U_0/\dot{Q}$. As a consequence of the quadratic dependence of the density of states, states with high energy are preferentially filled when the trap is loaded at temperatures where k_BT_0 is comparable

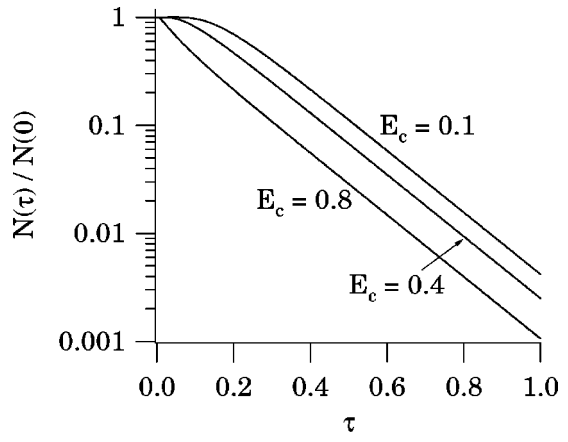

FIG. 5. Decay of the total trapped population vs $\tau=\dot{Q}t/U_0$ for fluctuations in the trap center ($\Gamma=0$).

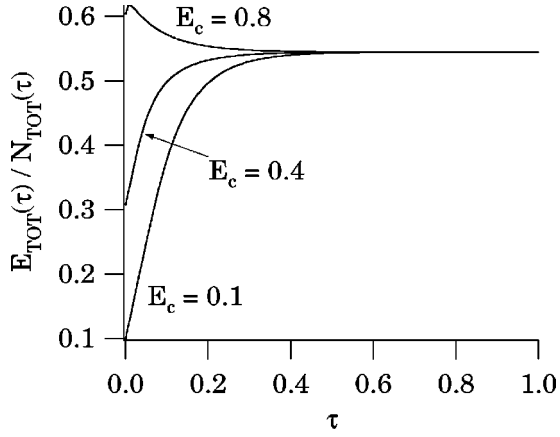

FIG. 6. Mean energy of the trapped atoms vs $\tau=\dot{Q}t/U_0$ for fluctuations in the trap center ($\Gamma=0$). The energy is in units of the well depth U_0.

to U_0. Hence, trap loss can occur on time scales much shorter than $U_0/\dot{Q}$. Figure 6 shows the mean energy per atom, $E_{\rm TOT}(\tau)/N_{\rm TOT}(\tau)$, which rapidly approaches $0.55\,U_0$ in agreement with the analytic solution. Again, at large times, the apparent temperature of the distribution does not change although the population decays as a consequence of the energy input.

The analytic solution shows that the energy distribution rapidly decays to the lowest eigenmode $\propto E\,J_2(z_{21}\sqrt{E/U_0})$. This mode decays as $\exp[-(z_{21}/2)^2\tau]=\exp[-6.6\tau]$, where $z_{21}=5.14$ is the first zero of $J_2(x)$.

Since the energy for a single atom increases generally according to $\dot{E}=\Gamma\,E+\dot{Q}$, the constant heating rate $\dot{Q}$ from fluctuations in the trap center can be exponentially enhanced by the rate Γ arising from fluctuations in the spring constant. Figure 7 shows the evolution of the total number that results for $\chi\equiv\Gamma\,U_0/\dot{Q}=0.5$, 2, and 5 under various loading conditions. The unit of time is $U_0/\dot{Q}$ and $\tau=\dot{Q}t/U_0$.

IV. DISCUSSION

We have derived a Fokker-Planck equation for the energy distribution of atoms in a three-dimensional fluctuating harmonic trap, under conditions of sufficient ergodicity. The expected trap lifetimes for a truncated harmonic potential are determined for initial energy distributions with different mean energies. The heating rates and rate constants which determine the trap lifetime are given in terms of the noise spectral densities of fluctuations in the trap spring constants and trap center position. The heating rates directly determine the time for atoms to leave the ground state of the well at very low temperature. Numerical solution of the Fokker-Planck equation for a variety of initial conditions shows that population loss from the trap can be quite rapid in some cases. The effective time constant can be significantly shorter than $1/\Gamma$ or $U_0/\dot{Q}$ as a result of the density of states, which predisposes atoms to be near the top of the trap when the occupation number is slowly varying. The results of the fluctuating harmonic oscillator model can be applied to magnetic, optical, and ion traps.

