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DYNAMICS OF PIECEWISE ISOMETRIES

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ABSTRACT. We begin a systematic study of Euclidean piecewise isometric dynamical systems (p.i.d.s.) with a particular focus on the interplay between geometry, symbolic dynamics, and the group of isometries associated with p.i.d.s. We investigate various aspects of the dynamical information contained in the coding: symbolic growth and the periodic behavior of codings and cells. This theoretical investigation is motivated by the many examples of piecewise isometric dynamical systems found recently in the literature. Piecewise isometric dynamical systems are direct generalizations of interval exchange transformations to non-invertible, higher dimensional maps.

1. Introduction

In this manuscript we begin underlying basic foundations of the theory of Euclidean piecewise isometric dynamical systems. The aim of this theory is to study the long term behavior of states in a dynamical system for which the local generating maps are isometries. Our motivation for this article includes the many intriguing examples of piecewise isometries with their mysterious behavior and beautiful computer graphics.

Systems of piecewise isometries have been recently linked to the dynamics of electronic components called digital filters [1], [3], [4], [8], [9], [10], [27]. Moreover, these systems generalize well known and well studied interval exchanges to a class of Euclidean two-dimensional piecewise isometries. Piecewise isometries appear in a variety of contexts and have been recently extensively studied as interval exchanges [2], [6], [11], [21], [23], [25], [31], [32], interval translations [7], rectangular exchanges [19], polygonal and polyhedron exchanges [17], and pseudogroup systems of rotations [24]. Piecewise isometric maps appear naturally in billiards [5], [16], dual billiards [30], [18], theory of foliations [26], and tilings [20], [28].

The structure of the article is as follows. In the first five sections, we study the growth of symbolic sequences associated to p.i.d.s. The main result is in Section 4 and it relates the growth of the associated semigroup of isometries to the growth of symbolic words.

THEOREM (Theorem 4.1). *Let $T\colon X \to X$ be a piecewise isometric system whose associated semigroup of isometries has polynomial growth. Also, suppose that the partition associated to T consists of sets that are a finite union of convex sets. Then the growth of symbolic words is also polynomial.*

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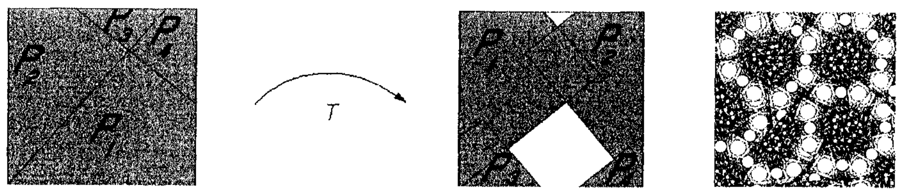

Figure 1. Piecewise rotation on the square: T: $[0,1] \times [0,1]$, $Tx = Mx \bmod \mathbb{Z}^2$ where $M \in SO(\mathbb{R}, 2)$ is an orthogonal matrix representing the rotation by an angle close to 40°. The right square is partitioned into the mosaic of "cells", sets following the same coding. White regions in the right figure follow eventually periodic codings.

The main idea in the proof of Theorem 4.1 is to show that the growth of the number of certain sets that are the building blocks of sets following the same codings (we call them *beans*) is polynomial. In the proof of this theorem, we use elements of the Euclidean theory of convex structures, and in particular, we use the Kakutani Separation Property.

It follows from our main result that two-dimensional piecewise rotations whose induced isometries are of finite order induce polynomial symbolic growths, and hence have zero entropy.

Further we study necessary condition for a p.i.d.s. to generate all possible finite words. We show that in order for a p.i.d.s. to induce a maximal growth, the induced isometries must have common fixed point (Proposition 5.1).

We conclude the article with remarks on the interplay between symbolic codings, periodic points, and the geometry of cells. This involves introducing a partition of X into cells (sets following the same coding pattern, the picture on the right, Figure 1) and the study of the relation between the symbolic codings and cells of positive measures. Finally, a number of results included in Section 6 are generalizations of well-known results for interval exchanges.

2. Preliminaries

In this section, we define our systems. We also define encoding of orbits via symbolic dynamics. Symbolic dynamics is used extensively in the description of the dynamics of p.i.d.s.

