Article

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**Abstract**

We propose a simple closed-form expression relating [...] the mean relative velocity of pairs of galaxies at fixed separation r—to the two-point correlation function of mass density fluctuations,[...]. Our Ansatz is an interpolation between the perturbative and stable clustering expressions for [...]. We compare our analytic model for [...] with N-body simulations and find excellent agreement in the entire dynamical range probed by the simulations ([...]). Our results can be used to estimate the cosmological density parameter, Ω, directly from redshift-distance surveys.

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# Dynamics of pairwise motions

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## ABSTRACT

We derive a simple closed-form expression, relating $v_{12}(r)$ – the mean relative velocity of pairs of galaxies at fixed separation $r$ – to the two-point correlation function of mass density fluctuations, $\xi(r)$. We compare our analytic model for $v_{12}(r)$ with N-body simulations, and find excellent agreement in the entire dynamical range probed by the simulations ($0.1 \lesssim \xi \lesssim 1000$). Our results can be used to estimate $\Omega^{0.6}{\sigma_8}^2$ directly from redshift-distance surveys, like Mark III or SFI. Combined with other observational constraints on $\beta \equiv \Omega^{0.6}\sigma_8$, such measurements can be used to break the degeneracy between $\Omega$ and $\sigma_8$.


*Subject headings:* Cosmology: theory – observation – peculiar velocities: large scale flows

## 1. Introduction

Recently, we have suggested a new method to determine pairwise velocities directly from peculiar velocity surveys, containing galaxy redshifts as well as redshift-independent distances by a simple alternative method, which has never been tried before (Ferreira et al. 1998). In this *Letter* we derive an very accurate approximate relation between the mean pairwise velocity and the amplitude of clustering, which can be used to extract cosmological parameters by comparing velocities and clustering amplitudes.

## 2. An analytical model for $v_{12}(r)$

Most dynamical estimates of the cosmological density parameter, $\Omega$, use the gravitational effect of departures from a strictly homogeneous distribution of objects which are considered as test particles. One such dynamical estimator can be constructed by using an equation expressing the conservation of particle pairs in a self-gravitating gas. This equation was derived by Davis & Peebles (1977) from the BBGKY theory (see also Peebles 1980, hereafter LSS). We will consider a pair of particles at a comoving separation vector $\vec{x}$ and cosmological time $t$, moving with a mean (pair-weighted) relative velocity $v_{12}(x,t)\,\vec{x}\,/x$. It's magnitude is related to the two-point correlation function of density fluctuations, $\xi(x,t)$, by the pair conservation equation (LSS),

$$\frac{a}{3[1+\xi(x,a)]}\,\frac{\partial\bar{\xi}(x,a)}{\partial a} \;=\; -\,\frac{v_{12}(x,a)}{Hr}\;, \tag{1}$$

where $a(t)$ is the expansion factor, $r = ax$ is the proper separation, $H(a)$ is the Hubble parameter, while $\bar{\xi}(x,a)$ is the two-point correlation function, averaged over a ball of comoving radius $x$ : $\bar{\xi}(x,a) = 3x^{-3}\int_0^x \xi(y,a)y^2\mathrm{d}y$. At the present cosmological time $a = 1$, $x = r$ and $H = 100\ h^{-1}$km s$^{-1}$Mpc$^{-1}$ The exact solution of (1) is unknown. However, it can be solved approximately in two limiting cases: on very small scales, where $\xi \gg 1$ (stable clustering regime), and on very large scales, where $|\xi| \ll 1$ (linear regime). The stable clustering solution is (LSS, §71) $v_{12}(x,a) \;=\; -Hr$ , as expected for virialized clusters on sufficiently small scales. The linear solution is given by the first terms in perturbative expansions for $v_{12}$ and $\xi$, which for the correlation is given by $\xi = \xi^{(1)} + \xi^{(2)} + \ldots$ , with $\xi^{(2)} = O\left[\xi^{(1)}\right]^2$, etc. The growing mode of the linear solution is $\xi^{(1)}(x,a) = \xi_1(x)D(t)^2$ , where the spatial part is determined by the initial spectrum of density fluctuations while $D(t)$ is the standard linear growing mode solution (see LSS, §11). Substituting the above expression into the linearized eq. (1), we get

