Acta Math., 178 (1997), 185–297

Dynamics of quadratic polynomials, I–II

by

MIKHAIL LYUBICH

SUNY
Stony Brook, NY, U.S.A.

Contents

1. Introduction

Rigidity is a fundamental phenomenon in hyperbolic geometry and holomorphic dynamics. Its meaning is that the metric properties of certain manifolds or dynamical systems are determined by their combinatorics. Celebrated works of Mostow, Thurston, Sullivan, Yoccoz, among others, provide us with examples of rigid objects. Moreover, this phenomenon is intimately linked to the universality phenomenon, to basic measure-theoretical and topological properties of systems, to the problem of describing typical systems.

In the setup of holomorphic dynamics the general rigidity problem can be posed as follows:

Rigidity problem. Any two combinatorially equivalent rational maps are quasi-conformally equivalent. Except for the Lattès examples, the quasi-conformal deformations come from the dynamics on the Fatou set.

Though the general problem is still far from being solved, there have been recently several breakthroughs in the quadratic case when the problem is equivalent to the famous MLC Conjecture ("the Mandelbrot set is locally connected"). In this case the problem has been directly linked to the renormalization theory. In 1990 Yoccoz proved MLC for all parameter values which are at most finitely renormalizable. In this paper we will prove MLC for a certain class of infinitely renormalizable maps. To this end we carry out a geometric analysis of Julia sets which has already found a number of other interesting applications.

Our analysis exploits a new powerful tool called "puzzle". It was introduced by Branner and Hubbard [BH] for cubic maps with one escaping critical point and by Yoccoz for quadratics (see [Hu], [Mi2]). The main geometric result of these works is the divergence property of moduli of a certain nest of annuli (provided the map is non-renormalizable). This implies that the corresponding domains ("puzzle pieces") shrink to points, which yields, for a non-renormalizable quadratic, local connectivity of the Julia set. Transferring this result to the parameter plane yields local connectivity of the Mandelbrot set at the corresponding parameter values.

The geometric results of Branner–Hubbard and Yoccoz do not contain information on the *rate* at which the pieces shrink to points. In this work we tackle this problem. We consider a smaller nest $V^0 \supset V^1 \supset \ldots$ of puzzle pieces called *principal*, and prove that the moduli of the annuli $A^n = V^{n-1} \setminus V^n$ grow at linear rate over a certain combinatorially specified subsequence of levels:

THEOREM III (moduli growth). *Let $n(k)$ count the non-central levels in the principal nest $\{V^n\}$. Then*

$$\operatorname{mod}(A^{n(k)+2}) \geqslant Bk,$$

where the constant B depends only on the first modulus $\mu_1 = \operatorname{mod}(A^1)$.

To gain control of the first principal modulus, $\operatorname{mod}(A^1)$, we consider a class $\mathcal{SL}$ of quadratics satisfying the *secondary limbs condition.* This class, in particular, contains

- maps which are at most finitely renormalizable and do not have non-repelling periodic points (Yoccoz class);
- infinitely renormalizable maps of bounded type;
- real maps which do not have non-repelling periodic points.

In §4, Theorem I, we construct, for maps of class $\mathcal{SL}$, a dynamical annulus A^1 with a definite modulus.

A basic geometric quality of infinitely renormalizable maps is a priori bounds. They provide a key to the renormalization theory, problems of rigidity and local connectivity. In this paper we prove a priori bounds for maps of class $\mathcal{SL}$ with sufficiently big combinatorial type (§7, Theorems IV and IV′).

Being specified for real quasi-quadratic maps of Epstein class, this result yields complex bounds on every renormalization level with sufficiently big "essential period" (§8, Theorem V). In a more recent work [LY] complex bounds were established for maps with essentially bounded combinatorics. Altogether this yields:

COMPLEX BOUNDS THEOREM (joint with Yampolsky). *Let f be an infinitely renormalizable quasi-quadratic map of Epstein class. Then for all sufficiently big m, the renormalization $R^m f$ is quadratic-like with a definite modulus*: $\operatorname{mod}(R^m f) \geqslant \mu > 0$, *with an absolute μ. If f is a quadratic polynomial, this occurs for all m.*

This result was independently proven by Levin and van Strien [LvS].

In Part II we use the above geometric information to prove the following result:

RIGIDITY THEOREM. *Any combinatorial class contains at most one quadratic polynomial satisfying the secondary limbs condition with a priori bounds.*

We also show that the quadratics satisfying the above assumptions have locally connected Julia sets (§9, Theorem VI). In particular, all real quadratics have locally connected Julia sets (see also [LvS]).

CONJECTURE. *The secondary limbs condition implies a priori bounds.*

Let $\mathcal{QC}(c) \subset \mathcal{T}op(c) \subset \mathcal{C}om(c) \subset \mathbf{C}$ stand respectively for the quasi-conformal, topological and combinatorial classes of the quadratic map P_c. A map P_c is called combinatorially, topologically or quasi-conformally rigid if $\mathcal{C}om(c)=\{c\}$, $\mathcal{T}op(c)=\{c\}$ or $\mathcal{QC}(c)=\{c\}$ respectively.

The strongest, combinatorial, rigidity of a map P_c turns out to be equivalent to the local connectivity of the Mandelbrot set M at c (see [DH1], [Sc1]). This property of M was conjectured by Douady and Hubbard under the name "MLC".

COROLLARY 1.1. *For a quadratic polynomial $P_c \in \mathcal{SL}$ of a sufficiently big type (i.e., satisfying the assumptions of Theorem* IV′) *the Julia set $J(f)$ is locally connected, and the Mandelbrot set is locally connected at c.*

In particular, this gives first examples of infinitely renormalizable parameter values $c \in M$ of *bounded type* where MLC holds (though one needs a minor part of Corollary 1.1 to produce some examples of such kind).

One might wonder: how big is the set of infinitely renormalizable parameter values satisfying the assumptions of Corollary 1.1? It is obviously dense on the boundary of the Mandelbrot set. We can show that this set has Lebesgue measure zero and Hausdorff dimension at least 1 [L10]. Note that $1=\frac{1}{2}\cdot 2$ where $2=\mathrm{HD}(\partial M)$ by Shishikura's theorem [Sh1].

Let us now dwell on the case of real parameter values $c\in\left[-2,\frac{1}{4}\right]$. Corollary 1.1 implies MLC (and thus complex rigidity) at real c with sufficiently big "essential period" on all renormalization levels (§12, Theorem VIII). For the rest of real parameters the Rigidity and Complex Bounds Theorems imply a weaker property, real rigidity. Let us say that a parameter value $c\in\mathbf{R}$ (or the corresponding quadratic polynomial P_c) is *rigid on the real line* if $\mathcal{C}om(c)\cap\mathbf{R}=\{c\}$. Thus we have:

DENSITY THEOREM. *Any real quadratic polynomial P_c without attracting cycles is rigid on the real line. Thus hyperbolic quadratics are dense on the real line.*

(The latter statement follows from the former by the Milnor–Thurston kneading theory [MT].)

Among other applications of the above results are the proof of the Feigenbaum–Coullet–Tresser renormalization conjecture [L9] and an advance in the problem of absolutely continuous invariant measures (joint with Martens and Nowicki [L8], [MN]).

Let us now describe the structure of the paper.

In §2, we overview the necessary preliminaries in holomorphic dynamics, particularly Douady–Hubbard renormalization and the Yoccoz puzzle.

In §3 we present our approach to the combinatorics of the puzzle. The main concepts involved are the *principal nest* of puzzle pieces, *generalized renormalization* and *central cascades*. As we indicated above, the principal nest $V^0\supset V^1\supset\ldots$ contains the key combinatorial and geometric information about the puzzle. We describe the combinatorics of this nest by means of *generalized renormalizations*, that is, appropriately restricted first return maps considered up to rescaling.

It may happen that a quadratic-like map $g_n\colon V^n\to V^{n-1}$ has "almost connected" Julia set. This phenomenon often requires a special treatment. Such a map generates a subnest of the principal nest called a *central cascade*. The number of central cascades in the principal nest is called the *height* $\chi(f)$ of a map f. In other words, $\chi(f)$ is the number of different quadratic-like *germs* among the g_n's. It will play a crucial role for our discussion.

In §4 we study the initial geometry of the puzzle. The main result of this section is the construction of an initial annulus $A^1=V^0\setminus V^1$ with definite modulus, provided the hybrid class of a map is selected from a truncated secondary limb (Theorem I).

In §5 we define a new geometric parameter (worked out jointly with J. Kahn), *the asymmetric modulus*, and prove that it is *monotonically non-decreasing* when we go through the principal nest (Theorem II). This already provides us with lower bounds for the principal moduli $\mu_n = \operatorname{mod}(A^n)$ (which, by the way, implies the Branner–Hubbard–Yoccoz divergence property), and upper bounds on the distortion. We reach these results by means of a purely combinatorial analysis plus the standard Grötzsch inequality.

Our main geometric result, Theorem III, is proven in §6. The above analysis does not always yield the linear growth of moduli. In particular, it is not good enough for the basic example called the *Fibonacci map*. The proof of the moduli growth for the Fibonacci combinatorics is the heart of the whole paper (§6.4). This crucial step is based on the *definite Grötzsch inequality*, estimates of hyperbolic distances between puzzle pieces and analysis of their shapes. The key observation is that sufficiently pinched pieces make a definite extra contribution to the moduli growth.

In the next section, §7, we prove a priori bounds for infinitely renormalizable quadratics of sufficiently big type (Theorems IV and IV′). The meaning of this condition is that certain combinatorial parameters of the renormalized maps $R^n f$ are sufficiently big. The main such parameter is the above mentioned height, but there are also a few others. These conditions together mean roughly that the *periods* of $R^n f$ are sufficiently big, except for a possibility of long "parabolic or Siegel cascades".

In the last section of Part I, §8, the above discussion is specified and refined for real maps of Epstein class. We define a notion of "essential period" and prove that $\operatorname{mod}(Rf)$ is big if and only if the essential period $\operatorname{per}_e(f)$ is big. This discussion exploits essentially Martens' real bounds [Mar] and complex bounds of [L4].

Let us now pass to Part II. In §9 we show that the secondary limbs condition and a priori bounds yield a definite space between the bouquets of little Julia sets. This provides us with special disjoint neighborhoods of little Julia bouquets with bounded geometry (called "standard"). Together with the work of Hu and Jiang [HJ], [J] and McMullen [Mc3] this yields local connectivity of the big Julia set (Theorem VI).

In the next two sections we prove the Rigidity Theorem. We start §10 with a discussion of reductions which boil the Rigidity Theorem down to the following problem: Two topologically equivalent maps (satisfying the assumptions of the theorem) are Thurston equivalent. Then we set up an inductive construction of approximations to the Thurston conjugacy. In particular, we adjust an approximate conjugacy in such a way that it respects the standard neighborhoods of little Julia bouquets.

The next section, §11, presents the proof of the Main Lemma. This lemma gives a uniform bound on the pseudo-Teichmüller distance between the generalized renormalizations of two combinatorially equivalent quadratic-like maps (the bound depends only on the selected secondary limbs and a priori bounds). The main geometric ingredient which makes this work is the linear growth of the principal moduli (Theorem III).

In the last section, §12, we discuss rigidity and deformations of real quasi-quadratic maps.

In Appendix A we collect necessary background material in conformal and quasi-conformal geometry.

In Appendix B we make further reference comments.

Acknowledgement. I would like to thank Jeremy Kahn for a fruitful suggestion (see §14) and for telling me his proof of the Yoccoz Rigidity Theorem. I thank Marco Martens, Curt McMullen, Yair Minsky, Mary Rees and Mitsuhiro Shishikura for useful comments on the results. I also had useful discussions with John Milnor on combinatorics of external rays. Feedback from the Stony Brook dynamics group during my course in the fall of 1994 was helpful for cleaning up the exposition. I also thank Scott Sutherland and Brian Yarrington for help with the computer pictures.

The main results of this work were obtained in June 1993 during the Warwick Workshop on hyperbolic geometry. I am grateful to the organizers, particularly David Epstein and Caroline Series, for that wonderful time. I also thank MSRI for their hospitality: this paper was partly prepared during the Complex Dynamics and Hyperbolic Geometry spring program 1995. This work has been partially supported by NSF Grants DMS-8920768 and DMS-9022140, and the Sloan Research Fellowship.

Part I. Combinatorics and geometry of the Yoccoz puzzle

2. Douady–Hubbard renormalization and Yoccoz puzzle

2.1. General terminology and notations

We will use the following notations:

$\mathbf{D}_r=\{z:|z|<r\}$ is the standard disk of radius r, $\mathbf{D}\equiv\mathbf{D}_1$ is the unit disk.

$\mathbf{T}_r=\partial\mathbf{D}_r$ is the standard circle of radius r, $\mathbf{T}\equiv\mathbf{T}_1$ is the unit circle.

$\mathbf{A}(r,R)=\{z:r<|z|<R\}$ is a standard annulus; similar notation is used for a closed annulus $\mathbf{A}[r,R]$ (or a semi-closed one).

Given two sets A and B, let $\operatorname{dist}(A,B)=\inf\{\operatorname{dist}(z,\zeta):z\in A,\zeta\in B\}$.

Given two subsets V and W of the complex plane, we say that V is *strictly contained* in W, $V\Subset W$, if $\operatorname{cl} V\subset\operatorname{int} W$.

By a *topological disk* we will mean a simply-connected region in $\mathbf{C}$. By an *annulus* we mean a doubly-connected region. A *horizontal curve* in an annulus A is a preimage of a circle centered at 0 by the Riemann mapping $A\to\{z:r<|z|<R\}$ (here $0\leqslant r<R\leqslant\infty$).

Let us consider a family of two topological nested disks $D_1 \subset D_2$ with $\Gamma_i = \partial D_i$ and $A = D_2 \setminus D_1$. The statement that $\mathrm{mod}(A) > \varepsilon$ with an $\varepsilon > 0$ uniform over the family will be freely expressed in the following ways: "the annulus A has a definite modulus", "D_1 is well inside D_2" or "there is a definite space in between Γ_1 and Γ_2".

Quasi-conformal and quasi-symmetric maps will be abbreviated as qc and qs correspondingly.

By $\mathrm{orb}\, z$ we denote the forward orbit $\{f^n z\}_{n=0}^{\infty}$ of z, and by $\omega(z)$ its ω-limit set. Let also $\mathrm{orb}_n\, z = \{f^m z\}_{m=0}^{n}$. Let $P_c: z \mapsto z^2 + c$.

2.2. Polynomials

By now there are many surveys and books on holomorphic dynamics. The reader can consult, e.g., [Bea], [CG], [L1], [Mi1] for general reference, and [Br], [DH1] for the quadratic case. Below we will remind the main definitions and facts required for discussion. However, we assume that the reader is familiar with the classification of periodic points as attracting, neutral, parabolic and repelling.

Let $f: \mathbf{C} \to \mathbf{C}$ be a monic polynomial of degree $d \geqslant 2$, $f(z) = z^d + a_1 z^{d-1} + \ldots + a_d$. The *basin of* ∞ is the set of points escaping to ∞:

$$D_f(\infty) \equiv D(\infty) = \{z \in \mathbf{C} : f^n z \to \infty\}.$$

Its complement is called *the filled Julia set*: $K(f) = \mathbf{C} \setminus D(\infty)$. The *Julia set* is the common boundary of $K(f)$ and $D(\infty)$: $J(f) = \partial K(f) = \partial D(\infty)$. The Fatou set $F(f)$ is defined as $\mathbf{C} \setminus J(f)$. The Julia set (and the filled Julia set) is connected if and only if none of the critical points escape to ∞, that is, all of them belong to $K(f)$.

Given a polynomial f, there is a conformal map (the *Böttcher function*)

$$B_f: U_f \to \{z : |z| > r_f \geqslant 1\}$$

of a neighborhood U_f of infinity onto the exterior of a disk such that $B_f(fz) = (B_f z)^d$ and $B_f(z) \sim z$ as $z \to \infty$. There is an explicit dynamical formula for this map:

$$B_f(z) = \lim_{n \to \infty} (f^n z)^{1/d^n} \tag{2.1}$$

with an appropriate choice of the branch of the d^nth root.

If the Julia set $J(f)$ is disconnected then ∂U_f contains a critical point b of f. Otherwise B_f coincides with the Riemann mapping of the whole basin of infinity $D(\infty)$ onto $\{z : |z| > 1\}$ (in this case $r_f = 1$).

The *external rays* $R^\theta \equiv R^\theta_f$ of angle θ and *equipotentials* $E^r \equiv E^r_f$ of level r are defined as the B_f-preimages of the straight rays $\{re^{i\theta}: r_f < r < \infty\}$ and the round circles $\{e^r e^{i\theta}: 0 \leqslant \theta \leqslant 2\pi\}$. They form two orthogonal invariant foliations of U_f. Moreover, even in the disconnected case, a ray R^θ can be infinitely extended towards the Julia set, unless it "bounces off" a critical point, and the Böttcher function can be analytically continued along this ray (see [GM, Appendix B] for a detailed discussion).

Let $\mathcal{R}^{\theta,(\varrho,r)} = \mathcal{R}_f^{\theta,(\varrho,r)}$ stand for the arc of the external ray of angle θ in between the equipotential levels $0 \leqslant \varrho < r \leqslant \infty$ (with the usual meaning of notations $[\varrho, r]$, $[\varrho, r)$, etc.). Note that if the ray lands at some point $a \in J(f)$ then $\mathcal{R}^{\theta,[0,r)}$ also makes sense.

Each ray comes together with the natural parametrization by the equipotential levels.

THEOREM 2.1 (see [Mi1, §18] or [Hu]). *Assume that $J(f)$ is connected. Then for any repelling periodic point a, there is at least one but at most finitely many external rays landing at a.*

Thus the external rays landing at a are organized in several cycles. The rotation number of these cycles is the same, and is called the *combinatorial rotation number* $\varrho(a)$ of a. Let $\mathcal{R}(a) \equiv \mathcal{R}_f(a)$ denote the union of the closed external rays landing at a, and

$$\mathcal{R}(\bar{a}) \equiv \mathcal{R}_f(\bar{a}) = \bigcup_{k=0}^{p-1} \mathcal{R}(f^k a)$$

(where p is the period of a and $\bar{a} = \operatorname{orb} a$ is the corresponding periodic cycle). This configuration, with the external angles marked at the rays, is called the *rays portrait* of the cycle $\bar{a}$. The class of isotopic portraits is called the *abstract rays portrait.*

2.3. Quadratic family

Let now $f \equiv P_c: z \mapsto z^2 + c$ be a quadratic map. In this case the rays portraits of periodic cycles have quite special combinatorial properties. The reader can consult [DH1], [At], [GM], [Sc2], [Mi4] for the proofs of the results quoted below.

PROPOSITION 2.2 (see [Mi4]). *Let $\bar{a} = \{a_k\}_{k=0}^{p-1}$ be a repelling periodic cycle such that there are at least two rays landing at each point a_k.*

(i) *Let S_1 be the components of $\mathbf{C} \setminus \mathcal{R}(\bar{a})$ containing the critical value c. Then S_1 is a sector bounded by two external rays.*

(ii) *Let S_0 be the component of $\mathbf{C} \setminus f^{-1}\mathcal{R}(\bar{a})$ containing the critical point* 0. *Then S_0 is bounded by four external rays: two of them land at a periodic point a_k, and two others land at the symmetric point $-a_k$.*

(iii) *The rays of $\mathcal{R}(\bar{a})$ form either one or two cycles under iterates of f.*

A particular situation of such kind is the following. Let $\bar{b}=\{b_k\}_{k=0}^{p-1}$ be an attracting cycle, $p>1$. Let D_k be the components of its basin of attraction containing b_k. Then the boundaries of D_k are Jordan curves, and the restrictions $f^p|\partial D_k$ are topologically conjugate to the doubling map $z\mapsto z^2$ of the unit circle. Hence there is a unique f^p-fixed point $a_k\in\partial D_k$. Altogether these points form a repelling periodic cycle $\bar{a}$ (whose period may be smaller than p), with at least two rays landing at each a_k. The portrait $\mathcal{R}(\bar{a})$ will also be called the rays portrait associated to the attracting cycle $\bar{b}$.

A case of special interest for what follows is the fixed points portraits. There is always a fixed point called β which is the landing point of the invariant ray $\mathcal{R}_0$. Moreover, this is the only ray landing at β, so that this point is non-dividing: the set $K(f)\setminus\{\beta\}$ is connected.

If the second fixed point called α is also repelling, it turns out to be dividing: there are at least two external rays landing at it, so that $K(f)\setminus\{\alpha\}$ is disconnected. These rays are cyclically permuted by dynamics with a certain combinatorial rotation number q/p.

The *Mandelbrot set* M is defined as the set of $c\in\mathbf{C}$ for which $J(P_c)$ is connected, that is, 0 does not escape to ∞ under iterates of P_c. If $c\in\mathbf{C}\setminus M$, then $J(P_c)$ is a Cantor set.

The Mandelbrot set itself is connected (see [DH1], [CG]). This is proven by constructing explicitly the Riemann mapping $B_M\colon\mathbf{C}\setminus M\to\{z:|z|>1\}$. Namely, let $D_c(\infty)$ be the basin of ∞ of P_c, and B_c be the Böttcher function (2.1) of P_c. Then

$$B_M(c)=B_c(c). \tag{2.2}$$

The meaning of this formula is that the "conformal position" of a parameter $c\in\mathbf{C}\setminus M$ coincides with the "conformal position" of the critical value c in the basin $D_c(\infty)$. This relation is a key to the similarity between dynamical and parameter planes.

Using the Riemann mapping B_M we can define the *parameter external rays* and *equipotentials* as the preimages of the straight rays going to ∞ and round circles centered at 0. This gives us two orthogonal foliations in the complement of the Mandelbrot set.

A quadratic polynomial P_c with $c\in M$ is called *hyperbolic* if it has an attracting cycle. The set of hyperbolic parameter values is the union of some components of $\operatorname{int} M$ called *hyperbolic components*. Conjecturally all components of $\operatorname{int} M$ are hyperbolic. This conjecture would follow from the MLC Conjecture asserting that the Mandelbrot set is locally connected (Douady and Hubbard [DH1]).

The *main cardioid* of M is defined as the set of points c for which P_c has a neutral fixed point α_c, that is, $|P_c'(\alpha_c)|=1$. It encloses the *main hyperbolic component* where P_c has an attracting fixed point. In the exterior of the main cardioid both fixed points are repelling.

Let $H\subset \operatorname{int} M$ be a hyperbolic component of the Mandelbrot set, and let $\bar{b}(c)=\{b_k(c)\}_{k=0}^{p-1}$ be the corresponding attracting cycle. On the boundary of H the cycle $\bar{b}$ becomes neutral, and there is a single point $d\in\partial H$ where $(P_d^p)'(b_0)=1$ [DH1]. This point is called the *root* of H.

If H is not the main component then for any $c\in H$ there is the rays portrait $\mathcal{R}_c$ associated to the corresponding attracting basin. Let θ_1 and θ_2 be the external angles of the two rays bounding the sector S_1 of Proposition 2.2.

THEOREM 2.3 (see [DH1], [Mi4], [Sc2]). *The parameter rays with angles θ_1 and θ_2 land at the root d of H. There are no other rays landing at d.*

The region W_d in the parameter plane bounded by the above two rays and containing H is called the *wake* of W_d. The part of the Mandelbrot set contained in the wake together with the root d is called the *limb* L_d of the Mandelbrot set originated at H. The root d is also referred to as the root of the wake W_d or the limb L_d.

Recall that for $c\in H$, $\bar{a}_c$ denotes the repelling cycle associated to the basin of the attracting cycle $\bar{b}_c$. The dynamical meaning of the wakes is reflected in the following statement.

PROPOSITION 2.4 (see [GM]). *Under the circumstances just described, the repelling cycle $\bar{a}_c$ stays repelling throughout the wake W_d originated at H. The corresponding rays portrait $\mathcal{R}(\bar{a}_c)$ preserves its isotopic type throughout this wake.*

The limbs attached to the main cardioid are called *primary*. Let H be a hyperbolic component attached to the main cardioid. The limbs attached to such a component are called *secondary*. More generally, if H is a hyperbolic component obtained from the main cardioid by means of n consecutive bifurcations, then the limbs originated at such a component will be called *limbs of order n*.

A *truncated limb* is obtained from a limb by removing a neighborhood of its root.

2.4. Douady–Hubbard polynomial-like maps

The main reference for the following material is [DH2]. Let $U'\Subset U$ be two topological disks. A branched covering $f\colon U'\to U$ is called a *DH polynomial-like map* (we will sometimes skip "DH" in case this does not cause confusion with "generalized" polynomial-like maps defined below). Every polynomial with connected Julia set can be viewed as a polynomial-like map after restricting it onto an appropriate neighborhood of the filled Julia set. Polynomial-like maps of degree 2 are called *(DH) quadratic-like*. Unless otherwise is stated, any quadratic-like map will be normalized so that the origin 0 is its critical point.

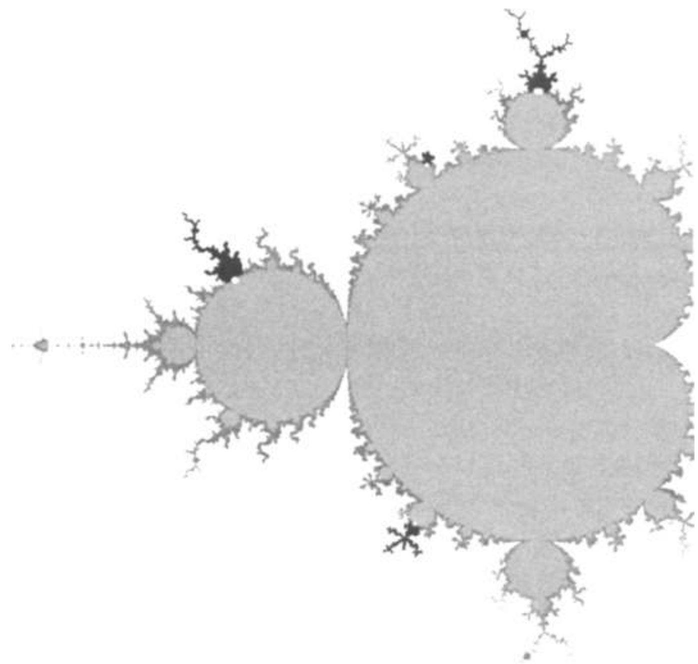

Fig. 1. Truncated secondary limbs of the Mandelbrot set

One can naturally define the filled Julia set of f as the set of non-escaping points:

$$K(f) = \{z : f^n z \in U', n = 0, 1, ...\}.$$

The Julia set is defined as $J(f) = \partial K(f)$. These sets are connected if and only if none of the critical points is escaping.

The choice of the domain U' and the range U of a polynomial-like map is not canonical. It can be replaced with any other pair $V' \Subset V$ such that $f: V' \to V$ is a polynomial-like map with the same Julia set (compare [Mc2, Theorem 5.11]).

Given a polynomial-like map $f: U' \to U$, we can consider a *fundamental annulus* $A = U \setminus U'$. It is certainly not a canonical object but rather depending on the choice of U' and U. Let

$$\operatorname{mod}(f) = \sup \operatorname{mod}(A),$$

where A runs over all fundamental annuli of f.

Two polynomial-like maps f and g are called *topologically* (*quasi-conformally, conformally, affinely*) *conjugate* if there is a choice of domains $f: U' \to U$ and $g: V' \to V$ and a homeomorphism $h: (U, U') \to (V, V')$ (qc map, conformal or affine isomorphism correspondingly) such that $h \circ f | U = g \circ h | U$.

If there is a qc conjugacy h between f and g with $\bar{\partial}h=0$ almost everywhere on the filled Julia set $K(f)$, then f and g are called *hybrid* or *internally* equivalent. A *hybrid class* $\mathcal{H}(f)$ is the space of DH polynomial-like maps hybrid equivalent to f modulo affine equivalence. According to Sullivan [S1], a hybrid class of polynomial-like maps can be viewed as an infinite-dimensional Teichmüller space. In contrast with the classical Teichmüller theory this space has a preferred point:

STRAIGHTENING THEOREM [DH2]. *Any hybrid class $\mathcal{H}(f)$ of DH polynomial-like maps with connected Julia set contains a unique (up to affine conjugacy) polynomial.*

In particular, any hybrid class of quadratic-like maps with connected Julia set contains a unique quadratic polynomial $z\mapsto z^2+c$ with $c=c(f)\in M$. So the hybrid classes of quadratic-like maps with connected Julia set are labeled by the points of the Mandelbrot set. In what follows we will freely identify such a hybrid class with its label $c\in M$.

Sullivan supplied any Teichmüller space of quadratic-like maps (with connected Julia set) with the following *Teichmüller metric* [S1]:

$$\operatorname{dist}_T(f,g)=\inf\log\operatorname{Dil}(h),$$

where h runs over all hybrid conjugacies between f and g, and $\operatorname{Dil}(h)$ denotes the qc dilatation of h. It is easy to see from the construction of the straightening that the Teichmüller distance from f to the quadratic $P_{c(f)}: z\mapsto z^2+c(f)$ in its hybrid class is controlled by the modulus of f:

PROPOSITION 2.5. *If* $\operatorname{mod}(f)\geqslant\mu>0$ *then* $\operatorname{dist}_T(f,P_{c(f)})\leqslant C$ *with a* $C=C(\mu)$ *depending only on* μ. *Moreover,* $C(\mu)\to 0$ *as* $\mu\to\infty$.

This is a reason why control of the moduli of polynomial-like maps is crucial for the renormalization theory (see [S2]).

Given a polynomial-like map with connected Julia set, we can define *external rays* and *equipotentials* near the filled Julia set by conjugating it to a polynomial and transferring the corresponding curves. This definition is certainly not canonical but rather depends on the choice of conjugacy. If $\operatorname{mod}(f)>\varepsilon$, then we can use a $K(\varepsilon)$-qc conjugacy. In what follows we always assume that the choice of the curves is made in such a way.

2.5. Douady–Hubbard renormalization

The reverse procedure under the name of *tuning* is discussed in [DH2], [D1] and [Mi3]. A more general point of view (but which is equivalent to the tuning, after all) is discussed in [Mc2].

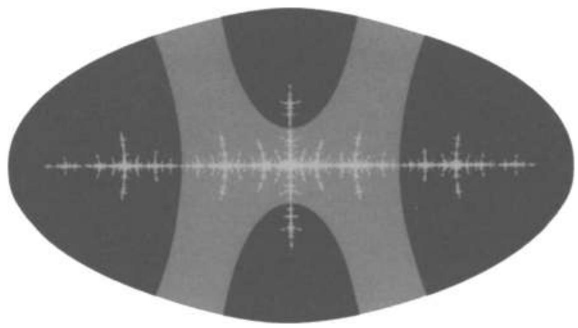

Fig. 2. Renormalization domain for the Feigenbaum polynomial

Let $f: U' \to U$ be a quadratic-like map. Let $\bar{a}$ be a dividing repelling cycle, so that there are at least two rays landing at each point of $\bar{a}$. Let $\mathcal{R} \equiv \mathcal{R}(\bar{a})$ denote the configuration of rays landing at $\bar{a}$, and let $\mathcal{R}' = -\mathcal{R}$ be the symmetric configuration. Let us also consider an arbitrary equipotential E. Let now Ω be the component of $\mathbf{C} \setminus (E \cup \mathcal{R} \cup \mathcal{R}')$ containing the critical point 0. By Proposition 2.2, it is bounded by four arcs γ_i of external rays and two pieces of the equipotential E.

Let p be the period of the above rays, and a be the point of the cycle $\bar{a}$ lying on $\partial\Omega$. Let us consider a domain $\Omega' \subset \Omega$, the component of $f^{-p}\Omega$ attached to a (see Figure 2). If $\Omega' \ni 0$ then $f^p: \Omega' \to \Omega$ is a double covering map (otherwise Ω' is a strip univalently mapped onto Ω).

A quadratic-like map f is called *DH renormalizable* if there is a repelling cycle $\bar{a}$ as above such that $\Omega' \ni 0$, and 0 does not escape Ω' under iterates of f^p. We will also say that this renormalization is associated with the periodic point a. We call f *immediately DH renormalizable* if a is the dividing fixed point α of f.

Note that the disks $\Omega', f\Omega', \ldots, f^{p-1}\Omega'$ have disjoint interiors. Indeed, otherwise $f^k\Omega'$ would be inside Ω for some $k<p$. But this is impossible since the external rays which bound $f^k\Omega'$ are outside of Ω.

In the DH renormalizable case one can extract a polynomial-like map $f^p: V' \to V$ by means of a "thickening procedure" (see [DH1] or [Mi2]). Namely, let us consider a little bit bigger domain $V \supset \Omega$ bounded by arcs of four external rays close to γ_i, two arcs of circles going around the point a and the symmetric point a' (i.e., $fa' = a$), and two arcs of E. Pulling V back by f^p, we obtain a domain $V' \Subset V$ such that the map $f^p: V' \to V$ is quadratic-like. This map considered up to rescaling (that is, up to affine conjugacy) is called the *DH renormalization* of f.

Let now $f: z \mapsto z^2 + c_0$ be a quadratic polynomial, $c_0 \in M$. If it is renormalizable then there is a homeomorphic copy $M_0 \ni c_0$ of the Mandelbrot set with the following properties (see [DH2], [D1]). For $z \in M_0' = M_0 \setminus \{\text{one point}\}$ the polynomial $P_c: z \mapsto z^2 + c$ is renormalizable. Moreover, there is the analytic parameter extension a_c of the periodic point a to a neighborhood of M_0' such that the above renormalization of P_c is associated to a_c. At the parameter value b removed from M_0 the periodic point a_c is becoming parabolic with multiplier one. This parameter value is called the *root* of M_0. We say that the component H_0 of M_0 corresponding to the main hyperbolic component of M "gives origin" to the copy M_0. Vice versa, any hyperbolic component H_0 of the Mandelbrot set gives origin to a copy of M. In particular, the copies corresponding to the immediate renormalization are attached to the main cardioid.

We will see below that among all renormalizations there is the *first* one, which we denote Rf (see §3.4). This renormalization corresponds to a *maximal* copy of the Mandelbrot set (that is, a copy which is not contained in any bigger copies except M itself). Let $\mathcal{M}$ denote the family of maximal Mandelbrot copies.

It may happen that Rf is also renormalizable, so that f is "twice renormalizable". In such a way we can associate to f a canonical finite or infinite sequence of renormalizations $f, Rf, R^2 f, \ldots$. Accordingly f can be classified as "at most finitely" or "infinitely renormalizable".

Given any sequence $\tau = \{M_0, M_1, \ldots\}$ of maximal copies of M, there is an infinitely renormalizable quadratic polynomial P_b such that $c(R^m P_b) \in M_m$, $m = 0, 1, \ldots$. Indeed, the sets

$$\mathcal{C}om_N(\tau) = \{b : c(R^m P_b) \in M_m,\ m = 0, 1, \ldots, N\}$$

form a nest of copies of M whose intersection $\mathcal{C}om(\tau)$ consists of the desired parameter values.

We say that these infinitely renormalizable quadratics have combinatorics τ. The MLC problem for infinitely renormalizable parameter values is equivalent to the assertion that there is only one quadratic with a given combinatorics, i.e., $\mathcal{C}om(\tau)$ is a single point for any τ (see Schleicher [Scl] for a detailed discussion of the combinatorial aspects of the MLC).

Let us say that f satisfies the *secondary limbs condition* if there is a finite family of truncated secondary limbs L_i of the Mandelbrot set such that the hybrid classes of all renormalizations $R^m f$ belong to $\bigcup L_i$. Let $\mathcal{SL}$ stand for the class of quadratic-like maps satisfying the secondary limbs condition.

Here are some examples of maps of class $\mathcal{SL}$:

• Maps which are at most finitely renormalizable and do not have non-repelling periodic points (Yoccoz class).

• Infinitely renormalizable maps of bounded type ("bounded type" means that there are only finitely many different Mandelbrot copies in the string $\tau=\{M_0, M_1, ...\}$).

• Real maps which do not have non-repelling periodic points.