Definition. Let X be a subset of $\mathbb{R}^{\mathbb{N}}$ and $\mathcal{P} = \{P_0, \ldots, P_{r-1}\}$ $(r > 1)$ be a finite partition of X, that is, $\bigcup_{0 \leq i < r} P_i = X$, and $P_i \cap P_j = \vee$ for $i \neq j$.

A *piecewise isometry* is a pair $(T, \mathcal{P})$, where T: $X \mapsto X$ is a map such that its restriction to each *atom* P_i, $i = 0, \ldots, r-1$ is a Euclidean isometry.

We also refer to the map T as a piecewise isometry. We assume that the partition $\mathcal{P}$ is minimal in the sense that T is not an isometry on the union of two distinct elements of $\mathcal{P}$. Frequently, the space X will be compact or at least bounded. Finally, note that a piecewise isometry need not be invertible (see Figure 1).

The partition $\mathcal{P} = \{P_0, \ldots, P_{r-1}\}$ associated to a piecewise continuous map T of X gives rise to a natural one-sided coding map ϕ: $X \to \Omega_r = \{1, \ldots, r\}^{\mathbb{N}}$ for T. The map ϕ encodes the forward orbit of a point by recording the indices of atoms visited by the orbit, that is, $\phi(x) = w_0 w_1 \cdots$, where $T^k x \in P_{w_k}$.

The coding map ϕ conjugates T with the (one-sided) *shift map* S: $\Omega_r \to \Omega_r$, $S(w_0 w_1 w_2 \cdots) = w_1 w_2 \cdots$:

(1)
$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \phi\downarrow & & \downarrow\phi \\ \Omega_r & \xrightarrow{S} & \Omega_r \end{array}$$

The shift map S: $\Omega_r \to \Omega_r$ is continuous with respect to the product topology of $\mathbb{N}$ copies of the finite set $\{1, \ldots, r\}$, each with discrete topology. This product topology can be also generated by the metric

$$d_{\Omega_r}\colon \Omega_r \times \Omega_r \to \mathbb{R}^+ \cup 0, \quad d_{\Omega_r}(\phi_1, \phi_2) = \sum_{i=0}^{\infty} \frac{|\phi_1(i) - \phi_2(i)|}{r^i}.$$

We equip Ω with the topology given by the metric d_Ω.

3. Graph of the coding map

The symbolic coding is of fundamental importance as it gives rise to a space $\hat{X}$ on which T has a continuous extension $\hat{T}$. This allows us to apply many standard results in dynamical systems. For example, using $\hat{T}$: $\hat{X} \to \hat{X}$, we propose a definition of the entropy of piecewise dynamical systems.

The idea of the construction of a continuous extension map $\hat{T}$ is to "separate" *cells* (sets of points encoded by the same sequences), by placing them at different levels in the set $X \times \Omega$.

Let $G = \{(x, \phi(x)), x \in X\}$ be the graph of ϕ: $X \to \Omega$ topologized by the product metric d: $(X \times \Omega)^2 \to \mathbb{R}^+ \cup 0$, $d((x_1, \phi(x_1)), (x_2, \phi(x_2))) = \max\{d_e(x_1, x_2)), d_{\Omega_r}(\phi(x_1), \phi(x_2))\}$, where d_e is the Euclidean metric in $\mathbb{R}^N$. With this notation, we obtain the following proposition [14].

PROPOSITION 3.1. *The extension map T_G: $G \to G$, $(x, \phi(x)) \to (T x, \phi(T x))$ is continuous.*

Let $\Pi_X \colon X \times \Omega_r \to X$ and $\Pi_{\Omega_r} \colon X \times \Omega_r \to \Omega_r$ be natural projections. We now have the diagrams

$$\begin{array}{ccc} G & \xrightarrow{T_G} & G \\ {\scriptstyle \Pi_X}\downarrow & & \downarrow{\scriptstyle \Pi_X} \\ X & \xrightarrow{T} & X \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{T_G} & G \\ {\scriptstyle \Pi_{\Omega_r}}\downarrow & & \downarrow{\scriptstyle \Pi_{\Omega_r}} \\ \phi(X) & \xrightarrow{S} & \phi(X) \end{array}$$

and a corollary that we will use in our definition of the entropy.