$$v_{12}(x,a) \;=\; -\,\tfrac{2}{3}Hrf(\Omega,\Lambda)\bar{\xi}^{(1)}(x,a) \;+\; O[\bar{\xi}^{(1)}]^2\;, \tag{2}$$

where $f \equiv d\ln D/d\ln a$, and $\bar{\xi}^{(1)}(x,a)$ is $\xi^{(1)}$ averaged over a ball of comoving radius $x$. For models with a vanishing cosmological constant ($\Lambda = 0$), and for zero curvature models with $\Lambda \neq 0$, $f \simeq \Omega^{0.6}$ (e.g. Peebles 1993). Since $\bar{\xi}^{(1)}$ is not a measurable quantity, it is more convenient to use $\bar{\xi}$ instead,

$$v_{12}(x,a) \;=\; -\tfrac{2}{3}\,Hrf\bar{\xi}(x,a) + O[\bar{\xi}\,]^2\;. \tag{3}$$

The difference between the above expression and equation (2) is of order $\bar{\xi}^2$, so they are equivalent at large separations, in the $\xi \to 0$ limit.

There are two obvious ways of improving the accuracy of the linear approximation solution above and extending its validity to smaller separations: (i) – by extending the perturbative series for $v_{12}$ to higher orders and (ii) – by matching the linear solution with the stable clustering solution. The latter approach was introduced by Peebles (LSS, §71). It amounts to replacing eq. (3) with

$$v_{12}(x,a) = -\,\tfrac{2}{3}\,Hrf\bar{\bar{\xi}}(x,a)\;, \tag{4}$$

where $\bar{\bar{\xi}} = \bar{\xi}(x,a)/[1+\xi(x,a)]$. In the limit of large separations, when $\xi$ is small, eq. (4) is identical to eq. (3). However, at small separations the two expressions differ significantly and the latter agrees with the stable clustering result up to the factor $2f/3$, which is of order unity for all meaningful choices of $\Omega$ and $\Lambda$, while the linear approximation overestimates $|v_{12}|$ by $2\bar{\xi}f/3 \gg 1$.

Let us now discuss the perturbative approach. The general technique for deriving $\xi^{(2)}$ for density fluctuations with Gaussian initial conditions was introduced by Juszkiewicz et al. (1980, 1984) and Vishniac (1983). The second order term in the expansion for $\xi$ can be written as $\xi^{(2)}(x,a) = D^4\,\xi_2(x)$, where $\xi_2$ is a function of $x$ alone, while $D(a,\Omega,\Lambda)$ is the usual linear order growing mode solution. For $\Omega = 1$, $\Lambda = 0$, this factorization of the $t-$ and $x-$ dependence is exact (Fry 1984; Juszkiewicz et al. 1984); for $\Omega \neq 1$ and $\Lambda \neq 0$, it provides a good approximation of the exact (formally non-separable) solution (Bouchet et al. 1992, 95; Juszkiewicz et al. 1993). Substituting $\xi^{(2)} = D^4\,\xi_2$ into eq. (1) and solving for $v_{12}$ we get[4]

$$v_{12} \;=\; -\tfrac{2}{3}\,Hrf\,\left[\bar{\xi}^{(1)} \;-\; \bar{\xi}^{(1)}\xi^{(1)} \;+\; 2\,\bar{\xi}^{(2)}\right] \;+\; O\,[\xi^{(1)}]^3\;. \qquad (5)$$

For a scale-free correlation function with a constant logarithmic slope $\gamma \;\equiv\; -\,d\ln\xi^{(1)}(x,a)/d\ln x$, $\bar{\xi}^{(2)}$ is related to $\bar{\xi}^{(1)}$ by a simple closed-form expression, $\bar{\xi}^{(2)} = \alpha(\bar{\xi}^{(1)})^2$, where $\alpha$ is a constant (Łokas et al. 1996, Scoccimarro & Frieman 1996). For $\gamma$ in the range from 0 to 2, $\alpha$ can be expressed in terms of gamma functions (Scoccimarro & Frieman 1996). The actual expression would however take half a page, so we will instead use a fitting formula, valid in the range $0 < \gamma < 2$:

$$\alpha \;=\; 1.843 \;-\; 1.1\gamma \;-\; 8.2\times 10^{-4}\gamma^{10}\;. \qquad (6)$$