• Select a finite family of (non-truncated) limbs L_j of order 3 (see §2.3). If

$$c(R^m f)\in\bigcup L_j, \quad m=0,1,...,$$

then $f\in\mathcal{SL}$. Unlike the $\mathcal{SL}$ assumption which involves truncation, this property is combinatorial.

All the above combinatorial notions are readily extended to quadratic-like maps via the straightening. A quadratic-like map f is said to have *a priori bounds* if there is an $\varepsilon>0$ such that $\operatorname{mod}(R^m f)\geqslant\varepsilon>0$ for all the renormalizations (note that maps of the Yoccoz class satisfy this condition by logic).

2.6. Yoccoz puzzle

Let $f: U'\to U$ be a quadratic-like map with both fixed points α and β repelling. As usual, α denotes the dividing fixed point with rotation number $\varrho(\alpha)=q/p$, $p>1$. Let E be an equipotential sufficiently close to $K(f)$ (so that both E and fE are closed curves). Let $\mathcal{R}_\alpha$ denote the union of external rays landing at α. These rays cut the domain bounded by E into p closed topological disks $Y_i^{(0)}$, $i=0, ..., p-1$, called *puzzle pieces of zero depth* (Figure 3). The main property of this partition is that $f\partial Y_j^{(0)}$ is outside of $\bigcup \operatorname{int} Y_i^{(0)}$.

Let us now define *puzzle pieces* $Y_i^{(n)}$ *of depth* n as the closures of the connected components of $f^{-n}\operatorname{int} Y_k^{(0)}$. They form a finite partition of the neighborhood of $K(f)$ bounded by $f^{-n}E$. If the critical orbit does not land at α, then for every depth there is a single puzzle piece containing the critical point. It is called *critical* and is labeled as $Y^{(n)}\equiv Y_0^{(n)}$.

Let $\mathcal{Y}_f$ denote the family of all puzzle pieces of f of all levels. It is *Markov* in the following sense:

(i) Any two puzzle pieces are either nested or have disjoint interiors. In the former case the puzzle piece of bigger depth is contained in the one of smaller depth.

(ii) The image of any puzzle piece $Y_i^{(n)}$ of depth $n>0$ is a puzzle piece $Y_k^{(n-1)}$ of the previous depth. Moreover, $f: Y_i^{(n)}\to Y_k^{(n-1)}$ is a two-to-one branched covering or a conformal isomorphism depending on whether $Y_i^{(n)}$ is critical or not.

We say that $f^k|Y_i^{(n)}$ *l-to-one covers* a union of pieces $\bigcup_{m,j} Y_j^{(m)}$ if $f^k|\operatorname{int} Y_i^{(n)}$ is an l-to-one covering map onto its image, and

$$f^k|(Y_i^{(n)}\cap J(f)) = \bigcup_{m,j} Y_j^{(m)}\cap J(f).$$

In this case $\bigcup Y_j^{(m)}$ is obtained from $f^k|Y_i^{(n)}$ by cutting with appropriate equipotential arcs.

On depth 1 we have $2p-1$ puzzle pieces: one central $Y^{(1)}$, $p-1$ non-central $Y_i^{(1)}$ attached to the fixed point α (cuts of $Y_i^{(0)}$ by the equipotential $f^{-1}E$), and $p-1$ symmetric ones Z_i^1 attached to α'. Moreover, $f|Y^{(1)}$ two-to-one covers $Y_1^{(1)}$, $f|Y_i^{(1)}$ univalently covers $Y_{i+1}^{(1)}$, $i=1, ..., p-2$, and $f|Y_{p-1}^{(1)}$ univalently covers $Y^{(1)} \cup \bigcup_i Z_i^{(1)}$. Thus $f^p Y^{(1)}$ truncated by $f^{-1}E$ is the union of $Y^{(1)}$ and $Z_i^{(1)}$ (Figure 3).

THEOREM 2.6 (Yoccoz, 1990). *Assume that both fixed points of a polynomial-like map f are repelling, and that f is DH non-renormalizable. Then the following divergence property holds*:

$$\sum_{n=0}^{\infty} \operatorname{mod}(Y^{(n)} \setminus Y^{(n+1)}) = \infty.$$

Hence $\operatorname{diam} Y^{(n)} \to 0$ *as* $n \to \infty$.

COROLLARY 2.7. *Under the circumstances of the above theorem the Julia set $J(f)$ is locally connected.*

The reader can consult [Hu], [L3], [Mil] for a proof (or go to Theorem II of this paper).

The Yoccoz puzzle provides us with a Markov family of puzzle pieces to play with. Two original ways of playing this game were by means of the Branner–Hubbard tableaux [BH] and by means of the Yoccoz τ-function (unpublished). Our way based on the idea of generalized renormalization is quite different.

2.7. Expanding sets

Let us consider Yoccoz puzzle pieces $Y_i^{(N)}$ of depth N, and let $\mathcal{Y}^{(N)}$ denote the family of puzzle pieces $Y_j^{(N+l)}$ such that

$$f^k Y_j^{(N+l)} \cap Y^{(N)} = \varnothing, \quad k = 0, ..., l-1.$$

Let $K^{(N)} = \{z : f^k z \notin Y^{(N)},\ k=0, 1, ...\}$. Recall that an invariant set K is called *expanding* if there exist constants $C>0$ and $\varrho \in (0,1)$ such that

$$|Df^k(z)| \geqslant C\varrho^k, \quad z \in K,\ k = 0, 1, ...$$

LEMMA 2.8. *For a given* N, $\operatorname{diam} Y_s^{(N+l)} \to 0$ *as* $Y_s^{(N+l)} \in \mathcal{Y}^{(N)}$ *and* $l \to \infty$. *Moreover, the set* $K^{(N)}$ *is expanding.*

Proof. Let us consider thickened puzzle pieces $\widehat{Y}_i^{(N)}$ as in Milnor [Mi2] or §2.5. Then $\operatorname{int}(f\widehat{Y}_i^{(N)})$ contains $\widehat{Y}_j^{(N)}$ whenever $fY_i^{(N)} \supset Y_j^{(N)}$ (recall that the $\widehat{Y}^{(N)}$ are closed). Hence the inverse map $f^{-1}: \widehat{Y}_j^{(N)} \to \widehat{Y}_i^{(N)}$ is contracting by a factor $\lambda < 1$ in the hyperbolic metrics of the pieces under consideration.

Let $Y_s^{(N+l)} \subset Y_i^{(N)}$. It follows that the hyperbolic diameter of $\widehat{Y}_s^{(N+l)}$ in $\widehat{Y}_i^{(N)}$ is at most λ^l, and the statement follows. □

3. Principal nest and generalized renormalization

In the rest of the paper we will assume, unless otherwise is stated, that both fixed points of the quadratic-like maps under consideration are repelling. Up to §3.6 quadratic-like maps and renormalization are understood in the sense of Douady and Hubbard.

3.1. Principal nest

Given a set $W = \operatorname{cl}(\operatorname{int} W)$ and a point z such that $f^l z \in \operatorname{int} W$, let us define the *pull-back* of W along the $\operatorname{orb}_l z$ as the chain of sets $W_0 = W$, $W_{-1} \ni f^{n-1} z, \ldots, W_{-l} \ni z$ such that W_{-k} is the closure of the component of $f^{-k}(\operatorname{int} W)$ containing $f^{l-k} z$. In particular if $z \in \operatorname{int} W$ and $l > 0$ is the moment of first return of $\operatorname{orb} z$ back to $\operatorname{int} W$ we will refer to the *pull-backs corresponding to the first return of* $\operatorname{orb} z$ to $\operatorname{int} W$.

Let us consider the puzzle pieces of depth 1 as described above: $Y^{(1)}$, $Y_i^{(1)}$ and $Z_i^{(1)}$, $i = 1, \ldots, p-1$ (Figure 3). If $z \in Y^{(1)}$ then $f^p z$ is either in $Y^{(1)}$ or in one of $Z_i^{(1)}$. Hence either $f^{pk} 0 \in Y^{(1)}$ for all $k = 0, 1, \ldots$, or there is a smallest $t > 0$ and a ν such that $f^{tp} 0 \in Z_\nu^{(1)}$. Thus either f is immediately DH renormalizable, or the critical point escapes through one of the non-critical pieces attached to α'.

In the immediately renormalizable case the principal nest of puzzle pieces consists of just the single puzzle piece $Y^{(0)}$ (which is not very informative). In the escaping case we will construct *the principal nest*

$$Y^{(0)} \supset V^0 \supset V^1 \supset \ldots \tag{3.1}$$

in the following way. Let $V^0 \ni 0$ be the pull-back of $Z_\nu^{(1)}$ along $\operatorname{orb}_{tp} 0$. Further, let us define V^{n+1} as the pull-back of V^n corresponding to the first return of the critical point 0 back to $\operatorname{int} V^n$. Of course it may happen that the critical point never returns back to $\operatorname{int} V^n$. Then we stop, and the principal nest turns out to be finite. This case is called *combinatorially non-recurrent.* If the critical point is *recurrent* in the usual sense, that is, $\omega(0) \ni 0$, it is also combinatorially recurrent, and the principal nest is infinite.

Let $l=l(n)$ be the first return time of the critical point back to $\operatorname{int} V^{n-1}$. Then the map $g_n = f^{l(n)}: V^n \to V^{n-1}$ is a two-to-one branched covering. Indeed, by the Markov property of the puzzle, $f^k V^n \cap \operatorname{int} V^{n-1} = \varnothing$ for $k=1, ..., l-1$, so that the maps $f: f^k V^n \to f^{k+1} V^n$ are univalent for those k's.

Let us call a return to level $n-1$ *central* if $g_n 0 \in V^n$. In other words $l(n)=l(n+1)$. Let us say that a sequence $n, n+1, ..., n+N-1$ (with $N \geqslant 1$) of levels (or corresponding puzzle pieces) of the principal nest form a (*central*) *cascade* if the returns to all levels $n, n+1, ..., n+N-2$ are central, while the return to level $n+N-1$ is non-central (see Figure 4). In this case

$$g_{n+k} | V^{n+k} = g_{n+1} | V^{n+k}, \quad k = 1, ..., N,$$

and $g_{n+1} 0 \in V^{n+N-1} \setminus V^{n+N}$. Thus all the maps $g_{n+1}, ..., g_{n+N}$ are the same quadratic-like maps with shrinking domains of definition (see the conventions in §2.4). We call the number N of levels in the cascade its *length.* Note that a cascade of length 1 consists of a single non-central level. Let us call the cascade *maximal* if the return to level $n-1$ is non-central. Clearly the whole principal nest, except the first element $Y^{(0)}$, is the union of disjoint maximal cascades. The number of such cascades is called the *height* $\chi(f)$ of f. In other words, $\chi(f)$ is the number of different quadratic-like maps among the g_n's. (If f is immediately renormalizable set $\chi(f)=0$.)

The annuli $A^n = V^{n-1} \setminus V^n$ and their moduli $\mu_n = \operatorname{mod}(A^n)$ will also be called *principal.*

Remark 1. The notion of the principal nest admits some useful modifications. First, there is a flexibility in the choice of the puzzle piece V^0 (compare §8). Second, one can modify the nest after passing through a long central cascade (see §3.6). The latter modification is useful, e.g., for studying the Hausdorff dimension of Julia sets (see Przytycki [Prz], Prado [Pra1]).

Remark 2. Given a quadratic polynomial $f: z \mapsto z^2 + c$, the principal nest determines a specific way to approximate c by superattracting parameter values. Namely, one should perturb c in such a way that the critical point becomes fixed under g_n, while the combinatorics on the preceding levels keeps unchanged, see [L8]. The number of points in this approximating sequence is equal to the height $\chi(f)$. This resembles the "internal addresses" of Lau and Schleicher [LS], but turns out to be different.

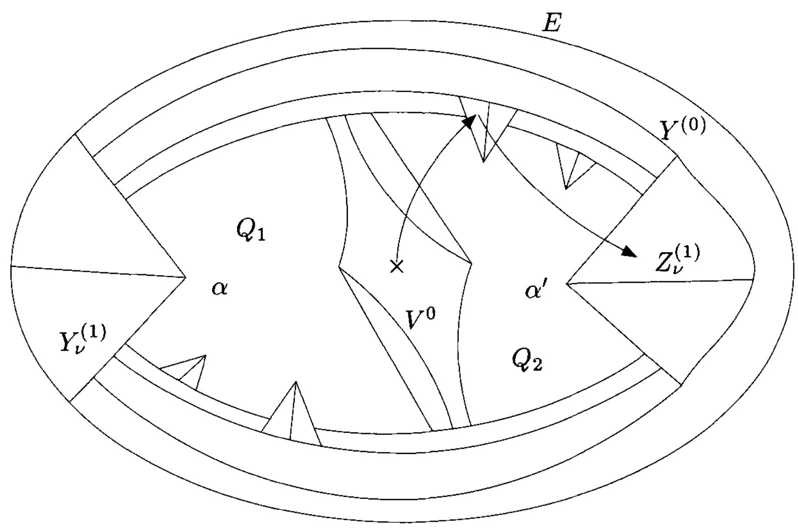

Fig. 3. Initial tiling

3.2. Initial Markov tiling

Let P_i be a finite or countable family of topological discs with disjoint interiors, and $g: \bigcup P_i \to \mathbf{C}$ be a map such that the restrictions $g|P_i$ are branched coverings onto their images. This map is called *Markov* if $gP_i \supset P_j$ whenever $\operatorname{int} gP_i \cap \operatorname{int} P_j \neq \varnothing$. Let us call it an *unbranched Markov map* if all the restrictions $g|P_i$ are one-to-one onto their images.

A Markov map is called *Bernoulli* if there is a topological disc D such that $gP_i \supset D \supset \bigcup P_j$ for all i. Any such D will be called a *range* of g. Similarly we can define an unbranched Bernoulli map.

We know that $f^p|Y^{(1)}$ two-to-one covers $Y^{(1)}$ and the puzzle pieces $Z_i^{(1)}$ attached to α'. If $f^p 0 \in Y^{(1)}$ (central return) then the pull-back of $Y^{(1)}$ by this map is the critical piece $Y^{(1+p)}$, while each $Z_i^{(1)}$ has two univalent pull-backs $Z_j^{(1+p)}$ (we label them by j in an arbitrary way) (see Figure 3).

Now, $f^p|Y^{(1+p)}$ two-to-one covers all these puzzle pieces. If we again have a central return, that is, $f^p 0 \in Y^{(1+p)}$, then the pull-back will give us one critical piece $Y^{(1+2p)}$, and $4(p-1)$ off-critical $Z_j^{(1+2p)}$.

Repeating this procedure t times (where $f^{tp} 0 \in Z_\nu^{(1)}$), we obtain the initial central nest

$$Y^{(1)} \supset Y^{(1+p)} \supset \ldots \supset Y^{(1+(t-1)p)}, \tag{3.2}$$

and a family of off-critical puzzle pieces $Z_j^{(1+sp)}$, $0\leqslant s\leqslant t-1$. Moreover,

$$f^p 0\in Z_\nu^{(1+(t-1)p)}, \tag{3.3}$$

where $f^{(t-1)p}Z_\nu^{(1+(t-1)p)}=Z_\nu^{(1)}$.

Let us say that a set D is *tiled into pieces* W_i rel $F(f)$ if the $\operatorname{int} W_i$ are disjoint, and $D\cap J(f)=\bigcup W_i\cap J(f)$.

Thus we have tiled $Y^{(0)}$ rel $F(f)$ into the pieces $Z_i^{(1+sp)}$, $0\leqslant s\leqslant t-1$, and $Y^{(1+(t-1)p)}$. Let us look closer at this last piece. Its image under f^p two-to-one covers all the above puzzle pieces of depth $1+(t-1)p$. The pull-back of $Z_\nu^{(1+(t-1)p)}$ from (3.3) gives us exactly $V^0\ni 0$, the first puzzle piece in the principal nest (3.1). The pull-backs of the other pieces $Z_j^{(1+(t-1)p)}$ provide some off-critical pieces $Z_i^{(1+tp)}$. Finally, we have two univalent pull-backs Q_1 and Q_2 of $Y^{(1+(t-1)p)}$. Altogether these pieces tile the piece $Y^{(1+(t-1)p)}$ rel $F(f)$.

To understand how the critical point returns back to V^0 we need to tile $Q_1\cup Q_2$ further. To this end let us iterate the unbranched Bernoulli map $f^p|Q_1\cup Q_2$ with range $Q_1\cup Q_2\cup V^0\cup Z_j^{(1+tp)}$. So take a point $z\in Q_1\cup Q_2$ and consider its f^p-orbit until it escapes $Q_1\cup Q_2$ (or iterate forever if it never escapes). It can escape through the piece V^0 or through a piece $Z_j^{(1+tp)}$. In any case pull the corresponding piece back to this point. In such a way we will obtain a tiling

$$Q_1\cup Q_2=\bigcup_{k>0}\bigcup_i X_i^k\cup\bigcup_{k>t}\bigcup_j Z_j^{(1+kp)}\cup R \text{ rel } F(f),$$

where X_i^k denote the pull-backs of V^0 under f^{kp}, Z_j^{1+kp} denote the pull-backs of the $Z_i^{(1+tp)}$ under $f^{(k-t)p}$, and R denotes the residual set of non-escaping points.

Altogether we have constructed the *initial Markov tiling*:

$$Y^{(0)}\setminus R=V^0\cup\bigcup_{k>0}\bigcup_i X_i^k\cup\bigcup_{k\geqslant 0}\bigcup_j Z_j^{(1+kp)} \text{ rel } F(f). \tag{3.4}$$

It is convenient (in order to reduce the number of iterates in what follows) to consider a Markov map

$$G\colon V^0\cup\bigcup_{k,i} X_i^k\cup\bigcup_{k,j} Z_j^{(1+kp)}\to\mathbf{C} \tag{3.5}$$

defined as follows. Observe that for any j there is an i such that $f^{sp}Z_j^{(1+sp)}$ univalently covers $Z_i^{(1)}$. Moreover, $f^{p-i}Z_i^{(1)}$ univalently covers $Y^{(0)}$. Let us set $G|Z_j^{1+sp}=f^{ps+(p-i)}$. The image of each piece $Z_j^{(1+sp)}$ under this map univalently covers $Y^{(0)}$. Similarly let us set $G|V^0=f^{tp+p-\nu}$, so that the image of this piece two-to-one covers $Y^{(0)}$. Finally $G|X_i^k=f^{kp}$, so that these pieces are univalently mapped onto V^0.

3.3. A non-degenerate annulus

Yoccoz has shown that if f is non-renormalizable then in the nest $Y^{(0)} \supset Y^{(1)} \ldots$ there is a non-degenerate annulus $Y^{(n)} \setminus Y^{(n+1)}$. However, the modulus of this annulus is not under control. We will construct a different non-degenerate annulus whose modulus we can control.

PROPOSITION 3.1. *Let f be a quadratic-like map which is not immediately renormalizable. Then all the principal annuli $A^n = V^{n-1} \setminus V^n$ are non-degenerate.*

Proof. Observe first that V^0 is strictly inside $Y^{(0)}$, that is, the annulus $Y^{(0)} \setminus V^0$ is non-degenerate. Indeed, V^0 is the pull-back of $Z_\nu^{(1)}$ which is strictly inside $Y^{(0)}$. As the iterates of $\partial Y^{(0)}$ stay outside $\operatorname{int} Y^{(0)}$, V^0 may not touch $\partial Y^{(0)}$.

For the same reason all other pieces $Z_j^{(1+kp)}$ and X_i^k of the initial Markov tiling (3.4) are strictly inside $Y^{(0)}$ as well.

Let us consider the orbit of the critical point 0 under iterates of the map G (see (3.5)) until it returns back to V^0. It first goes through the Z-pieces of the initial Markov tiling, then at some moment $l \geqslant 1$ it lands at either V^0 or some X_i^s. In the latter case, it lands at V^0 at the next moment.

Since the map $G: V^0 \cup Z_j^{(1+kp)} \to \mathbf{C}$ is Bernoulli with range $Y^{(0)}$, there is a topological disc $P \subset V^0$, such that $G^l | P$ two-to-one covers $Y^{(0)}$. Clearly V^1 is the pull-back of either V^0 or X_i^s by $G^l: P \to Y^{(0)}$. Since both V^0 and X_i^s are strictly inside $Y^{(0)}$, we conclude that $V^1 \Subset P$.

Now it is easy to see that all the annuli A^n are non-degenerate as well. Indeed, it follows that the orbit of ∂V^1 stays away from V^1. Hence V^2 cannot touch ∂V^1, for otherwise there would be a point on ∂V^1 which returns back to V^1. So A^2 is non-degenerate. Now we can proceed inductively. □

3.4. Renormalization and central cascades

PROPOSITION 3.2. *A quadratic-like map is renormalizable if and only if it is either immediately renormalizable, or the principal nest $V^0 \supset V^1 \supset \ldots$ ends with an infinite cascade of central returns. Thus the height $\chi(f)$ is finite if and only if f is either renormalizable or combinatorially non-recurrent.*

Proof. Let the principle nest end with an infinite central cascade $V^{m-1} \supset V^m \supset \ldots$. Then the return times stabilize, $l_m = l_{m+1} = \ldots \equiv l$, and $g_n | V_n = g_m | V_n$, $n \geqslant m$. Moreover, by Proposition 3.1, $V^m \Subset V^{m-1}$, and hence $g \equiv g_m \equiv g^l: V^m \to V^{m-1}$ is a quadratic-like map. We conclude that $\bigcap V^k$ consists of all points which never escape V^m under iterates of g, that is, $\bigcap V^k = K(g)$. Since $0 \in \cap V^k$, $K(g)$ is connected.

Take now the non-dividing fixed point b of g. Let us show that b is dividing for the big Julia set $K(f)$. To this end let us consider the configuration of the *full* external rays whose segments bound V^n. They divide the plane into the central component Ω^n containing V^n, and a family $\mathcal{S}^n=\{S_i^n\}$, $i\in\mathcal{I}^n$, of disjoint sectors each bounded by two external rays. Since the critical puzzle pieces V^n are symmetric (with respect to the involution $z\mapsto z'$ such that $fz'=fz$), the families $\mathcal{S}^n$ are symmetric as well. It follows that every sector has external angle less than π.

Observe that $\mathbf{C}\setminus(V^n\cup\bigcup_i S_i^n)$ does not intersect the big Julia set $J(f)$. Hence every sector S_i^n is contained in some sector $S_{\tau(i)}^{n+1}$ of the next level. Moreover, since rays are mapped to rays, $g(\partial S_i^{n+1})=\partial S_{\varkappa(i)}^n$. (Warning: however, gS_i^{n+1} does not necessarily coincide with $S_{\varkappa(i)}^n$ but can be the whole complex plane. This makes the argument below somewhat involved.) So we have two families of maps $\tau:\mathcal{I}^n\to\mathcal{I}^{n+1}$ and $\varkappa:\mathcal{I}^{n+1}\to\mathcal{I}^n$ (hopefully skipping label n in the notation of these maps will not lead to confusion).

Let us show that these two maps commute. Indeed, by definition $S_i^n\subset S_{\tau i}^{n+1}$. Let us consider a domain $D=S_{\tau i}^{n+1}\cap V^n$. Then $gD=S_{\varkappa(\tau i)}^n\cap V^{n-1}$. Since ∂D contains an arc of ∂S_i^n, $\partial(gD)$ contains an arc of $g(\partial S_i^n)=\partial S_{\varkappa(i)}^{n-1}$. Hence $S_{\varkappa(\tau i)}^n\supset S_{\varkappa(i)}^{n-1}$, so that $\varkappa(\tau i)=\tau(\varkappa i)$.

Let $\sigma=\tau\circ\varkappa:\mathcal{I}^n\to\mathcal{I}^n$, $n\geqslant m$. This map commutes with τ. Let $A\subset\mathcal{I}^m$ be a set of r indices which are cyclically permuted by $\sigma:\mathcal{I}^m\to\mathcal{I}^m$. By the commutation property, the set $\tau^k A\subset\mathcal{I}^{m+k}$ is cyclically permuted by $\sigma:\mathcal{I}^{m+k}\to\mathcal{I}^{m+k}$ as well. Thus for $i\in A$ we have: $\varkappa^r(\tau^r i)=\sigma^r i=i$. Applying $\tau^{(l-1)r}$ to this equation and taking into account the commutation law we conclude that

$$\varkappa^r(\tau^{lr})i=\tau^{(l-1)r}i,\quad l\geqslant 1. \tag{3.6}$$

Let $T_i^l=S_{\tau^{lr}i}^{m+lr}$, $i\in A$, $l\geqslant 0$. Then $T_i^0\subset T_i^1\subset\ldots$, and by (3.6)

$$g^r(\partial T_i^l)=\partial T_i^{l-1}. \tag{3.7}$$

Let us consider the union of these sectors, $T_i\equiv\bigcup_i T_i^l$. Clearly all the sectors T_i and the symmetric sectors T_i' have pairwise disjoint interiors. Moreover, by (3.7), the boundary of each T_i is g^r-invariant and consists of two external rays (limits of the external rays which bound T_i^l) and a piece of the Julia set $J(f)$.

By [DH1] these boundary rays land at some periodic points. Actually, they land at the same point. Indeed, otherwise the piece of the Julia set $J(f)$ contained in the ∂T_i would correspond to an invariant arc of the ideal boundary $\mathbf{T}$ of $\mathbf{C}\setminus K(f)$ ($\mathbf{T}$ is the boundary of the unit disc uniformizing $\mathbf{C}\setminus K(f)$). This arc would not coincide with the whole circle $\mathbf{T}$ since the boundary of T_i cannot contain the whole Julia set $J(f)$ (as one of the symmetric sectors $\operatorname{int} T_i'$ contains a piece of the Julia set). But such arcs do not exist.

Thus each T_i is bounded by two g^r-invariant rays landing at the same periodic point.

Observe finally that the period r of this point must be equal to 1, so that it actually coincides with the fixed point b. Indeed, by construction all the sectors T_i and the symmetric sectors T_i' have pairwise disjoint interiors. If $r>1$ then this situation would contradict Proposition 2.2 (ii).

So the periodic point b is dividing for $J(f)$. Let $\Omega'\subset\Omega$ be the corresponding domains constructed in §2.5. Recall that Ω is bounded by the rays landing at b and b' and two equipotential arcs, and Ω' is the connected component of $(f^l|\Omega)^{-1}$ attached to b. Then $K(g)\subset\Omega$ since the external rays landing at b and b' do not cut through $K(g)$. Since $K(g)$ is connected, $\Omega'\supset K(g)$, and hence $\Omega'\ni 0$. It follows that $g\colon\Omega'\to\Omega$ is a double covering. Moreover, $g^n0\in K(g)\subset\Omega'$, $n=0,1,\dots$. Thus f is renormalizable.

Vice versa, assume that f is renormalizable. Let $Rf=f^l\colon\Omega'\to\Omega$ be the corresponding double covering.

Then the fixed point α may not lie in $\operatorname{int}\Omega'$, for otherwise $\operatorname{int} f\Omega'$ would intersect $\operatorname{int}\Omega'$. Hence α does not cut the filled Julia set $K(Rf)$. But then the preimages of α do not cut $K(Rf)$ either. Hence given a puzzle piece $Y_i^{(n)}$, either $K(Rf)$ is contained in $Y_i^{(n)}$, or $K(Rf)\cap\operatorname{int} Y_i^{(n)}=\varnothing$. In particular $V^m\supset K(Rf)$. But then $f^l0\in V^m$ for all m, so that the first return times to V^m are uniformly bounded. Hence this nest must end up with a central cascade. □

The above discussion shows that there is a well-defined first renormalization Rf with the biggest Julia set, and it can be constructed in the following way. If f is immediately renormalizable, then Rf is obtained by thickening $Y^{(1)}\to Y^{(0)}$. Otherwise the principal nest ends up with the infinite central cascade $V^{m-1}\supset V^m\supset\dots$. Then $Rf=g_m\colon V^m\to V^{m-1}$.

The internal class $c(Rf)$ of the first renormalization belongs to a maximal copy M_0 of the Mandelbrot set.

3.5. Return maps and Koebe space

Let f be a quadratic-like map, and let $V\in\mathcal{Y}_f$ be a puzzle piece.

LEMMA 3.3. *Let z be a point whose orbit passes through* $\operatorname{int} V$. *Let l be the first positive moment of time for which $f^lz\in\operatorname{int} V$. Let $U\ni z$ be the puzzle piece mapped onto V by f^l. Then $f^l\colon U\to V$ is either a univalent map or a two-to-one branched covering depending on whether U is off-critical or otherwise.*

Proof. Let $U_k = f^k U$, $k=0,1,\dots,l$. Since $f^k z \notin \operatorname{int} V$ for $0<k<l$, by the Markov property of the puzzle, $U_k \cap \operatorname{int} V = \varnothing$ for those k's. Hence $f: U_k \to U_{k+1}$ is univalent for $k=1,\dots,l-1$, and the conclusion follows. □

Let $z \in \operatorname{int} V$ be a point which returns back to $\operatorname{int} V$, and let $l>0$ be the first return time. Then there is a puzzle piece $V(z) \subset V$ containing z such that $f^l V(z) = V$. It follows that the first return map $A_V f$ to $\operatorname{int} V$ is defined on the union of disjoint open puzzle pieces $\operatorname{int} V_i$. Moreover, if

$$f^m \partial V \cap V = \varnothing, \quad m = 1, 2, \dots, \tag{3.8}$$

then it is easy to see that the closed pieces V_i are pairwise disjoint and are contained in $\operatorname{int} V$. Indeed, otherwise there would be a boundary point $\zeta \in \partial V$ whose orbit would return back to V, despite (3.8).

Somewhat loosely, we will call the map

$$A_V f : \bigcup V_i \to V \tag{3.9}$$

the *first return map* to V. (Warning: it may happen that a point $z \in \partial V$ returns back to V but does not belong to $\bigcup V_i$; it may also happen that a point $z \in \partial V_i$ returns to V earlier than prescribed by the map A_f.)

Let V_0 denote the critical ("central") puzzle piece (provided the critical point returns back to V). Now Lemma 3.3 immediately yields

LEMMA 3.4. *The first return map A_V univalently maps all the off-critical pieces V_i onto V, and maps the critical piece V_0 onto V as a double-branched covering.*

Thus the first return map $A_V f$ is Bernoulli, and is unbranched on $\bigcup_{i \neq 0} V_i$.

Let us now state an important improvement of Lemma 3.3 which will provide us later on with a "Koebe space" and distortion control.

LEMMA 3.5. *Let z be a point whose orbit passes through the central domain* $\operatorname{int} V_0$ *of the first return map* (3.9), *and $l \geqslant 0$ be the first moment when $f^l z \in V_0$. Then there is a puzzle piece $\Omega \ni z$ mapped univalently by f^l onto V.*

Proof. Let s be the first moment when $f^s z \in V$. Then $f^l z = (A_V f)^k (f^s z)$ for some $k \geqslant 0$. Moreover, $(A_V f)^r (f^s z) \notin V_0$ for $r<k$.

Since the return map is unbranched Bernoulli outside of the central piece, there is a piece $X \subset V$ containing $f^s z$ which is univalently mapped by $(A_V f)^k$ onto V. On the other hand, by Lemma 3.3 there is a domain $D \ni z$ which is univalently mapped by f^s onto V. Hence the domain $(f|D)^{-s} X$ is univalently mapped by f^l onto V. □

Let us now consider the principal nest (3.1) of f. Let

$$g_n: \bigcup V_i^n \to V^{n-1} \tag{3.10}$$

be the first return map to V^{n-1}, where $V_0^n \equiv V^n \ni 0$. We will call it a *principal return map*. We will also let $g_0 \equiv f$.

COROLLARY 3.6. *For $n \geqslant 2$ the pieces V_i^n are pairwise disjoint, and the annuli $V^{n-1} \setminus V_i^n$ are non-degenerate. Moreover, the map $g_n|V_i^n$ can be decomposed as $h_{n,i} \circ f$ where $h_{n,i}$ is a univalent map with range V^{n-2}.*

Proof. As $V^{n-1} \Subset V^{n-2}$ (Proposition 3.1) and

$$f^m(\partial V^{n-1}) \cap \operatorname{int} V^{n-2} = \varnothing, \quad m = 1, 2, \ldots,$$

condition (3.8) is satisfied for $V = V^{n-1}$, and the first statement follows.

Take a piece V_i^n, and let $g_n|V_i^n = f^l$. By Lemma 3.5 the map $f^{l-1}: fV_i^n \to V^{n-1}$ can be extended to a univalent map with range V^{n-2}, and the second statement follows as well. □

Let $\Phi: z \mapsto z^2$ be the quadratic map. Since $f: U' \to U$ is a double covering with the critical point at 0, it can be decomposed as $\Phi \circ h$ where $h: U' \to \mathbf{C}$ is a univalent map. Let $U'' = f^{-1}U$. If $\operatorname{mod}(U' \setminus U) > \varepsilon > 0$ then by the Koebe Theorem, $h|U''$ has an $L(\varepsilon)$-bounded distortion. Thus f is "quadratic up to bounded distortion". Moreover, once we know that the $\operatorname{mod}(A^n)$ are bounded away from 0 (see Theorem II below), we can conclude by Corollary 3.6 that all the maps g_n are quadratic up to bounded distortion.

3.6. Solar system: Bernoulli scheme associated to a central cascade

In the case of a central cascade we need a more precise analysis of the Koebe space. Let us consider a central cascade $\mathcal{C} \equiv \mathcal{C}^{m+N}$:

$$V^m \supset V^{m+1} \supset \ldots \supset V^{m+N-1} \supset V^{m+N}, \tag{3.11}$$

where $g_{m+1} 0 \in V^{m+N-1} \setminus V^{m+N}$. Set $g = g_{m+1}|V^{m+1}$. Then $g: V^k \to V^{k-1}$ is a double-branched covering, $k = m+1, \ldots, m+N$.

Let us consider the first return map $g_{m+1}: \bigcup V_i^{m+1} \to V^m$, see (3.10). Let us pull the pieces V_i^{m+1} back to the annuli $A^k = V^{k-1} \setminus V^k$ by iterates of g, $k = m+1, \ldots, m+N$. We obtain a family $\mathcal{W}(\mathcal{C}) \equiv \mathcal{W}^{m+N}$ of pieces W_j^k. By construction, $W_j^k \subset A^k$ and g^{k-m-1} univalently maps each W_j^k onto some $V_i^{m+1} \equiv W_i^{m+1}$.

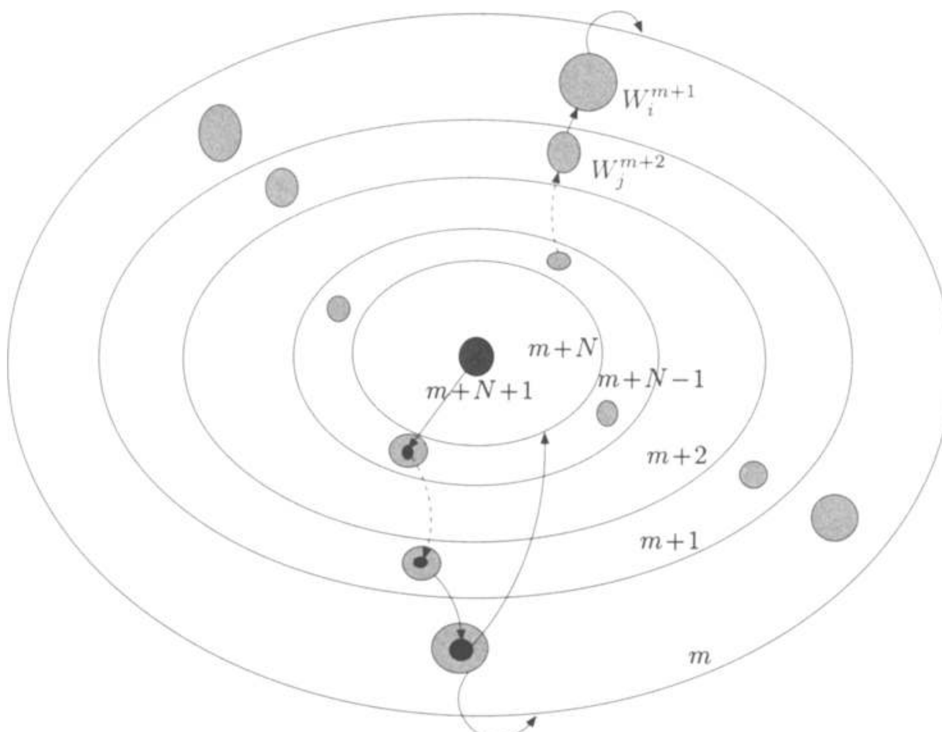

Fig. 4. Solar system

Let us define an unbranched Bernoulli map

$$G \equiv G_{m+N}\colon \bigcup_{\mathcal{W}(\mathcal{C})} W_j^k \to V^m \tag{3.12}$$

as follows: $G|W_j^k = g_{m+1} \circ g^{k-m-1}$ (see Figure 4).