COROLLARY 3.1. *Let* $\overline{G_T} = \hat{X}$. *Let* $\hat{T} \colon \hat{X} \to \hat{X}$ *be the continuous extension of* $T_G \colon G_T \to G_T$. *Then* $\Pi_{\Omega_r}|_{\hat{X}} \colon \hat{X} \to \overline{\phi(X)}$ *is a continuous, surjective mapping and* $S|_{\overline{\phi(X)}} \colon \overline{\phi(X)} \to \overline{\phi(X)}$ *is a topological factor map of* $\hat{T}$. *In particular, the following diagram commutes*:

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{T}} & \hat{X} \\ {\scriptstyle \Pi_{\Omega_r}}\downarrow & & \downarrow{\scriptstyle \Pi_{\Omega_r}} \\ \overline{\phi(X)} & \xrightarrow{S} & \overline{\phi(X)}. \end{array}$$

If X is compact, then $\hat{X}$ is compact. Since $\hat{T} \colon \hat{X} \to \hat{X}$ is a continuous map, it is natural to define the *entropy of a piecewise isometry* $T \colon X \to X$ (for compact X) as the topological entropy of $\hat{T} \colon \hat{X} \to \hat{X}$.

PROPOSITION 3.2. *The entropy of a piecewise isometry* $T \colon X \to X$ *(X is compact) is equal to* $h(S|_{\overline{\phi(X)}})$, *the topological entropy of the shift map restricted to* $\overline{\phi(X)}$.

Let W denote all (finite) words generated by $\{0, \dots, r-1\}$ that appear as a subword of an element in $\phi(X)$, and let $W(n) \subset W$ be the collection of the words of length n; thus $W = \bigcup_{n \geq 1} W(n)$. Since $W(n)$ is also the set of words that appear as a subword of an element in $\overline{\phi(X)}$ of length n, $h(S|_{\overline{\phi(X)}})$ is equal to the exponential growth rate of the cardinality of $W(n)$. Proposition 3.2 thus yields:

COROLLARY 3.2. *The entropy of a piecewise isometry* $T \colon X \to X$ *(X is compact) is equal to*

$$\limsup_{n \to \infty} \frac{\log \|W(n)\|}{n}.$$

Corollary 3.2 can also serve as the definition of the entropy for a piecewise isometric system. Thus, in order to study the entropy of p.i.d.s., one needs to study the exponential growth rate of $W(n)$. We will state some results on this subject in the next section.

Proof of Proposition 3.2. Since $S|_{\overline{\phi(X)}}$ is a topological factor of $\hat{T}$ (Corollary 3.1), $h(\hat{T}) \geq h(S|_{\overline{\phi(X)}})$. We show that the opposite inequality follows from the definition of the entropy that uses (n, ϵ)-separated sets [29]. It is enough if we observe that for any fixed $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{\log W(n)}{n} \geq \limsup_{n\to\infty} \frac{\log r(n,\epsilon)}{n}, \tag{2}$$

where $r(n, \epsilon) = \max\{\|A\|, A \subset \hat{X}$ is an (n, ϵ) – separated set$\}$.

Fix $\epsilon > 0$. Since $\hat{X}$ is compact, there is some k such that every $(0, \epsilon)$-separated set has cardinality smaller than k. We show that $r(n, \epsilon) \leq k\|W(n)\|$ (then inequality (2) follows). Suppose otherwise. Let $A \subset \hat{X}$ be an $(n, \epsilon)-$separated set. By the pigeon-hole principle, there exists a subset $Z \subset A$ of cardinality $(k + 1)$ and a finite word $w_0w_1 \cdots w_{n-1} \in W(n)$ such that for every $(x, \omega) \in Z$, $(\omega)_i = w_i$ for $i \in \{0, \dots, n - 1\}$. Since $\|Z\| > k$, Z is not $(0, \epsilon)$-separated, and thus there are two points $(x, \omega), (y, \eta) \in Z$ such that $d((x, \omega), (y, \eta)) \leq \epsilon$. Since the codings ω and η agree on the first n slots, $d(\hat{T}^i(x, \omega), \hat{T}^i(x, \omega)) = d((x, \omega), (y, \eta)) \leq \epsilon$ for all $i \in \{0, \dots, n - 1\}$. This means that (x, ω) and (y, η) are not $(n, \epsilon)-$separated, hence $r(n, \epsilon) \leq k\|W(n)\|$ and thus inequality (2) follows. □

Remark 1. An alternative approach to define the entropy for our systems could be to take the supremum of the measure-theoretic entropies of T: $X \to X$ over all invariant Borel probability measures (a theorem in [1] may implicitly suggest this definition). We do not know whether this entropy is the same as the entropy that we defined.