In recent BEC experiments, magnetic traps are employed

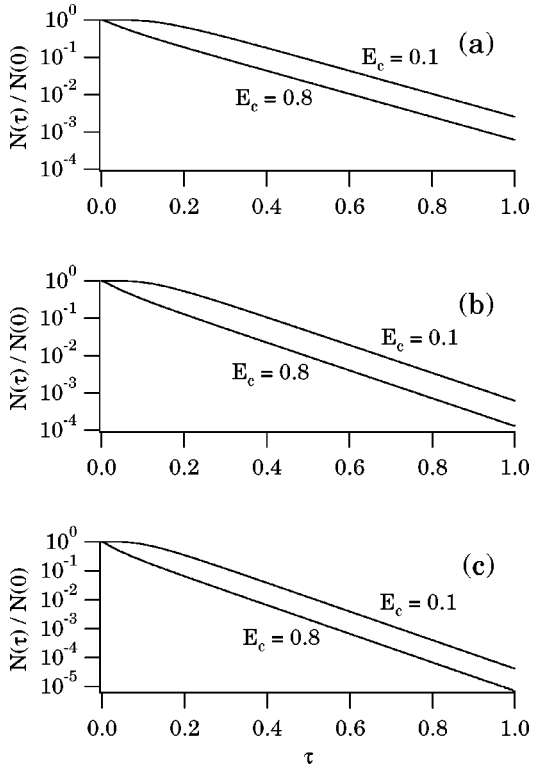

FIG. 7. Decay of the total trapped population vs $\tau = \dot{Q}t/U_0$ for fluctuations in both the trap center and the spring constant. Two initial conditions are shown: $E_c = 0.1\,U_0$ and 0.8. The ratio of the heating rates arising from fluctuations in the spring constant to that from fluctuations in the trap center is defined by the parameter $\chi \equiv \Gamma\, U_0/\dot{Q}$. (a) $\chi = 0.5$. (b) $\chi = 2.0$. (c) $\chi = 5.0$.

to achieve evaporative cooling using a radio-frequency (rf) scalpel [1–3]. These traps are very well approximated by the truncated harmonic-oscillator model when the rf scalpel is on. The harmonic-oscillator model predicts time scales for the energy to increase in terms of the trap fluctuation spectra. In Refs. [1,2], electromagnets are used to create the trapping potential. It is of interest to estimate the power supply stability required to attain lifetimes exceeding 100 sec, as used in the first demonstration of BEC's [1]. For a radial oscillation frequency of 320 Hz [2], Eq. (10) shows that achieving 100 sec requires a fractional stability of $S_k = 10^{-4}/\sqrt{\text{Hz}}$. Assuming that the high current power supply noise has a bandwidth of 1 kHz, the required rms stability is 0.3%, which is not too stringent.

In a previous paper, we investigated heating rates in far-off-resonance optical traps [10]. For tightly confining traps, very high trap oscillation frequencies are possible, in some cases as high as 100 kHz. In this case, to achieve 100-sec storage times, the fractional stability in the laser intensity must be smaller than $S_k = 3\times 10^{-7}/\sqrt{\text{Hz}}$. For atom distributions localized to a μm at low temperatures, the position stability of the trap must be $S_x = 3\times 10^{-7}\,\mu/\sqrt{\text{Hz}}$.