LEMMA 3.7. *Let us consider the central cascade* (3.11). *Let z be a point whose orbit passes through V^{m+N}, and l be the first moment for which $f^l z \in V^{m+N}$. Then there is a piece $\Omega \ni z$ which is univalently mapped by f^l onto V^m.*

Proof. Let s be the first moment for which $f^s z \in V^m$. Then $f^l z = G^k(f^s z)$, where G is the Bernoulli map (3.12). Now repeat the argument of Lemma 3.5 just using G instead of the first return map. □

COROLLARY 3.8. *Let us consider the central cascade* (3.11). *Then the map*

$$g_{m+N+1}\colon V^{m+N+1} \to V^{m+N}$$

can be represented as $h_{m+N+1} \circ f$ where h_{m+N+1} is a univalent map with range V^m.

Proof. Repeat the proof of Corollary 3.6 using Lemma 3.7 instead of Lemma 3.5. □

Remark. The modification of the principal nest after passing a long central cascade mentioned in Remark 1 of §3.1 is the following. Let $g_{m+1}0 \in W_j^{m+N}$. Define $\widetilde{V}^{m+N+1}$ as the pull-back of W_j^{m+N} by $g_{m+1}: V^{m+N} \to V^{m+N-1}$. Then continue the nest by the first return pull-backs beginning with $\widetilde{V}^{m+N+1}$ modifying it each time after passing a long cascade. Note that the construction of the first piece V^0 described in §3.2 is similar to this modification after passing the initial degenerate central cascade (3.2).

3.7. Generalized polynomial-like maps and renormalization

Let $\{U_i\}$ be a finite or countable family of topological discs with disjoint interiors strictly contained in a topological disk U. We call a map $g: \bigcup U_i \to U$ a *(generalized) polynomial-like map* if $g: U_i \to U$ is a branched covering of finite degree which is univalent on all but finitely many U_i.

Let us say that a polynomial-like map g is *of finite type* if its domain consists of finitely many disks U_i. In this case we define the filled Julia set $K(g)$ as the set of all non-escaping points, and the Julia set $J(g)$ as its boundary. The DH polynomial-like maps correspond to the case of a single disk U_0.

GENERALIZED STRAIGHTENING THEOREM. *Any generalized polynomial-like map of finite type is qc conjugate to a polynomial with the same number of non-escaping critical points.*

Proof. For the case of two discs U_0, U_1 the proof is given in [LM]. In general let us proceed inductively in the number of the discs. Enclose two of the discs by a figure eight, and make a qc surgery which creates a new escaping critical point at the singularity of the figure eight, see [LM]. This surgery decreases by one the number of the discs. □

Let us call a generalized polynomial-like map *generalized quadratic-like* if it has a single (and non-degenerate) critical point. In such a case we will assume, unless otherwise is stated, that 0 is the critical point, and label the discs U_i in such a way that $U_0 \ni 0$. In what follows we will deal exclusively with quadratic-like maps, namely with the principal sequence g_n of the first return maps (3.10).

Given a V_j^{n+1}, $n \geqslant 1$, let l be its first return time back to V^n under iterates of g_n, that is, $g_{n+1}|V_j^{n+1} = g_n^l$. Then

$$g_n^k V_j^{n+1} \subset V_{i(k)}^n, \quad k=0,1,\dots,l,$$

with $i(0)=i(l)=0$. Moreover, $g_n^k V_j \Subset V_{i(k)}^n$ for $k<l$. The sequence $0=i(0), i(1), \dots, i(l)=0$ is called the *itinerary* of V_j^{n+1} through the domains of the previous level. A piece V_j^{n+1} is called *precritical* if $g_n V_j^{n+1} = V_0^n$, so that it has the shortest possible itinerary: $l=1$.

Let us define the n-fold *generalized renormalization* $T^n f$ of f as the first return map g_n restricted to the union of puzzle pieces V_i^n meeting the critical set $\omega(0)$, and considered up to rescaling. In the most interesting situations these maps are of finite type:

LEMMA 3.9. *If f is a DH renormalizable quadratic-like map, then all the maps $T^n f$ are of finite type.*

Proof. In the tail of the principal nest the maps $T^n f$ are DH quadratic-like, and their domains consist of just one component. So we should take care only of the initial piece of the cascade.

Let us take the renormalization $Rf = f^l: V^{t+1} \to V^t$ with $t \geqslant n$. Since 0 is non-escaping under iterates of Rf, we have the following property: the first landing time of any point $f^k 0$ back to V^{t+1} is at most l. All the more, the landing time to the bigger domain $V^{n-1} \supset V^t$ is bounded by l. Hence the components of $f^{-t} V^{n-1}$, $t=0,1,\dots,l-1$, cover the whole postcritical set. For sure, there are only finitely many of these components. But the domain of $T^n f$ consists of the pull-backs of these components by $f|V^{n-1}$. □

3.8. Return graph

Let $\mathcal{I}^n$ be the family of puzzle pieces V_i^n intersecting $\omega(0)$, that is, the pieces in the domain of the generalized renormalization

$$T^n f: \bigcup_{\mathcal{I}^n} V_i^n \to V^{n-1}.$$

Let us consider a graded graph Υ_f whose vertices of level n are the pieces $V_j^n \in \mathcal{I}^n$, $n=0,1,\dots$, where V_j^0 stand for the pieces of the initial tiling (3.4). Let us take a vertex $V_j^{n+1} \in \mathcal{I}^{n+1}$, and let $i(1),\dots,i(t)=0$ be its itinerary through the pieces of the previous level under the iterates of g_n. Then join V_j^{n+1} with V_i^n by k edges, provided the symbol i appears in the above itinerary k times. This means that the piece V_j^{n+1} under iterates of g_n passes through V_i^n k times before the first return back to V^n. Let us order the edges joining two vertices V_j^{n+1} and V_i^n so that the first edge represents the first return of V_j^{n+1} to V_i^n, the second one represents the second return, etc.

Note that for any vertex V_j^{n+1} there is exactly one edge joining it to the critical vertex V_0^n of the previous level. Note also that by Lemma 3.9 in the DH renormalizable case the number of vertices on a given level is finite. In any case there are clearly only finitely many edges leading from a V_j^{n+1} to the previous level n. Let $\tau(V_j^{n+1})$ denote the number of such edges, which is equal to the first return time of V_j^{n+1} back to V^n under iterates of g_n.

By a *path* in the graded graph Υ_f we mean a sequence of consecutively adjacent vertices

$$V^n_{i(n)}, \quad n=l, l+1, ..., l+k,$$

up to reversing the order. So we do not endow a path with orientation, and can go along it either strictly upwards or strictly downwards.

Diverse combinatorial data can be easily read off this graph. For example, given $n \geqslant m$, the number of paths joining V^{n+1}_j to V^m_i is equal to the number of times which the g_m-orbit of V^{n+1}_j passes through V^m_i before the first return back to V^n. Hence the return time of V^{n+1}_j back to V^n under iterates of g_m is equal to the total number of paths in Υ_f leading from V^{n+1}_j up to level m. For $m=0$ we obtain the return time under iterates of the original map $f=g_0$.

Assume now that the map f is DH renormalizable, and let s be a renormalization level in the principal nest, that is, $g_{s+1}: V^{s+1} \to V^s$ is a quadratic-like map with non-escaping critical point. Then there is a single vertex V^{s+1} at level $s+1$, and below it the return graph is just the "vertical path" through the critical vertices. By the above discussion, the total number of paths in the graph Υ_f joining the top level to the bottom vertex V^{s+1} is equal to the *renormalization period* $\mathrm{per}(f)$ (i.e., the return time of V^{s+1} back to V^s under iterates of f).

It follows that $\mathrm{per}(f)$ is bounded if and only if the DH level s is bounded, and all the return times $\tau(V^{m+1}_i)$ are bounded for $1 \leqslant m \leqslant s$ and any i. For instance, the "if" statement means: If $s \leqslant \bar{s}$ and $\tau(V^{m+1}_i) \leqslant \bar{\tau}$ for all vertices V^{m+1}_i, then $\mathrm{per}(f) \leqslant p(\bar{s}, \bar{\tau})$. Indeed, the total number of paths in the graph is bounded by τ^s.

Note that central cascades correspond to the vertical paths through the critical vertices. We say that a path γ passes through a central cascade (3.11) if $\gamma \ni V^n_j$ with $n \in [m+1, m+N]$.

Let us now define one more combinatorial notion, the *rank* (compare [L4, §3]). Let $D^n \subset V^{n-1}$ be a puzzle piece of the full Markov family $\mathcal{Y}_f$ (see §2.6) containing at least one piece V^n_i of level n. Let us consider the shortest path γ leading from D^n (i.e., from one of the pieces $V^n_i \subset D^n$) down to a critical piece V^{n+s}. The number of central cascades this γ passes through will be called the rank of D^n.

This notion is motivated by the following consideration. Let us consider two adjacent puzzle pieces $V^n_i \subset D^n$ and V^{n+1}_j, and an edge γ joining them. Let t be the return time represented by γ, i.e., $g^t_n V^{n+1}_j \subset V^n_i$. The piece $D^{n+1} \supset V^{n+1}_j$ in V^n such that $g^t_n D^{n+1} = D^n$ will be called the pull-back of D^n along the edge γ. More generally, let us define the pull-back of D^n along a path γ leading from D^n downwards by composing the pull-backs along the edges.

LEMMA 3.10. *Let γ be the shortest path leading from D^n down to a critical piece V_0^{n+s}, and let D^{n+s} be the pull-back of D^n along this path. Then*

$$V^{n+s} \subset D^{n+s} \subset V^{n+s-1},$$

and the map $D^{n+s} \to D^n$ is a double-branched covering.

Proof. Follows easily from the definitions. □

3.9. Full principal nest

Let f be a non-immediately DH renormalizable quadratic-like map. Then its principal nest

$$Y^{(0,0)} \supset V^{0,0} \supset V^{0,1} \supset \ldots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \supset \ldots$$

ends up with an infinite cascade of central returns (we call this nest "short" and label it by two indices for the reason which will become clear in a moment). Let us select a level $t(0)$ of this cascade, so that the return map $Rf = g_{0,t(0)+1}: V^{0,t(0)+1} \to V^{0,t(0)}$ is DH quadratic-like. We will call such a level DH. (The particular choice of DH levels in what follows will depend on the geometry.)

If Rf is non-immediately renormalizable, let us cut the puzzle piece $V^{0,t(0)+1}$ by the external rays, and construct its short principal nest:

$$Y^{(1,0)} \supset V^{1,0} \supset V^{1,1} \supset \ldots \supset V^{1,t(1)} \supset V^{1,t(1)+1} \supset \ldots .$$

If Rf is DH renormalizable, then this nest also ends up with an infinite central cascade. Then select a DH level $t(1)+1$, and pass to the next short nest.

If f is infinitely DH renormalizable but none of the renormalizations are immediate, then in such a way we construct the *full principal nest*

$$\begin{aligned}
&Y^{(0,0)} \supset V^{0,0} \supset V^{0,1} \supset \ldots \supset V^{0,t(0)} \supset V^{0,t(0)+1} \\
&\supset Y^{(1,0)} \supset V^{1,0} \supset V^{1,1} \supset \ldots \supset V^{1,t(1)} \supset V^{1,t(1)+1} \supset \\
&\ldots \\
&\supset Y^{(m,0)} \supset V^{m,0} \supset V^{m,1} \supset \ldots \supset V^{m,t(m)} \supset V^{m,t(m)+1} \supset \\
&\ldots .
\end{aligned} \tag{3.13}$$

Here $Y^{(m,0)}$ is the first critical Yoccoz puzzle piece for the m-fold DH renormalization $R^m f$, while the pieces $V^{m,n}$ form the corresponding short principal nest. Moreover, for $m>1$, $Y^{(m,0)}$ is obtained by cutting $V^{m-1,t(m-1)+1}$ with the external rays of $R^m f: V^{m-1,t(m-1)+1} \to V^{m-1,t(m-1)}$.

The annuli $A^{m,n} = V^{m,n-1} \setminus V^{m,n}$ will be called the *principal annuli.*

3.10. Big type: special families of Mandelbrot copies

Assume that we associated to any quadratic-like map a "combinatorial parameter" $\tau(f)$ which depends only on the hybrid class $c(f)$ and is constant over any maximal copy of the Mandelbrot set. (Keep in mind the height function $\chi(f)$ or the period $\operatorname{per}(f)$.) Thus we can use the notation $\tau(M')$.

Let $\mathcal{S}\subset\mathcal{M}$ be a family of maximal copies of the Mandelbrot set. Let us call it τ-*special* if it satisfies the following property: for any truncated secondary limb L there is a τ_L such that $\mathcal{S}$ contains all maximal copies $M'\subset L$ of the Mandelbrot set with $\tau(M')\geqslant\tau_L$.

Let f be an infinitely renormalizable quadratic-like map. Let us say that it is of $\mathcal{S}$-type if all the internal classes $c(R^n f)$ belong to copies M' from $\mathcal{S}$.

4. Initial geometry

The goal of this section is to give a bound on the first principal modulus depending only on the choice of the secondary limbs and $\operatorname{mod}(f)$:

THEOREM I. *Let f be a quadratic-like map with internal class $c(f)$ ranging over a truncated secondary limb L_b^{tr}. If $\operatorname{mod}(f)\geqslant\mu>0$ then*

$$\operatorname{mod}(A^1)\geqslant C(\mu)\nu(L_b^{tr})>0,$$

where $C(\mu)>0$ and $C(\mu)\nearrow 1$ as $\mu\nearrow\infty$.

4.1. Geometry of rays

Let us consider a parameter region D. Assuming that the rays $\mathcal{R}_c^{\theta,(\varrho,r)}$ (see §2.2) do not bounce off the critical point for $c\in D$, let us consider their natural parametrization $\psi_c\colon(\varrho,r)\to\mathcal{R}_c^{\theta,(\varrho,r)}$. Continuous/smooth/real-analytic dependence of the ray on $c\in D$ is defined as the corresponding property of the function $(t,c)\mapsto\psi_c(t)$. The same definitions are applied to equipotentials.

Let $B(a,\delta)=\{z:|z-a|<\delta\}$.

LEMMA 4.1. (i) *Assume that a ray $\mathcal{R}_c^\theta$ and an equipotential E_c^ϱ do not hit the critical point, $c\in D$. Then $\mathcal{R}_c^\theta$ and E_c^ϱ depend real-analytically on c;*

(ii) *Let a_c be a repelling periodic point of P_c continuously depending on $c\in D$. Let the ray $\mathcal{R}_c^\theta$ land at a_c. Then the closure of this ray, $\overline{\mathcal{R}}_c^\theta$, depends continuously on c.*

Proof. (i) The first statement follows from the fact that the Bötcher function analytically depends on c, which is clear from the explicit formula (2.1).

(ii) Let us check continuity at some $d \in D$. By (i), we only need to check that for $r>0$ sufficiently small, the arc $\mathcal{R}_c^{\theta,[0,r]}$ is uniformly close to a_d. Indeed, for any $\varepsilon>0$ there exist $\delta>0$ and $r>0$ such that

- for $|c-d|<\varepsilon$, P_c univalently maps $B_c \equiv B(a_c, \delta)$ onto a strictly bigger disc;
- $\mathcal{R}_c^{\theta,[r,2r]} \subset B_c$ (this follows from (i)).

Pulling back the arc $\mathcal{R}_c^{\theta,[r,2r]}$ by $P_c|B_c$, we conclude that $\mathcal{R}_c^{\theta,[0,2r]} \subset B_c$. □

Given a configuration $\mathcal{C}_0$ of finitely many parametrized curves and points in $\mathbf{C}$, let us consider the space $\mathrm{QC}(\mathcal{C}_0)$ of all configurations qc equivalent to $\mathcal{C}_0$. There is a natural Teichmüller (pseudo-)distance on this space:

$$\operatorname{dist}_T(\mathcal{C}_1, \mathcal{C}_2) = \inf \log \operatorname{Dil}(h),$$

where h runs over all qc equivalences $h: (\mathbf{C}, \mathcal{C}_1) \to (\mathbf{C}, \mathcal{C}_2)$.

We say that configurations of a certain family have *bounded geometry* if they stay at bounded Teichmüller distance from a reference configuration $\mathcal{C}_0$ whose curves are smooth and intersect transversally.

LEMMA 4.2. *Let a_c, $c \in D$, be a repelling periodic point as in Lemma* 4.1. *Let us consider a configuration $\mathcal{R}(a_c)$ of finitely many rays $\mathcal{R}_c^{\theta_i}$ landing at a_c, $c \in D$. Then $\mathcal{R}(a_c)$ has bounded geometry when c ranges over any compact subset of D.*

Proof. Take a $d \in D$. For any nearby $c \in B(d, \delta)$, let us truncate the configuration $\mathcal{R}_c$ with an equipotential E_c^{ϱ}, where ϱ is selected big enough so that E_c^{ϱ} is a Jordan curve. We obtain an inner configuration $\mathcal{R}_c^1$ and an outer one $\mathcal{R}_c^2$. The latter one has bounded geometry over $B(d, \delta)$, as it is conformally equivalent to the configuration consisting of the circle of radius ϱ and radial rays of angles θ_i.

Take a small $\varepsilon>0$. Lemma 4.1 implies that for $c \in B(d, \delta)$ with a small δ, there exists a smooth parametrized Jordan curve $\gamma \subset B(a_c, \varepsilon) \equiv B_c$ enclosing a_c which transversally intersects every ray of $\mathcal{R}_c$ at a single point. It truncates $\mathcal{R}_c^1$ into the inner configuration $\mathcal{R}_c^i$ and the outer one $\mathcal{R}_c^o$. The latter one has bounded geometry over $B(d, \delta)$ since by Lemma 4.1 it smoothly depends on c.

Let us consider $\mathcal{R}_c^i$. The parametrized curves $\gamma_{-N} = (f|B_c)^{-N} \gamma$ also intersect every ray of $\mathcal{R}_c$ at a single point. Moreover, for sufficiently big N (locally uniform), γ_{-N} lies strictly inside γ with a definite space in between.

Let us consider a configuration $\mathcal{C}_c$ consisting of the annulus bounded by γ and γ_{-N} with the arcs of the rays $\mathcal{R}^{\theta_i}$ in between (with the natural parametrization). Since this configuration smoothly depends on c (by Lemma 4.1), it has bounded geometry near d. Thus $\mathcal{C}_c$ stays at bounded Teichmüller distance from a standard configuration $\mathcal{C}_0$, the round annulus $\mathbf{A}\left(\frac{1}{2}, 1\right)$ with p equally spaced radial intervals inside.

So for c near d, there is a K-qc map $h: \mathcal{C}_c \to \mathcal{C}_0$ with locally uniform dilatation K, which conjugates f to $z \mapsto 2z$ on the inner boundaries of the configurations. Pulling this map back by the dynamics, we obtain a K-qc equivalence between the configuration $\mathcal{R}$ and a standard configuration $\mathcal{I}$ consisting of the unit circle and p equally spaced radial intervals emanating from 0.

Since the dilatation K is locally uniform, it is uniform over any compact set. □

Let the α-fixed point of f have rotation number q/p. Then there is a single periodic point $\gamma \in \operatorname{int} Y^{(1)}$ of period p. Let $\mathcal{C}(f)$ stand for the configuration of the rays landing at α, γ and the symmetric points α', γ'.

COROLLARY 4.3. *The configuration $\mathcal{C}(P_c)$ has a bounded geometry while c ranges over a truncated secondary limb L_b^{tr}.*

4.2. Fundamental domain near the fixed point

The goal of this subsection is to construct a combinatorially defined fundamental domain with bounded geometry near the fixed point α. It is where the secondary limbs condition comes into the scene.

Let γ and γ' be the periodic and co-periodic points defined prior to Corollary 4.3. Consider the family $\mathcal{R}(\gamma')$ of rays landing at γ'. Let $D = D_f$ be the component of $Y^{(1)} \setminus \mathcal{R}(\gamma')$ attached to the fixed point α (see Figure 5). Then f^p univalently maps D onto a domain containing the component of $Y^{(0)} \setminus \mathcal{R}(\gamma)$ attached to α. Note that $\partial D \cap \partial (f^p D)$ is contained in the union of two rays landing at α.

Hence there is a univalent branch of f^{-p} which fixes α and maps D inside itself. It is now easy to see that $f^{-pn} D$ shrink to α as $n \to \infty$. So we can select $Q = Q_f = D \setminus f^{-p} D$ as a fundamental domain for f^p near α: any trajectory which starts near α must pass through $Q = Q_f$. Now Corollary 4.3 yields

LEMMA 4.4. *Geometry of the fundamental domain Q_f is bounded if $c(f)$ ranges over a truncated secondary limb and f has a definite modulus.*

4.3. Modulus of the first annulus

LEMMA 4.5. *Let P_c be a quadratic polynomial with c outside the main cardioid but not immediately renormalizable. If c ranges over a truncated secondary limb L_b^{tr}, then all the pieces W of the initial Markov tiling* (3.4) *are well inside $Y^{(0)}$:*

$$\operatorname{mod}(Y^{(0)} \setminus W) > \nu(L_b^{tr}) > 0.$$

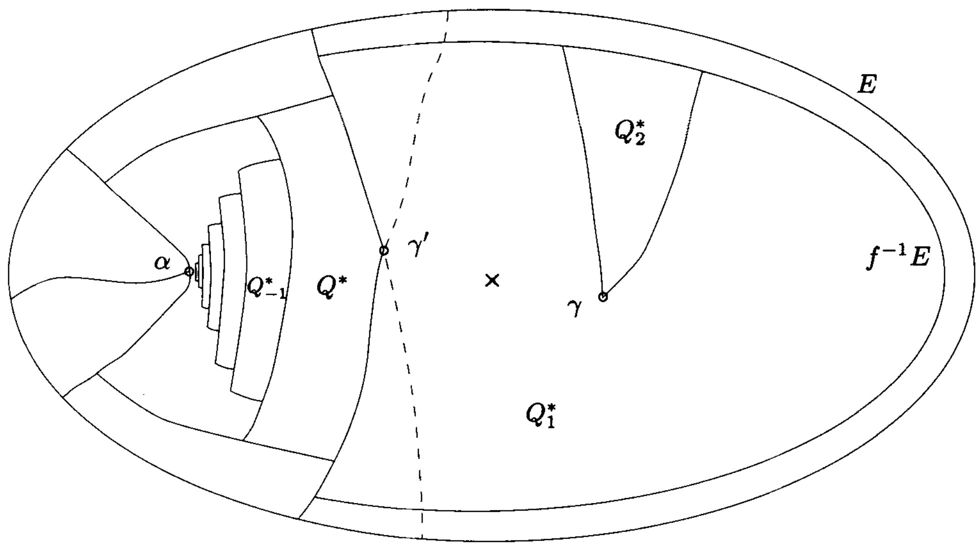

Fig. 5. The fundamental domain near α

Proof. Let $Y^{(0)}$ be bounded by the equipotential $E\equiv E^1$ of level 1 (together with two rays). Let $U^r\supset J(f)$ be the domain bounded by the equipotential E^r.

Take a little $\varepsilon>0$. Then there exist N and $\delta>0$ such that the distance from $U^{1/2^N}\setminus B(\alpha,\varepsilon)$ to $\partial Y^{(0)}$ is at least δ (for all $c\in L_b^{tr}$).

The statement is obviously true for all the pieces W of depth $\leqslant N$.

Any other piece W is contained in $U^{1/2^N}$. Then $\operatorname{dist}(W,\partial Y^{(0)})\geqslant\delta$ if $\operatorname{dist}(W,\alpha)>\varepsilon$. As $\operatorname{diam} W$ is uniformly bounded, we conclude that W is well inside $Y^{(0)}$.

Assume now that $\operatorname{dist}(W,\alpha)<\varepsilon$. Then W intersects the domain $D=D_f$. Since $\partial D\cap\partial W=\varnothing$, $W\subset D$. Let us consider the iterates $f^{pk}W$, $k=0,1,\ldots$, until the last moment l such that $f^{pl}W\subset D$. At this moment $f^{pl}W$ must intersect the fundamental domain Q. Since their boundaries do not intersect, we conclude that $f^{pl}W\subset Q$.

Let us consider the domain $Q^*=Q\cap U^{1/2^p}\subset Q$ obtained by truncating Q with the equipotential $f^{-p}E\equiv E^{1/2^p}$. This domain has a bounded geometry since the fundamental domain Q does (Lemma 4.4). Hence Q^* is well inside f^pD. Moreover, $f^{pl}W\subset Q^*$ since all the puzzle pieces of (3.4) which belong to D are enclosed by the equipotential $f^{-p}E$ (see Figure 5). Hence $f^{pl}W$ is well inside f^pD as well.

We conclude that there is always a definite space around $f^{pl}W$ in f^pD. Pulling this space back by iterates of the univalent branch $f^{-p}\colon f^pD\to D$, we obtain a definite space around W in D. $\square$

We are now ready to prove the theorem stated in the beginning of this section:

Proof of Theorem I. Assume first that $f=P_c$ is a polynomial. Let us go through the proof of Proposition 3.1. We found an l and a puzzle piece $P\subset V^0$ such that G^lP two-to-one covers $Y^{(0)}$, where G is the Markov map (3.5). Moreover, $G^l0\in W$ where $W=V^0$ or $W=X_i^s$. Then V^1 is the pull-back of W by $G^l|P$. But by Lemma 4.5 W is well inside $Y^{(0)}$. Hence V^1 is well inside V^0.

If f is quadratic-like then its straightening yields the desired estimate by Proposition 2.5. The constant $C(\mu)$ can certainly be selected so that it is monotone in μ. □

5. Bounds on the moduli and distortion

In this section we introduce the asymmetric moduli and prove that they do not decrease under the generalized renormalization. This yields a priori bounds on the principal moduli and distortion. The precise formulation (Theorem II) is given at the end of the section. Note that already this result yields the Yoccoz divergence property (Theorem 2.6).

5.1. First estimates

Let $\mathcal{V}^n\subset\mathcal{Y}_f$ stand for the family of all pieces V_i^n of level n.

Let us start with a lemma which partly explains the importance of the principal nest: the principal moduli control the distortion of the first return maps (see the appendix for the definition of distortion). Let us consider the decomposition:

$$g_n|V^n=h_n\circ f, \tag{5.1}$$

where h_n is a diffeomorphism of fV^n onto V^{n-1}.

LEMMA 5.1. *Let $D\in\mathcal{Y}_f$ be a puzzle piece such that $f^lD=V^n$, while $f^kD\cap V^n=\varnothing$, $k=0,...,l-1$. If $\mu_n\geqslant\bar\mu$ then the distortion of f^l on D is $O(\exp(-\mu_{n-1}))$ with a constant depending only on $\bar\mu$. Hence the distortion of h_n is $O(\exp(-\mu_{n-2}))$.*

Proof. This follows from Lemma 3.5, Corollary 3.6 and the Koebe Theorem. □

Let us fix a level $n>0$, denote $V^{n-1}=\Delta$, $V_i=V_i^n$, $g=g_n$, $A=A^n=\Delta\setminus V_0$, $\mu=\mu_n$, and mark the objects of the next level $n+1$ with a prime. Thus $\Delta'\equiv V\equiv V_0$ and $g'\colon\bigcup V_i'\to\Delta'$. (We restore the index n whenever we need it.)

LEMMA 5.2. *Let $D'\subset\Delta'$ be a puzzle piece such that $g^kD'\subset V_{i(k)}$, $k=1,...,l$, with $i(k)\neq0$ for $0<k<l$. Then*

$$\operatorname{mod}(\Delta'\setminus D')\geqslant\tfrac12\sum_{k=1}^{l}\operatorname{mod}(\Delta\setminus V_{i(k)}).$$

Proof. Let us consider the following nest of topological disks:

$$\Delta' \equiv W_1 \supset \dots \supset W_l \supset W_{l+1} \supset D',$$

where W_{k+1} is defined inductively as the pull-back of $V_{i(k)}$ under $g^k\colon W_k \to \Delta$, $k=1, \dots, l$. Since $\deg(g^k\colon W_k \to \Delta)=2$,

$$\operatorname{mod}(W_k \setminus W_{k+1}) = \tfrac{1}{2} \operatorname{mod}(\Delta \setminus V_{i(k)}) \quad (1 \leqslant k \leqslant l).$$

But by the Grötzsch inequality

$$\operatorname{mod}(\Delta' \setminus D') \geqslant \sum_{k=1}^{l} \operatorname{mod}(W_k \setminus W_{k+1}),$$

and the desired estimate follows. □

COROLLARY 5.3. *Given a puzzle piece V_j', we have*

$$\operatorname{mod}(\Delta' \setminus V_j') \geqslant \tfrac{1}{2}\mu.$$

Moreover, if the return to level n is non-central, that is, $g0 \in V_i$ with an $i \neq 0$, then

$$\operatorname{mod}(\Delta' \setminus V_j') \geqslant \tfrac{1}{2}(\mu + \operatorname{mod}(\Delta \setminus V_i)).$$

So, a definite principal modulus on some level produces a definite space around all the puzzle pieces of the next level.

5.3. Isles and asymmetric moduli

Let $\{V_i\}_{i \in \mathcal{I}} \subset \mathcal{V}^n$ be a finite family of disjoint puzzle pieces consisting of at least two pieces (that is, $|\mathcal{I}| \geqslant 2$) and containing a critical puzzle piece V_0. Let us call such a family *admissible*. We will freely identify the label set $\mathcal{I}$ with the family itself.

Given a puzzle piece D, let $\mathcal{I}|D$ denote the family of puzzle pieces of $\mathcal{I}$ contained in D. Let D be a puzzle piece containing at least two pieces of family $\mathcal{I}$. For $V_i \subset D$ let

$$R_i \equiv R_i(\mathcal{I}|D) \subset D \setminus \bigcup_{j \in \mathcal{I}|D} V_j$$

be an annulus of maximal modulus enclosing V_i but not enclosing other pieces of the family $\mathcal{I}$. Such an annulus exists by the Montel Theorem (see Figure 6). We will briefly call it the *maximal annulus* enclosing V_i in D (rel the family $\mathcal{I}$).

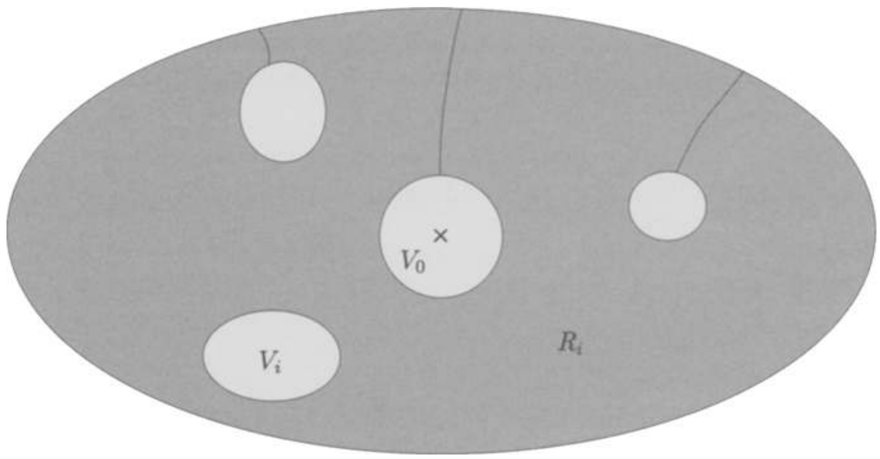

Fig. 6. Annulus R_i

Let us define *the asymmetric modulus of the family $\mathcal{I}$ in D* as

$$\sigma(\mathcal{I}|D)=\sum_{i\in\mathcal{I}}\frac{1}{2^{1-\delta_{i0}}}\operatorname{mod}(R_i(\mathcal{I}|D)),$$

where δ_{ji} is the Kronecker symbol. So the critical modulus is supplied with weight 1, while the off-critical moduli are supplied with weights $\frac{1}{2}$ (if D is off-critical then all the weights are actually $\frac{1}{2}$).

For $D=V^{n-1}$, let $\sigma_n(\mathcal{I})\equiv\sigma(\mathcal{I}|V^{n-1})$. The asymmetric modulus of level n is defined as follows:

$$\sigma_n=\min_{\mathcal{I}}\sigma_n(\mathcal{I}),$$

where $\mathcal{I}$ runs over all admissible subfamilies of $\mathcal{V}^n$.

The principal moduli μ_n and the asymmetric moduli σ_n are the main geometric parameters of the renormalized maps g_n. Again, in what follows the label n will be suppressed as long as the level is not changed.

Let $\{V_i'\}_{i\in\mathcal{I}'}$ be an admissible subfamily of $\mathcal{V}'$. Let us organize the pieces of this family in *isles* in the following way. A puzzle piece $D'\subset\Delta'$ is called an *island* (for the family I') if

- D' contains at least two puzzle pieces of family $\mathcal{I}'$;
- there is a $t\geqslant 1$ such that $g^kD'\subset V_{i(k)}$, $k=1,\ldots,t-1$, with $i(k)\neq 0$, while $g^tD=\Delta$.

Given an island D', let $\phi_{D'}=g^t\colon D'\to\Delta$. This map is either a double covering or a biholomorphic isomorphism depending on whether D' is critical or not. In the former case, $D'\supset V_0'$ (for otherwise $D'\subset V_0'$ contradicting the first part of the definition of isles).

We call a puzzle piece $V'_j \subset D'$ *$\phi_{D'}$-precritical* if $\phi_{D'}(V'_j)=V_0$. There are at most two precritical pieces in any D'. If there are actually two of them, then they are off-critical and symmetric with respect to the critical point 0. In this case D' must also contain the critical puzzle piece V'_0.

Let $\mathcal{D}'=\mathcal{D}(\mathcal{I}')$ be the family of isles associated with $\mathcal{I}'$. Let us consider the asymmetric moduli $\sigma(\mathcal{I}'|D')$ as a function on this family. This function is clearly monotone:

$$\sigma(\mathcal{I}'|D') \geqslant \sigma(\mathcal{I}'|D'_1) \quad \text{if } D' \supset D'_1, \tag{5.2}$$

and superadditive:

$$\sigma(\mathcal{I}'|D') \geqslant \sigma(\mathcal{I}'|D'_1)+\sigma(\mathcal{I}'|D'_2),$$

provided D'_i are disjoint subisles in D'.

Let us call an island D' *innermost* if it does not contain any other isles of the family $\mathcal{D}(\mathcal{I}')$. As this family is finite, innermost isles exist.

5.2. Non-decreasing of the moduli

LEMMA 5.4. *Let $\mathcal{I}'$ be an admissible family of puzzle pieces. Let D' be an innermost island associated to the family $\mathcal{I}'$, and let $\mathcal{J}'=\mathcal{I}'|D$. For $j\in\mathcal{J}'$, let us define $i(j)$ by the property $\phi_{D'}(V'_j)\subset V_{i(j)}$, and let $\mathcal{I}=\{i(j):j\in\mathcal{J}'\}\cup\{0\}$. Then $\{V_i\}_{i\in\mathcal{I}}$ is an admissible family of puzzle pieces, and*

$$\sigma(\mathcal{I}'|D') \geqslant \frac{1}{2}\Bigg((|\mathcal{J}'|-s)\mu+s\,\mathrm{mod}(R_0)+\sum_{j\in\mathcal{J}',\,i(j)\neq 0}\mathrm{mod}(R_{i(j)})\Bigg), \tag{5.3}$$

where $s=\#\{j:i(j)=0\}$ is the number of $\phi_{D'}$-precritical pieces, and R_i are the maximal annuli enclosing V_i in Δ rel $\mathcal{I}$.