Remark 2. In a more restrictive context, the entropy of a piecewise isometry was also defined in [17].

4. Polynomial symbolic growth

In this section, we explore the growth of all possible symbolic sequences of finite length generated by the dynamics associated with T. The rate of growth of symbolic words is one way of measuring the complexity of the dynamics of p.i.d.s. The main result of this section is Theorem 4.1, which states that if the group associated to T has a polynomial growth, then the cardinality of $W(n)$ (the set of allowable words of length n in the symbolic dynamics associated to T) is bounded by a polynomial in n. It then follows from Corollary 3.2 that T has zero entropy.

Recall that a semigroup $\Gamma = \langle g_1, \ldots g_l \rangle^-$ has *polynomial growth* if the number of distinct elements in G that can be written as compositions of not more than n elements from $\{g_1^{-1}, \ldots g_l^{-1}\}$ is bounded by a fixed polynomial in n.

THEOREM 4.1. *Let $T\colon X \to X \subset \mathbb{R}^N$ be a piecewise isometry with partition $\mathcal{P}$. (In this theorem, we allow X to be unbounded.) Suppose that every atom in $\mathcal{P}$ is a disjoint union of a finite number of convex sets. Suppose that the semigroup $\langle T_0, \ldots, T_{r-1} \rangle^-$ generated by induced isometries has polynomial growth. Then the cardinality of words in $W(n)$ is bounded by a polynomial in n.*

Theorem 4.1 immediately yields a result for piecewise translations:

COROLLARY 4.1. *Piecewise translations in $\mathbb{R}^N$ with convex partitions induce polynomial symbolic word growths.*

Another application of Theorem 4.1 is a result for two-dimensional piecewise rotations. A *two-dimensional piecewise rotation* is a piecewise isometry $T\colon X \to X \subset \mathbb{R}^2$ whose induced isometries are orientation preserving isometries.

COROLLARY 4.2. *A two-dimensional piecewise rotation with convex partition all of whose induced isometries are rotations of finite order induces polynomial symbolic word growth.*

Corollary 4.2 can be applied, for example, to piecewise rotations studied in [1], [15], [14].

Remark 1. In Theorem 4.1 some hypothesis on the atoms is necessary in order for the conclusion to hold. Because if the partition $\mathcal{P}$ is totally disconnected, then one can show that there are examples [12], [13] for which the group of induced isometries is even commutative (hence has polynomial growth) and yet all finite codings are realizable (hence the cardinality of words in $W(n)$ grows exponentially in n).

Remark 2. Theorem 4.1 generalizes some partial and very particular remarks found in the literature. This theorem (and in particular Corollary 4.1) can be viewed as a generalization of a remark in [7] on the growth of possible codings in one-dimensional interval translation maps. A related statement in the context of partially defined rotations on the circle can be found in [24]. Also, a particular case of Corollary 4.2 with a different proof can be found in [1].

Remark 3. The techniques used in the proof of Theorem 4.1 also work for a larger class of systems called piecewise affine maps [1], [17]. Given a set X and its finite partition $\mathcal{P}$, we define a a *piecewise affine map* to be a map $T\colon X \mapsto X$ whose restriction to each element of $\mathcal{P}$ is an affine map.

Proof of Theorem 4.1. The proof consists of two steps. We first define a partition into "n-beans" which are the building blocks of n-cells (Lemma 4.1). Then, using a classic result about the maximal number of regions determined by k half-spaces (Lemma 4.2), we show that the number of n-beans is bounded by a polynomial in n.

Step 1. In order to define n-beans, we first show that the atoms are elements of some algebra generated by a finite number of half-spaces intersected with X.

Let $\mathcal{Q}$ be a finite convex sub-partition of $\mathcal{P} = \{P_0, \dots, P_{r-1}\}$. Define a *half-space* of $\mathbb{R}^N$ to be a convex set $H \subset R^N$, $H \notin \{\vee, \mathbb{R}^N\}$, whose complement $\mathbb{R}^N - H$ is convex, too. By the Kakutani Separation Property (for example, see [33]), every pair $\{Q_i, Q_j\} \subset \mathcal{Q}$ $(Q_i \neq Q_j)$ can be separated by some half-space $H_{Q_iQ_j}$, that is, $Q_i \subset H_{Q_iQ_j}$ and $Q_j \in \mathbb{R}^N - H_{Q_iQ_j}$.