Note that the Peebles approximation (4), expanded to second order reproduces all terms in eq. (5) except for the factor of 2 in the term $\propto \bar{\xi}^{(2)}$. For $\gamma \approx 1.6$, $\bar{\xi}^{(2)}$ vanishes, hence the error in the Peebles approximation is of order $\bar{\xi}^3$, not $\bar{\xi}^2$ as one might expect. This is important for practical applications because the slope of the observed galaxy correlation function is close to 1.6 over a wide range of separations.

We now combine the second-order perturbative solution with the Peebles idea of replacing $\bar{\xi}$ with $\bar{\bar{\xi}}$. We consider the following ansatz:

$$v_{12} \;=\; -\tfrac{2}{3}\,Hrf\bar{\bar{\xi}}\,\left[1 \;+\; \alpha\bar{\bar{\xi}}\,\right]\;. \qquad (7)$$

The above expression agrees with the perturbative expansion (5) when $\xi \to 0$, and it behaves like the stable clustering limit, $v_{12} = -Hr$ on small scales up to the factor $(2/3)f(\Omega)(1+\alpha)$, which is of order unity in the real universe and for the models considered here. In the following section, we compare our approximation with N-body simulations.

## 3. N-body simulations

We analyze high-resolution AP$^3$M simulations of $256^3$ dark matter particles in periodic boxes of comoving volume $(240\,h^{-1}\mathrm{Mpc})^3$, kindly provided to us by the Virgo collaboration (Jenkins et al. 1998). We consider four variants with linear input spectra inspired by cold dark matter cosmologies: A 'standard SCDM model with $\Omega = 1$, Hubble parameter $h = 0.5$ and normalization $\sigma_8 = 0.6$, a variant of this model with more large-scale power ($\tau$CDM), and two low-density models with $\Omega = 0.3$ and $h = 0.7$, one open (OCDM, $\sigma_8 = 0.85$), and one spatially flat ($\Lambda$CDM, $\sigma_8 = 0.9$). The simulations have been normalized to the observed abundance of rich clusters of galaxies (for details, see Jenkins et al. 1998).

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[4]Here and for the linear solution, derived earlier, we have used the boundary condition $v_{12} = 0$ at $x = 0$. This condition holds at every order of the expansion.

Since CDM-like models are not scale-free, we first need to address the problem of finding an effective slope $\gamma_{\rm eff}$. In Fig. 1, we show the logarithmic slope of the linear theory correlation function, and compare it to the slope of the measured non-linear correlation function of the simulations. As expected, both curves agree at large separations, apart from small differences arising from noise in the measurement (we use only a finite number of bins and pairs to measure $\xi$), and from finite box-size and cosmic-variance effects (Jenkins et al. 1998). However, there is a well-defined scale at which the non-linear slope turns away from the linear theory prediction, marking the onset of the non-linear regime. We can hope to use perturbation theory with success at most up to this scale, hence we take $\gamma_{\rm eff}$ to be the logarithmic slope of $\bar{\xi}^{(1)}$ at that scale. Judging from Fig. 1, we identify the points marked with an asterisk as these 'turn-away' scales. They correspond to $\gamma_{\rm eff} = 1.53$ (SCDM), $\gamma_{\rm eff} = 1.46$ ($\Lambda$CDM), $\gamma_{\rm eff} = 1.46$ (OCDM), and $\gamma_{\rm ef} = 1.39$ ($\tau$CDM), respectively. With the exception of SCDM model, where $\gamma$ becomes larger than 2 and our second order perturbation theory formula (6) based on simple power law spectra do no longer apply, these values are in good agreement with $\gamma_{\rm eff} \equiv -(d\ln\xi/d\ln x)|_{\xi=1}$. The advangage of the latter definition being that all of the quantities involved are observable.