Proof. Let $\phi\equiv\phi_{D'}$. Let us show first that the family $\mathcal{I}$ is admissible. This family is finite since $\mathcal{J}'\subset\mathcal{I}'$ is finite. The critical puzzle piece belongs to $\mathcal{I}$ by definition. So the only property to check is that $|\mathcal{I}|\geqslant 2$. But otherwise $\mathcal{J}'$ would consist of two precritical puzzle pieces. But then D' would be critical, and thus should also have contained the critical piece V'_0, which is a contradiction.

Let us observe next that

$$\mathrm{mod}(V_{i(j)}\setminus\phi V'_j) \geqslant \mu \quad \text{if } i(j)\neq 0. \tag{5.4}$$

Indeed, in this case $g^m(\phi V'_j)=V_0$ for some $m>0$. Let $W\subset V_{i(j)}$ be the pull-back of Δ under g^m. Then the annulus $W\setminus\phi V'_j$ is univalently mapped by g^m onto the annulus $\Delta\setminus V_0$. Hence $\mathrm{mod}(W\setminus\phi V'_j)=\mathrm{mod}(\Delta\setminus V_0)=\mu$, and (5.4) follows.

Given an $i\in I$, let us consider a topological disk $Q_i=R_i\cup V_i\subset\Delta$ ("filled annulus R_i"). By the Grötzsch inequality and (5.4),

$$\operatorname{mod}(Q_{i(j)}\setminus\phi V_j)\geqslant\operatorname{mod}(R_{i(j)})+(1-\delta_{0,i(j)})\mu. \tag{5.5}$$

For a $j\in J'$, let us consider an annulus $B'_j\subset D'$, the component of $\phi^{-1}R_{i(j)}$ enclosing V'_j. This annulus does not enclose any other pieces $V'_k\in\mathcal{J}'$, $k\neq j$. Indeed, otherwise the inner component of $\mathbf{C}\setminus B'_j$ would be an island contained in D', despite the assumption that D' is innermost.

Let us now consider a topological disk P'_j obtained by filling the annulus B'_j. Then

$$\operatorname{mod}(R'_j)\geqslant\operatorname{mod}(P'_j\setminus V'_j), \tag{5.6}$$

where $R'_j\subset D'$ is the maximal annulus enclosing V'_j rel $\mathcal{J}'$. Moreover, $\phi\colon P'_j\to Q_{i(j)}$ is univalent or a double covering depending on whether $j\neq0$ or $j=0$. Hence

$$\operatorname{mod}(P'_j\setminus V'_j)\geqslant\frac{1}{2^{\delta_{j0}}}\operatorname{mod}(Q_{i(j)}\setminus\phi V_j). \tag{5.7}$$

Inequalities (5.5)–(5.7) yield

$$\operatorname{mod}(R'_j)\geqslant\frac{1}{2^{\delta_{j0}}}(\operatorname{mod}(R_{i(j)})+(1-\delta_{0,i(j)})\mu). \tag{5.8}$$

Summing up the estimates (5.8) over $\mathcal{J}'$ with weights $1/2^{1-\delta_{j0}}$, we obtain the desired inequality. □

COROLLARY 5.5. *For any island D' of the family $\mathcal{D}'$ the following estimates hold:*

$$\sigma(\mathcal{I}'|D')\geqslant\tfrac{1}{2}\mu\quad\textit{and}\quad\sigma(\mathcal{I}'|D')\geqslant\sigma(\mathcal{I})\geqslant\sigma.$$

Proof. By monotonicity (5.2), it is enough to check the case of an innermost island D'. Let us use the notations of the previous lemma. Since the family $\mathcal{I}$ is admissible, it contains an off-critical piece. Hence $|\mathcal{J}'|$ is always strictly greater than the number s of precritical pieces in D', and (5.3) implies the first of the above inequalities.

Furthermore, as $\mu\geqslant\operatorname{mod}(R_0)$ and $|\mathcal{J}'|\geqslant2$, the right-hand side in (5.3) is bounded from below by

$$\frac{1}{2}\Bigl(|\mathcal{J}'|\operatorname{mod}(R_0)+\sum_{i\in\mathcal{I},\,i\neq0}\operatorname{mod}(R_i)\Bigr)\geqslant\sigma(\mathcal{I}).$$

(Note that $\sigma(\mathcal{I})$ makes sense since $\mathcal{I}$ is admissible.) Finally $\sigma(\mathcal{I})\geqslant\sigma$, and the second inequality follows. □

Let us fix a "big" integer quantifier $N_*>0$. We say that a level n is in the "tail of a cascade" if all levels $n-1$, $n-N_*$ belong to a cascade (note that level $n-1$ itself may be non-central). Cascades of length at least N_* we call "long".

THEOREM II. *Given a generalized quadratic-like map g_1, we have the following bounds of the geometric parameters within its principal nest:*

- *The asymmetric moduli σ_n grow monotonically and hence stay away from* 0 *on all levels*: $\sigma_n \geqslant \bar{\sigma} > 0$.
- *The principal moduli μ_n stay away from* 0 (*that is*, $\mu_n \geqslant \bar{\mu} > 0$) *everywhere except for the case when $n-1$ is in the tail of a long cascade* (*the bound $\bar{\mu}$ depends on the choice of N_**).
- *The off-critical puzzle pieces V_i^n are well inside V^{n-1}* (*that is*, $\mathrm{mod}(V^{n-1} \setminus V_i^n) \geqslant \bar{\mu} > 0$) *except for the case when V_i^n is precritical and $n-2$ is the last level of a long cascade.*
- *The distortion of h_n from* (5.1) *is uniformly bounded on all levels by a constant $\bar{K}$.*

All the bounds depend only on the first principal modulus μ_1 and (*as $\bar{\mu}$ is concerned*) *on the choice of N_*.*

Proof. The first assertion follows from the second inequality of Corollary 5.5. Together with Corollary 5.3 it implies the second one (note that the second inequality of this corollary implies that $\mu' \geqslant \frac{1}{2}\sigma$ in the non-central case). One more application of Corollary 5.3 yields the next assertion.

Let us check the last statement. If $n-2$ is not in the tail of a central cascade, then $\mu_{n-1} \geqslant \bar{\mu}$ by the second statement, and the desired result follows from Lemma 5.1.

Let $n-2$ be in the tail of a central cascade $V^m \supset \ldots \supset V^{n-2} \supset \ldots$. If this is not the last level of this cascade then $g_n | V^n = g_{m+2} | V^n$, so that h_n is just a restriction of the map h_{m+2} with bounded distortion.

Finally, if $n-2$ is the *last* level of a central cascade, then by Corollary 3.8 h_n can be extended to a univalent map with range V^m, and the Koebe Theorem implies the distortion bound. □

Theorems I and II imply

COROLLARY 5.6. *Let f be a renormalizable quadratic-like map whose internal class $c(f)$ belongs to a truncated secondary limb L. Then*

$$\mathrm{mod}(Rf) \geqslant \nu_L(\mathrm{mod}(f)) > 0.$$

Remark. Though we believe that Theorem II is still true for higher degree complex unimodal polynomials $z \mapsto z^d + c$, $c \in \mathbf{C}$, the above argument does not work. However, it is worthwhile to notice that the following estimate is still valid: $\mu_{n(k)+2} \geqslant \phi(\mu_{n(k-1)+2})$, where $n(k)$ is the subsequence of non-central levels and $\phi > 0$ is a function depending only on d (which can be easily written down explicitly). In particular, in the renormalizable case,

$$\operatorname{mod}(Rf) \geqslant \nu(\operatorname{mod}(f), \chi(f)) > 0,$$

where the function ν depends on d and the choice of truncated secondary limbs.

6. Linear growth of the moduli

In this section we will prove the central result of the paper:

THEOREM III. *Let $n(k)$ counts the non-central levels in the principal nest $\{V^n\}$. Then*

$$\operatorname{mod}(A^{n(k)+2}) \geqslant Bk,$$

where the constant B depends only on the first modulus $\mu_1 = \operatorname{mod}(A^1)$.

6.1. Proof of Theorem III

This proof will occupy the rest of this section. Our goal is to prove that $\sigma' \geqslant \sigma + a$ with a definite $a > 0$ (that is, dependent only on $\operatorname{mod}(A_0)$) at least on every other level, except for the tails of long cascades and a couple of the following levels. (Theorem II shows the reason why these tails play a special role: In the tails the principal moduli become tiny which slows down the growth rate of asymmetric moduli.)

Clearly it is enough to show that for any innermost island D'

$$\sigma(\mathcal{I}'|D') \geqslant \sigma + a \tag{6.1}$$

with a definite $a > 0$. The analysis will be split into a tree of cases.

6.2. Let D' contain at least three puzzle pieces

PROPOSITION 6.1. *If an innermost island D' contains at least three puzzle pieces V'_j, $j \in \mathcal{J}'$, then*

$$\sigma(\mathcal{J}'|D') \geqslant \sigma(I) + \tfrac{1}{2}\mu.$$

Proof. Let us split off $\frac{1}{2}\mu$ in (5.3) and estimate all other μ's by $\operatorname{mod}(R_0)$. This estimates the right-hand side by

$$\tfrac{1}{2}\mu + \tfrac{1}{2}(|\mathcal{J}| - 1)\operatorname{mod}(R_0) + \tfrac{1}{2}\sum_{i \in \mathcal{I}, i \neq 0} \operatorname{mod}(R_i),$$

which immediately yields what is claimed. □

Hence under the circumstances of Proposition 6.1 we observe a definite growth of the asymmetric modulus provided level $n-1$ is not in the tail of a long cascade. Indeed, then by Theorem II μ is bounded away from 0, and (6.1) follows.

6.3. Let D' contain two puzzle pieces

The further analysis needs some preparation in the geometric function theory summarized in Appendix A.

Assume that the island D' contains two puzzle pieces V'_j, $j\in\mathcal{J}'$. Let $\phi\equiv\phi_{D'}$ and let $\phi V'_j\subset V_i$ with $i=i(j)$. Fix a quantifier $L_*>0$. When we say that something is "big", this means that it is at least $C(L_*)$ where $C(L_*)\to\infty$ as $L_*\to\infty$. Similarly "small" means an upper bound by $\varepsilon(L_*)\to 0$ as $L_*\to\infty$. The sign $\approx$ will mean an equality up to a small (in the above sense) error, while the sign $\succ$ will mean the inequality up to a small error.

Case (i). *Assume that there is a non-critical puzzle piece $V_{i(j)}$ whose Poincaré distance in Δ from the critical point is less than L_*.* Then by Lemma A.1

$$\mu\geqslant \operatorname{mod}(R_0)+\alpha \tag{6.2}$$

with a definite $\alpha=\alpha(L_*)>0$. But observe that when we passed from Lemma 5.4 to Corollary 5.5 we estimated μ by $\operatorname{mod}(R_0)$. Using the better estimate (6.2), we obtain a definite increase of σ.

Case (ii). *Assume now that the hyperbolic distance in Δ from any non-critical puzzle piece $V_{i(j)}$ to the critical point is at least L_*.* Assume also that levels $k\in[n-3,n]$ do not belong to the tail of a long cascade (for the sake of linear growth it is enough to prove definite growth on such levels). Then V_0 may not belong to any non-trivial island together with some off-critical piece $V_{i(j)}$. Indeed, by Theorem II all puzzle pieces of level $n-1$ are well inside V^{n-2}. But then by Lemma 5.2 all non-trivial isles of level n are well inside of $V^{n-1}\equiv\Delta$. (The quantifier L_* should be chosen bigger than the a priori bound on the hyperbolic diameters of the isles.)

Subcase (ii-a). *Assume that both $V_{i(j)}$ are non-critical.* Then by Corollary 5.5 $\sigma(\mathcal{J}'|D')$ is estimated by $\sigma_n(\mathcal{I})$ where the family $\mathcal{I}$ consists of three puzzle pieces: two pieces $V_{i(j)}$ and the central puzzle piece V_0.

If the puzzle pieces $V_{i(j)}$, $j\in\mathcal{J}'$, do not belong to the same non-trivial island, then by Proposition 6.1 $\sigma(\mathcal{I})\geqslant\sigma_{n-1}+a$ with a definite $a>0$, and we are done.

Otherwise the puzzle pieces $V_{i(j)}$ belong to an island W. Since by Lemma 5.2 W is well inside of Δ, it stays on the big Poincaré distance from the critical point (namely, on distance $L_*-O(1)$). Hence $\operatorname{mod}(R_0)\approx\mu$ and

$$\sigma(\mathcal{I})\geqslant\sigma(\mathcal{I}|W)+\operatorname{mod}(R_0)\succ\sigma_{n-1}+\mu,$$

where $\mu\equiv\mu_n$ is bounded away from 0, since level $n-1$ is not in the tail of a long cascade. So we have gained some extra growth, and can pass to the next case.

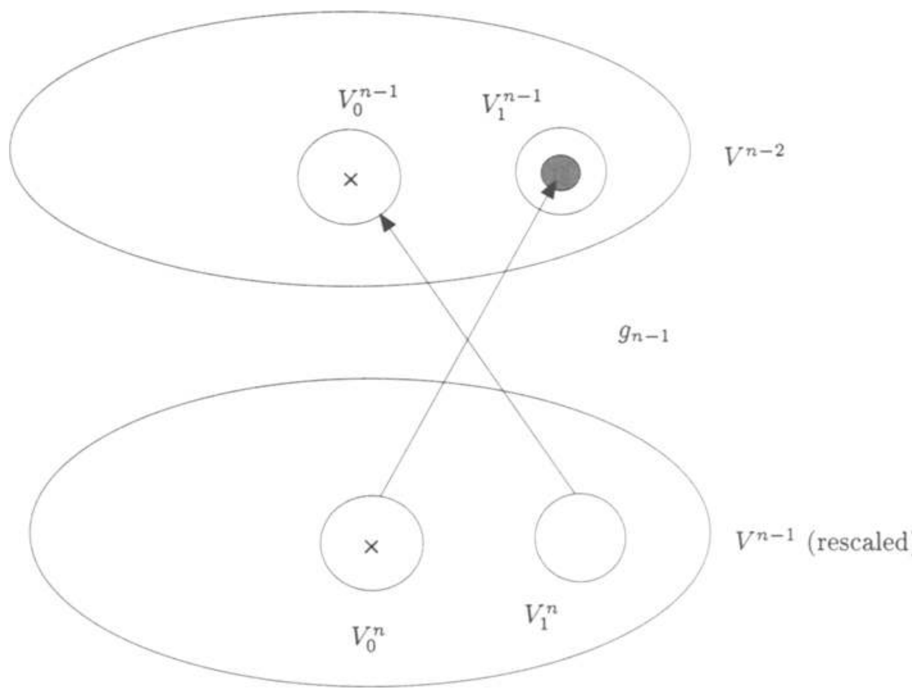

Fig. 7. Fibonacci scheme

Below we will restore labels n and $n+1$ since many levels will be involved in the consideration.

Subcase (ii-b). *Let one of the puzzle pieces $V_{i(j)}^n$ be critical.* So we have the family $\mathcal{I}^n$ of two puzzle pieces V_0^n and V_1^n. Remember that we also assume that *the hyperbolic distance between these pieces is at least L_*.* Hence, V^{n-1} is the only island containing both of them, so that $g_{n-1}V_0^n$ and $g_{n-1}V_1^n$ belong to different puzzle pieces of level $n-1$. For the same reason we can assume that one of these puzzle pieces is critical. Denote them by V_0^{n-1} and V_1^{n-1}. Thus one of the following two possibilities on level $n-2$ can occur:

(1) *Fibonacci return* when $g_{n-1}V_0^n \subset V_1^{n-1}$ and $g_{n-1}V_1^n = V_0^{n-1}$ (see Figure 7);

(2) *Central return* when $g_{n-1}V_0^n = V_0^{n-1}$ and $g_{n-1}V_1^n \subset V_1^{n-1}$.

We can assume that one of these schemes occurs on several previous levels $n-3$, $n-4$, ... as well (otherwise we gain an extra growth by the previous considerations). To fix the idea, let us first consider the following particular case, which plays the key role for the whole theorem.

6.4. Fibonacci cascades

Assume that on both levels $n-2$ and $n-3$ the Fibonacci returns occur. Let us look more carefully at the estimates of Lemma 5.4. In the Fibonacci case we just have:

$$\operatorname{mod}(R_1^n) \geqslant \operatorname{mod}(R_0^{n-1}), \tag{6.3}$$

$$\operatorname{mod}(R_0^n) \geqslant \tfrac{1}{2} \operatorname{mod}(Q_1^{n-1} \backslash g_{n-1} V_0^n), \tag{6.4}$$

where $Q_i^n = V_i^n \cup R_i^n$. Applying g_{n-2} we see that

$$\operatorname{mod}(Q_1^{n-1} \backslash g_{n-1} V_0^n) \geqslant \operatorname{mod}(Q_0^{n-2} \backslash V_0^{n-1}).$$

But since V_1^{n-2} is hyperbolically far away from the critical point (the assumption of Case (ii) is still effective),

$$\operatorname{mod}(Q_0^{n-2} \backslash V_0^{n-1}) \approx \operatorname{mod}(V_0^{n-3} \backslash V_0^{n-1}).$$

By the Grötzsch inequality there is an $a \geqslant 0$ such that

$$\operatorname{mod}(V_0^{n-3} \backslash V_0^{n-1}) = \mu_{n-1} + \mu_{n-2} + a. \tag{6.5}$$

Clearly

$$\mu_{n-1} \geqslant \operatorname{mod}(R_0^{n-1}).$$

Furthermore, let $P_1^{n-1} \subset V^{n-2}$ be the pull-back of Q_0^{n-2} by g_{n-2}. Since ∂P_1^{n-1} is hyperbolically far away from V_1^{n-1}, we have

$$\mu_{n-2} \geqslant \operatorname{mod}(R_0^{n-2}) = \operatorname{mod}(P_1^{n-1} \backslash V_1^{n-1}) \approx \operatorname{mod}(V^{n-2} \backslash V_1^{n-1}) \geqslant \operatorname{mod}(R_1^{n-1}). \tag{6.6}$$

Combining estimates (6.4) through (6.6) we get

$$\operatorname{mod}(R_0^n) \succ \tfrac{1}{2}(\operatorname{mod}(R_0^{n-1}) + \operatorname{mod}(R_1^{n-1}) + a). \tag{6.7}$$

We see from (6.3) and (6.7) that we need to check that the constant a in (6.5) is definitely positive. Assume that this is not the case, that is, for any $\delta > 0$ we can find a level n in the Fibonacci cascade as above such that $a < \delta$. Set $\Gamma_n = \partial V^n$. Then by the definite Grötzsch inequality (see Appendix A), the width(Γ_{n-2}) in the annulus $T = V^{n-3} \backslash V^{n-1}$ is at most $\xi(\delta)$ with $\xi(\delta) \to 0$ as $\delta \to 0$. Since Γ_{n-2} is well inside of T, we conclude by the Koebe Distortion Theorem that Γ_{n-2} is contained in a narrow neighborhood of a curve γ with a bounded geometry. Hence there is a $k = k(\delta) \to 0$ as $\delta \to 0$ and $\varepsilon = \varepsilon(\delta, k) > 0$ such that the curve Γ_{n-2} is not (k, ε)-pinched.

On the other hand, the hyperbolic distance from the puzzle piece V_1^{n-1} to the critical point 0 in V^{n-2} is at least L_*. Hence by Lemma A.4 it must be located in the Euclidean sense very close to Γ_{n-2} relative to the Euclidean distance to the critical point (that is, the relative distance is at most $\beta(\delta)\to 0$ as $\delta\to 0$). Hence the critical value $g_{n-1}0$ is also very close to Γ_{n-2} relative to the distance to the critical point, that is,

$$\frac{\operatorname{dist}(g_n 0, \Gamma_{n-2})}{\operatorname{dist}(g_n 0, 0)} \leqslant \varepsilon(L_*),$$

where $\varepsilon(L_*)\to 0$ as $L_*\to\infty$.

By the last statement of Theorem II, g_{n-1} is a quadratic map up to a bounded distortion. Hence the curve Γ_{n-1} which is the pull-back of Γ_{n-2} by g_{n-1} must have a huge eccentricity around the critical point. But then by Lemma A.2 the width of Γ_{n-1} in $V^{n-2}\setminus V^n$ is also big, which by the above considerations gives a definite linear growth on the next level.

Remark. The actual shape of a deep level puzzle piece for the Fibonacci cascade is shown on Figure 8. There is a good reason why it resembles the filled Julia set for $z\mapsto z^2-1$ (see [L5]). As the geodesic in V_0^{n-1} joining the puzzle pieces V_0^n and V_1^n goes through the pinched region, the Poincaré distance between these puzzle pieces is, in fact, big.

It is time now to look closer at central cascades.

6.5. Central cascades

Let $N\geqslant 2$, $n=m+N$, and let us consider a nest $\mathcal{C}^{m+N}$ of puzzle pieces

$$V^m \supset V^{m+1} \supset \ldots \supset V^{m+N-1} \supset V^{m+N} \supset D^{m+N} \tag{6.8}$$

satisfying the following properties (see Figure 9):

- The return on level $m-1$ is non-central: $g_m 0\notin V_0^m$;
- Central returns occur on levels $m, m+1, ..., m+N-2$, that is, $g_{m+1}0\in V^{m+N-1}$;
- D^{m+N} is an island with a family $\mathcal{I}^{m+N+1}$ of two puzzle pieces inside, V_0^{m+N+1} and V_1^{m+N+1}. Let us denote by $\phi\equiv\phi_{m+N}\equiv\phi_{D^{m+N}}$ the corresponding double covering $D^{m+N}\to V^{m+N-1}$;
- One of the puzzle pieces $\phi_{m+N}V_0^{m+N+1}$, $\phi_{m+N}V_1^{m+N+1}$ is critical.

We would like to analyze when

$$\sigma(\mathcal{I}^{m+N+1}|D^{m+N}) \geqslant \sigma_{m+1}+a \tag{6.9}$$

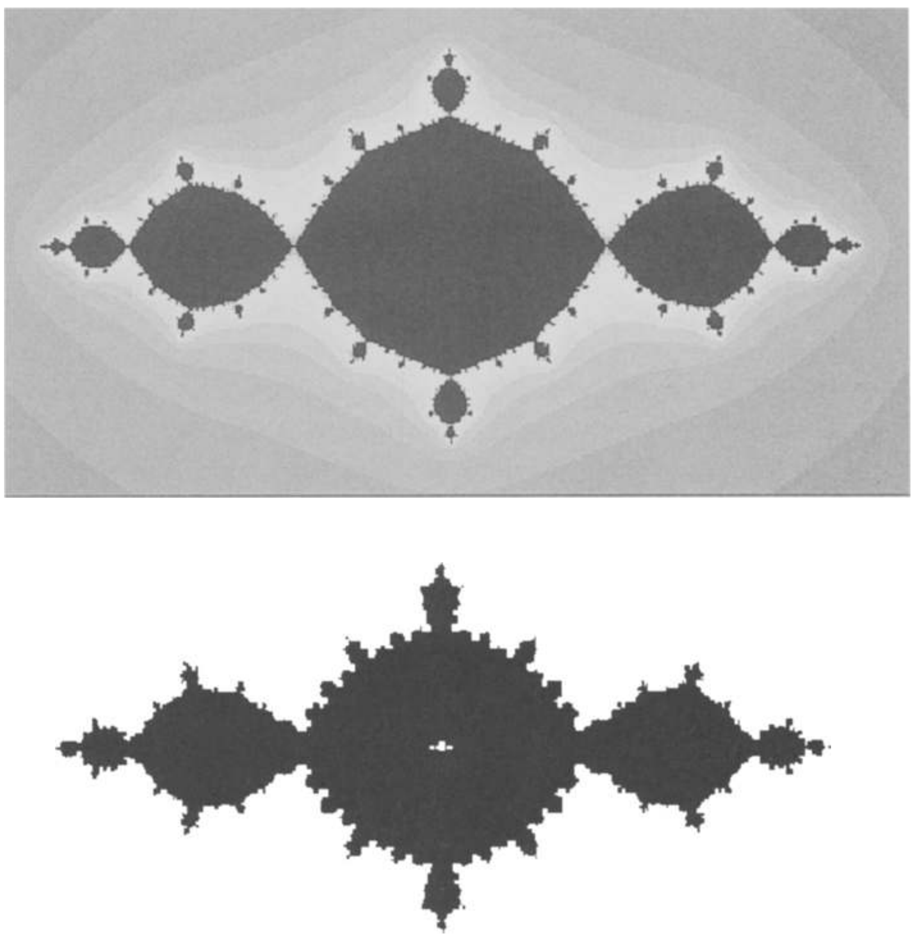

Fig. 8. Fibonacci puzzle piece (below) vs the Julia set of $z \mapsto z^2 - 1$ (above)

with a definite $a > 0$. To this end we need to pass from level $m+N$ all way up to level m.

Let $V_*^{m+N+1} \subset D^{m+N}$ be a non-precritical piece of the family $\mathcal{I}^{m+N+1}$, and

$$\phi V_*^{m+N+1} \subset V_1^{m+N} \subset W_i^{m+N}$$

for some $i \neq 0$. Then the return map $g_{m+N+1}: V_*^{m+N+1} \to V^{m+N}$ can be decomposed as $G^l \circ \phi$ for an appropriate $l \geqslant 1$, where $G: \bigcup W_i^k \to V^m$ is the Bernoulli map (3.12) associated with the central cascade. Since G has range V^m,

$$\operatorname{mod}(W_i^{m+N} \setminus \phi V_*^{m+N+1}) \geqslant \operatorname{mod}(V^m \setminus V^{m+N}). \tag{6.10}$$

Let $\Gamma^k = \partial V^k$ and

$$w_k = \operatorname{width}(\Gamma^k | V^{k-1} \setminus V^{k+1}).$$

For $k\in[m+1, m+N]$ let V_1^k denote the puzzle piece of level k which contains $g_{m+1}^{m+N-k}\phi V_*^{m+N+1}$, and $\mathcal{I}^k$ denote the family of two puzzle pieces: V_0^k and V_1^k. Moreover, let $R_i^k\subset V^{k-1}$ denote an annulus of maximal modulus going around V_i^k but not going around the other piece of family $\mathcal{I}^k$, $i=0,1$.

By the definite Grötzsch inequality and the second part of Theorem II, there is an $a=a(w_{m+1})$ such that

$$\begin{aligned}\operatorname{mod}(V^m\setminus V^{m+N}) &\geqslant \sum_{k=m+1}^{m+N}\operatorname{mod}(A^k)+a\\ &=\sum_{k=0}^{N-1}\frac{1}{2^k}\operatorname{mod}(A^{m+1})+a\geqslant\left(2-\frac{1}{2^{N-1}}\right)\operatorname{mod}(R_0^{m+1})+a.\end{aligned}\tag{6.11}$$

Let S_0^{m+N} and S_1^{m+N} denote the pull-backs of the annuli R_0^{m+1} and R_1^{m+1} by the map $g_{m+1}^{N-1}\colon V^{m+N-1}\to V^m$. Then

$$\operatorname{mod}(S_0^{m+N})\geqslant\frac{1}{2^{N-1}}\operatorname{mod}(R_0^{m+1})\quad\text{and}\quad\operatorname{mod}(S_1^{m+N})\geqslant\operatorname{mod}(R_1^{m+1}).\tag{6.12}$$

Note that the inner boundary of S_1^{m+N} coincides with the outer boundary of $W_i^{m+N}\setminus\phi V_*^{m+N+1}$. Let Q_1^{m+N} denote the union of these two annuli. This annulus goes around ϕV_*^{m+N+1} but not around V_0^{m+N}. Now estimates (6.10), (6.11), (6.12) yield

$$\operatorname{mod}(S_0^{m+N})+\operatorname{mod}(Q_1^{m+N})\geqslant 2\operatorname{mod}(R_0^{m+1})+\operatorname{mod}(R_1^{m+1})+a\geqslant 2\sigma(\mathcal{I}^{m+1})+a.$$

Finally, pulling S_0^{m+N} and Q_1^{m+N} back by $\phi=\phi_{m+N}$ to the island D^{m+N} we obtain

$$\sigma(\mathcal{I}^{m+N+1}|D^{m+N})\geqslant\tfrac{1}{2}(\operatorname{mod}(S_0^{m+N})+\operatorname{mod}(Q_1^{m+N}))\geqslant\sigma(\mathcal{I}^{m+1})+\tfrac{1}{2}a.$$

So we come up with the following statement:

STATEMENT 6.2. *There is an increasing function* $a\colon\mathbf{R}_+\to\mathbf{R}_+$, $a(0)=0$, *such that for the cascade* $\mathcal{C}^{m+N}$ *estimate* (6.9) *holds with* $a=a(w_{m+1})$, *where*

$$w_{m+1}=\operatorname{width}(\Gamma^{m+1}|V^m\setminus V^{m+2}).$$

Let us fix a quantifier w_* which distinguishes "small width" w from a "definite" one. For further analysis let us go several levels up. Let $m-1-l$ be the highest non-central level preceding $m-1$, $l\geqslant 1$. We are going to study when

$$\sigma(\mathcal{I}^{m+N+1}|D^{m+N})\geqslant\sigma_{m-l}+a\tag{6.13}$$

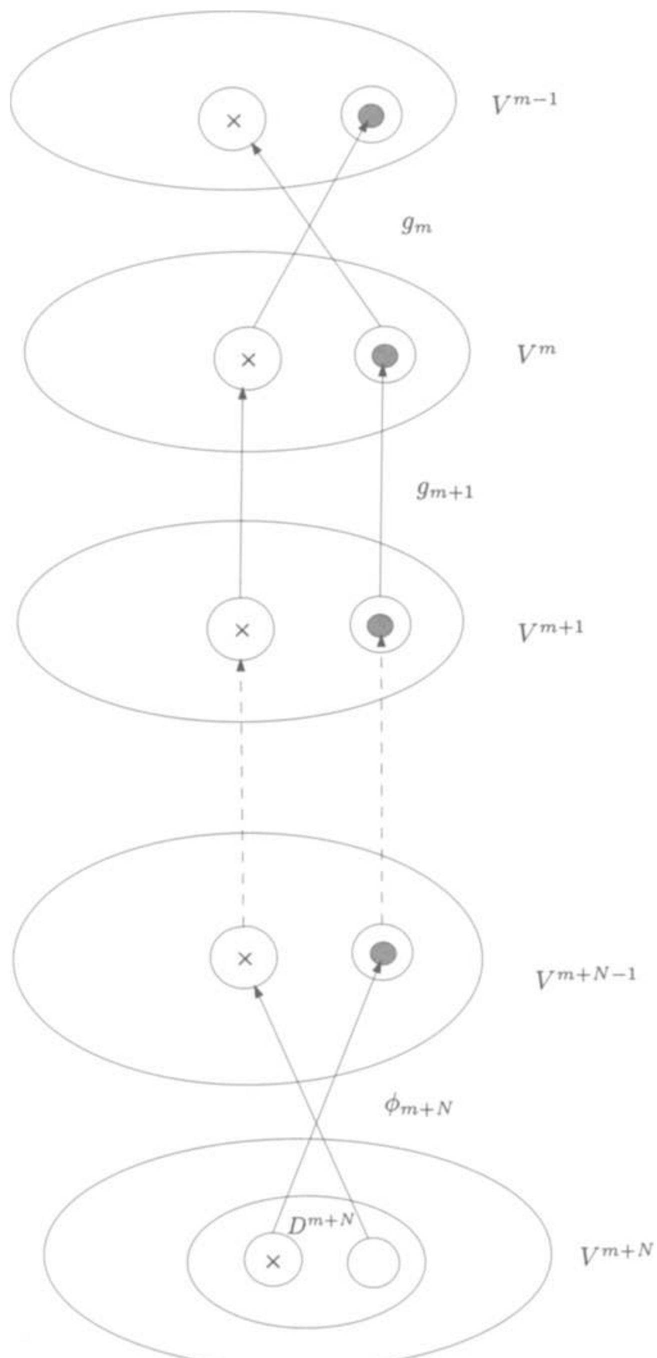

Fig. 9. Central cascade (with Fibonacci returns on the top and the bottom)

with a definite $a>0$. We cannot now assume that l is bounded, so we face a possibility of a long cascade $\mathcal{C}^{m-1}$: $V^{m-l}\supset\ldots\supset V^{m-1}$. Set $g=g_{m-l+1}$; then $g0\in V^{m-1}$.

— *Assume first that* $m-2$ *is not the last level of a long cascade* (in particular, this is the case when the central return occurs on level $m-2$, that is, $l\geqslant 2$). Then by the third part of Theorem II all non-central pieces of level m are well inside V^{m-1}:

$$\operatorname{mod}(V^{m-1}\setminus V_j^m)\geqslant\bar{\mu},\quad j\neq 0.$$

Hence $g_m V_0^{m+1}$ and $g_m V_1^{m+1}$ belong to different pieces of level m. Indeed, otherwise the hyperbolic distance between V_0^{m+1} and V_1^{m+1} in V^m would be bounded by a constant $L(\bar{\mu})$. But according to our assumption this distance is at least L_*. So this situation is impossible if L_* was a priori selected bigger than $L(\bar{\mu})$.

For the same reason the pieces $g^k\circ g_m V_i^{m+1}$, $i=0,1$, also belong to different pieces V_j^{m-k} for $0\leqslant k\leqslant l-3$. Indeed, assume that they belong to the same piece V_j^{m-k}. Clearly this piece is non-central, that is, $j\neq 0$. Then it is contained in a piece W_j^{m-k} of the Bernoulli family $\mathcal{W}(\mathcal{C}^{m-1})$ associated to the central cascade $\mathcal{C}^{m-1}$. Hence $g_m V_0^{m+1}$ and $g_m V_1^{m+1}$ belong to W_j^m, the pull-back of W_j^{m-k} by g^k. As $\operatorname{mod}(W_j^m\setminus g_m V_i^{m+1})\geqslant\bar{\mu}$, the hyperbolic distance between V_0^{m+1} and V_1^{m+1} in V^m is at most $L(\bar{\mu})$ contradicting our assumptions.

Let us show now that (6.13) holds if both $g_m V_i^{m+1}$ are non-central. Indeed let us then consider the family $\mathcal{I}^m$ of three pieces: two pieces of level m containing $g_m V_i^{m+1}$ and the central piece V^m. Let $\mathcal{I}^{m-k}$ denote the family of puzzle pieces of level $m-k$ containing the pieces of $g^k\mathcal{I}^m$. By the previous two paragraphs, $\mathcal{I}^{m-k}$ consists of three puzzle pieces. Then by Corollary 5.5 and Proposition 6.1,

$$\sigma(\mathcal{I}^{m+1})\geqslant\sigma(\mathcal{I}^m)\geqslant\ldots\geqslant\sigma(\mathcal{I}^{m-l+2})\geqslant\sigma(\mathcal{I}^{m-l+1})+\tfrac{1}{2}\bar{\mu},$$

and we are done.

Thus let us assume that the Fibonacci return occurs on level $m-1$. In this case let $\mathcal{I}^{m-k}$ denote the family of two puzzle pieces V_0^{m-k} and V_1^{m-k} containing $g^k\circ g_m V_i^{m+1}$, $i=1,2$, $k\leqslant l-1$.

Note that in order to have (6.13) it is enough to have a definite increase of the $\sigma(\mathcal{I}^{m-k})$ in the beginning of the cascade $\mathcal{C}^{m-1}$. By Statement 6.2 applied to this cascade this is the case if $\operatorname{width}(\Gamma^{m-l+1}|V^{m-l}\setminus V^{m-l+2})\geqslant w_*$. So assume that the opposite inequality holds. Similarly, we can assume that the hyperbolic distance from V_1^{m-l+2} to 0 in V^{m-l+1} is at least L_* (for otherwise we are fine: see Case (i) above).