In addition to determining trap lifetimes arising from noise-induced heating processes, the fluctuating harmonic-oscillator model also offers useful insights for optical heating processes, such as the recoil heating rate in a far-off-resonance trap, $\dot{Q}_{\text{rec}} = 2\,R_s \epsilon_{\text{rec}}$ [14]. Here R_s is the optical scattering rate, and ϵ_{rec} is the recoil energy, $\hbar^2 q^2/(2M)$, where q is the optical wave vector. The trap lifetime is well known to be of order $U_0/\dot{Q}_{\text{rec}}$ [14]. However, the recoil heating rate is equivalent to the heating caused by a position-independent fluctuating force $\delta\vec{f}(t)$, with a short correlation time of the order of the spontaneous lifetime τ_{sp}. In one dimension, this force is described by an effective potential $H' = -x\,\delta f(t)$. In a harmonic well, this is equivalent to a displacement in the well center, by $\epsilon_x(t) = \delta f_x(t)/(M\omega_x^2)$. Using Eq. (14) in the limit $\omega_x \tau_{\text{sp}} \ll 1$, one can show that this leads to $\dot{Q}_x = D_p^x/M$, where D_p^x is the momentum diffusion constant for the **x** direction, as it should be. For a short correlation time, the force changes the momentum, but not the position. The potential energy remains constant during the fluctuation, and the free particle heating rate is obtained. In three dimensions, one obtains the recoil heating rate. Since the scattering force is equivalent to a fluctuation in the center of the trap, the average heating rate, Eq. (30) is $\dot{Q} = \dot{Q}_{\text{rec}}/3$. The corresponding first moment, diffusion constant, and Fokker-Planck equation are identical in form to those for fluctuations in the trap center. The recoil heating rate simply can be added to the heating rate for position fluctuations. The analytic solution shows that the lowest eigenmode of the energy distribution decays as $\exp[-6.6\,\dot{Q}t/U_0]$ as described above. Hence, recoil heating causes trap loss according to $\exp[-2.2\,\dot{Q}_{\text{rec}}t/U_0]$. The decay is exponential and the rate is more than twice as fast as that determined from the heating rate neglecting energy diffusion.

Optical scattering also can induce fluctuations in the spring constant of a far-off-resonance trap. This is already implicit in the general results for the optical heating obtained in Ref. [14], that includes the position dependence of the momentum diffusion constant. The dipole force contribution [14,15] in a far-off-resonance trap is equivalent to three-photon scattering that causes real transitions from the ground state to the excited state. This causes fluctuations in the restoring force and hence in the spring constant over a short correlation time, of order $\tau_{\text{sp}} = \gamma_{\text{sp}}^{-1}$. In far-off-resonance traps, usually the detuning is large enough that dipole force heating is negligible, and even the lowest-order recoil scattering contributions are small. However, when near-resonant fields are present, for example, optical repumping beams for loading the trap from a low-intensity magneto-optical trap or for Raman cooling, leakage fields, etc., real transitions to the excited state can occur. Then the near-resonant fields will cause heating both from recoil and from induced fluctuations in the spring constant of the trap. In the two-level approximation, the excited- and ground-state potentials have identical shapes and opposite signs. For small excitation probability, the net heating rate from the near-resonant beam will be dominated by induced spring constant fluctuations when the force from the trap $-M\omega_x^2 x$ exceeds the maximum radiation pressure force $\gamma_{\text{sp}}\hbar q/2$.

Our treatment has been restricted to nearly harmonic traps in order to obtain simple analytical results. Of course, real optical traps often employ focused laser beams with a Gaussian intensity distribution. Since the force decreases near the top of a Gaussian trap, one expects reduced heating when $E = U_0$, at least for trap fluctuations with a white noise fluc-

tuation spectrum. The numerical modeling shows that the initial energy distribution in the trap can significantly affect the trap lifetime. Since the harmonic-oscillator density of states is $\propto E^2$, high-lying states may be preferentially loaded in some cases. For three-dimensional Gaussian traps, the density of states at $E=U_0$ is 6.4 times as large as that of a harmonic well, further favoring the loading of a Gaussian trap near the top. Modeling of noise-induced trap dynamics and loading for Gaussian potentials is currently under investigation.

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