In Figure 2 we compare our N-body measurements of the mean relative velocity $v_{12}$ with four approximate closed-form solutions of the pair conservation equation. We consider two versions of linear approximation, eq. (2) and eq. (3), the Peebles approximation, (eq. 4), and finally the ansatz (7) proposed in this paper. For the latter, we use the effective slopes given above. Based on these curves, we find that the deviations from linear theory are small at large separations, as they should, but the linear approximations break down completely in the strongly non-linear regime. The range of validity of the Peebles formula is already considerably wider than that of linear theory. However, our new ansatz provides by the far the best approximation. In fact, it covers the entire dynamical range probed by the simulations.

## 4. Velocity bias

So far we considered the dynamics of pairwise motions of dark matter particles. However, for practical applications, it is necessary to understand the relation between $v_{12}(r)$ and the relative pairwise velocity of galaxies, $v_{12{\rm g}}(r)$. We define the galaxy clustering bias as the square root of the ratio between the galaxy and the dark matter correlation functions: $b(r)^2 = \xi_{\rm g}(r)/\xi(r)$. In the simplest analytical toy model for bias, $b$ is a time- and scale-independent constant, and it is usually assumed that $b = 1/\sigma_8$. Moreover, in this model, also known as linear biasing, the galaxy density contrast at position $\vec{r}_A$ is simply given by $\delta_{gA} = b\delta_A$, where $\delta_A \equiv \rho_A/\langle\rho\rangle - 1$ is the mass density fluctuation, and $A = 1, 2, \ldots$ enumerate galaxy positions. For the dark matter particles, the mean relative pairwise velocity is

$$\vec{v}_{12}(r) = \frac{\langle(\vec{v}_1 - \vec{v}_2)(1+\delta_1)(1+\delta_2)\rangle}{1+\xi(r)} \ , \tag{8}$$

where $\vec{v}_A$ is the peculiar velocity at a point $\vec{r}_A$, $r = |\vec{r}_1 - \vec{r}_2|$ is the separation, and $\xi(r) = \langle\delta_1\delta_2\rangle$. For the galaxy pair density-weighted relative velocity, $v_{12{\rm g}}$, the matter density field in the above expression, $\delta$, has to be replaced by $\delta_{\rm g}$. In the limit of large separations ($\xi \to 0$), the linear biasing model then gives $v_{12{\rm g}}(r) = bv_{12}(r)$ and since $v_{12} \propto {\sigma_8}^2\Omega^{0.6}$ one obtains $v_{12{\rm g}} \propto \sigma_8\Omega^{0.6}$ (Fisher et al. 1994). On small scales, where $1+\delta \sim \delta$ and $1+\xi \sim \xi$, the factors of $b$ cancel and there is no velocity bias. This very unphysical result shows the limitations of the linear biasing model. It seems more natural that bias is scale and time dependent. One can actually show that Newtonian clustering, which we assume to be the relevant mechanism on large scales, actually tends to erase bias (Fry 1984). Also large numerical simulations using different biasing models find no significant velocity bias (Kauffmann et al. 1998). The same result is obtained from actual data: Splitting the Mark III catalogue into subsamples of elliptical and spiral galaxies has no effect on the velocities. We therefore conclude the velocities are not biased and $v_{12g} = v_{12}$. On large