It follows from Lemma A.4 from Appendix A that the piece V_1^{m-l+2} stays at Euclidean distance at most $\varepsilon\operatorname{diam}\Gamma^{m-l+1}$ from Γ^{m-l+1} where $\varepsilon=\varepsilon_{\bar{\mu}}(w_*,L_*)\to 0$ as $w_*\to 0$,

$L_*\to\infty$ (for a fixed $\bar{\mu}>0$). Hence the Euclidean distance from V_1^{m-l+2} to Γ^{m-l+1} is relatively small as compared with its distance to Γ^{m-l} and Γ^{m-l+2}. More precisely, there is a $\delta=\delta_{\bar{\mu}}(w_*,L_*)$ with the same properties as ε above such that for any $z\in V_1^{m-l+2}$,

$$\operatorname{dist}(z,\Gamma^{m-l+1})\leqslant\delta\operatorname{dist}(z,\partial(V^{m-l}\setminus V^{m-l+2})).\tag{6.14}$$

Take $z_0\in V_1^{m-l}$, and let $r=\operatorname{dist}(z_0,\partial(V^{m-l}\setminus V^{m-l+2}))$. Note that the disk $B(z_0,r)$ can be univalently pulled by g^{l-3} to the annulus $V^{m-3}\setminus V^{m-1}$. By the Koebe Distortion Theorem and (6.14), for any $\zeta\in V_1^{m-3}$,

$$\operatorname{dist}(\zeta,\Gamma^{m-2})\leqslant C\delta\operatorname{dist}(\zeta,\partial(V^{m-3}\setminus V^{m-1}))\leqslant C\delta\operatorname{dist}(\zeta,0)$$

with an absolute constant C. All the more,

$$\operatorname{dist}(\zeta,\Gamma^{m-2})\leqslant C\delta\operatorname{diam}V^{m-2},$$

so that Γ^{m-2} has a big eccentricity about V_1^{m-2} (that is, this eccentricity is at least $e(w_*,L_*)$, where $e(w_*,L_*)\to\infty$ as $w_*\to 0$, $L_*\to\infty$).

Pulling Γ^{m-2} back by $g_{m+1}\circ g_m\circ g_{m-1}$, we conclude that Γ^{m+1} has a big eccentricity about 0. Hence it has big width in the annulus $V^m\setminus V^{m+2}$, and Statement 6.2 yields the desired result.

Let us summarize the information which will be useful in what follows:

STATEMENT 6.3. *If the width w_{m-l+1} is at most w_* and the Poincaré distance from V_1^{m-l+2} to 0 in V^{m-l+1} is at least L_*, then the eccentricity Γ^m about the origin is at least $e(w_*,L_*)$, where $e(w_*,L_*)\to\infty$ as $w_*\to 0$ and $L_*\to\infty$.*

— *Let us assume now that $m-2$ is the last level of a long cascade $\mathcal{C}^{m-2}$:*

$$V^{m-2-t}\supset\ldots\supset V^{m-2},\quad t\geqslant N_*.$$

Then the non-central return occurs on level $m-2$. We will show that

$$\sigma(\mathcal{I}^{m+N+1}|D^{m+N})\geqslant\sigma_{m-2-t}+a\tag{6.15}$$

with a definite $a>0$.

Let $D^m\subset V^m$ be an island containing V_0^{m+1} and V_1^{m+1}, and $\phi_m: D^m\to V^{m-1}$ be the corresponding two-to-one map. Note that in the case under consideration this island may be non-trivial and still the Poincaré distance between V_0^{m+1} and V_1^{m+1} be big (since the precritical puzzle pieces in V^{m-1} are not well inside V^{m-1}). Moreover, the map ϕ_m is not necessarily a bounded perturbation of the quadratic map. These are the circumstances which make this case special.

As $m-2$ is a non-central level, $\mu_{m+1}\leqslant\bar{\mu}$, and by the previous considerations we are done unless

• the return on level $m-1$ is Fibonacci, that is, $\phi_m V_1^{m+1}=V_0^m$ and $\phi_m V_0^{m+1}\subset V_1^m$ for some puzzle piece V_1^m;

• the hyperbolic distance between the puzzle pieces V_0^m and V_1^m is at least L_*;

• the return on level $m-2$ is also Fibonacci: $g_{m-1}V_1^m=V_0^{m-1}$ and $g_{m-1}V_0^m=V_1^{m-1}$ for some puzzle piece V_1^{m-1}.

Let V_0^k and V_1^k be the pieces containing the corresponding push forwards of V_0^{m-1} and V_1^{m-1} along the cascade $\mathcal{C}^{m-2}$, $m-1\leqslant k\leqslant m-t-1$. Then (6.15) follows unless

• the width w_{m-t-3} is at most w_*, and the distance between V_0^{m-t-4} and V_1^{m-t-4} in V^{m-t-3} is at least L_*.

But then by Statement 6.3 applied to the cascade $\mathcal{C}^{m-2}$ the eccentricity of Γ^{m-1} about 0 is at least $e=e(w_*, L_*)$. As g_m is a bounded perturbation of the quadratic map, by Lemma A.5 the curve Γ^m is $(0.1, \varepsilon)$-pinched, where $\varepsilon=\varepsilon_{\bar{\mu}}(e)\to 0$ as $e\to\infty$. (Note that the pinched region is not necessarily around V_1^m, since ϕ_m may differ from g_m.) Applying Lemma A.5 again, we conclude that the curve Γ^{m+1} is $((10C)^{-1}, C\sqrt{\varepsilon})$-pinched. By Lemma A.3, Γ^{m+1} has a definite width inside $V^m\setminus V^{m+2}$. Now Statement 6.2 yields (6.15). Theorem III is proven.

7. Big type yields big space

Below we will analyze a variety of combinatorial factors (like height, return time, length of a central cascade) which yield a big modulus of the renormalized map. Altogether they are quite close to a "big renormalization period", except that "parabolic or Siegel cascades" may interfere. This is summarized in Theorem IV$'$ stated at the end of the section which loosely says that if the periods of $R^n f$ are sufficiently big and there are no "parabolic" or "Siegel cascades" in the principal nests then there are a priori bounds.

7.1. Big height yields big modulus

Let us start with a quick consequence of Theorem III. We refer to §3.10 for the terminology used below.

THEOREM IV. *For any Q, there is a χ-special family $\mathcal{S}$ of the Mandelbrot copies with the following property. Let f be an infinitely renormalizable quadratic of $\mathcal{S}$-type. Then* $\operatorname{mod}(R^m f)\geqslant Q$, $m=0,1,\ldots$.

Proof. Let us fix a $Q>0$. Take a truncated secondary limb $L\equiv L_b^{tr}$, and find $q=C(Q)\nu(L)$ from Theorem I. Note that $q\geqslant\frac{1}{2}\nu(L)$ for sufficiently big Q (independently of L). Let us now select all copies M' of the Mandelbrot set with the height $\chi(M') \geqslant Q/B$, where $B=B(q)$ is the constant from Theorem III. Taking the union of all these copies over all truncated limbs, we obtain a special family $\mathcal{S}$.

Let us now consider an infinitely renormalizable quadratic-like map f of $\mathcal{S}$-type with $\operatorname{mod}(f) \geqslant Q$ (to start, take a quadratic polynomial). Then by Theorem I, $\operatorname{mod}(A^1) \geqslant q$. Hence by Theorem III, $\operatorname{mod}(Rf) \geqslant B\chi(f) \geqslant Q$.

By induction, $\operatorname{mod}(R^n f) \geqslant Q$ for all n. □

7.2. Big return time implies big modulus

The simplest possible way to create a big modulus is the following. By Lemma 2.8, if the return time l of the critical point to a puzzle piece $Y^{(k)}$ of a given depth k is big then the pull-back $Y^{(k+l)}$ of $Y^{(k)}$ along $\operatorname{orb}_l(0)$ has a small diameter (uniformly over a truncated primary limb), so that $\operatorname{mod}(Y^{(k)} \setminus Y^{(k+l)})$ is big. We can now start a principal nest from $Y^{(k)}$. By Theorem II (§5) we will observe big moduli on all levels down except those in the tails of cascades. In particular, $\operatorname{mod}(Rf)$ is also big if f is renormalizable.

Below we describe more involved situations creating a big modulus. We will rely on the combinatorial considerations of §3.8. Let $\mathcal{I}^n \subset \mathcal{V}^n$ stand for the family of puzzle pieces V_i^n intersecting $\omega(0)$. Consider an edge γ^{n+1} of the return graph with vertices at $V_j^{n+1} \in \mathcal{I}^{n+1}$ and $V_i^n \in \mathcal{I}^n$, and let t be the corresponding landing time, so that $g_n^t V_j^{n+1} \subset V_i^n$. Then we will use the notation $\operatorname{mod}(\gamma)$ for $\operatorname{mod}(V_i^n \setminus g_n^t V_j^{n+1})$. If $i \neq 0$ then $\operatorname{mod}(\gamma^n) \geqslant \mu_n$.

LEMMA 7.1. *Let $D^n \subset V^{n-1}$ be a puzzle piece containing at least one piece of $\mathcal{I}^n$. Let Γ be a path leading from D^n down to some critical piece V^{n+t}, and let D^{n+t} be the pull-back of D^n along this path. Then*

$$\operatorname{mod}(D^{n+t} \setminus V^{n+t}) \geqslant \tfrac{1}{2} \bar{\mu} \operatorname{rank}(D^n).$$

Proof. Let $\{\gamma^{n+1}, ..., \gamma^{n+t}\}$ be the edges of Γ. By Lemma 3.10, all these edges except the last one represent univalent maps, and the last one represents a double covering. Hence

$$\operatorname{mod}(D^{n+t} \setminus V^{n+t}) \geqslant \tfrac{1}{2} \sum \operatorname{mod}(\gamma^{n+k}) \geqslant \tfrac{1}{2} \sum_{k=1}^{t} \mu_{n+k},$$

and Theorem II (together with the definition of the rank) completes the proof. □

LEMMA 7.2. *Assume that $n-2$ is not in the tail of a long central cascade. Assume that for a puzzle piece $V_j^{n+1}\in\mathcal{I}^{n+1}$, $r\equiv\tau(V_j^{n+1})\geqslant R$. Then there is a level m such that $\mu_m\geqslant L(R)$, where $L(R)\to\infty$ as $R\to\infty$.*

Proof. Let $M>0$. We need to find a level m with $\mu_m\geqslant M$, provided r is sufficiently big. If $\operatorname{rank}(V_j^{n+1})>N\equiv 2M/\bar{\mu}$, then Lemma 7.1 yields the desired. So let us assume that

$$\operatorname{rank}(V_j^{n+1})\leqslant N.$$

Let $0=i(0), i(1), ..., i(r)=0$ be the itinerary of V_j^{n+1} through the pieces of the previous level. Let us consider a nest of puzzle pieces

$$V^n\equiv U_r\supset U_{r-1}\supset ...\supset U_0\equiv V_j^{n+1}, \tag{7.1}$$

where $g_n^{r-k}U_k=V_{i(r-k)}^n$. Then

$$U_{k+1}\backslash U_k\geqslant \tfrac{1}{2}\bar{\mu},\quad k=0,...,r-1. \tag{7.2}$$

Let us pull the pieces U_k, $k<r$, down along a path Γ joining V_j^{n+1} with a critical vertex V^{n+t}. Denote the corresponding pull-backs by U_k^{n+s}. If these pull-backs turn out to be double-branched then by (7.2)

$$\mu_{n+t}\geqslant\tfrac{1}{2}\operatorname{mod}(U_{r-1}\backslash U_0)\geqslant\tfrac{1}{4}(r-1)\bar{\mu},$$

which is greater than M for sufficiently big r. Otherwise let us consider the first level $n+s$ where U_{r-1}^{n+s} hits the critical point. Let us find such an l that $0\in U_l^{n+s}\backslash U_{l-1}^{n+s}$.

If $r-1-l>N$ then by (7.2) $\mu_{n+s}\geqslant M$, and we are done. Otherwise

$$\operatorname{mod}(U_{l-1}^{n+s}\backslash U_0^{n+s})\geqslant\tfrac{1}{2}(r-N-2)\bar{\mu}.$$

Then let us repeat the same procedure with U_{l-1}^{n+s} instead of U_{r-1}. Note that $\operatorname{rank}(U_{l-1}^{n+s})<\operatorname{rank}(U_{r-1})$, since the pull-back of U_{r-1} through the top central cascade is univalent. Hence this procedure can be repeated at most N times, and the principal modulus at the end will be at least M, provided $\tfrac{1}{2}(r-N(N+2))\bar{\mu}>M$. □

7.3. Parabolic and Siegel cascades

We will show that we usually observe a big principal modulus after just one long central cascade. Let us consider a central cascade (3.11): The double covering $g_{m+1}\colon V^{m+1}\to V^m$ can be viewed as a small perturbation of a quadratic-like map g_* with a definite modulus and with non-escaping critical point.

To make this precise, let us consider the space $\mathcal{Q}$ of quadratic-like maps modulo affine conjugacy supplied with the *Carathéodory topology* (see [Mc2]). Convergence in this topology means Carathéodory convergence of the domains and uniform convergence of the maps on compact subsets. Given a $\mu>0$, let $\mathcal{Q}(\mu)$ denote the set of quadratic-like maps $g\in\mathcal{Q}$ with $\operatorname{mod}(g)\geqslant\mu$. By Theorem II, the return maps $g_{m+1}: V^{m+1}\to V^m$ of the principal nest belong to $\mathcal{Q}(\bar{\mu})$.

COMPACTNESS LEMMA (see [Mc2]). *The set $\mathcal{Q}(\mu)$ is Carathéodory compact.*

Let $\mathcal{Q}_N$ (or $\mathcal{Q}_N(\mu)$) denote the space of quadratic-like maps $g: U'\to U$ from $\mathcal{Q}$ (or $\mathcal{Q}(\mu)$) such that $g^n 0\in U$, $n=0,1,\dots,N$.

As $\bigcap_N \mathcal{Q}_N(\mu)=\mathcal{Q}_\infty(\mu)$, for any neighborhood $\mathcal{U}\supset\mathcal{Q}_\infty(\mu)$, there is an N such that $\mathcal{Q}_N(\mu)\subset\mathcal{U}$. In this sense any double map $g\in\mathcal{Q}_N(\mu)$ is close to some quadratic-like map g_* with connected Julia set. In particular, this concerns the return map g_{m+1} generating a cascade (3.11) of big length N. Moreover, since g_{m+1} has an escaping fixed point, the neighborhood of g_* containing g_{m+1} also contains a quadratic-like map with hybrid class $c(g_*)\in\partial M$.

If we have a sequence of maps $f_n\in\mathcal{Q}_N$ converging to a map $g_*\in\mathcal{Q}_\infty$, we also say that the *f_n-central cascades converge to g_**.

Let us say that the principal nest is minor-modified if a piece V^m is replaced by a piece $\widetilde{V}^n\subset V^n$ such that $\operatorname{cl} V_i^{n+1}\subset\widetilde{V}^n$ for all pieces $V_i^{n+1}\in\mathcal{I}^{n+1}$.

LEMMA 7.3. *Let g_* be a quadratic-like map with $c(g_*)\in\partial M$ which does not have neither parabolic points, nor Siegel disks. Let g_{m+1} be the return map of the principal nest generating cascade* (3.11). *Take an arbitrary big $M>0$. If g_{m+1} is sufficiently close to g_* (depending on the a priori bound $\bar{\mu}$ from Theorem* II) *then the principal nest can be minor-modified in such a way that $\tilde{\mu}_n\geqslant M$ for some $n>m+N$.*

Proof. Take a big number $e>0$.

By the above assumptions, the Julia set $J(g_*)$ has empty interior. If g_{m+1} is sufficiently close to g_* then $\Gamma^{m+N-1}=\partial V^{m+N-1}$ is close in the Hausdorff metric to the Julia set $J(g_*)$. Hence Γ^{m+N-1} has an eccentricity at least e with respect to any point $z\in V^{m+N-1}$.

As the g_m are purely quadratic up to bounded distortion (Theorem II), the curves Γ_{m+N}, Γ_{m+N+1} and Γ_{m+N+2} also have big eccentricity with respect to any enclosed point. Moreover, by the same theorem, there is a definite space in between these two curves. Hence by Lemma A.2, $\operatorname{mod}(V^{m+N+1}\setminus V^{m+N+3})$ is at least $M(e)$ where $M(e)\to\infty$ as $e\to\infty$.

Let us assume that the non-central return occurs on level $m+N+1$: $g_{m+N+2}0\in V_i^{m+N+2}$ with $i\neq 0$. As the map $g_{m+N+2}: V_i^{m+N+2}\to V^{m+N+1}$ is quadratic up to bounded distortion, the curve $\Gamma_i^{m+N+2}=\partial V_i^{m+N+2}$ has a big eccentricity e' about any enclosed point (that is, e' can be made arbitrary big by a sufficiently big choice of e, depending on the a priori bound $\bar\mu$). By Lemma A.2,

$$\operatorname{mod}(V^{m+N+1}\setminus g_{m+N+2}V^{m+N+3})\geqslant M(e),$$

where $M(e)\to\infty$ as $e\to\infty$. Hence $\operatorname{mod}(A^{m+N+3})\geqslant\frac{1}{2}M(e)$, and we are done.

Let the central return occur on level $m+N+1$ but this is not yet a DH-renormalizable level. Then the corresponding central cascade is finite. Let $m+N+T$ be the last level of this cascade. Then by Lemma A.2 and Statement 6.2, $\mu_{m+N+T+2}\geqslant M(e)$, where $M(e)\to\infty$ as $e\to\infty$.

Assume finally that $m+N+1$ is a DH-renormalizable level. Then let us take a horizontal curve $\Gamma\subset A^{m+N+2}$ which divides this annulus into two subannuli of moduli at least $\frac{1}{2}\bar\mu$. Let $\Gamma'\subset A^{m+N+3}$ be its pull-back by g_{m+N+2}, and $\tilde A$ be the annulus bounded by Γ and Γ'. Then by Lemma A.2 $\operatorname{mod}(\tilde A)\geqslant M(e)$ with $M(e)$ as above. As this is a minor modification of the nest, we are done. □

7.4. Variation

Let us now improve Theorem IV by taking into account not only the height but also the other factors yielding big space.

THEOREM IV′. *Let $f\in\mathcal{SL}$ be an infinitely renormalizable quadratic polynomial, and let $P_m: z\mapsto z^2+c_m$ be the straightened $R^m f$. Assume that*

- *the set $\mathcal{A}\subset\mathcal{Q}$ of accumulation points of the central cascades of P_m (of lengths growing to ∞) does not contain parabolic or Siegel maps;*
- $\operatorname{per}(R^m f)\geqslant p$.

Then $\liminf_{n\to\infty}\operatorname{mod}(R^n f)\geqslant Q(p)$, *where the function $Q(p)$ depends on the choice of the limbs and the accumulation set $\mathcal{A}$, and $Q(p)\to\infty$ as $p\to\infty$.*

Proof. By Theorem II the top modulus of the central cascades of P_m is bounded from below by some $\bar\mu$. Hence the set $\mathcal{A}\subset\mathcal{Q}(\bar\mu)$ is compact. By Lemma 7.3, for any Q there is a neighborhood $\mathcal{U}\supset\mathcal{A}$ such that: If $f\in\mathcal{U}$ is renormalizable then $\operatorname{mod}(Rf)>Q$.

As $\mathcal{A}$ is the accumulation set for the central cascades of the P_m, there is an N such that all but finitely many of these cascades of length $\geqslant N$ belong to $\mathcal{U}$. Hence if the principal nest of P_m contains a cascade of length $\geqslant N$ then $\operatorname{mod}(R(P_m))\geqslant Q$ (for sufficiently big m).

Further, by Theorems I and III, there is a χ such that if the height $\chi(P_m)\geqslant\chi$ then $\operatorname{mod}(R(P_m))\geqslant Q$. Let us also find a T such that if for some cascade the return time from Lemma 7.2 is at least T, then $\operatorname{mod}(R(P_m))\geqslant Q$.

It is easy to see that there is a p such that: If $\mathrm{per}(P_m)\geqslant p$ then either P_m has a central cascade of length at least N, or $\chi(P_m)\geqslant\chi$, or one of the above return times is at least T. In any case $\mathrm{mod}(R(P_m))\geqslant Q$.

Now the same argument as for Theorem IV yields a priori bounds. □

8. Geometry of quasi-quadratic maps

In this section we discuss real unimodal maps of Epstein class. We introduce a notion of essential period, and prove that $\mathrm{mod}(Rf)$ is big if and only if the corresponding essential period is big. This discussion naturally continues [L4].

We assume that the reader is familiar with some basics of one-dimensional dynamics including the real Koebe principle (see the book of de Melo and van Strien [MS] for the reference).

8.1. Essential period

Below we will adjust the combinatorial discussion of §3 to the real line (see [L4] for details). Let $I'\subset I$ be two nested intervals. A map $f:(I',\partial I')\to(I,\partial I)$ is called *quasi-quadratic* if it is S-unimodal and has quadratic-like critical point $0\in\mathrm{int}\, I'$.

Let us also consider a more general class $\mathcal{A}$ of maps $g:\bigcup J_i\to J$ defined on a finite union of disjoint intervals J_i strictly contained in an interval J. Moreover, $g|J_i$ is a diffeomorphism onto J for $i\neq 0$, while $g|J_0$ is unimodal with $g(\partial J_0)\subset\partial J$. We also assume that the critical point $0\in J_0$ is quadratic-like, and that $Sg<0$. Maps of class $\mathcal{A}$ are real counterparts of generalized quadratic-like maps of finite type. To simplify the exposition, let us also assume that $g|J_0$ is symmetric, i.e., $g(x)=g(-x)$. Then $g|J_0=h\circ\Phi$, where $\Phi(x)=x^2$ and h is a diffeomorphism of an appropriate interval $K\supset\phi(J_0)$ onto J. By definition, this map belongs to *Epstein class* $\mathcal{E}$ (see [E], [S2], [L4]) if the inverse branches $f^{-1}: J\to J_i$ for $i\neq 0$ and $h^{-1}: J\to K$ admit the analytic extension to the slit complex plane $\mathbf{C}\setminus(\mathbf{R}\setminus J)$ (such functions are called Herglotz).

Let $I^0=[\alpha,\alpha']$ be the interval between the dividing fixed point α and the symmetric one. Let $\mathcal{Y}\equiv\mathcal{Y}_f$ denote the full Markov family of pull-backs of the interval I^0. Given a critical interval $J\in\mathcal{M}$ (that is, $J\ni 0$), we can define a (generalized) renormalization T_Jf on J as the first return map to J restricted to the components of its domain meeting the post-critical $\omega(0)$. If f admits a unimodal renormalization $Rf\equiv T_Jf$ for some J, then there are only finitely many such components, so that we have a map of class $\mathcal{A}$. Moreover, if f is a map of Epstein class or a quadratic-like map, the renormalizations T_Jf inherit the corresponding property.

Let $I^0 \supset I^1 \supset ... \supset I^{t+1}$ be the real principal nest of intervals until the next quadratic-like level (that is, I^{n+1} is the pull-back of I^n corresponding to the first return of the critical point). Let us use the same notation $g_n: \bigcup I_j^n \to I^{n-1}$ for the real generalized renormalizations as we used for the complex ones.

For $\sigma \in (0,1)$, let $\mathcal{E}_\sigma$ stand for the space of quasi-quadratic maps $f: I' \to I$ of Epstein class with $|I'| \leqslant \sigma |I|$. In this section we will assume that $f \in \mathcal{E}_\sigma$. All the bounds below depend on σ but become absolute after skipping the first $k(\sigma)$ central cascades.

THEOREM 8.1 (Martens [Mar]). *The following real bounds hold*:

- I^{m+1} *is well inside* I^m *unless* I^m *is in the tail of a long central cascade*;
- *the return maps* $g_m: I^m \to I^{m-1}$ *can be decomposed as* $h_m \circ \Phi$ *where* $h_m: L_m \to I^{m-1}$ *is a diffeomorphism of an interval* L_m *onto* I^{m-1} *with bounded distortion.*

(See also Guckenheimer and Johnson [GJ] for related earlier results on bounds and distortion.)

Let us look closer at real cascades of central returns. The return to level $n-1$ is called *high* or *low* if $g_n I^n \supset I^n$ or $g_n I^n \cap I^n = \varnothing$ correspondingly. Let us classify a central cascade $\mathcal{C} \equiv \mathbf{C}^{m+N}$,

$$I^m \supset ... \supset I^{m+N}, \quad g_{m+1} 0 \in I^{m+N-1} \setminus I^{m+N}, \tag{8.1}$$

as *Ulam–Neumann* or *saddle-node* according as the return to the level $m+N-1$ is high or low. In the former case the map $g_{m+1}: I^{m+1} \to I^m$ is combinatorially close to the Ulam–Neumann map $z \mapsto z^2 - 2$, while in the latter it is close to the saddle-node map $z \mapsto z^2 + \frac{1}{4}$. There is a fundamental difference between these two types of cascades.

Remark. Unlike the complex situation, on the real line we observe only two types of cascades. The reason is that there are only two boundary points in the "real Mandelbrot set" $\left[-2, \frac{1}{4}\right]$ (compare §7.3).

Consider the return graph Υ (see §3.8). Let $\Upsilon(I^n)$ stand for the part of this graph growing up from the vertex I^n (i.e., restrict Υ to the set of vertices I_j^k, $k \leqslant n$, which can be joined with I^n).

Let us consider the orbit $J_k \equiv f^k I^n$, $k=0, ..., l(n)$, of I^n until its first return to I^{n-1}, i.e., $f^{l(n)} I^n \subset I^{n-1}$. Let us watch how this orbit passes through a saddle-node cascade (8.1). Let us say that a level $m+s$ of the cascade is "branched" if for some interval $J_k \subset I^m \setminus I^{m+1}$ we have: $g_{m+1} J_k \subset I^{m+s-1} \setminus I^{m+s}$ (note that this can be expressed in terms of branching of the graph $\Upsilon(I^n)$).

Let us eliminate from each saddle-node cascade of the graph $\Upsilon(I^n)$ the maximal string of levels $m+d, ..., m+N-d$ which do not contain branched vertices of the graph.

Call the remaining graph $\Upsilon_e(I^n)$. Let us define the *essential return time* $l_e(I^n)$ as the number of paths in $\Upsilon_e(I^n)$ joining I^n with the top level. The essential period $\mathrm{per}_e(f)$ of a renormalizable map f is defined as the essential return time of an interval I^n of the renormalizable level.

Let us define the *scaling factors*

$$\lambda_n \equiv \lambda_n(f) = \frac{|I^n|}{|I^{n-1}|}.$$

Let us call the geometry of f *essentially K-bounded* until the next renormalization level if the scaling factors λ_n bounded below by K^{-1}, while the configurations $(I^{n-1}\setminus I^n, I^n_k)$ have K-bounded geometry (that is, all the intervals I^n_j, $j\neq 0$, and all the components of $I^{n-1}\setminus\bigcup I^n_k$ ("gaps") are K-commensurable). Note that the scaling factors λ_n are allowed to be close to 1.

8.2. Complex bounds

THEOREM V. *Assume that f admits a unimodal renormalization. Then:*

- *If* $\mathrm{per}_e(f)$ *is sufficiently big then the unimodal renormalization Rf admits a quadratic-like extension to the complex plane. Moreover,* $\mathrm{mod}(Rf)\geqslant\mu(\mathrm{per}_e(f))$, *where $\mu(p)\to\infty$ as $p\to\infty$.*
- *The real geometry of f is essentially K-bounded until the next renormalization level, with $K=K(\mathrm{per}_e(f))$.*

In [LY] complex bounds have been proven for infinitely renormalizable maps with essentially bounded combinatorics (which means that the essential periods $\mathrm{per}_e(R^m f)$ are uniformly bounded). This yields the Complex Bounds Theorem stated in the Introduction.

Remark. To get a bound for $\mathrm{mod}(R^m f)$ we never go beyond level m, so that our bounds are still valid for m times renormalizable maps.

Given an interval I, let $|I|$ denote its length, and let $D(I)$ denote the Euclidean disk based upon I as a diameter.

The rest of the section will be occupied with the proof of Theorem V. It relies on the following geometric fact:

SCHWARZ LEMMA. *Let I and J be two real intervals. Let $\phi\colon \mathbf{C}\setminus(\mathbf{R}\setminus I)\to\mathbf{C}\setminus(\mathbf{R}\setminus J)$ be an analytic map which maps I to J. Then $\phi(D(I))\subset D(J)$.*

Proof. Just notice that $D(I)$ is the hyperbolic r-neighborhood of I in the slit plane $\mathbf{C}\setminus(\mathbf{R}\setminus I)$ (with r independent of I). Since analytic maps are hyperbolic contractions, the statement follows. □

LEMMA 8.2. *If a scaling factor* λ_n *is sufficiently small then the generalized renormalization* $g_{n+1}: \bigcup I_j^{n+1} \to I^n$ *admits a (generalized) polynomial-like extension to the complex plane,* $g_{n+1}: \bigcup V_j^{n+1} \to V^n$. *Moreover,* $\mathrm{mod}(V^n \setminus V^{n+1}) \geqslant \mu(\lambda) \to \infty$ *as* $\lambda \to 0$.

Proof. Let us select V^n as the Euclidean disc $D \equiv D(I^n)$. Let us pull it back by the inverse branches of g_{n+1}. We obtain domains V_j^{n+1} based on the intervals I_j^{n+1}. Moreover, $g_{n+1}: V_j^{n+1} \to D$ is a double-branched covering for $j=0$ and is univalent otherwise.

By the Schwarz Lemma, $V_j^{n+1} \subset D(I_j^{n+1}) \Subset D$ for $j \neq 0$.

Let us estimate the size of V_0^{n+1}. Let t be the first return time of the critical point back to I^n under iterates of g_n. Let $J \ni g_n(0)$ be the interval which is monotonically mapped onto I^n under g_n^{t-1}. Then by the real Koebe principle,

$$|J|/\operatorname{dist}(J, \partial I^{n-1}) = O(\lambda_n). \tag{8.2}$$

Let us consider the decomposition

$$g_{n+1}|I^{n+1} = g_n^{t-1} \circ h \circ \Phi,$$

where $h: (K, L) \to (I^{n-1}, J)$ is a diffeomorphism of an appropriate interval K onto I^{n-1}. Using the real Koebe principle once more, we derive from (8.2) that

$$|L|/\operatorname{dist}(L, \partial K) = O(\lambda_n). \tag{8.3}$$

By the Schwarz Lemma, the pull-back U of D by the inverse branch of $g^{t-1} \circ h: L \to I^n$ is contained in $D(L)$. Hence $V^{n+1} \subset \Phi^{-1} D(L)$ and by (8.3),

$$\operatorname{diam} V^{n+1}/|I^n| = O\big(\sqrt{\lambda_n}\big).$$

It follows that V^{n+1} lies well inside $D = D(I^n)$, and we have a generalized polynomial-like map with the desired properties. □

In the following two lemmas we analyze the geometry of long central cascades. Let us call a unimodal map saddle-node or Ulam–Neumann if it is topologically conjugate to $z \mapsto z^2 + \frac{1}{4}$ or $z \mapsto z^2 - 2$ correspondingly.

LEMMA 8.3. *Let us consider an Ulam–Neumann cascade* (8.1). *If it is sufficiently long then the generalized renormalization* g_{m+N+1} *admits a polynomial-like extension to the complex plane with a definite modulus. Moreover,* $\mathrm{mod}(g_{m+N+1}) \to \infty$ *as* $N \to \infty$.

Proof. Take the Euclidean disk $D = D(I^{m+N})$ and pull it back by the inverse branches of g_{m+N+1}. We obtain domains V_j^{m+N+1} based upon the intervals I_j^{m+N+1}. By the

Schwarz Lemma, all the off-critical domains V_j^{m+l+1}, $j\neq 0$, are contained in the round discs $D(I_j^{m+N+1})$, and hence are strictly contained in D.

Let us estimate the size of the central domain $V\equiv V_0^{m+N+1}$. By Theorem 8.1, I^{m+1} is well inside I^m. If the scaling factor λ_{m+1} is small then the statement follows from Lemma 8.2 and Theorem II. So we can assume that I^{m+1} is commensurable with I^m. By a compactness argument, if the cascade is long enough then the map $g\equiv g_{m+1}: I^{m+1}\to I^m$, with the domain rescaled to unit size, is C^1-close to an Ulam–Neumann map. It follows that $|I^{m+k}\setminus I^{m+N}|$ decrease with k at a uniformly exponential rate. Hence for a sufficiently long cascade, $I^{m+N-1}\setminus I^{m+N}$ is ε-tiny as compared with I^{m+N}.

Let $g(0)\in I\equiv I_j^{m+N}$, $j\neq 0$. Let U be the pull-back of $D(I^{m+N-1})$ by the inverse branch of $g_{m+N}^{-1}: I^{m+N-1}\to I$ extended to the complex plane. By the Schwarz Lemma, $U\subset D(I)$.

Furthermore, by Theorem 8.1, there is an interval $L\supset \Phi(I^{m+N})$ such that $g|I^{m+N}=h\circ\Phi$, where $h: L\to I^{m+N-1}$ is a diffeomorphism of bounded distortion. Hence the image $\Phi I^{m+N}=h^{-1}gI^{m+N}\supset h^{-1}I^{m+N}$ occupies at least a $(1-O(\varepsilon))$-portion of L.

It follows that $h_{-1}U\subset D(h^{-1}I)$ is of size $O(\varepsilon)$ as compared with $|L|$. Hence $V\subset \Phi^{-1}D(h^{-1}I)$ is of size $O(\sqrt{\varepsilon}\,)$ as compared with $|I^{m+N}|$, and the lemma follows. □

LEMMA 8.4. *All saddle-node patterns* (8.1) *of the same length with commensurable* I^m *and* I^{m+1} *are* $\varkappa$*-qs equivalent, with an absolute* $\varkappa$.

Proof. Let $g: I'\to[0,1]$ be a quasi-quadratic map of Epstein class (and perhaps escaping critical point): $g\in\mathcal{E}$. By definition, $g=h\circ\Phi$ with a diffeomorphism h whose inverse admits the analytic extension to $\mathbf{C}\setminus[0,1]$. Let us supply this space with the Montel topology on the h^{-1}.

Take a $\delta\in\left(0,\frac{1}{2}\right)$. The set of maps $g\in\mathcal{E}$ with $\delta\leqslant|I'|\leqslant 1-\delta$ is compact. Hence given a long saddle-node cascade (8.1), the map G obtained from $g_{m+1}: I^{m+1}\to I^m$ by rescaling I^m to the unit size must be close to a saddle-node quasi-quadratic map. Hence we can reduce G to a form $z\mapsto z+\varepsilon+\psi(z)$ where $\psi(z)>0$ is uniformly comparable with z^2 (here the fixed point of the nearby saddle-node map is selected as the origin). Moreover, we will see in a moment that ε is determined, up to a bounded error, by the length of the cascade.

Take a big $a>0$. When $|z|<a\sqrt{\varepsilon}$, the step $G(z)-z$ is of order ε. Otherwise $\psi(z)$ dominates over ε, and in the chart $\zeta=1/z$ the step is of order 1. It follows that the qs class of the cascade is determined by ε, which in turn is related to the length of the cascade by $N\asymp 1/\sqrt{\varepsilon}$. □

The following lemma refines Lemma 7.2 in the case of real cascades.

LEMMA 8.5. *Let us consider a cascade* (8.1). *Let* $t=t_{m+N}$ *be the maximal return time of the intervals* $g_{m+N+1}I_i^{m+N+1}\subset I^{m+N-1}$ *back to* I^{m+N} *under iterates of the Bernoulli map* G_{m+N}, *see* (3.12). *Then there exists a level* l *such that* $\lambda_l\leqslant\lambda(t)$ *where* $\lambda(t)\to 0$ *as* $t\to\infty$.

Proof. If some interval I_j^{m+1} is not well inside I^m then by Theorem 8.1 and Lemma 8.4, the level m follows a long Ulam–Neumann cascade. Then the scaling factor λ_{m+1} is small. It follows that λ_{m+N+1} is small as well (see [L4, §2], for the estimate of λ_{m+N+1} via λ_{m+1}).

If all the intervals I_j^{m+1} are well inside of I^m then repeat the argument of Lemma 7.2 on the real line using the negative Schwarzian in place of conformality and the Bernoulli map G_{m+N} in place of g_n. □

Let $\chi(m)$ stand for the height of I^m, that is, the number of central cascades preceding it.

LEMMA 8.6. *If the height* $\chi(f)$ *is sufficiently big then there is an interval* $J\in\mathcal{M}$ *such that the generalized renormalization* T_Jf *admits a polynomial-like extension to the complex plane with a definite principle modulus* μ. *Moreover,* J *lies on a bounded height, i.e.,* $J\supset I^m$ *with a bounded* $\chi(m)$.