It follows that for every $Q_i \in \mathcal{Q}$,

$$Q_i = X \cap \bigcap_{Q_j \in \mathcal{Q}, Q_i \neq Q_j} H_{Q_iQ_j}. \tag{3}$$

Since every atom $P_j \in \mathcal{P}$ is the union of some elements in $\mathcal{Q}$, from (3) it follows that $P_j = X \cap A$, where A an element of the algebra $\mathcal{A}(\mathcal{H}_0)$ generated by the finite collection of half-spaces $\mathcal{H}_0 = \{H_{Q_iQ_j}\colon Q_i, Q_j \in \mathcal{Q}, Q_i \neq Q_j\}$.

Let $\langle T_0, \dots T_{r-1}\rangle_n^-$ denote the collection of isometries that are the compositions of not more than n elements from $\{T_0^{-1}, \dots, T_{r-1}^{-1}\}$. For $n > 0$, let

$$\mathcal{H}_n = \{G(H)\colon G \in \langle T_0, \dots, T_{r-1}\rangle_n^-;\ H \in \mathcal{H}_0\}.$$

Finally, the sets in the collection

$$\mathcal{B}_n = \left\{X \cap \bigcap_{H \in \mathcal{H}_n} H^*\colon H^* = H \text{ or } H^* = \mathbb{R}^N - H\right\}$$

will be called *n-beans*.

LEMMA 4.1. *Every n-cell is the union of non-empty n-beans.*

Proof. In order to prove this lemma, first observe the following:

1. Every n-bean is contained in exactly one atom.
2. For every $B \in \mathcal{B}_n$ and induced isometry T_{s_0}, $T_{s_0}^{-1}B$ is the union of a collection of $(n+1)$-beans.
3. For every n-cell $\langle s_0 s_1 \cdots s_n\rangle$, $(\langle s_0 s_1 \cdots s_n\rangle = \{x \in X\colon T^k x \in P_{s_k}, 0 \le k \le n\})$, $\langle s_0 s_1 \cdots s_n\rangle = T_{s_0}^{-1}\langle s_1 s_2 \cdots s_n\rangle \cap P_{s_0}$.

We prove Lemma 4.1 by induction. By observation 1, and because the union of all 0-beans contains X, Lemma 4.1 follows for $n = 0$. Let $\langle s_0 s_1 \cdots s_n\rangle$ be any $(n+1)$-cell. By inductive hypothesis, $\langle s_1 \cdots s_n\rangle$ is the union of n-beans. We note that by observation 2, and by definition of the bean, $T_{s_0}^{-1}\langle s_1 s_2 \cdots s_n\rangle$ is the union of $(n+1)$-beans. By observation 1, also $T_{s_0}^{-1}\langle s_1 s_2 \cdots s_n\rangle \cap P_{s_0}$ is the union of $(n+1)$-beans. Hence, from observation 3, we conclude that $\langle s_0 s_1 \cdots s_n\rangle$ is the union of $(n+1)$-beans, which concludes the proof of Lemma 4.1. □

Step 2. In this step, we show that the cardinality of $\mathcal{B}_n$ is bounded by a polynomial in n. We use the following lemma (which is probably a classic result in the theory of convex structures).

LEMMA 4.2. *Let $\{L_1, \dots, L_k\}$ denote a collection of half-spaces in $\mathbb{R}^N$. Let*

$$\mathcal{D}_{L_1,\dots,L_k} = \left\{ \bigcap_{i=1}^{k} L_i^* \text{ where } L_i^* = L_i \text{ or } L_i^* = \mathbb{R} - L_i \right\}.$$

Then the cardinality of $\mathcal{D}_{L_1,\dots,L_k}$ is bounded by a polynomial in k. In particular, $\|\mathcal{D}_{L_1,\dots,L_k}\| \le ck^N$, where c is a constant that depends only on N.