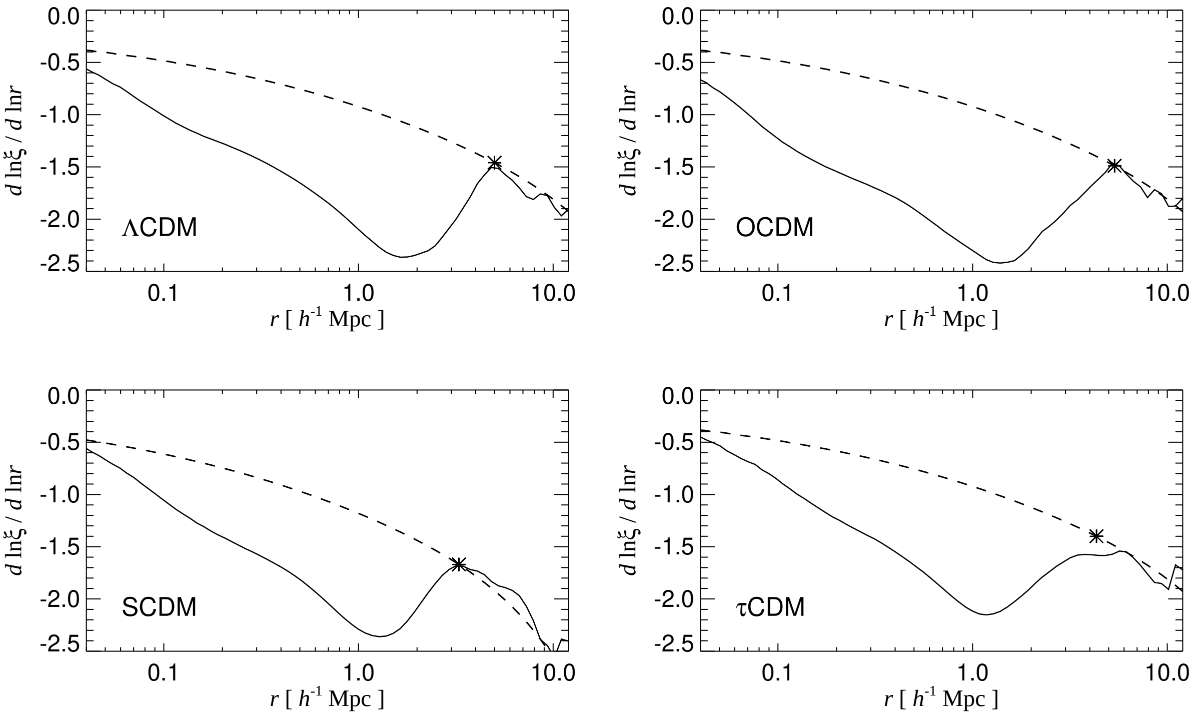


Fig. 1.— Logarithmic slope $d\ln\xi/d\ln r$ of the linear theory correlation function (dashed), and the measured non-linear correlation function (solid) for the four Virgo simulations which we have analyzed. The asterisks mark the effective slopes $\gamma_{\rm eff}$ used in equations (6) and (7), respectively.