Proof. Take small $\varepsilon>0$ and $\delta>0$, and consider the inequality

$$\lambda_l>(1-\delta)\lambda_{l-1}. \tag{8.4}$$

If (8.4) fails to happen on the first $s=\log\varepsilon/\log(1-\delta)+1$ levels then we come up with an ε-small scaling factor, and Lemma 8.2 yields the desired statement, provided ε is small enough.

Otherwise a desired interval J exists by [L4, §4] (provided δ is small enough). □

LEMMA 8.7. *Assume that* f *is renormalizable. If* $\mathrm{per}_e(f)$ *is sufficiently high, then the renormalization* Rf *is polynomial-like. Moreover,* $\mathrm{mod}(Rf)>\mu(\mathrm{per}_e(f))$, *where* $\mu(p)\to\infty$ *as* $p\to\infty$.

Proof. Big essential period amounts to one of the following circumstances:

(i) the height of $\chi(f)$ is big; or

(ii) for some cascade (8.1) (maybe of length 1) the return time t_{m+N} of Lemma 8.5 is big; or

(iii) there is a long Ulam–Neumann cascade; or

(iv) there is a saddle-node cascade (8.1) and an interval I_j^{m+1} which lands deep inside the cascade under one iterate of g_m.

Assume that (i) occurs. Then the statement follows from Lemma 8.6 and Theorem III.

If (ii) happens then by Lemma 8.5 we observe a small scaling factor on some level, and Lemma 8.2 yields the statement.

If (iii) occurs then the desired statement follows from Lemma 8.3 and Theorem II.

Assume finally that (iv) happens. Let $J\equiv I_j^{m+1}$. Then $g_{m+1}J\subset I^{m+i}\setminus I^{m+i+1}$ for $d\leqslant i\leqslant N-d$, with big d. Then by Lemma 8.4, $I^{m+i}\setminus I^{m+i+1}$ is tiny in I^m. It follows that J is tiny as compared with the $\operatorname{dist}(J,\partial I^m)$. By [L4], this produces a small scaling factor several levels down (if $\operatorname{rank}(J)$ is big, use Lemma 3.6 of [L4]; otherwise use Lemma 2.12 of that paper). Now Lemma 8.2 and Theorem II complete the proof. □

LEMMA 8.8. *If* $\operatorname{per}_e(f)$ *is bounded, then the geometry of* f *is essentially bounded until the next renormalization level.*

Proof. Assume that the geometry is bounded on level $n-1$, and let us see what happens on the next level. Given an $x\in\omega(c)\cap(I^{n-1}\setminus I^n)$, let $J(x)$ denote the pull-back of I^n corresponding to the first landing of $\operatorname{orb}(x)$ at I^n. As the landing time under iterates of g_n is bounded, $J(x)$ is commensurable with I^{n-1}.

To create the intervals I_j^{n+1}, we should pull all intervals $J(x)$ back by $g_n\colon I^n\to I^{n-1}$. As g_n is a quasi-quadratic map, all non-central intervals I_j^{n+1} and the gaps in between are commensurable with I^n.

The only possible problem is that the central interval I^{n+1} may be tiny in I^n. This may happen only if the critical value $g_n 0\in J(x)$ is very close to $\partial J(x)$. Let l be such that $g_n^l J(x)=I^n$. Since $g_n^l\colon J(x)\to I^n$ is qs, $g_{n+1}0=g_n^{l+1}$ turns out to be very close to ∂I^n ("very low return"). But then $g_{n+1}0$ belongs to some non-central interval I_j^{n+1} whose Poincaré length in I^n is definite (as we have shown above). This is a contradiction.

So when we pass from one level to the next, the geometric bounds change gradually. But the same is also true when we pass through a saddle-node cascade (8.1). Let us consider the Bernoulli map $G\colon\bigcup K_j^{m+i}\to I^m$ associated with this cascade (see §3.6), where the $K_j^{m+i}\subset I^{m+i-1}\setminus I^{m+i}$ are the pull-back of the I_k^{m+2}.

Observe that for $i<N$ the transit maps

$$g_{m+1}^{i-2}\colon I^{m+i-1}\setminus I^{m+i}\to I^{m+1}\setminus I^{m+2}$$

have bounded distortion, as its Koebe space spreads over the appropriate components of $I^m\setminus I^{m+3}$. Moreover, the passages from the level m to $m+1$ and from $m+N-2$ to $m+N-1$ have bounded distortion by Theorem 8.1 and Lemma 8.4.

Hence if the geometry of the configuration $(I^m\setminus I^{m+1},\{I_i^{m+2}\})$ on level m is bounded, then the geometry of the configuration $(I^{m+N-1}\setminus I^{m+N},K_j^{m+N})$ is bounded as well.

Moreover, by Lemma 8.4, I^{m+N} is commensurable with $I^{m+N-1}\setminus I^{m+N}$. Thus the configuration $(I^{m+N-1}, \{K_i^{m+N}\})$ of level $m+N-1$ has bounded geometry.

Let us now define the intervals $J(x)$, $x\in\omega(c)\cap I^{m+N-1}$, as the pull-backs of I^{m+N} corresponding to the first landing of $\operatorname{orb}(x)$ at I^{m+N}. Then it follows from the bounded geometry on level $m+N-1$ together with the bounded return G-times and the landing depths that the configuration of the intervals $J(x)$ has bounded geometry in I^{m+N-1}.

Let us now pull these intervals back to the next level $m+N$. Then the same argument as in the beginning of the proof shows that the geometry on level $m+N$ is still bounded. □

Now Theorem V follows from the last two lemmas.

Remark. Theorem V is still valid for higher degree real unimodal polynomials $z\mapsto z^d+c$, $c\in\mathbf{R}$, except for the growing of $\mu(p)$. The same proof works, with the following adjustment of logic. The proof of Lemma 8.6 shows that generalized quadratic-like maps with a definite modulus can be created on a sequence of levels m_i with bounded $\chi(m_{i+1})-\chi(m_i)$. Together with the remark at the end of §5 this implies that all the generalized renormalizations $T^{n(k)+1}f$ have a definite principal modulus (where $n(k)$ counts the non-central levels). In particular, $\operatorname{mod}(Rf)$ is definite.

Part II. Rigidity and local connectivity

9. Space between Julia bouquets

In this section we will prove local connectivity of the Julia sets satisfying the secondary limbs condition with a priori bounds (Theorem VI).

9.1. Space and unbranching

Let J_i^m denote the little Julia sets of level m, that is, $J^m\equiv J_0^m=J(R^mf)$ and $J_i^m=f^iJ^m$, $i=0,\dots,r_m-1$. They are organized in the pairwise disjoint bouquets $B_j^m=B_j^m(f)$ of the Julia sets touching at the same periodic point. Namely, if level $m-1$ is immediately renormalizable with period l then each B_j^m consists of l little Julia sets J_i^m touching at their β-fixed points. Otherwise the bouquets B_j^m just coincide with the little Julia sets J_j^m. By $B^m\equiv B_0^m$ we will denote the *critical* bouquet containing the critical point 0. Let $\mathbf{J}^m=\mathbf{J}^m(f)=\bigcup_i J_i^m=\bigcup_j B_j^m$. Finally let K_i^m be little filled Julia sets.

We will use the notation F_m for the quadratic-like map f^{r_m} near any little Julia set J_i^m (it should be clear from the context which one is considered). In particular, $F_m=R^mf$ near the critical Julia set $J^m\ni 0$.

Recall that $\mathcal{Q}(\mu)$ stands for the space of quadratic-like maps f with $\operatorname{mod}(f)\geqslant\mu>0$ supplied with the Carathéodory topology (§7.3). Take a little copy $M'\subset M$ of the Mandelbrot set truncated at the root. Let $\mathcal{Q}(\mu, M')$ denote the subspace of $\mathcal{Q}(\mu)$ consisting of renormalizable quadratic-like maps f whose hybrid class belongs to M'.

Let us have a family $\mathcal{F}$ of sets $X_a\subset\mathbf{C}$ depending on some parameter a ranging over a topological space $\mathcal{T}$. This dependence is said to be (sequentially) *upper semi-continuous* if for any $a(i)\to a$, the Hausdorff limit of $X_{a(i)}$ is contained in X_a. For example it is easy to see that the filled Julia set $K(f)$ of a quadratic-like map f depends upper semi-continuously on f. Let us say that a family $\mathcal{F}$ of sets $X_f\subset\mathbf{C}$ is (upper) *semi-compact* if any sequence X_n of these sets contains a subsequence $X_{n(i)}$ converging in Hausdorff topology to a subset of some $X\in\mathcal{F}$.

LEMMA 9.1. *The little filled Julia sets $K_i^1(f)$ form a semi-compact family of sets as f ranges over the space $\mathcal{Q}(\mu, M')$.*

Proof. By the Compactness Lemma (see §7.3), the space $\mathcal{Q}(\mu, M')$ is compact. Moreover, the quadratic-like map F_1 depends continuously on $f\in\mathcal{Q}(\mu, M')$ near any K_i^1. In turn, the little filled Julia sets K_i^1 depend upper semi-continuously on F_1. $\square$

LEMMA 9.2. *Let f be a quadratic-like map of class $\mathcal{SL}$ with complex a priori bounds. Then there is a definite space in between its bouquets B_j^m.*

Proof. Let us take a bouquet B^m. Let $\mathcal{I}^m$ stand for the set of indices j such that $B_j^{m+1}\subset B^m$. We will show first that there is a definite annulus

$$T^m\subset\mathbf{C}\setminus\bigcup_{j\in\mathcal{I}^m} B_j^{m+1},$$

which goes around B^{m+1} but does not go around other bouquets B_j^{m+1}, $j\in\mathcal{I}^m$.

If $R^m f$ is not immediately renormalizable, then this follows from Theorem II (ii). So assume that $R^m f$ is immediately renormalizable.

If $B^m=J^m$, then it is nothing to prove as there is only one bouquet B^{m+1} inside B^m. Otherwise there are only finitely many renormalization types producing the bouquet B^m (which correspond to the little Mandelbrot sets attached to the main cardioid and belonging to the selected secondary limbs). By Lemma 9.1, the bouquets B_j^{m+1} contained in B^m belong to a compact family of sets. As they do not touch each other, there is a definite space in between them.

Let $N(L,\varepsilon)$ denote an $(\varepsilon \operatorname{diam} L)$-neighborhood of a set L (that is, the set of points on distance at most $\varepsilon\operatorname{diam} L$ from L). We have shown that there is an $\varepsilon>0$ such that the neighborhood $N(B^{m+1},\varepsilon)$ does not intersect other bouquets B_j^{m+1} contained in the same B^m. In particular, $N(B^1, \varepsilon)$ does not intersect any other B^1_j (as all of them are contained in $B^0 \equiv J(f)$).

Let us show by induction that

$$N(B^m, \varepsilon) \cap B^m_k = \varnothing, \quad k \neq 0. \tag{9.1}$$

Assuming this for m, we should show that

$$N(B^{m+1}, \varepsilon) \cap B^{m+1}_j = \varnothing, \quad j \neq 0. \tag{9.2}$$

As we already know (9.2) for $j \in \mathcal{I}^m$, let $j \notin \mathcal{I}^m$. Then $B^{m+1}_j \subset B^m_k$ for some $k \neq 0$, while $N(B^{m+1}, \varepsilon) \subset N(B^m, \varepsilon)$, and (9.2) follows from (9.1).

What is left, is to show that there is a definite space around any bouquet B^{m+1}_j (not only around the critical one). But there is an iterate f^l which univalently maps B^{m+1}_j onto B^{m+1}. Pulling back the space around B^{m+1} we obtain the desired space about B^{m+1}_j. □

An infinitely renormalizable map f is said to satisfy an *unbranched a priori bounds* condition (see [Mc3]) if for infinitely many levels m, there is a definite space in between J^m and the rest of the postcritical set, $\omega(0) \setminus J^m$.

LEMMA 9.3. *A map $f \in \mathcal{SL}$ with a priori bounds satisfies an unbranched a priori bounds condition.*

Proof. We will show that the unbranched condition can fail only if the level m is not immediately renormalizable, while $m-1$ is immediately renormalizable. As the complimentary sequence of levels is infinite, the lemma will follow.

If $R^{m-1}f$ is not immediately renormalizable then the bouquet B^m coincides with the little Julia set J^m. By Lemma 9.2, there is a definite space in between J^m and $\mathbf{J}^m \setminus J^m$. As $\omega(0) \setminus J^m \subset \mathbf{J}^m \setminus J^m$, the unbranched condition holds on level m.

Assume now that both levels $m-1$ and m are immediately renormalizable. Then we will show that there is a definite space in between J^m and $\mathcal{B}^{m+1} \equiv \bigcup_{j \neq 0} B^{m+1}_j$.

By Lemma 9.2, there is a definite space in between $B^m \supset J^m$ and $\mathcal{B}^{m+1} \setminus B^m$. So we should check that there is a definite space in between J^m and $\mathcal{B}^{m+1} \cap B^m$ (that is, the union of non-critical bouquets B^{m+1}_j contained in B^m). But J^m does not touch any such B^{m+1}_j. Indeed, the only point where they can touch could be the β-fixed point β_m of J^m. But one can easily see that the little Julia sets of level $m+1$ never contain β_m. By Lemma 9.1 there is a desired space.

Finally, as $\omega(0) \setminus J^m \subset \mathcal{B}^{m+1}$, the statement follows. □

Remark. If $R^m f$ is not immediately renormalizable, while $R^{m-1}f$ is immediately renormalizable, then the unbranched condition can fail. Indeed in this case there are several Julia sets J_i^m which touch at the common fixed point $\beta_m \in J^m$. But the postcritical set $\omega(0) \cap J_i^m$ can come arbitrarily close to β_m (when $R^m f$ is a small perturbation of a map whose critical orbit eventually lands at β_m).

9.2. Local connectivity of Julia sets

Using Sullivan's a priori bounds Hu and Jiang [HJ] proved that the Feigenbaum quadratic polynomial has locally connected Julia set. Then a more general result of this kind was worked out: Any infinitely renormalizable quadratic map with unbranched a priori bounds has locally connected Julia set (see [J], [Mc3]). Together with Lemma 9.3 this yields

THEOREM VI. *Let $f \in \mathcal{SL}$ be an infinitely renormalizable quadratic polynomial with a priori bounds. Then the Julia set $J(f)$ is locally connected. In particular, all maps from Theorems* IV *and* IV′ *of* §7 *have locally connected Julia sets.*

Proof. I learned the argument given below from J. Kahn (Durham 93).

A priori bounds imply that the "little" Julia sets J^m shrink down to the critical point. Indeed let $f_m \equiv R^m f \equiv f^{r_m} : U'_m \to U_m$ where $\operatorname{mod}(U_m \setminus U'_m) \geqslant \varepsilon > 0$, with an ε independent of m. Clearly U_m does not cover the whole Julia set.

Let $\Gamma_m \subset U_m \setminus U'_m$ be a horizontal curve in the annulus $U_m \setminus U'_m$ which divides it into two subannuli of modulus at least $\frac{1}{2}\varepsilon$, and let $\Gamma'_m \subset U'_m$ be its pull-back by f_m. By the Koebe Theorem, these curves have a bounded eccentricity about 0 (with a bound depending on ε). Since the inner radius of curve Γ'_m about 0 tends to 0 as $m \to \infty$ (it follows from the fact that the sufficiently high iterates of any disk intersecting $J(f)$ cover the whole $J(f)$), the $\operatorname{diam} \Gamma'_m \to 0$ as well. All the more, $\operatorname{diam}(J_m) \to 0$ as $m \to \infty$.

Let us take a $\delta > 0$, and find an m such that J_m is contained in the $\mathbf{D}_\delta$.

Let us now inscribe into $\mathbf{D}_\delta$ a domain bounded by equipotentials and external rays of the original map f. Let β_m denote the non-dividing fixed point of the Julia set J^m, and $\beta'_m = -\beta_m$ be the symmetric point. Let us consider a puzzle piece $P^{m,0} \ni 0$ bounded by any equipotential and four external rays of the *original map* f landing at β_m and β'_m. This is a "degenerate" domain of the renormalized map F_m (see §2.5). By definition of the renormalized Julia set, the preimages $P^{m,k} \equiv F_m^{-k} P^{m,0}$ shrink down to J^m. Hence there is a puzzle piece $P^{m,l}$ contained in the $\mathbf{D}_\delta$. As $J(f) \cap P^{m,l}$ is clearly connected, the Julia set $J(f)$ is locally connected at the critical point.

Let us now prove local connectivity at any other point $z \in J(f)$. This is done by a standard spreading of the local information near the critical point around the whole dynamical plane. Let us consider two cases.

Case (i). Let the orbit of z accumulate on all Julia sets J^m. Let m be an unbranched level. Then there is an $l=l(m)$ such that the puzzle piece $P^{m,l}$ is well inside $\mathbf{C}\setminus(\omega(0)\setminus J^m)$.

Take now the first moment $k=k(m)\geqslant 0$ such that $f^k z\in P^{m,l}$. Let us consider the pull-backs $Q^{m,l}\ni z$ of $P^{m,l}$ along the orbit $\mathrm{orb}_k(z)$. By Lemma 3.3, this pull-back is univalent. Moreover, it allows a univalent extension to a definitely bigger domain.

By the Koebe Theorem, $Q^{m,l}$ has a bounded eccentricity about z. Since the inner radius of this domain about z tends to 0 as $m\to\infty$, $\operatorname{diam} Q^{m,l}\to 0$ as well. As $Q^{m,l}\cap J(f)$ are connected, the Julia set is locally connected at z.

Case (ii). Assume now that the orbit of z does not accumulate on some J^m. Hence it accumulates on some point $a\notin\omega(0)$. Let us consider the puzzle associated with the periodic point β_m (so that the initial configuration consists of a certain equipotential and the external rays landing at β_m). Since the critical puzzle pieces shrink to J^m, the puzzle pieces $Y_i^{(l)}$ of sufficiently big depth l containing a are disjoint from $\omega(0)$ (there are several such pieces if a is a preimage of β_m). Take such an l, and let X be the union of these puzzle pieces. It is a closed topological disk disjoint from $\omega(0)$ whose interior contains a.

Consider now the moments $k_i\to\infty$ when the orbit of z lands at $\operatorname{int} X$, and pull X back to z. By the same Koebe argument as in Case (i) we conclude that these pull-backs shrink to z. It follows that $J(f)$ is locally connected at z. □

9.3. Standard neighborhoods

In this section we will construct some special fundamental domains near little Julia bouquets. Let us consider first the non-immediately renormalizable case when the construction can be done in a particularly nice geometric way.

LEMMA 9.4. *Let f be an m times renormalizable quadratic map. Assume that the space in between the little Julia sets J_i^m is at least $\mu>0$. Then there are disjoint fundamental annuli A_i^m around little Julia sets J_i^m, with* $\operatorname{mod}(A_i^m)\geqslant\nu(\mu)>0$.

Proof. Let us consider the Riemann surfaces $S=\mathbf{C}\setminus\mathbf{J}^m$ and $S'=\mathbf{C}\setminus f^{-1}\mathbf{J}^m\subset S$. Then $f\colon S'\to S$ is a double covering. Let us uniformize S, that is, represent it as the quotient $\mathcal{H}^2/\Gamma$ of the hyperbolic plane modulo the action of a Fuchsian group. In this conformal representation S admits a compactification $S\cup\partial S$ to a bordered Riemann surface, with the components ∂S_i^m of the "ideal boundary" ∂S corresponding to the little Julia sets J_i^m.

Let $\widehat{S}=S\cup\partial S\cup\bar{S}$ be the double of S, that is, $(\mathbf{C}\setminus\Lambda(\Gamma))/\Gamma$, where $\Lambda(\Gamma)\subset S^1$ is the limit set of Γ. The boundary components ∂S_i^m are geodesics in $\widehat{S}$. Moreover, these geodesics have hyperbolic length bounded by a constant $L=L(\mu)$ independent of m.

Let $\sigma: S\to S$ be the natural anti-holomorphic involution of S. Let $\bar{S}'=\sigma S'$ and $\widehat{S}'=S'\cup\partial S\cup\bar{S}'\subset\widehat{S}$ be the double of S' inside S. Then f admits an extension to a holomorphic double covering $\hat{f}:\widehat{S}'\to\widehat{S}$ commuting with the involution σ. Its restriction $\hat{f}|\partial S_0^m\to\partial S_1^m$ is a double covering, while the restrictions to the other boundary components $\partial S_i^m\to\partial S_{i+1}^m$ are diffeomorphisms.

Let $C_i^m(r)\supset\partial S_i^m$ stand for the hyperbolic r-neighborhood of the geodesic ∂S_i^m. By the Collar Lemma (see [Ab]), there is an $r=r(L)$ (independent of the particular Riemann surface and geodesics) such that the collars $C_i^m\equiv C_i^m(r)$ are pairwise disjoint. Moreover, $\operatorname{mod}(C_i^m)\geqslant\mu(L)>0$.

Let us now take such a collar $C=C_i^m$, and let $\gamma=\partial S_i^m$. Let $C'\subset S'\cap C$ be the component of $\hat{f}^{-p}C$ containing γ (where p is the period of the little Julia sets). Then $\hat{f}^p: C'\to C$ is a double covering preserving γ. As we have in the hyperbolic metric of S that

$$\int_\gamma \|D\hat{f}^p\|=2l(\gamma),$$

there is a point $z\in\gamma$ such that $\|D\hat{f}^p(z)\|\geqslant 2$. This easily implies that $\|D\hat{f}^{-p}(\zeta)\|\leqslant q(a)<1$ if the hyperbolic distance between $\hat{f}^p z$ and ζ does not exceed a. In particular, $\|D\hat{f}^{-p}\|(\zeta)\leqslant q=q(L,r)<1$ for all $\zeta\in C$.

It follows that C' is contained in the hyperbolic (r/q)-neighborhood of γ, and hence $\operatorname{mod}(C\setminus C')\geqslant\varrho(r,q)=\varrho(\mu)$. Let now $A_i^m=(C\setminus C')\cap S$. □

Note that in the above lemma we do not assume a priori bounds but just a definite space between the Julia sets (which thus implies a priori bounds). Assuming a priori bounds, let us now give a different construction which works in the immediately renormalizable case as well.

Let us consider a bouquet $B_j^m=\bigcup_i J_i^m$ of level m, where J_i^m touch at point α_{m-1}. Let $b_i^m\in J_i^m$ be the points F_m-symmetric to α_{m-1}, that is, $F_m b_i^m=\alpha_{m-1}$ ("co-fixed points"). Let us consider the domain Υ_j^m bounded by the pairs of rays landing at these points (defined via a straightening of F_{m-1}), and p_m arcs of equipotentials. Let us then thicken this domain near the points b_i^m as described in §2.5 (that is, replace the rays landing at b_i^m by nearby rays and little circle arcs around b_i^m). Denote the thickened domains by U_j^m (see Figure 10). We also require that these domains are naturally related by dynamics so that $f\Upsilon_j^m=\Upsilon_k^m$ and $fU_j^m=U_k^m$ whenever $fB_j^m=B_k^m$ and B_j^m is non-critical. Let us call U_j^m a *standard neighborhood* of the bouquet B_j^m. Let $\mathbf{U}^m=\bigcup U_j^m$.

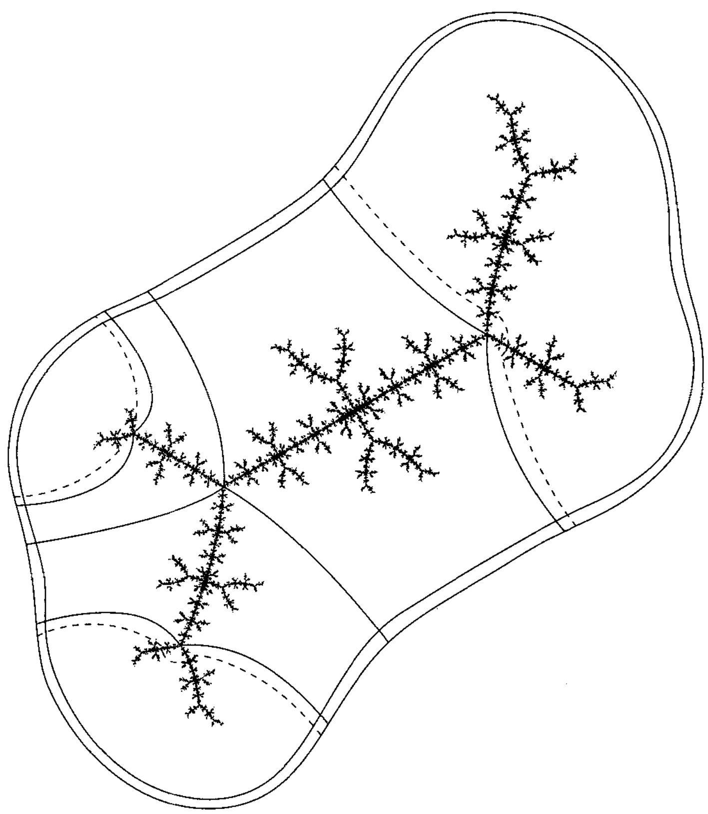

Fig. 10. Standard neighborhood of a Julia bouquet

LEMMA 9.5. *Let f be an m times renormalizable quadratic map of class $\mathcal{SL}$ with a priori bounds. Then there exist disjoint standard neighborhoods U_j^m of B_j^m with bounded geometry, and such that the annuli* $\operatorname{mod}(U_i^m \setminus B_i^m)$ *have a definite modulus.*

Proof. By the Straightening Theorem, the renormalization $R^{m-1}f\colon J_k^{m-1} \to J_k^{m-1}$ is K-qc conjugate to a quadratic polynomial $P_c\colon z \mapsto z^2 + c$, with K dependent only on *a priori bounds.*

Let $B \subset J(P_c)$ be the critical bouquet of little Julia sets of RP_c. Let $\Omega(\varepsilon)$ be its neighborhood bounded by arcs of equipotentials of level $1-\varepsilon$, circle arcs of radius ε, and rays with arguments $\theta_i + t(\varepsilon)$ (see §2.5). Here θ_i are the arguments of the rays landing at the co-fixed points, and $t(\varepsilon) \in (-\varepsilon, \varepsilon)$ is selected in such a way that $\Omega(\varepsilon)$ is a renormalization domain for any P_c from selected truncated secondary limbs.

The geometry of these domains depends only on the selected limbs and ε. Also, the Hausdorff distance $d_c(\varepsilon)$ of $\partial\Omega(\varepsilon)$ to B tends to 0 as $\varepsilon \to 0$ uniformly over c belonging to the selected truncated limbs. Indeed, this is clearly true for a given parameter value c. Take a little $\delta>0$, and find an $\varepsilon=\varepsilon_c$ such that $d_c(\varepsilon_c)<\delta$. Then for all b sufficiently close to c, $d_b(\varepsilon_c)<2\delta$. Compactness of the truncated limbs completes the argument.

It follows that for all sufficiently small ε (depending only on the selected limbs and a priori bounds), $\Omega(\varepsilon)$ belongs to the range of the straightening map. Hence these neighborhoods can be transferred to the dynamical f-plane. We obtain neighborhoods $U^m(\varepsilon)$ of the corresponding bouquet B^m with bounded geometry (depending on parameter ε).

Moreover, as quasi-conformal maps are quasi-symmetric (see Appendix A), the Hausdorff distance from $\partial U^m(\varepsilon)$ to the bouquet B^m is at most $\varrho(\varepsilon)\cdot\operatorname{diam} B^m$, where $\varrho(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence for all sufficiently small ε, the neighborhood $U(\varepsilon)$ is well inside the domain $\mathbf{C}\setminus\bigcup_{j\neq 0} B^m_j$.

Let us now pull this neighborhood back by dynamics to obtain standard neighborhoods $U^m_j(\varepsilon)$ of other bouquets B^m_j. Since $U^m(\varepsilon)$ is well inside $\mathbf{C}\setminus\bigcup_{j\neq 0} B^m_j$, these pull-backs have a bounded distortion. Hence the Hausdorff distance from $\partial U^m_j(\varepsilon)$ to the bouquet B^m_j is at most $\varrho(\varepsilon)\cdot\operatorname{diam} B^m$, where $\varrho(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Since by Lemma 9.2 there is a definite space between the bouquets, there is also a definite space between the neighborhoods $U^m_j(\varepsilon)$, for all $\varepsilon\in(0,\varepsilon_*]$ (with ε_* depending only on the selected limbs and a priori bounds). Also, the moduli of $U^m_j(\varepsilon)\setminus B^m_j$ depend only on the limbs, a priori bounds and ε. So they are definite, for instance in the range $\varepsilon\in(0.01\varepsilon_*,\varepsilon_*]$. □

We keep using the notations B^m_j, Υ^m_j, etc., introduced before Lemma 9.5, and we also assume that the standard neighborhoods U^m_j satisfy the conclusions of Lemma 9.5. We will define a special qc map

$$S_m\colon (U^m_j\setminus B^m_j)\to \mathbf{A}(1,4), \tag{9.3}$$

with bounded dilatation. This map will be called a *standard straightening* or a *standard local chart* near the bouquet B^m_j.

It follows from the a priori bounds assumption that for any Julia set J^l_i there exist Jordan disks $\Omega^l_i\supset\Pi^l_i\supset J^l_i$ such that $F_l\colon\Pi^l_i\to\Omega^l_i$ is a quadratic-like map, and there exists a qc map

$$\Psi_{l,i}\colon (\Omega^l_i\setminus J^l_i, \Pi^l_i\setminus J^l_i)\to(\mathbf{A}(1,4),\mathbf{A}(1,2)) \tag{9.4}$$

with bounded dilatation conjugating $F_l\colon\Pi^l_i\to\Omega^l_i$ and $P_0\colon\mathbf{A}(1,2)\to\mathbf{A}(1,4)$, $P_0\colon z\mapsto z^2$.

If J^m_i does not touch other Julia sets of level m (that is, F_{m-1} is not immediately renormalizable) then one can select the standard neighborhood U^m_i as Ω^m_i. In this case let us define the standard straightening (9.3) as $\Psi_{m,i}$.

If F_{m-1} is immediately renormalizable, then let us consider a family of little Julia sets and bouquets:

$$\bigcup_i J_i^m = B_j^m \subset J_k^{m-1}. \tag{9.5}$$

Let us cut Υ_j^m by the rays landing at the fixed point α_{m-1} into components $\Xi_i^m \supset J_i^m$. Since the hybrid class of F_{m-1} may belong to a bounded number of little Mandelbrot sets (attached to the main cardioid and intersecting the selected secondary limbs), the domains Ξ_i^m have a bounded geometry. Hence the maps S_m can be selected in such a way that they have bounded dilatation and

$$S_m \Big| \bigcup_i \Xi_i^m = \Psi_{m-1,k}.$$

Thus they glue together into a single qc map (9.3).

By the rays and equipotentials near the bouquet we will mean the S_m-preimages of the vertical intervals and horizontal circles in the cylinder $\mathbf{A}(1,4)$. This will be also referred to as the *standard coordinate system* near B_j^m.

Let us show in conclusion that the little Julia bouquets and the corresponding standard neighborhoods exponentially decay. Let $\operatorname{diam}(X)$ stand for the Euclidean diameter of a set X.

LEMMA 9.6. *Let $f \in \mathcal{SL}$ be a quadratic-like map with a priori bounds. Then there exist constants $\lambda < 1$ and $l_0 > 0$ depending on the choice of limbs and a priori bounds such that for any two Julia bouquets $B_j^{m+l} \subset B_i^m$,*

$$\operatorname{diam} B_j^{m+l} \leqslant \lambda^l \operatorname{diam} J_i^m, \quad l \geqslant l_0.$$

Proof. Let us straighten the renormalization $R^m f$ near J_i^m to a quadratic polynomial P_c. The dilatation K of the straightening depends only on the a priori bounds, and K-qc maps are Hölder continuous with exponent $1/K$ (see [Ah]). Hence it is enough to show that for the quadratic map P_c, there exist constants $\lambda < 1$ and $l_0 > 0$ depending on the choice of limbs and a priori bounds such that

$$\operatorname{diam} B_j^l \leqslant \lambda^l, \quad l \geqslant l_0. \tag{9.6}$$

(Now B_j^m, $\mathbf{J}^m$, etc., stand for the objects associated to P_c.)

Note that $J(P_c) \subset \mathbf{D}_2$. Let ϱ_l be the hyperbolic metric on $\mathbf{D}_3 \setminus \mathbf{J}^l$. Let γ_i^m, $m \leqslant l$, be the hyperbolic geodesic in $\mathbf{D}_3 \setminus \mathbf{J}^l$ homotopic to a curve $\Gamma_i^m \subset \mathbf{C} \setminus \mathbf{J}^m$ going once around B_i^m but not going around other Julia bouquets B_k^m, $k \neq i$.

By Lemma 9.2, there are annuli $A_i^m \subset \mathbf{D}_3 \setminus \mathbf{J}^l$ in the homotopy class of Γ_i^m with a definite modulus, $\operatorname{mod}(A_i^m) \geqslant \nu > 0$. Let us pick Γ_i^m as the hyperbolic geodesic in A_i^m.

Then the hyperbolic length of this geodesic in A_i^m is at most π/ν. All the more, the hyperbolic length of γ_i^m in $\mathbf{D}_3\backslash\mathbf{J}^l$ is bounded by the same constant.

By the Collar Lemma (see [Ab]), there exist disjoint annuli $\Lambda_i^m\subset\mathbf{D}_3\backslash\mathbf{J}^l$ in the homotopy class of γ_i^m with $\operatorname{mod}(\Lambda_i^m)\geqslant\eta=\eta(\nu)>0$. By the Grötzsch inequality we have $\operatorname{mod}(\mathbf{D}_3\backslash B_j^l)\geqslant l\eta$. Hence there is an absolute constant C such that $\operatorname{diam} B_j^l\leqslant Ce^{-l\eta}$ (see Appendix A), and (9.6) follows. □

COROLLARY 9.7. *Under the assumptions of Lemma* 9.6, *there exist constants* $\lambda<1$ *and* $l_0>0$ *such that for the standard neighborhoods* $U_j^{m+l}\subset U_i^m$ *the following estimates hold*:

$$\operatorname{diam} U_j^{m+l}\leqslant\lambda^l\operatorname{diam} U_i^m,\quad l\geqslant l_0.$$

Proof. Indeed, the standard neighborhoods U_i^m are commensurable with the corresponding Julia sets J_i^m. □

9.4. Removability of certain dynamical sets

The reader is referred to Appendix A for the definition and a discussion of removability.

LEMMA 9.8 ([Mc3]). *Under the assumptions of Lemma* 9.6, *the post-critical set* $\omega(0)$ *is a removable Cantor set coinciding with* $\bigcap\mathbf{J}^m$.

Proof. It was shown in the proof of Lemma 9.6 that for any $z\in\omega(0)\subset\bigcap\mathbf{J}^m$, there is a nest of disjoint annuli around z with a definite modulus. Thus the first statement follows from the removability criterion (see Appendix A).

Clearly, $\omega(0)\subset\bigcap\mathbf{J}^m\subset\bigcap\mathbf{U}^m$. Vice versa, by Lemma 9.6, $\bigcap\mathbf{J}^m$ is covered by the uniformly shrinking bouquets B_i^m. As every B_i^m contains a postcritical point, $\operatorname{orb}0$ is dense in $\bigcap\mathbf{J}^m$. □

Let us finish this section with stating a standard fact on removability of expanding Cantor sets. Let $\{U_i\}$ be a finite family of closed topological disks with disjoint closures. Let us consider a Markov map $g:\bigcup U_i\to\mathbf{C}$ satisfying the following property: If $\operatorname{int}(gU_i\cap U_j)\neq\varnothing$ then $gU_i\supset U_j$. As usual, let

$$K(g)=\{z:g^nz\in\bigcup U_i,\,n=0,1,...\}$$

stand for the Julia set of g.