Proof of Lemma 4.2. Let $f_N(k) = \max \|\mathcal{D}_{L_1,\dots,L_k}\|$ where the maximum extends over all possible collections of k half spaces. We show that following recursive relation holds:

$$f_{N+1}(k+1) \le f_{N+1}(k) + f_N(k). \tag{4}$$

In order to show (4), among all collections of $(k+1)$ half-spaces, let $\{L_1, \dots, L_{k+1}\}$ be such that $\|\mathcal{D}_{L_1,\dots,L_k}\| = f_N(k)$. Given the collection $\mathcal{D}_{L_1,\dots,L_k}$ of regions determined by the first k half-spaces, by adding the half-space L_{k+1}, only the sets in $\{D \in \mathcal{D}_{L_1,\dots,L_k}\colon D \cap L_{k+1} \ne \emptyset$ and $D \cap L_k \ne \emptyset\}$ are subdivided into two subsets. Hence,

$$\begin{aligned}
\|\mathcal{D}_{L_1,\dots,L_{k+1}}\| &= \|\mathcal{D}_{L_1,\dots,L_k}\| + \|\{D \in \mathcal{D}_{L_1,\dots,L_k}\colon D \cap L_{k+1} \ne \emptyset \\
&\qquad\qquad \text{and } D \cap (\mathbb{R}^{N+1} - L_{k+1}) \ne \emptyset\}\| \\
&\le \|\mathcal{D}_{L_1,\dots,L_k}\| + \|\{D \in \mathcal{D}_{L_1,\dots,L_k}\colon D \cap \partial L_{k+1} \ne \emptyset\}\| \\
&= \|\mathcal{D}_{L_1,\dots,L_k}\| + \left\| \left\{ D = \bigcap_{i=1}^{k} L_i^* \cap \partial L_{k+1}\colon L_i^* = L_i \right.\right. \\
&\qquad\qquad \left.\left. \text{or } L_i^* = \mathbb{R}^{N+1} - L_i,\ D \cap \partial L_{k+1} \ne \emptyset \right\} \right\| \\
&= \|\mathcal{D}_{L_1,\dots,L_k}\| + \left\| \left\{ \bigcap_{i=1}^{k} (L_i^* \cap \partial L_{k+1})\colon L_i^* = L_i \text{ or } L_i^* = \mathbb{R}^{N+1} - L_i \right\} \right\| \\
&\le f_{N+1}(k) + f_N(k).
\end{aligned}$$

The last inequality followed from the inductive hypothesis applied to k half-spaces in $\mathbb{R}^{N+1}$ and also applied to k half hyper-planes in the hyper-plane ∂L_{k+1}. Hence inequality (4) holds.

From inequality (4), we obtain

$$f_{N+1}(k+1) \leq f_{N+1}(1) + \Sigma_{i=1}^{k} f_N(i), \tag{5}$$

and the conclusion of Lemma 4.2 follows from the induction applied to (5) and the following integral estimate: $\Sigma_{i=1}^{k} i^N < \frac{1}{N+1}(k+1)^{N+1}$. □

Finally, in order to conclude the proof of Theorem 4.1, note that since the semi-group $\langle T_0, \ldots, T_{r-1}\rangle^-$ has polynomial growth, the cardinality of $\mathcal{H}_n$ is bounded by a polynomial in n. By Lemma 4.2, $\|\mathcal{B}_n\|$ is bounded by a polynomial in n, and since, by Lemma 4.1, the number of n-cells does not exceed $\|\mathcal{B}_n\|$, the conclusion of Theorem 4.1 follows. □

Proof of Corollary 4.2. By Theorem 4.1, it is enough to show that the induced isometries $\{T_0, \ldots, T_{r-1}\}$ of a piecewise isometry T generate a semi-subgroup of the group of two-dimensional isometries with polynomial growth. This is a well-known result and it can be observed, for example, by explicitly writing the composition of isometries. Let $\Gamma = \langle T_0, \ldots, T_{r-1}\rangle^-$, where in the complex notation, $T_i^{-1}x = \rho_i x + z_i$, and ρ_i is a root of unity. Then (by induction) an element in Γ of length m can be written as

$$T_{i_0}^{-1} T_{i_1}^{-1} \cdots T_{i_{m-1}}^{-1} x = R x + \tau, \tag{6}$$

where

$$R = \rho_{i_0} \rho_{i_1} \cdots \rho_{i_{m-1}} \text{ and } \tau = \rho_{i_0} \rho_{i_1} \cdots \rho_{i_{m-2}} z_{i_{m-1}} + \cdots + \rho_{i_0} z_{i_1} + z_{i_0}.$$