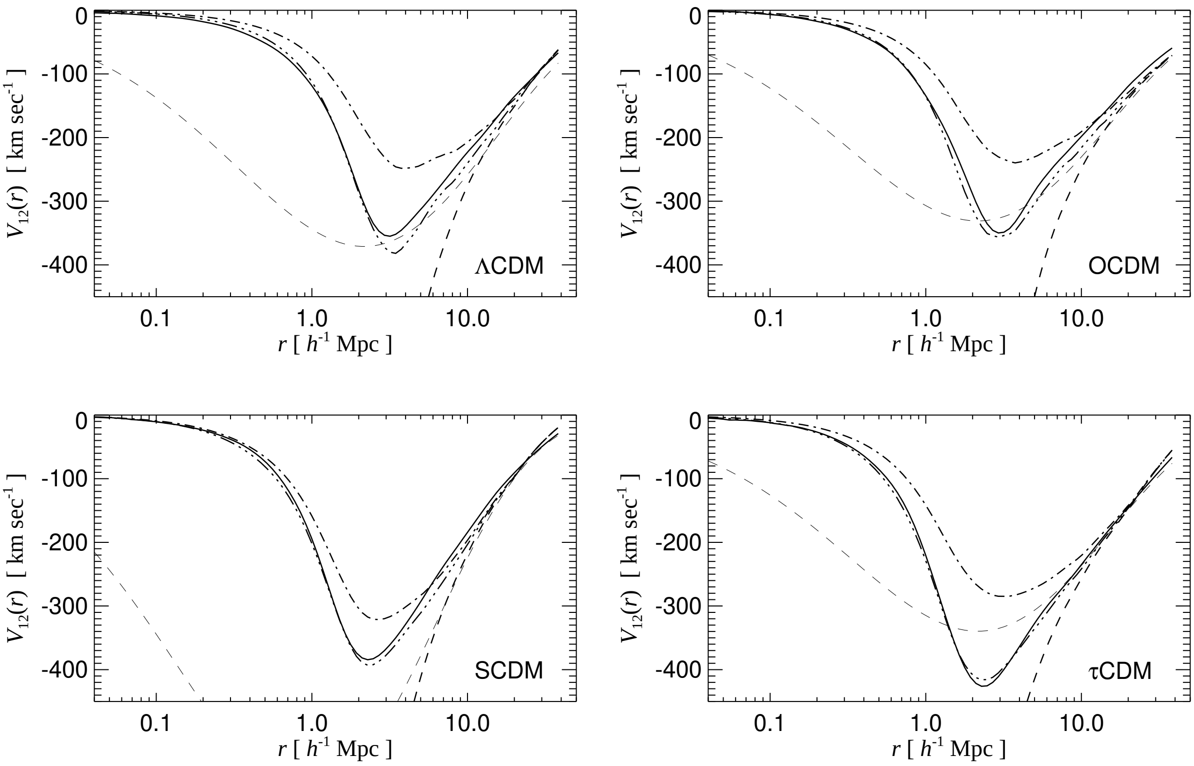


Fig. 2.— The mean pairwise velocity $v_{12}$ of the Virgo simulations (solid lines) compared with four closed-form approximations to solutions of the pair conservation equation: the two versions of the linear approximation, eq. (2), and eq. (3), are plotted as thin and thick dashed curves respectively; the Peebles approximation, eq. (4), is shown as dot-dashed curve; and eq. (7) – the Ansatz proposed in this paper – is drawn as dot-dot-dot-dashed line.

scales we then obtain from the linearized pair conservation, eq. (3),

$$v_{12}(r) = -\frac{2}{3} H r \Omega^{0.6} \sigma^2(r) \ . \tag{9}$$

which yields the combination $\Omega^{0.6}\sigma_8^2$ from peculiar velocity measurements (Ferreira et al. 1998). If the galaxy correlation function can be approximated by $\xi_g(r) = \xi(r)/\sigma_8$, the non-linearity in relation (7) actually removes the degeneracy and allows the determination of $\Omega$ and $\sigma_8$ separately in the mildly non-linear regime.

## 5. Conclusions

We have found an analytic formula, (7), relating pairwise velocities of galaxies to an integral of the two-point correlation function. Our formula provides an excellent fit to numerical simulations on all scales from the strongly non-linear to the linear regime. Its comparison with observations can provide an important test of the gravitational instability hypothesis as pairwise velocities clearly separate linear from non-linear scales.

On very large scales, pairwise velocities measure the combination $\Omega^{0.6}\sigma_8^2$. On intermediate scales (the mildly non-linear regime) the degeneracy is removed and $\Omega$ and $\sigma_8$ can be measured separately. This differs from other estimators: The POTENT method (Sigad et al. 1998) and the cluster abundances (Bahcall & Fan, 1998, Eke et al. 1998) are sensitive to $\Omega^{0.6}\sigma_8$, the super-novae Ia (Riess et al. 1998, Perlmutter et al. 1998) distances measure $\Omega_{\rm matter} - \Omega_\Lambda$, and the position of the acoustic peaks in the CMB power spectrum (Doroshkevich et al. 1978) is sensitive to $\Omega_{\rm matter} + \Omega_\Lambda$. The advantage of our estimator over the last method is its model independence; and unlike the first method it is related to observations by a straight forward procedure (Ferreira et al. 1998). Our formula provides a new, independent and powerful way to measure $\Omega$ and (together, *e.g.* with cluster abundances) the dark matter fluctuation amplitude, $\sigma_8$.

Finally, it may be worth investigating why our simple formula allows such an accurate prediction of the full curve of $v_{12}(r)$ based on the two-point correlation function of mass fluctuations *alone*. This is due to the generic shape of the non-linear correlation function in CDM cosmologies. Non-linearities develop a kink in the effective spectral index $\gamma(r)$ which allows an identification of $\gamma_{\rm eff}$ as the maximum slope of $\xi(r)$ realized on scales $1\,h^{-1}{\rm Mpc} \le r \le 10\,h^{-1}{\rm Mpc}$ (see Fig. 1). A good approximation to $\gamma_{\rm eff}$ is also $\gamma(r_{\rm nl})$, where $r_{\rm nl}$ is defined by $\xi(r_{\rm nl}) = 1$. In addition, the fact that second order corrections are relatively weak and that the result is not very sensitive to the definition of $\gamma_{\rm eff}$, is related to the slope of the linear power spectrum being close to $\gamma = 1.6$.

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