LEMMA 9.9. *For a Markov map as above, the Julia set* $K(g)$ *is removable.*

Proof. Let us select a family of annuli $A_j \subset gU_j \setminus \bigcup U_i$ homotopic to $\partial(gU_i)$ in $gU_j \setminus \bigcup U_i$. Let consider cylinder sets $U^m_{i(0),i(1),\dots,i(m-1)}$ defined by the following property:

$$g^k U^m_{i(0),i(1),\dots,i(m-1)} \subset U_{i(k)}, \quad k=0,1,\dots,m-2; \quad g^{m-1} U^m_{i(0),i(1),\dots,i(m-1)} = U_{i(m-1)}.$$

The pull-back of the annulus $A_{i(m-1)}$ to

$$U^m_{i(0),i(1),\dots,i(m-1)} \setminus \bigcup_i U^{m+1}_{i(0),i(1),\dots,i(m-1),i}$$

by the univalent map $g^m: U^m_{i(0),i(1),\dots,i(m-1)} \to gU_{i(m-1)}$ has the same modulus as $A_{i(m-1)}$. This provides us with a nest of disjoint annuli with definite moduli about any $z \in K(g)$. The removability criterion concludes the proof. □

10. Rigidity: beginning of the proof

This and the next sections will be occupied with the proof of the Rigidity Theorem stated in the Introduction.

10.1. Reductions

In this section we begin to prove the Rigidity Theorem stated in the Introduction. Since quadratic polynomials label hybrid classes of quadratic-like maps, this theorem can be stated in the following way:

RIGIDITY THEOREM (equivalent statement). *Let $f, \tilde{f} \in \mathcal{L}$ be two quadratic-like maps with a priori bounds. If f and $\tilde{f}$ are combinatorially equivalent then they are hybrid equivalent.*

The proof is split into three steps:

Step 1. f and $\tilde{f}$ are topologically equivalent;

Step 2. f and $\tilde{f}$ are qc equivalent;

Step 3. f and $\tilde{f}$ are hybrid equivalent.

The first step (passage from combinatorial to topological equivalence) follows from the local connectivity of the Julia sets (Theorem VI). Indeed, a locally connected Julia set is homeomorphic to its combinatorial model (see [D2]). Since the combinatorial model is the same over the combinatorial class, the conclusion follows.

The last step (passage from qc to hybrid equivalence) is taken care of by McMullen's Rigidity Theorem [Mc2]. Indeed, it asserts that an infinitely renormalizable quadratic-like map with a priori bounds does not have invariant line fields on the Julia set. It follows that if h is a qc conjugacy between f and $\tilde{f}$ then $\partial\bar{h}=0$ almost everywhere on the Julia set. Thus h is a hybrid conjugacy between f and $\tilde{f}$.

So, our task is to take care of Step 2:

THEOREM VII. *Let $f, \tilde{f}\in\mathcal{L}$ be two quadratic-like maps with a priori bounds. If f and $\tilde{f}$ are topologically equivalent then they are qc equivalent.*

In what follows we will mark with tilde the objects for $\tilde{f}$ corresponding to those for f. When we introduce some objects for f, we assume that the corresponding tilde objects are automatically introduced as well.

Remark. The proof of the Rigidity Theorem given below comes through for higher degree maps $z\mapsto z^d+c$, $c\in\mathbf{C}$, of *bounded* type with a priori bounds. The geometric preparation needed for this is the weak form of Theorem II (see the remark at the end of §5).

10.2. Thurston's equivalence

Let $f: U'\to U$ and $\tilde{f}: \widetilde{U}'\to\widetilde{U}$ be two topologically equivalent quadratic-like maps. Let us say that f and $\tilde{f}$ are *Thurston equivalent* if for appropriate choice of domains U, U', $\widetilde{U}$, $\widetilde{U}'$, there is a qc map $h: (U, U', \omega(0))\to(\widetilde{U}, \widetilde{U}', \omega(0))$ which is homotopic to a conjugacy $\psi: (U, U', \omega(0))\to(\widetilde{U}, \widetilde{U}', \omega(0))$ relative $(\partial U, \partial U', \omega(0))$. Note that h conjugates $f: \omega(0)\cup\partial U'\to\omega(0)\cup\partial U$ and $\tilde{f}: \omega(0)\cup\partial\widetilde{U}'\to\omega(0)\cup\partial\widetilde{U}$. A qc map h as above will be called a *Thurston conjugacy.*

Remark. It is enough to assume that h is homotopic to ψ rel postcritical sets. Then one can extend it to a qc map $U\to\widetilde{U}$ which is homotopic to a conjugacy rel the bigger set as required above.

The following result comes from the work of Thurston (see [DH3], [Mc1]) and Sullivan (see [MS], [S2]). It originates the "pull-back method" in holomorphic dynamics.

LEMMA 10.1. *If two quadratic-like maps are Thurston equivalent then they are qc equivalent.*

Proof. We will use the notations for the domains and maps preceding the statement of the lemma. Let U^n be the preimages of U under the iterates of f, and let $c=f(0)$. Let h have dilatation K.

Since $h(c)=\tilde{c}$, we can lift h to a K-qc map $h_1: U^1\to\widetilde{U}^1$ which is homotopic to ψ rel $(\partial U^1, \partial U^2, \omega(0))$. (Note that the dilatation of h_1 is the same as the dilatation of h, since the lift is analytic.) Hence $h_1=h$ on ∂U^1, and we can extend h_1 to $U\setminus U^1$ as h (keeping the same notation h_1). By the Gluing Lemma from Appendix A this extension has the same dilatation K. Moreover, this map is homotopic to ψ rel $\left(\omega(0), \bigcup_{1\leqslant k\leqslant 2} \partial U^k\right)$. Also, it conjugates $f\colon \omega(0)\cup(U^1\setminus U^2)\to\omega(0)\cup(U^0\setminus U^1)$ to the corresponding tilde map (notice that h_1 is a conjugacy on a bigger set than h).

Let us now replace h with h_1 and repeat the procedure. We will construct a K-qc map $h_2\colon U\to\widetilde{U}$ which is homotopic to ψ rel $\left(\omega(0), \bigcup_{1\leqslant k\leqslant 3} \partial U^k\right)$ and conjugates $f\colon \omega(0)\cup(U^1\setminus U^3)\to\omega(0)\cup(U\setminus U^2)$ to the corresponding tilde map.

Proceeding in this way we construct a sequence of K-qc maps h_n homotopic to ψ rel $\left(\omega(0), \bigcup_{1\leqslant k\leqslant n+1} \partial U^k\right)$ and conjugating $f\colon \omega(0)\cup(U^1\setminus U^{n+1})\to\omega(0)\cup(U\setminus U^n)$ to the corresponding tilde map. By the Compactness Lemma from Appendix A, we can select a converging subsequence $h_{n(l)}\to h$. The limit map h is a desired qc conjugacy. □

The method used in the above proof is called "the pull-back argument". The idea is to start with a qc map respecting some dynamical data, and then pull it back so that it will respect some new data on each step. At the end it becomes (with some luck) a qc conjugacy.

Remark. For infinitely renormalizable maps of bounded type with a priori bounds, McMullen proved that the postcritical set $\omega(0)$ has bounded geometry [Mc3]. It easily follows that there is a qc map $h\colon (\mathbf{C}, \omega_f(0))\to(\mathbf{C}, \omega_{\tilde f}(0))$ conjugating f to $\tilde f$ on their postcritical sets. This is close to being a Thurston conjugacy but not the same, as h may be in a wrong homotopy class.

10.3. Approximating sequence of homeomorphisms

So we need to construct a Thurston conjugacy. We will construct it as a limit of an appropriate sequence of maps. Take a sufficiently small $\varepsilon>0$, and consider the corresponding sequence of standard neighborhoods $\mathbf{U}^m=\bigcup_i U_i^m\equiv\bigcup_i U_i^m(\varepsilon)$ (see §9.3). By Corollary 9.7 there is an l such that $\mathbf{U}^m$ is well inside $\mathbf{U}^{m-l}$. Moreover, by Lemma 9.8, $\bigcap \mathbf{J}^m=\omega(0)$.

We will consecutively construct a sequence of homeomorphisms

$$h_m\colon (\mathbf{C}, \mathbf{U}^m, \mathbf{J}^m)\to(\mathbf{C}, \widetilde{\mathbf{U}}^m, \tilde{\mathbf{J}}^m) \tag{10.1}$$

such that

(i) h_0 is a topological conjugacy;

(ii) h_m is homotopic to h_{m-1} rel $(\mathbf{J}^m\cup(\mathbf{C}\setminus\mathbf{U}^{m-l}))$. In particular $h_m|\mathbf{J}^m=h_{m-1}|\mathbf{J}^m$ and $h_m|(\mathbf{C}\setminus\mathbf{U}^{m-l})=h_{m-1}|(\mathbf{C}\setminus\mathbf{U}^{m-l})$;

(iii) the h_m are K_*-qc on $\mathbf{U}^{m-1}\setminus\mathbf{J}^m$, with dilatation K_* depending only on the choice of limbs and a priori bounds;

(iv) $\operatorname{Dil}(h_m|\mathbf{U}^{m-l}\setminus\mathbf{J}^m)\leqslant 4K_*^4 \operatorname{Dil}(h_{m-1}|\mathbf{U}^{m-l}\setminus\mathbf{J}^{m-1})$.

Such a sequence will do the job:

LEMMA 10.2. *A sequence h_m satisfying the above three properties uniformly converges to a Thurston conjugacy.*

Proof. By the second property, this sequence eventually stabilizes outside $\bigcap \mathbf{J}^m$, and thus it pointwise converges to a homeomorphism $h: \mathbf{C}\setminus\bigcap \mathbf{J}^m \to \mathbf{C}\setminus\bigcap \tilde{\mathbf{J}}^m$. By the last two properties, the dilatation of h_m on $\mathbf{U}^{m-l}\setminus\mathbf{J}^m$ is uniformly bounded. Hence h is quasi-conformal on $\mathbf{C}\setminus\bigcap \mathbf{J}^m$. But by Lemma 9.8, $\bigcap \mathbf{J}^m = \omega(0)$ is a removable Cantor set. Hence h admits a qc extension across $\omega(0)$.

Further, h is homotopic to h_0 rel $\omega(0)$. Indeed, let h^t, $1-2^{-m}\leqslant t\leqslant 1-2^{-(m+1)}$, be a homotopy between h_m and h_{m+1} given by (ii). Let $\varepsilon_m = \max_i \operatorname{diam} U_i^m$. As the $\mathbf{U}^m$ shrink to a Cantor set, $\varepsilon_m \to 0$. As $h(U_i^{m-l}) = h^t(U_i^{m-l}) = \widetilde{U}_i^{m-l}$, $1-2^{-m}\leqslant t<1$, the uniform distance between h and h^t is at most ε_{m-l}. It follows that the h^t uniformly converge to h as $t\to 1$. Hence h is homotopic to h_0 rel $\omega(0)$.

Since h_0 is a topological conjugacy by (i), h is a Thurston conjugacy. □

10.4. Construction of h_0

Let us supply the exterior $\mathbf{C}\setminus \operatorname{cl}\mathbf{D}$ of the unit disk with the hyperbolic metric ϱ. The hyperbolic length of a curve γ will be denoted by $l_\varrho(\gamma)$, while its Euclidean length will be denoted by $|\gamma|$.

LEMMA 10.3. *Let A and $\tilde{A}$ be two (open) annuli whose inner boundaries are the circle $\mathbf{T}$. Let $\omega: A\to\tilde{A}$ be a homeomorphism commuting with $P_0: z\mapsto z^2$ near $\mathbf{T}$. Then ω admits a continuous extension to a map $A\cup\mathbf{T}\to\tilde{A}\cup\mathbf{T}$ identical on the circle.*

Proof. Given a set $X\subset A$, let $\widetilde{X}$ denote its image by ω. Let us take a configuration consisting of a round annulus $L^0 = \mathbf{A}[r, r^2]$ contained in A, and an interval $I_0 = [r, r^2]$. Let $L^n = P_0^{-n}L^0$, and I_k^n denote the components of $P_0^{-n}I^0$, $k=0,1,\dots,2^n-1$. The intervals I_k^n subdivide the annulus L^n into 2^n "Carleson boxes" Q_k^n.

Since the (multi-valued) square root map P_0^{-1} is infinitesimally contracting in the hyperbolic metric, the hyperbolic diameters of the boxes $\widetilde{Q}_k^n$ are uniformly bounded by a constant C.

Let us now show that ω is a hyperbolic quasi-isometry near the circle, that is, there exist $\varepsilon>0$ and $A, B>0$ such that

$$A^{-1}\varrho(z,\zeta)-B\leqslant \varrho(\tilde{z},\tilde{\zeta})\leqslant A\varrho(z,\zeta)+B, \tag{10.2}$$

provided $z,\zeta\in\mathbf{A}(1,1+\varepsilon)$, $|z-\zeta|<\varepsilon$.

Let γ be the arc of the hyperbolic geodesic joining z and ζ. Clearly it is contained in the annulus $\mathbf{A}(1,r)$, provided ε is sufficiently small. Let $t>1$ be the radius of the circle $\mathbf{T}_t$ centered at 0 and tangent to γ. Let us replace γ with a *combinatorial geodesic* Γ going radially up from z to the intersection with $\mathbf{T}_t$, then going along this circle, and then radially down to ζ. Let N be the number of the Carleson boxes intersected by Γ. Then one can easily see that

$$\varrho(z,\zeta)=l_\varrho(\gamma)\asymp l_\varrho(\Gamma)\asymp N,$$

provided $\varrho(z,\zeta)\geqslant 10\log(1/r)$ (here $\log(1/r)$ is the hyperbolic size of the boxes Q_k^n).

On the other hand

$$\varrho(\tilde{z},\tilde{\zeta})\leqslant l_\varrho(\widetilde{\Gamma})\leqslant CN,$$

so that $\varrho(\tilde{z},\tilde{\zeta})\leqslant C_1\varrho(z,\zeta)$, and (10.2) follows.

But quasi-isometries of the hyperbolic plane admit continuous extensions to $\mathbf{T}$ (see, e.g., [Th]). Finally, it is an easy exercise to show that the only homeomorphism of the circle commuting with P_0 is identical. □

LEMMA 10.4. *Let f be a quadratic-like map. Let A and $\tilde{A}$ be two (open) annuli whose inner boundaries are $J(f)$. Let $\omega: A\to\tilde{A}$ be a homeomorphism commuting with f near $J(f)$. Then ω admits a continuous extension to a map $A\cup J(f)\to\tilde{A}\cup J(f)$ identical on the Julia set.*

Proof. By the Straightening Theorem, we can assume without loss of generality that $f=P_c: z\mapsto z^2+c$ is a quadratic polynomial. Let $R: \mathbf{C}\setminus K(f)\to\mathbf{C}\setminus\operatorname{cl}\mathbf{D}$ be the Riemann mapping normalized by $R(z)\sim z$ near infinity. It conjugates P_c to $P_0: z\mapsto z^2$.

Let $\omega^\#=R\circ\omega\circ R^{-1}: \mathbf{C}\setminus\operatorname{cl}D\to\mathbf{C}\setminus\operatorname{cl}D$. Then $\omega^\#$ commutes with P_0 in an open annulus attached to the circle $\mathbf{T}$. By Lemma 10.3, $\omega^\#$ continuously extends to $\mathbf{T}$ as id. Hence for any $\varepsilon>0$ there is an $r>1$ such that $|\omega^\#(z)-z|<\varepsilon$ for $z\in\mathbf{A}(1,r)$.

Let us show that the hyperbolic distance $\varrho(\omega^\#(z),z)$ is bounded if $|z|<2$. Clearly $\varrho(\omega^\#(z),z)\leqslant C(r)$, provided $1<r\leqslant|z|<2$. Let $r^{1/2}\leqslant|z|\leqslant r$, $\zeta=\omega^\#(z)$. Let us consider the hyperbolic geodesic γ joining P_0z and $P_0\zeta$. Clearly $|\gamma|=O(\varepsilon)$. Then $P_0^{-1}\gamma$ consists of two symmetric curves σ and $-\sigma$ of Euclidean length $O(\varepsilon)$. One of these curves, say σ, joins z with a preimage u of $P_0(\zeta)$. Then $|z+u|>2-O(\varepsilon)>\varepsilon$, so that $-u\neq\zeta$. Thus $u=\zeta$.

As the square root map P_0^{-1} is infinitesimally contracting in the hyperbolic metric,

$$\varrho(z,\zeta)\leqslant l_\varrho(\sigma)\leqslant l_\varrho(\gamma)=\varrho(P_0(z),P_0(\zeta))\leqslant C(r).$$

Take now any point z in the annulus $\mathbf{A}[r^{1/4},r^{1/2}]$. Using the same argument we conclude that $\varrho(z,\omega^\#(z))\leqslant C(r)$ (with the same $C(r)$). By induction, the same bound holds for all z.

Now we can complete the proof. Since the Riemann mapping R is a hyperbolic isometry, the hyperbolic distance between $\omega(z)$ and z in $\mathbf{C}\setminus J(P_c)$ is also bounded near $J(P_c)$. Hence the Euclidean distance $|z-\omega(z)|$ goes to 0 as $z\to J(f)$. It follows that the extension of ω as the identity on the Julia set is continuous. $\square$

COROLLARY 10.5. *Let f and $\tilde f$ be two topologically equivalent quadratic-like maps, and let ψ be a topological conjugacy between them. Let A and $\tilde A$ be two open annuli whose inner boundaries are $J(f)$ and $J(\tilde f)$ respectively. Let $h: A\to\tilde A$ be a homeomorphism conjugating f and $\tilde f$ on these annuli. Then h matches with ψ on the Julia set, that is, h admits a continuous extension to a map $A\cup J(f)\to\tilde A\cup J(\tilde f)$ coinciding with ψ on the Julia set.*

Proof. Apply Lemma 10.4 to the homeomorphism $\omega=\psi^{-1}\circ h$ commuting with f. $\square$

LEMMA 10.6 ([DH2]). *If two quadratic-like maps f and $\tilde f$ are topologically conjugate then there is a conjugacy h_0 which is quasi-conformal outside the Julia sets.*

Proof. Given an annulus R, let $\partial_o R$ and $\partial_i R$ stand for its outer and inner boundary components. Let us select a closed fundamental annulus R for f with smooth boundary, and let $R^n=f^{-n}R$. Let $\widetilde R$ and $\widetilde R^n$ be similar objects for $\tilde f$. Then there is a diffeomorphism $\phi: R\to\widetilde R$ such that

$$\phi(fz)=\tilde f(\phi z), \quad z\in\partial_i R.$$

This diffeomorphism can be lifted to a diffeomorphism $\phi_1: R^1\to\widetilde R^1$ with the same qc dilatation and such that

$$\phi_1(z)=\phi(z),\ z\in\partial_o R^1, \quad \text{and} \quad \phi_1(fz)=\tilde f(\phi_1 z),\ z\in\partial_i R^1.$$

In turn, ϕ_1 can be lifted to a diffeomorphism $\phi_2: R^2\to R^2$ with the same dilatation, which matches with ϕ_1 on $\partial_o R^2$ and respects dynamics on $\partial_i R^2$, etc.

By the Gluing Lemma from the appendix, these diffeomorphisms glue together into a single quasi-conformal map $h_0: A\setminus J(f)\to\tilde A\setminus J(\tilde f)$ conjugating f and $\tilde f$, where $A=\bigcup R^n$.

On the other hand, let ψ be a topological conjugacy between f and $\tilde f$ near the Julia sets. Then by Corollary 10.5, h_0 matches with ψ on $J(f)$. $\square$

10.5. Adjustment of h_m

Recall that r_m is the period of the little Julia sets J_j^m, and $F_m=f^{r_m}$ is the corresponding quadratic-like map near J_j^m. Let $\mathbf{U}^m=\bigcup U_j^m$ be a standard neighborhood of the little Julia orbit $\mathbf{J}^m=\bigcup B_j^m=\bigcup J_i^m$, with a definite space in between the U_j^m and definite annuli $U_j^m\setminus B_j^m$, and let $S_m: \mathbf{U}_j^m\setminus\mathbf{J}_j^m\to\mathbf{A}(1,4)$ be the standard straightenings (9.3). Its dilatation is bounded by a constant K_* depending only on the choice of secondary limbs and a priori bounds. Let $U_j^m(t)=S_m^{-1}\mathbf{A}(1,t)$ (note that $U_j^m \equiv U_j^m(4)$). The notation $\mathbf{U}^m(t)$ is self-evident.

We say that a homeomorphism $\phi: U_j^m(2)\setminus B_j^m \to \tilde{U}_j^m(2)\setminus \tilde{B}_j^m$ is *standard* near the bouquet B_j^m if it is identical in the standard coordinates on $U_j^m(2)$, that is,

$$\tilde{S}_m \circ \phi | U_j^m(2) = S_m. \tag{10.3}$$

The dilatation of such a map is bounded by K_*^2. Note also that by Corollary 10.5, the standard map admits a homeomorphic extension across the Julia bouquet.

We will now adjust the map h_m so that it will become standard near $\mathbf{J}^m$.

LEMMA 10.7. *Take an l as in §10.3. Let a homeomorphism $h_m: (\mathbf{C}, \mathbf{J}^m) \to (\mathbf{C}, \tilde{\mathbf{J}}^m)$ be a conjugacy on $\mathbf{J}^m$ and be K_m-qc on $\mathbf{U}^{m-l}\setminus \mathbf{J}^m$. Then there is a homeomorphism*

$$\hat{h}_m: (\mathbf{C}, \mathbf{U}^m, \mathbf{J}^m) \to (\mathbf{C}, \tilde{\mathbf{U}}^m, \tilde{\mathbf{J}}^m),$$

homotopic to h_m rel $(\mathbf{J}^m \cup (\mathbf{C}\setminus \mathbf{U}^{m-l}))$, such that $\mathrm{Dil}(\hat{h}^m | (\mathbf{U}^{m-1}\setminus \mathbf{J}^m)) \leqslant 4K_*^4 \cdot K_m$, *and $\hat{h}_m: \mathbf{U}^m(2)\setminus \mathbf{J}^m \to \tilde{\mathbf{U}}^m(2)\setminus \tilde{\mathbf{J}}^m$ is standard.*

Proof. In what follows we skip the subscript m. Let us consider a retraction ψ_j^t: $U_j(4)\setminus B_j \to U_j(4)\setminus U_j(t)$ which is the affine vertical contraction in the standard coordinates. Its dilation is bounded by $2K_*^2$. Let us extend the ψ_j^t to a homeomorphism $\psi^t: \mathbf{C}\setminus \mathbf{J} \to \mathbf{C}\setminus \mathbf{U}(t)$ by identity on $\mathbf{C}\setminus \mathbf{U}(4)$. By the Gluing Lemma from the appendix, ψ^t is also $2K_*^2$-qc.

Let us now define a homeomorphism $h^t: (\mathbf{C}, \mathbf{U}(t), \mathbf{J}) \to (\mathbf{C}, \tilde{\mathbf{U}}(t), \tilde{\mathbf{J}})$ as follows:

$$h^t | (\mathbf{C}\setminus \mathbf{U}(t)) = \tilde{\psi}^t \circ h \circ (\psi^t)^{-1},$$

while $h^t: \mathbf{U}(t) \to \tilde{\mathbf{U}}(t)$ is standard. Then h^1 is a desired adjusted map (homotopic to $h^0=h$ via the $\{h^t\}$). ☐

In what follows we will assume that h_m is adjusted as in Lemma 10.7, and will skip the "hat" in the notation for the adjusted map.

10.6. Beginning of the construction of h_{m+1}

We keep using the notations of §9.3.

Let p_m denote the number of rays landing at the α-fixed of the Julia sets J_i^m. Consider the configurations $\mathcal{R}_i^m$ of p_m rays landing at the α-fixed points of the J_i^m. Let

$\Omega_s^{m,0} \equiv \Omega_s^m$ stand for the components of $\Xi_i^m \setminus \mathcal{R}_i^m$, and let $\Omega_s^{m,1} \subset \Omega_s^m$ be the corresponding components of $F_m^{-p_m} \Omega_s^m$, so that

$$G_m \equiv F_m^{p_m} : \Omega_s^{m,1} \to \Omega_s^m \tag{10.4}$$

is a double-branched covering. The boundaries of these domains are naturally marked with the standard coordinates. (*Marking* of a curve means its preferred parametrization.) As the map

$$h_m : (\mathbf{C}, U_j^m, \Omega_s^m, \Omega_s^{m,1}, J_i^m) \to (\mathbf{C}, \widetilde{U}_j^m, \widetilde{\Omega}_s^m, \widetilde{\Omega}_s^{m,1}, \tilde{J}_i^m)$$

is standard on the U_j^m, it respects this marking.

Since the configurations $(\bigcup \mathcal{R}_s^m, \bigcup \partial\Omega_s^{m,1})$ have bounded geometry (see §4), there is a qc map with a bounded dilatation

$$\Psi_m : (\mathbf{C}, U_j^m, \Omega_s^m, \Omega_s^{m,1}) \to (\mathbf{C}, \widetilde{U}_j^m, \widetilde{\Omega}_s^m, \widetilde{\Omega}_s^{m,1}) \tag{10.5}$$

coinciding with h_m on $\mathbf{C} \setminus \bigcup_s \Omega_s^m$ and respecting the boundary marking (in particular, it conjugates $F_m : \partial\Omega_s^{m,1} \to \partial\Omega_s^m$ and $\widetilde{F}_m : \partial\widetilde{\Omega}_s^{m,1} \to \partial\widetilde{\Omega}_s^m$). Moreover, Ψ_m is homotopic to h_m rel $((\mathbf{C} \setminus \mathbf{U}^m) \cup \partial\Omega_s^m \cup \partial\Omega_s^{m,1})$, since all regions complementary to this set are simply-connected Jordan domains.

Note however that unlike h_m, the map Ψ_m does not respect dynamics on the little Julia sets. We need to pay temporarily this price in order to make Ψ_m globally quasi-conformal.

10.7. Construction of h_{m+1} in the immediately renormalizable case

Let us consider the double covering (10.4). In the immediately renormalizable case,

$$G_m^n 0 \in \Omega_s^{m,1}, \quad n = 0, 1, 2, \ldots .$$

Moreover, there is a nest of topological disks

$$\Omega_s^{m,0} \supset \Omega_s^{m,1} \supset \Omega_s^{m,2} \supset \ldots$$

shrinking to the little Julia set J_s^{m+1}, and such that $G_m : \Omega_s^{m,n} \to \Omega_s^{m,n-1}$ is a branched double covering. The complement $Q_s^{m,n} = \Omega_s^{m,n-1} \setminus \Omega_s^{m,n}$ consists of 2^n quadrilaterals.

As $G_m : Q_s^{m,n} \to Q_s^{m,n-1}$ is an unbranched covering, the map $\Psi_m : Q_s^{m,1} \to \widetilde{Q}_s^{m,1}$ can be lifted to a qc map

$$\Psi_{m,n} : Q_s^{m,n} \to \widetilde{Q}_s^{m,n}$$

with the same dilatation homotopic to h_m rel the boundary. Hence all these maps glue together in a single qc map with the same dilatation

$$h_{m+1}: \Omega_s^m \setminus J_s^{m+1} \to \widetilde{\Omega}_s^m \setminus \tilde{J}_s^{m+1} \tag{10.6}$$

equivariantly homotopic to h_m rel $\bigcup_n \partial\Omega_s^{m,n}$.

Let ψ^t be the corresponding homotopy, and let ϱ be the hyperbolic metric in $\widetilde{\Omega}^m \setminus \tilde{J}^{m+1}$. Then by equivariancy $\varrho(\psi^t(z), h_m(z)) \leqslant C$. Hence $|\psi^t(z) - h_m(z)| \to 0$ as $z \to J_s^{m+1}$ uniformly in t. It follows that the homotopy ψ^t extends across the little Julia set J_s^{m+1}. Thus the map (10.6) extends across J_s^{m+1} to a homeomorphism homotopic to h_m rel $(\partial\Omega_s^m \cup J_s^{m+1})$.

Outside the $\bigcup_s \Omega_s^m$ let h_{m+1} coincide with h_m. This provides us with the desired map h_{m+1}.

11. Through the principal nest

In what follows we will assume that $R^m f \equiv F_m$ is not immediately renormalizable.

11.1. Teichmüller distance between the configurations of puzzle pieces

Let us make a choice of a standard neighborhood U^m of the Julia bouquet B^m and the corresponding standard straightening S_m, see (9.3). When F_{m-1} is not immediately renormalizable, this provides us with a family $\mathcal{Y}$ of puzzle pieces $Y_i^{(k)}$, see §2.6.

As the puzzle pieces $Y_i^{(k)}$ are bounded by equipotentials and rays, they bear the *standard boundary marking*, i.e., the parametrization S_m^{-1} by the corresponding straight intervals or circle arcs.

Since $h_m: U^m \to \widetilde{U}^m$ is the standard conjugacy (see (10.3)), it maps the pieces $Y_i^{(k)}$ to the corresponding tilde pieces $\widetilde{Y}_i^{(k)}$ respecting the boundary marking. Given some family of puzzle pieces $P_i \in \mathcal{Y}$ contained in some $Y \in \mathcal{Y}$, let us say that a homeomorphism

$$\phi: (Y, \bigcup P_i) \to (\widetilde{Y}, \bigcup \widetilde{P}_i)$$

is a *pseudo-conjugacy* if it is homotopic to h_m rel the boundary $(\partial Y, \bigcup \partial P_i)$. Note that if $f^l: P_i \to Y$ (or $f^l: P_i \to P_j$) for some iterate of f and some puzzle pieces of our family, then the pseudo-conjugacy ϕ is a true conjugacy between the boundary maps $f^l: \partial P_i \to \partial Y$ and $\tilde{f}^l | \partial\widetilde{P}_i \to \partial\widetilde{Y}$ (or ∂P_j instead of ∂Y).

In particular, the above terminology will be applied to the principal nest of puzzle pieces (see §3):

$$Y^{(m,0)} \supset V^{m,0} \supset V^{m,1} \supset \ldots, \quad V_0^{m,n} \equiv V^{m,n}, \quad \bigcap_n V^{m,n} = J^{m+1}, \tag{11.1}$$

and the corresponding generalized renormalizations $g_{m,n}\colon \bigcup_i V_i^{m,n} \to V^{m,n-1}$.

The Teichmüller distance dist_T between $(V^{m,n-1}, V_i^{m,n})$ and $(\tilde{V}^{m,n-1}, \tilde{V}_i^{m,n})$ is defined as $\inf_\phi \log K_\phi$ as ϕ runs over all qc pseudo-conjugacies $(V^{m,n-1}, \bigcup_i V_i^{m,n}) \to (\tilde{V}^{m,n-1}, \bigcup_i \tilde{V}_i^{m,n})$.

MAIN LEMMA. *The configurations $(V^{m,n-1}, V_i^{m,n})$ and $(\tilde{V}^{m,n-1}, \tilde{V}_i^{m,n})$ stay bounded Teichmüller distance apart (independently of m and n).*

The rest of this section, except the two final subsections, will be occupied with the proof of this lemma. As the level m is fixed, *we skip in what follows the label m in the notations of $V_i^{m,n} \equiv V_i^n$, $g_{m,n} \equiv g_n$, $h_m \equiv h$, etc.* (unless it may lead to a confusion). In what follows, by referring to a qc-map we will mean that it has a definite dilatation (depending only on the selected limbs and a priori bounds).

11.2. A point set topology lemma

In the statement below, the objects involved need not have any dynamical meaning.

LEMMA 11.1. *Let P_i be a family of closed Jordan discs with disjoint interiors contained in a domain Y, such that* $\operatorname{diam} P_i \to 0$. *Let $\tilde{P}_i$, $\tilde{Y}$ be another family of discs with the same properties.*

- *Let $h\colon (Y, \bigcup P_i) \to (\tilde{Y}, \bigcup \tilde{P}_i)$ be a one-to-one map which is a homeomorphism on $\bigcup P_i$ and on $X \equiv Y \setminus (\bigcup \operatorname{int} P_i)$. Then h is a homeomorphism.*
- *Let $h^j\colon (Y, \bigcup P_i) \to (\tilde{Y}, \bigcup \tilde{P}_i)$, $j=0,1$, be two homeomorphisms which coincide on $Y \setminus (\bigcup \operatorname{int} P_i)$. Then the h^j are homotopic rel $Y \setminus (\bigcup \operatorname{int} P_i)$.*

Proof. Given an $\varepsilon > 0$, there exists an N such that $\operatorname{diam}(\tilde{P}_n) < \varepsilon$ for $n > N$. Let $T = \bigcup_{1 \leqslant i \leqslant N} P_i$. Note that h is continuous on $X \cup T$.

Given a point $z \in Y$, let us show that h is continuous at it. This is certainly true if $z \in \bigcup \operatorname{int} P_i$, so let $z \in X$. We will show that

$$|h(z) - h(\zeta)| < 2\varepsilon \tag{11.2}$$

for any nearby point $\zeta \in Y$. Indeed, if $\zeta \in X \cup T$ it follows from the above remark. Otherwise $\zeta \in P_j$ for some $j > N$, and there is a point $u \in [z, \zeta] \cap \partial P_j$. Then

$$|h(z) - h(\zeta)| \leqslant |h(z) - h(u)| + |h(u) - h(\zeta)|.$$

If ζ is sufficiently close to z then the first term is at most ε by continuity of $h|X$. As the second term is bounded by $\operatorname{diam}(P_j) < \varepsilon$, (11.2) follows.

Let us now prove the second statement. As each P_i is simply-connected, $h^0|P_i$ is homotopic to $h^1|P_i$ rel ∂P_i. Let $h^t: \bigcup P_i \to \widetilde{P}_i$ be a corresponding homotopy. Extend it to the whole domain Y as h^0. We should check that this extension $h^t(z): (Y, \bigcup P_i) \to (\widetilde{Y}, \bigcup \widetilde{P}_i)$ is continuous in two variables.

Note first that for $z \notin \bigcup_{1 \leqslant i \leqslant N} P_i \equiv T$,

$$|h^t(z) - h^0(z)| < \varepsilon. \tag{11.3}$$

Given a pair (z, t), we will show that $|h^t(z) - h^\tau(\zeta)| < 3\varepsilon$ as (ζ, τ) is sufficiently close to (z, t). To this end let us consider a few cases:

- If $z \in \operatorname{int} \bigcup P_i$, it is true since $h^t|P_i$ is a homotopy.
- If $z, \zeta \in T$, it is true since $h^t|T$ is a homotopy.
- If $z \in \partial T$ but $\zeta \notin T$, then for ζ sufficiently close to z,

$$|h^t(z) - h^\tau(\zeta)| = |h^0(z) - h^\tau(\zeta)| \leqslant |h^0(z) - h^0(\zeta)| + |h^\tau(\zeta) - h^0(\zeta)| < 2\varepsilon$$

by continuity of h^0 and (11.3).

- Let $z \notin T$. Then sufficiently close points ζ do not belong to T either. Hence by (11.3) and continuity of h^0,

$$|h^t(z) - h^\tau(\zeta)| \leqslant |h^0(z) - h^0(\zeta)| + |h^t(z) - h^0(z)| + |h^\tau(\zeta) - h^0(\zeta)| < 3\varepsilon. \qquad \square$$

11.3. First landing maps

Let us have a family of puzzle pieces P_i with disjoint interiors contained in a puzzle piece X, where as usual $P_0 \ni 0$ stands for the critical puzzle piece. Let us also have a Markov map $G: \bigcup P_i \to X$ which is univalent on all non-critical pieces P_i, $i \neq 0$, and the double-branched covering on the critical one, P_0. The Markov property means that if $\operatorname{int}(GP_i \cap P_j) \neq \varnothing$ then $GP_i \supset P_j$. Let A be the corresponding Markov matrix: $A_{ij} = 1$ if $\operatorname{int}(GP_i \cap P_j) \neq \varnothing$, and $A_{ij} = 0$ otherwise.

Let $P \equiv P^0$. A string of labels $\bar{\imath} = (i(0), ..., i(l-1))$ is called *admissible* if $A_{i(k), i(k+1)} = 1$ for $k = 0, ..., l-2$, and $i(k) \neq 0$ for $k < l-1$. The length l of the string will be denoted by $|\bar{\imath}|$. To any admissible string corresponds a *cylinder* of rank l defined by

$$G^k P^l_{\bar{\imath}} \subset P_{i(k)}, \ k = 0, ..., l-2; \quad G^{l-1} P^l_{\bar{\imath}} = P_{i(l-1)}. \tag{11.4}$$

Note that G^{l-1} univalently maps $P^l_{\bar{\imath}}$ onto $P_{i(l-1)}$.