Since the multiplicative group generated by the roots of unity $\rho_0, \rho_1, \ldots, \rho_{m-1}$ has a finite order, the rotational part R of the isometry (6) may be at most one out of g different rotations, where g is some fixed number that does not depend on m. The translational part τ of the isometry (6) can assume at most $r\, m^r$ values. Hence, in Γ, the there are at most $g\, r\, m^r$ distinct elements of length m. □

5. Maximal symbolic growth

A natural question is whether there exist systems that give rise to richer than polynomial symbolic growths. In this section, we list necessary conditions for p.i.d.s. to give rise to maximal exponential growths.

PROPOSITION 5.1. *Let $X \subset \mathbf{R}^N$ be a bounded set, and let T: $X \to X$ be a piecewise isometry with maximal symbolic word growth (the cardinality of the* *set* $W(n)$ *is* r^n). *Then the induced isometries*: $T_0, \dots, T_{r-1}$ *have a common fixed point.*

Remark 1. Given the set of induced isometries with a common fixed point, there are piecewise isometries that induce a maximal growth [12], [13]. However, these examples are "very artificial" since the atoms of the partition are totally disconnected and they satisfy $\overline{P_i} = \overline{P_j}$ (but $P_i \cap P_j = \vee$). From the main result in [12], if follows that there do not exist piecewise isometries with convex partition and surjective coding map (a stronger condition than in Proposition 5.1).

Remark 2. We do not know any examples of piecewise isometries with convex partition for which the growth rate of $W(n)$ is exponential. In [17], there is a result that two-dimensional invertible piecewise isometries have zero entropy, though the definition of the entropy in [17] is different.

Proof of Proposition 5.1. Suppose $T\colon X \to X$ induces a maximal symbolic word growth. Fix a point $p \in X$. Let $\langle T_0, \dots, T_{r-1}\rangle^+$ be the semigroup generated $T_0, \dots, T_{r-1}$. The key idea in the proof is to show that the orbit of a ball centered at p under the semigroup $\langle T_0, \dots, T_{r-1}\rangle^+$ is bounded.

Let $w = \sigma_0\sigma_1 \cdots \sigma_{n-1} \in W(n)$ be any word of length n ($\sigma_i \in \{0, \dots, r-1\}$). Let $T_w = T_{\sigma_{n-1}} \cdots T_{\sigma_0}$. Since every finite code is realizable, there is a point $x_w \in X$ such that $\phi(x_w) = \sigma_0 \cdots \sigma_n \cdots$. In particular, $|Tx_w - x_w| \le \operatorname{diam}(X)$. We use this and the fact that T_w is an isometry in the following key estimate:

$$\begin{aligned} |T_w p - p| &\le |T_w p - T_w x_w| + |T_w x_w - x_w| + |x_w - p| \\ &\le |p - x_w| + |Tx_w - x_w| + |x_w - p| \le 3 \operatorname{diam}(X). \end{aligned}$$

Since $w \in W$ was arbitrary, the above inequality implies that the set

$$U = \bigcup_{G \in \langle T_0, \dots, T_{r-1}\rangle^+} G(B(p, 1))$$

is bounded.

Since the set U has finite and positive measure, its center of mass o is well defined. Because T_i preserves measure and $T_i U \subset U$, $T_i U \cup Z_i = U$ where Z_i is some of zero measure. It follows that the center of mass o must be fixed by each T_i. □

6. Convexity and coding partition Σ

In this section, we remark on elementary relations between coding and Lebesgue measure, and we briefly discuss a number of applications that are derived from convexity of the atoms. Many of the statements here are generalizations of well-known results for interval exchanges.

The symbolic coding naturally induces a refinement of the partition $\mathcal{P}$, which is the *coding* partition Σ of X (right square in Figure 1). The collection Σ is induced by the equivalence relation $x \sim y$ if and only if $\phi(x) = \phi(y)$. After [22], the elements of Σ will be called *cells*.

The first proposition is an application of the Poincaré argument [13].

PROPOSITION 6.1. *If X has finite Lebesgue measure, then every cell of positive Lebesgue measure has rational (eventually periodic) coding.*

Proposition 6.1 suggests that the dynamics of cells of positive measure resemble the dynamics of the Fatou components of rational functions in the complex plane:

COROLLARY 6.1. *Suppose I is a cell of positive measure, then there exist nonnegative integers m and n such that $T^n I \subset M$ where M is a periodic cell in Σ of period m.*

In general, cells of positive measure may not exist. For example, the interval exchange of two intervals of rationally independent lengths does not have any cells of positive measure as all points have irrational codings. This example generalizes to N-dimensional interval translation maps. We say that a piecewise isometry T is an *N-dimensional piecewise translation* of X if the induced isometries of T are translations: $T_i x = x + v_i$, where $\{v_i\} \subset \mathbf{R}^N$ will be called the *translation vectors* of T.

The following proposition can be paraphrased that for a "typical" piecewise translation, the unstable set is the entire space.

PROPOSITION 6.2. *Let $T \colon X \to X$ (X is compact) be a piecewise translation map with rationally independent translation vectors. Then every point in X has irrational coding.*

In contrast, if the rational rank of the vectors $(v_0, v_1, \ldots, v_{r-1})$ is the same as the dimension of the space, then the translates of a point form a lattice, hence all points have rational coding, and if X is bounded, then there are only a finite number of different cells of positive measures.

Proof of Proposition 6.2. Suppose that there is a point $x \in X$ with rational code

$$\phi(x) = s_0 \cdots s_k s_{k+1} \cdots s_{k+l}\, s_{k+1} \cdots s_{k+l} \cdots .$$

Then $T^l(T^k x) = T_{s_{k+l}} \cdots T_{s_{k+1}}(T^k x)$. Since X is bounded and $T_{s_{k+l}} \cdots T_{s_{k+1}}$ is a translation, $T_{s_{k+l}} \cdots T_{s_{k+1}}$ is the identity. Therefore, there exists a nonzero integer linear combination of the translation vectors $(v_0, \ldots, v_{r-1})$ that is zero, a contradiction to the rational independence of $(v_0, \ldots, v_{r-1})$. $\square$

A number of additional remarks can be derived under an additional assumption that the elements of the partition $\mathcal{P}$ are convex. We conclude the article with these remarks that relate periodicity of points, codings, and the topology and size of cells.

Every cell can be written as an intersection of isometric images of the atoms [14]. Therefore, if the atoms are convex, then every cell is convex as well. Hence, using Proposition 6.1, we can describe the topology of a set of points of the same irrational code:

COROLLARY 6.2. *Suppose that the partition $\mathcal{P}$ consists of convex sets. Then the inverse image of an irrational coding under the coding map ϕ is a convex set contained in a hyper-plane.*

In particular, if the space X is a bounded subset of $\mathbb{R}^2$, then the inverse image of an irrational code $\phi \in \Omega$ must be either a single point or a line segment.

One of the applications of the corollary is a generalization of a well-known result for interval exchange maps.

PROPOSITION 6.3. *Let T be an N-dimensional piecewise translation map with convex partition. If there are $N+1$ points $x_1, x_2, \ldots, x_{N+1} \in X$ that do not lie in one hyper-plane and whose codings are the same, then all points in the simplex $(x_1, x_2, \ldots, x_{N+1})$ have the same rational code, and are thus eventually periodic.*

The following proposition gives a relation between points of rational code and periodic points.

PROPOSITION 6.4. *Let T be a piecewise isometry acting on the bounded space X with convex partition $\mathcal{P}$. Then in the space X, there exist points of rational codings if and only if there are periodic points.*

Proof. The existence of points with rational codings given periodic points is obvious. Conversely, suppose that $x \in X$ is a point whose code is rational of period m. Let $K \in \Sigma$ be the cell of all points whose code is equal to the periodic part of $\phi(x)$. Then $T^m K = K$ and T^m is an isometry on K. If K contains only one point, then that point is periodic. Suppose that K contains at least two points. As an element of Σ, K must be convex. Hence, there is some d-dimensional subspace of $\mathbb{R}^N$ $(1 \le d \le N)$ in which K is contained, and K has positive Lebesgue d-measure. Then in that subspace, K has positive Lebesgue d-measure. Thus the center of mass o of K is well defined (X is bounded) and o must be a periodic point of period m. $\square$

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