Let us denote by $\Omega_{\bar{\imath}} \equiv P^l_{\bar{\imath}}$ the cylinders mapped onto the critical puzzle piece (so that $i(l-1) = 0$). The *first landing map*

$$T: \bigcup \Omega_{\bar{\imath}} \to P_0 \tag{11.5}$$

is defined as follows: $Tz = G^{l-1} z$ for $z \in \Omega_{\bar{\imath}}$, $|\bar{\imath}| = l$. This map is univalent on all pieces Ω_i (identical on the critical piece Ω_0).

LEMMA 11.2. *Let us have a K-qc pseudo-conjugacy $H\colon (X,\bigcup P_i)\to(\widetilde{X},\bigcup \widetilde{P}_i)$ between G and $\widetilde{G}$. Then there is a K-qc pseudo-conjugacy $\phi\colon (X,\bigcup \Omega_j)\to(\widetilde{X},\bigcup\widetilde{\Omega}_j)$ which conjugates the first landing maps T and $\widetilde{T}$.*

Proof. Let us pull H back to the pieces P_i, $i\neq 0$, that is, let us consider the map

$$H_1\colon (P_i,\bigcup_j P^l_{ij})\to(\widetilde{P}_i,\bigcup_j \widetilde{P}^l_{ij})$$

such that $\widetilde{G}\circ H_1|P_i=h\circ G|P_i$. Since H is a pseudo-conjugacy, H_1 matches with H on $\bigcup_{i\neq 0}\partial P_i$. Hence these maps glue together into a single K-qc map equal to H_1 on $\bigcup P_i$, and equal to H outside of it. We will keep the notation H_1 for this map.

Let us do the same pull-back with H_1. We will obtain a K-qc pseudo-conjugacy

$$H_2\colon (P,\bigcup P^1_i,\bigcup P^2_{ij},\bigcup P^3_{ijk})\to(\widetilde{P},\bigcup \widetilde{P}^1_i,\bigcup \widetilde{P}^2_{ij},\bigcup \widetilde{P}^3_{ijk}).$$

Repeating this procedure over again, we obtain a sequence of K-qc pseudo-conjugacies

$$H_s\colon \bigcup_{l\leqslant s}\ \bigcup_{|\bar{\imath}|=l} P^l_{\bar{\imath}}\to\bigcup_{l\leqslant s}\ \bigcup_{|\bar{\imath}|=l}\widetilde{P}^l_{\bar{\imath}}.$$

By the Compactness Lemma from Appendix A we can pass to a limit K-qc map

$$\phi\colon \bigcup_{l,\bar{\imath}} P^l_{\bar{\imath}}\to\bigcup_{l,\bar{\imath}}\widetilde{P}^l_{\bar{\imath}}.$$

By Lemma 11.1 this map is homotopic to h rel $(\partial X\cup\partial\Omega_j)$, and hence is a desired pseudo-conjugacy. $\square$

Let us now do a bit more (assuming a bit more). Consider the first return map $g\colon\bigcup V_j\to P_0$, and let $b=g(0)=G^t0$ be its critical value.

LEMMA 11.3. *Let us have two K-qc pseudo-conjugacies $H_0\colon (X,\bigcup P_i)\to(\widetilde{X},\bigcup\widetilde{P}_i)$ and $H_1\colon (P_0,b)\to(\widetilde{P}_0,\tilde{b})$. Then there exists a K-qc pseudo-conjugacy $\psi\colon (P_0,\bigcup V_i)\to(\widetilde{P}_0,\bigcup\widetilde{V}_i)$ between g and $\tilde{g}$.*

Proof. As H_0 and H_1 match on ∂P_0, they glue together into a singe K-qc pseudo-conjugacy $H\colon (X,\bigcup P_i,b)\to(\widetilde{X},\bigcup\widetilde{P}_i,\tilde{b})$ coinciding with H_1 on P_0 and coinciding with H_0 on $X\setminus P_0$ (see the Gluing Lemma in Appendix A). By Lemma 11.2, there is a K-qc map $\phi\colon (X,\bigcup\Omega_j)\to(\widetilde{X},\bigcup\widetilde{\Omega}_j)$ homotopic to h rel $(\partial X\cup\partial\Omega_j)$, and conjugating the first landing maps. As $H\colon b\mapsto\tilde{b}$, we have: $\phi\colon G^k0\mapsto\widetilde{G}^k0$, $k=1,\dots,t$. In particular, ϕ respects the G-critical values: $G(0)\mapsto\widetilde{G}(0)$.

Recall that the domains V_i are the pull-backs of the Ω_j by $G\colon P_0\to X$, that is, the components of $(G|P_0)^{-1}\Omega_j$. It follows that ϕ can be lifted to a K-qc map $\psi\colon (P_0,\bigcup V_i)\to(\widetilde{P}_0,\bigcup\widetilde{V}_i)$ homotopic to h rel $(\partial P_0\cup\partial V_i)$. (This lift is uniquely determined by the diagram $\widetilde{G}\circ\psi|P_0=\phi\circ G|P_0$ and the homotopy condition.)

This map ψ is the desired pseudo-conjugacy. $\square$

11.4. Initial constructions

Now the reader should consult §3.2 of this paper where the initial Markov tiling (3.4) of the Yoccoz puzzle piece $Y^{(0)}$ is constructed. We will apply it to the renormalized map F. Let us recall some notations. The first piece of the partition, $Y \equiv Y^{(0)}$, is bounded by the external rays landing at the fixed point α, and the equipotential E. The central piece of this partition, V^0, is the first piece of the principal nest. It is obtained by pulling back a puzzle piece $Z_\nu^{(1)}$ attached to the co-fixed point α'. There is a double-branched covering $F^s: V^0 \to Z_\nu^{(1)}$. All the puzzle pieces of the initial partition intersecting the Julia set $J(F)$ are univalent pull-backs of either Y or V^0. Let us denote the pieces of this partition by P_i, in such a way that $P_0 \equiv V^0$, $P_i \equiv Z_i^{(1)}$, $i=1, ..., p-1$, where p is the number of external rays of F landing and α. With these notations,

$$Y \cap J(F) = \bigcup (P_i \cap J(F)) \cup K, \tag{11.6}$$

where K is the residual Cantor set (of the points whose orbits never land at $\bigcup_{0 \leqslant i \leqslant p-1} P_i$).

LEMMA 11.4. *In the decomposition* (11.6), $\operatorname{diam} P_i \to 0$ *and the set* K *is a removable Cantor set.*

Proof. The first statement follows from Lemma 2.8. To prove removability of K, let us consider the domains Q_1 and Q_2 defined in §3.2. Then $F^p Q_i$ covers $Q_1 \cup Q_2$, and K is the set of points which never escape $Q_1 \cup Q_2$. By a little thickening of these domains, we obtain a Bernoulli map $F^p: \widehat{Q}_1 \cup \widehat{Q}_2 \to \mathbf{C}$ (so that $\operatorname{int}(F^p \widehat{Q}_i)$ contains $\widehat{Q}_i$). By Lemma 9.9, the Julia set $\widehat{K}$ of this map is removable. All the more, $K \subset \widehat{K}$ is removable (one can actually see that $K = \widehat{K}$). $\square$

Let us now go back to §4.2 where the fundamental domain Q near the fixed point α is constructed. Recall that $\gamma \in Y^{(1)}$ is the periodic point of period p, $\gamma' = -\gamma$ is the "co-periodic" point, and $\mathcal{R}(\gamma')$ is the family of rays landing at γ'. Also, let $X = Y^{(1)} \cup \bigcup_{1 \leqslant i \leqslant p-1} P_i$. This domain is bounded by the rays landing at α and the equipotential $F^{-1}E$.

Furthermore, D is the connected component of $Y^{(1)} \setminus \mathcal{R}(\gamma')$ attached to α, F^{-p}: $D \to F^{-p} D$ is the branch of the inverse map fixing α, and $Q = D \setminus F^{-p} D$.

Let us also consider quadrilaterals $D^* = D \cap Y^{(1+p)}$ and $Q^* = Q \cap Y^{(1+p)}$ obtained by cutting D and Q with the equipotential $F^{-p-1}E$. Note that $D \setminus D^* = Q \setminus Q^*$ consists of two quadrilaterals which do not contain points of the Julia set $J(F)$. Let $Q^*_{-k} = F^{-pk} Q^*$, $k=-1, 0, 1, ...,$ and $Q^*_{-2} = X \setminus F^p D^*$ (see Figure 5). Note that $J(F) \cap X$ is tiled into the pieces Q^*_{-k}, $k=-2, -1, ...$.

LEMMA 11.5. *The hyperbolic diameter of the domains Q^*_{-k}, $k=-2,-1,0,\ldots$, in Y is bounded. Moreover, if $|k-j|>1$ then there is a definite space in between Q_{-k} and Q_{-j} in Y.*

Proof. By the secondary limbs and a priori bounds assumptions, the geometry of the configuration $(Y, Y^{(1)}, Y^{(1+p)}, \mathcal{R}(\gamma), \mathcal{R}(\gamma'))$ is bounded (see §4.1). Hence Q^*_{-2} and Q^*_1 have a bounded hyperbolic diameter in Y. For the same reason, Q^* has a bounded hyperbolic diameter in F^pD^*. As $F^{-p}: F^pD^* \to F^pD^*$ is a hyperbolic contraction, the diameters of Q^*_{-k} in F^pD are bounded by the same constant. All the more, they are bounded in a bigger domain Y.

To prove the second statement, note that by bounded geometry of the initial ray-equipotential configurations, there is a definite annulus $T_0 \subset F^pD^*$ about Q^*_0 which does not intersect Q^*_{-k}, $k=2,3\ldots$. Then $T_{-i}=F^{-ip}T_0 \subset F^pD^*$ is the annulus with the same modulus going around Q^*_{-i} and disjoint from Q^*_{-k} with $|k-i|>1$. □

Our first essential step towards the Main Lemma is

LEMMA 11.6. *The Teichmüller distance between the configurations $(Y, \bigcup P_i, \bigcup Q^*_{-k})$ and $(\widetilde{Y}, \bigcup \widetilde{P}_i, \bigcup \widetilde{Q}^*_{-k})$ is bounded.*

Proof. Recall that $F^s(V^0)=P_\nu$, and $F(P_i)$ univalently covers Y. Let us consider a point $a=F^{s+1}0 \in X$. We will construct a qc map $(Y,a) \to (\widetilde{Y}, \tilde{a})$ respecting the boundary marking.

By §4.1, the geometry of the configuration $(Y, Y^{(1)}, Y^{(1+p)}, \mathcal{R}(\gamma), \mathcal{R}(\gamma'))$ (and the corresponding tilde one) is bounded, so that there is a qc pseudo-conjugacy

$$\phi: (Y, Y^{(1)}, Y^{(1+p)}, \mathcal{R}(\gamma), \mathcal{R}(\gamma')) \to (\widetilde{Y}, \widetilde{Y}^{(1)}, Y^{(1+p)}, \widetilde{\mathcal{R}}(\gamma), \widetilde{\mathcal{R}}(\gamma')).$$

In particular, this map conjugates $F^p: Q^* \to F^pQ^*$ to the corresponding tilde map.

As F^p univalently maps Q^*_{-k-1} onto Q^*_{-k}, ϕ can be redefined on the Q^*_{-k}, $k \geqslant 0$, in such a way that it becomes the pseudo-conjugacy between the configurations

$$\phi: (Y, Y^{(1)}, \bigcup Q^*_{-k}) \to (\widetilde{Y}, \widetilde{Y}^{(1)}, \bigcup \widetilde{Q}^*_{-k}) \tag{11.7}$$

with the same dilatation. (Just let $\phi(z)=\widetilde{F}^{-kp} \circ \phi \circ F^{kp}(z)$ for $z \in F^{-pk}Q$.)

It follows that $\phi(a)$ and $\tilde{a}$ belong to the same piece of the family $\{Q^*_{-k}\}_{k=-2}^{\infty}$. By Lemma 11.5 the hyperbolic distance between $\phi(a)$ and $\tilde{a}$ in $\widetilde{Y}$ is bounded.

By the Moving Lemma from Appendix A, there is a qc map $\psi: \widetilde{Y} \to \widetilde{Y}$ identical on the boundary and carrying $\phi(a)$ to $\tilde{a}$. Hence $\phi_1=\psi \circ \phi: (Y,a) \to (\widetilde{Y}, \tilde{a})$ is a qc map (with a definite, though bigger, dilatation) respecting the boundary marking.

Consider now the double-branched covering $F^{s+1}: (V^0, 0) \to (Y, a)$ with the critical point at 0, and the corresponding tilde map. As $\phi_1: (Y, a) \to (\widetilde{Y}, \tilde{a})$ respects the critical values for these maps, it can be lifted to a map $\phi_2: V^0 \to \widetilde{V}^0$ with the same dilatation respecting the boundary marking.

Let us now construct a qc pseudo-conjugacy ϕ_3 between corresponding non-critical puzzle pieces P_i and $\widetilde{P}_i$. It is easy as any non-central puzzle piece P_i under some iterate F^{l_i} is univalently mapped onto either Y or V^0. In the first case let ϕ_3 be the pull-back of $\phi: Y \to \widetilde{Y}$; in the second let it be the pull-back of ϕ_2. This pull-back preserves the dilatation and respects the boundary marking. This provides us with a qc map $\phi_3: \bigcup P_i \to \bigcup \widetilde{P}_i$ respecting the boundary marking of the puzzle pieces.

The latter property means that ϕ_3 matches with h on $\bigcup \partial P_i$. By the first part of Lemma 11.4 and Lemma 11.1, these maps glue together into a single homeomorphism coinciding with ϕ_3 on $\bigcup P_i$ and with h outside, homotopic to h rel $\partial Y \cup \partial P_i$ (we will still denote it ϕ_3).

By the Gluing Lemma from Appendix A, this homeomorphism is qc on $Y \setminus K$. By the second part of Lemma 11.4, the residual set K is removable, and thus ϕ_3 is automatically quasi-conformal across it (with the same dilatation). □

The next step towards the Main Lemma is

LEMMA 11.7. *The configurations* $(V^0, \bigcup V_i^1)$ *and* $(\widetilde{V}^0, \bigcup \widetilde{V}_i^1)$ *stay bounded Teichmüller distance apart.*

Proof. Let us consider the first return $b = g_1 0$ of the critical point back to V^0. We will construct a qc map

$$H: (V^0, b) \to (\widetilde{V}^0, \tilde{b}) \tag{11.8}$$

respecting the boundary marking.

Let $u = F^{s+1} b \in X$ (where F^s maps V^0 onto P_ν). Let ϕ be a pseudo-conjugacy given by Lemma 11.6. Then $\phi(u)$ and $\tilde{u}$ belong to the same piece of the tiling $X \cap J(F) = \bigcup_{-\infty < k \leqslant 2} (Q_{-k}^* \cap J(F))$. By Lemma 11.5, the hyperbolic diameters of these pieces in Y (and the corresponding tilde pieces) are bounded by a constant ϱ dependent only on the selected limbs and a priori bounds. Hence $\varrho_{\widetilde{Y}}(\phi(u), \tilde{u}) \leqslant \varrho$.

Let $a = F^{s+1} 0$, as in the proof of Lemma 11.6. Assume that $a \in Q_k$, $u \in Q_j$. Let us consider two cases:

- Let $|k - j| \leqslant 1$. Then $\varrho_Y(u, a) \leqslant 2\varrho$. Hence there is an annulus $C \subset Y$ going around a and u with $\operatorname{mod}(C) \geqslant \mu(\varrho) > 0$. As $F^{s+1}: (V^0, 0, b) \to (Y, a, u)$ is a double-branched covering with critical point at 0, the pull-back C_0 of this annulus to V_0 has modulus at least $\frac{1}{2}\mu(\varrho)$. Hence $\operatorname{mod}(\phi(C_0)) \geqslant K^{-1} \mu(\varrho)$, where $K = \operatorname{Dil}(\phi)$ depends only on the selected limbs and a priori bounds. Hence $\varrho_{\widetilde{V}^0}(\phi b, 0)$ is bounded. For the same reason, $\varrho_{\widetilde{V}^0}(\tilde{b}, 0)$ is bounded, and hence $\varrho_{\widetilde{V}^0}(\phi(b), \tilde{b})$ is bounded.

By the Moving Lemma from the appendix, there is a qc map $\psi: (\widetilde{V}^0, \phi(b)) \to (\widetilde{V}^0, \tilde{b})$, identical on the boundary. Then $H = \psi \circ \phi$ is a desired map (11.8).

• Let now $|k-j|>1$. Then by Lemma 11.5, there is a definite space in between Q_k^* and Q_j^* (and between the corresponding tilde sets). By the Moving Lemma, there is a qc map $\psi: (\widetilde{Y}, \phi(a), \phi(u)) \to (\widetilde{Y}, \tilde{a}, \tilde{u})$, identical on $\partial\widetilde{Y}$. This map lifts to a qc map (11.8) (with the same dilatation).

So, we have constructed a qc map (11.8) which carries the critical value $b = g_1(0)$ to the critical value $\tilde{b} = \tilde{g}_1(0)$. Lemma 11.3 completes the proof. □

11.5. Inductive step (non-central case)

Let us now inductively estimate the Teichmüller distance between the configurations $(V^{n-1}, \bigcup V_i^n)$ and $(\widetilde{V}^{n-1}, \bigcup \widetilde{V}_i^n)$. Let τ_n stand for the maximum of this Teichmüller distance and $\log \mathrm{Dil}(h)$, where as above, h stands for the conjugacy between F and $\widetilde{F}$. Recall that $\mu_n = \mathrm{mod}(V^{n-1} \setminus V^n)$ denote the principal moduli.

The following lemma is the main step of our construction.

Lemma 11.8. *Let $\mu_n \geqslant \bar{\mu} > 0$ and $\tau_n \leqslant \bar{\tau}$. Assume that $g_n(0) \in V_k^n$ with $k \neq 0$, that is, the return to level $n-1$ is non-central. Then $\tau_{n+1} \leqslant \tau_n + O(\exp(-\frac{1}{4}\mu_n))$, with a constant depending only on $\bar{\mu}$.*

Remark 1. We actually do not need to assume that the maps are topologically equivalent. As the proof below shows, it is enough to assume that the maps are combinatorially equivalent up to level n (that is, the configurations $(V^n, \bigcup V_i^{n+1})$ and $(\widetilde{V}^n, \bigcup \widetilde{V}_i^{n+1})$ are topologically pseudo-conjugate) and the critical values $g_{n+1}0$ and $\tilde{g}_{n+1}0$ belong to the corresponding pieces V_j^{n+1} and $\widetilde{V}_j^{n+1}$.

Remark 2. We do not assume that the off-critical puzzle pieces V_i^n, $i \neq 0$, are well inside V^{n-1}, since this is not the case on the levels which immediately follow long central cascades (see Theorem II).

Proof. Let us skip n in the notations of the objects of level n, so that $V_i^n \equiv V_i$, $g_n \equiv g$, $\mu_n \equiv \mu$, etc. Also, let $V^{n-1} \equiv \Delta$ and $g(0) \equiv c_1$. As above, the corresponding objects for $\tilde{f}$ are marked with tilde. Thus we have two generalized polynomial-like maps $g: \bigcup V_i \to \Delta$ and $\tilde{g}: \bigcup \widetilde{V}_i \to \tilde{\Delta}$, which are pseudo-conjugate by a $K = e^{\tau}$-qc map

$$\phi: (\Delta, \bigcup V_i) \to (\tilde{\Delta}, \bigcup \widetilde{V}_i). \tag{11.9}$$

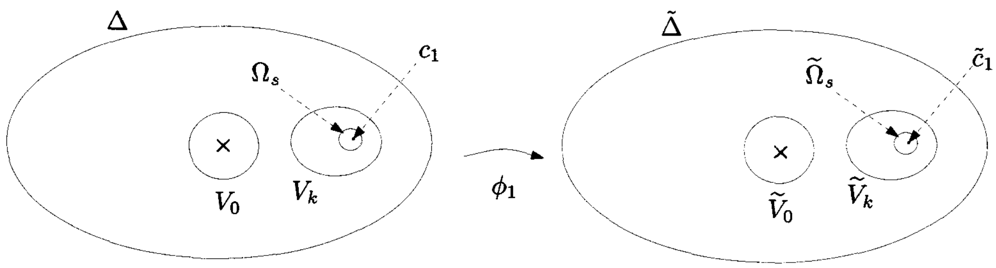

Fig. 11. Localization of the critical values

The objects on the next level, $n+1$, will be marked with prime: $V^{n+1} \equiv V'$, $g' \equiv g_{n+1}$, etc. (where g' is *not* the derivative of g). So $g' : \bigcup V_j' \to \Delta'$, $\Delta' \equiv V_0$.

Let $\lambda(\nu)$ be the maximal hyperbolic distance between the points in the hyperbolic plane enclosed by an annulus of modulus ν. Note that $\lambda(\nu) = O(e^{-\nu})$ as $\nu \to \infty$ (see Appendix A). Set $\lambda = \lambda(\mu)$.

Our goal is to lift ϕ to a $K(1+O(\lambda^{1/4}))$-qc pseudo-conjugacy

$$\phi' : (\Delta', \bigcup V_i') \to (\tilde{\Delta}', \bigcup \widetilde{V}_i'). \tag{11.10}$$

The problem is that ϕ need not respect the positions of the critical values: $\phi(c_1) \neq \tilde{c}_1$.

Let us consider the first landing map $T : \bigcup \Omega_j \to V^0$. By Lemma 11.2, the pseudo-conjugacy ϕ admits the pull-back to a K-qc pseudo-conjugacy

$$\phi_1 : (\Delta, \bigcup \Omega_j) \to (\tilde{\Delta}, \bigcup \widetilde{\Omega}_j). \tag{11.11}$$

This *localizes* the positions of the critical values in the sense that $\phi_1(c_1)$ and $\tilde{c}_1$ belong to the same domain $\Omega_s \subset V_k$ (see Figure 11) and hence the hyperbolic distance between these points in $\widetilde{V}_k$ is at most λ.

By the Moving Lemma from Appendix A, we can find a $(1+O(\lambda))$-qc map

$$\psi : (\widetilde{V}_k, \phi_1(c_1)) \to (\widetilde{V}_k, \tilde{c}_1), \tag{11.12}$$

identical outside $\widetilde{V}_k$. Then the map

$$\phi_2 = \psi \circ \phi_1 : (\Delta, \bigcup V_i, c_1) \to (\Delta, \bigcup V_i, \tilde{c}_1)$$

is a $K(1+O(\lambda))$-qc pseudo-conjugacy respecting the critical values.

Let $\{U_j'\}$ be the family of the components of the $\{(g|\Delta')^{-1} V_i\}$. The map ϕ_2 can be lifted to a $K(1+O(\lambda))$-qc pseudo-conjugacy

$$H : (\Delta', U_i') \to (\tilde{\Delta}', \widetilde{U}_i'). \tag{11.13}$$

However, U_i' are not the same as V_j' (the components of $\{(g|\Delta')^{-1}\Omega_i\}$), so we have to do more: We will localize the positions of the critical values $a=g'0$ and $\tilde{a}$ in Δ', and construct a $K(1+O(\lambda))$-qc map

$$\phi_0' \colon (\Delta', a) \to (\tilde{\Delta}', \tilde{a}) \tag{11.14}$$

respecting the boundary marking. The argument depends on the position of these a-points. Let $a_1=g(a)\in V_j$.

Case (i). Assume that V_j is off-critical and different from V_k. Let $a_1\in\Omega_l$. Then the annuli $V_j\setminus\Omega_l\subset V_j$ and $V_k\setminus\Omega_s\subset V_k$ are disjoint (recall that $c_1\in\Omega_s$). Hence by the Moving Lemma, there is a $(1+O(\lambda))$-qc map

$$\psi_1\colon (\tilde{\Delta}, \phi_1(c_1), \phi_1(a_1)) \to (\tilde{\Delta}, \tilde{c}_1, \tilde{a}_1)$$

identical outside $(\widetilde{V}_k\cup\widetilde{V}_j)$ (where ϕ_1 is the map (11.11)). With this map instead of (11.12), the above construction leads to a map (11.13) which already respects the critical values: $H(a)=\tilde{a}$. Then we can let $\phi_0'=H$.

Case (ii). Assume that $V_j=V_k$.

- Assume first that the hyperbolic diameter of the set of four points

$$\{\tilde{c}_1, \tilde{a}_1, \phi_1(c_1), \phi_1(a_1)\}$$

in $\widetilde{V}_k$ does not exceed $\sqrt{\lambda}$. Then the hyperbolic distance between the points $\tilde{a}_1$ and $H(a_1)$ in $\tilde{\Delta}'$ is $O(\sqrt{\lambda})$ (where H is the map (11.13)). Hence there is a $(1+O(\lambda^{1/4}))$-qc map $\psi_2\colon (\tilde{\Delta}', H(a_1))\to(\tilde{\Delta}', \tilde{a})$ identical on $\partial\tilde{\Delta}'$. Define now the map (11.14) as $\psi\circ H$.

- Otherwise the hyperbolic distance between the pairs $(\phi_1(a_1), \tilde{a}_1)$ and $(\phi_1(c_1), \tilde{c}_1)$ in $\widetilde{V}_k$ is greater than $\sigma\sqrt{\lambda}$ (since there is an annulus of modulus μ separating one pair from another). Then there are separating annuli S_i about these pairs with $\operatorname{mod}(S_i)\geqslant q\sqrt{\lambda}$ (where $\sigma>0$ and $q>0$ depend only on the choice of limbs and a priori bounds). By the Moving Lemma, we can simultaneously move these points to the right positions by a $\big(1+O(\sqrt{\lambda})\big)$-qc map

$$\psi_2\colon (\tilde{\Delta}, \widetilde{V}_k, \phi(a_1), \phi(c_1)) \to (\tilde{\Delta}, \widetilde{V}_k, \tilde{a}_1, \tilde{c}_1),$$

identical on $\tilde{\Delta}\setminus\widetilde{V}_k$. Using this map instead of (11.12) we come up with a $(1+O(\sqrt{\lambda}))$-qc map (11.13) respecting the critical values of g': $H(a)=\tilde{a}$.

Case (iii). Let us finally assume that $V_j=V_0$ is critical. Then a belongs to a precritical puzzle piece $V_t'\subset\Delta'$. Since $\operatorname{mod}(\Delta'\setminus V_t')\geqslant\frac{1}{2}\mu$, the hyperbolic distance between

$H(a)$ and $\tilde{a}$ in Δ' is $O(\sqrt{\lambda})$ (where H is the map (11.13)). By the Moving Lemma, there is a $(1+O(\sqrt{\lambda}))$-qc map

$$\psi_3\colon (\tilde{\Delta}', \phi(a)) \to (\tilde{\Delta}', \tilde{a}).$$

Let us now define a map (11.14) as follows: $\phi_0' = \psi_3 \circ H$.

So in all cases we have constructed a $(1+O(\lambda^{1/4}))$-qc map (11.14). Now Lemma 11.3 completes the proof. □

11.6. Through a central cascade

Let $V^m \supset V^{m+1} \supset \ldots \supset V^{m+N}$ be a cascade of central returns, so that the critical value $g_{m+1}0$ belongs to V_k, $k=m+1, \ldots, m+N-1$, but escapes V^{m+N}.

LEMMA 11.9. *Let $\mu_{m+1} \geqslant \bar{\mu} > 0$ and $\tau_{m+1} \leqslant \bar{\tau}$. Then for $k \leqslant N+1$,*

$$\tau_{m+k} \leqslant \tau_{m+1} + O\big(\exp(-\tfrac{1}{4}\mu_{m+1})\big),$$

with a constant depending only on $\bar{\mu}$.

Proof. We will adjust the proof of Lemma 11.8 to this situation. Let $g = g_{m+1}$, $\mu = \operatorname{mod}(V^m \setminus V^{m+1})$, etc. By definition, there is a $K = e^{\tau}$-qc pseudo-conjugacy

$$\phi\colon (V^m, \bigcup V_i^{m+1}) \to (\widetilde{V}^m, \bigcup \widetilde{V}_i^{m+1}).$$

Let us consider the first landing map $T\colon \bigcup \Omega_j \to V^{m+1}$ corresponding to g, $\Omega_0 = V^{m+1}$. By Lemma 11.2, T and $\widetilde{T}$ are pseudo-conjugate by a K-qc map

$$\phi_1\colon (V^m, \bigcup \Omega_j) \to (\widetilde{V}^m, \bigcup \widetilde{\Omega}_j).$$

Let us take a family of puzzle pieces $V_i^{m+1} \subset A^{m+1} = V^m \setminus V^{m+1}$ and pull them back to the annuli $A^{m+2}, \ldots, A^{m+N}$. We obtain a family of puzzle pieces W_i^{m+k}, together with the Bernoulli map

$$G\colon V^{m+N} \cup \bigcup_{k,i} W_i^{m+k} \to V^m \tag{11.15}$$

(see §3.6). Similarly let Ω_l^{m+k} stand for the pull-backs of the $\Omega_j \equiv \Omega_j^{m+1}$, $j \neq 0$, to the A^{m+k}, $k=1, \ldots, N$. If $W_i^{m+k} \supset \Omega_l^{m+k}$ then

$$\operatorname{mod}(W_i^{m+k} \setminus \Omega_l^{m+k}) \geqslant \mu,$$

so that the dynamically defined points are well localized by these puzzle pieces.

Let us now lift ϕ_1 to the annuli $A^{m+k} \to \tilde{A}^{m+k}$, $k=2, ..., N$. We obtain a K-qc map

$$\phi_2: (V^m \backslash V^{m+N}, \bigcup W_i^{m+k}, \bigcup \Omega_l^{m+k}) \to (\widetilde{V}^m \backslash \widetilde{V}^{m+N}, \bigcup \widetilde{W}_i^{m+k}, \bigcup \widetilde{\Omega}_l^{m+k}) \qquad (11.16)$$

respecting the boundary marking.

Let $c_1 \equiv g(0) \in \Omega_l^{m+N} \subset V_k^{m+N}$. By the Moving Lemma, there is a $(1+O(e^{-\mu}))$-qc map

$$\psi: (\widetilde{V}^m, \widetilde{V}_k^{m+N}, \phi_2(c_1)) \to (\widetilde{V}^m, \widetilde{V}_k^{m+N}, \tilde{c}_1),$$

identical outside $\widetilde{V}_k^{m+N}$. Then the map

$$\begin{aligned} \phi_3 = \psi \circ \phi_2 \colon & \big(V^m \backslash V^{m+N}, \bigcup_{1 \leqslant k \leqslant N} \bigcup_{i \neq 0} W_i^{m+k}, c_1\big) \\ & \to \big(\widetilde{V}^m \backslash \widetilde{V}^{m+N}, \bigcup_{1 \leqslant k \leqslant N} \bigcup_{i \neq 0} \widetilde{W}_i^{m+k}, \tilde{c}_1\big) \end{aligned} \qquad (11.17)$$

is $K(1+O(e^{-\mu}))$-qc, respects the boundary marking and positions of the critical values.

Consider now the topological discs Q_1 and Q_2 in V^{m+N} univalently mapped by g onto V^{m+N}. The Bernoulli map $g: Q_1 \cup Q_2 \to V^{m+n}$ produces a family of cylinders $Q_{\bar{\imath}}^t$, $\bar{\imath} = (i(0), i(1), ..., i(t-1))$, such that

$$g^j Q_{\bar{\imath}}^t \subset Q_{i(j)}, \;\; j = 0, ..., t-1; \quad g^t Q_{\bar{\imath}} = V^{m+N}.$$

Let $\mathbf{Q}^t = \bigcup_{\bar{\imath}} Q_{\bar{\imath}}^t$, $Q^0 \equiv V^{m+N}$. Moreover, by Lemma 9.9, the residual set $K = \bigcap \mathbf{Q}^t$ is removable.

The map ϕ_3 can be consecutively lifted to the maps

$$\omega_t: \mathbf{Q}^{t-1} \backslash \mathbf{Q}^t \to \widetilde{\mathbf{Q}}^{t-1} \backslash \widetilde{\mathbf{Q}}^t, \quad t = 1, 2, ...,$$

with the same dilatation respecting the boundary marking. By the Gluing Lemma, they are organized in a single qc map

$$\omega: V^{m+N} \backslash K \to \widetilde{V}^{m+N} \backslash \widetilde{K}$$

with the same dilatation. As K is removable, this map automatically extends across K:

$$H: (V^{m+N}, \bigcup U_i^{m+N+1}, Q_1, Q_2) \to (\widetilde{V}^{m+N}, \bigcup \widetilde{U}_i^{m+N+1}, \widetilde{Q}_1, \widetilde{Q}_2), \qquad (11.18)$$

where $U_i^{m+N+1} \subset V^{m+N}$ are the components of $g^{-1} W_j^{m+N}$, $U_0^{m+N+1} \equiv V_0^{m+N+1}$. Note that $\operatorname{mod}(V^{m+N} \backslash U_i^{m+N+1}) \geqslant \frac{1}{2}\mu$.

The maps (11.18) and (11.17) glue together into a single $K(1+O(e^{-\mu}))$-qc map

$$\phi_4: \big(V^m, \bigcup_{1 \leqslant k \leqslant N} \bigcup_{i \neq 0} W_i^{m+k}, V^{m+N}\big) \to \big(\widetilde{V}^m, \bigcup_{1 \leqslant k \leqslant N} \bigcup_{i \neq 0} \widetilde{W}_i^{m+k}, \widetilde{V}^{m+N}\big).$$

Take now a family of cylinders $W_{\bar{\imath}}^{m+k}$ of the Bernoulli map (11.15) (where $\bar{\imath}$ are finite strings of symbols). The map ϕ_4 is naturally lifted to a qc pseudo-conjugacy Φ with the same dilatation which respects this family of cylinders. Moreover, every $W_{\bar{\imath}}^{m+k}$ contains a piece $V_{\bar{\imath}}^{m+k}$ such that

$$G^l V_{\bar{\imath}}^{m+k} = V^{m+k-1}, \quad \text{where } l = |\bar{\imath}|,$$

and all puzzle pieces V_j^{m+k} are obtained in such a way. As ϕ_4 respects the V^{m+k-1}-pieces, $k \leqslant N$, the new map Φ respects the V_j^{m+k}-pieces. Thus Φ is a $K(1+O(e^{-\mu}))$-qc pseudo-conjugacy between g_{m+k} and $\tilde{g}_{m+k}$, so that $\tau_{m+k} \leqslant \log K + O(e^{-\mu})$, $k=1, ..., m+N$.

Let us proceed with the estimate of τ_{m+N+1}. Take the first return a of the critical point back to V^{m+N}, and construct a $K(1+O(e^{-\mu/4}))$-qc map

$$\phi_0' \colon (V^{m+N}, a) \to (\widetilde{V}^{m+N}, \tilde{a}). \tag{11.19}$$

To this end let us go through Cases (i), (ii), (iii) of the proof of Lemma 11.8 using the $\{W_i^{m+N}\}$ in place of $\{V_i\}$ and V^{m+N} in place of $V^{m+1} \equiv \Delta'$.

In the first two cases the argument is the same as above. However, the last case is different since the precritical puzzle pieces Q_1 and Q_2 are not necessarily well inside of V^{m+N}. To take care of this problem let us consider the first "escaping moment" t when $b \equiv g^t a \notin Q_1 \cup Q_2$. Then $b \in U_i^{m+N+1}$ for some U-domain from (11.18). Then there is a domain $\Lambda \subset Q_1 \cup Q_2$ containing a which is univalently mapped onto U_i^{m+N+1} by g^t. Moreover,

$$\operatorname{mod}(Q \backslash \Lambda) \geqslant \operatorname{mod}(V^{m+N} \backslash U_i^{m+N+1}) \geqslant \tfrac{1}{2}\mu.$$

By means of $g \colon Q_1 \cup Q_2 \to V^{m+N}$, the map (11.18) can be turned into a qc map (with the same dilatation)

$$H_1 \colon (V^{m+N}, \Lambda) \to (\widetilde{V}^{m+N}, \tilde{\Lambda})$$

(coinciding with H outside $Q_1 \cup Q_2$). This gives us an appropriate localization of the a-points. The Moving Lemma turns H_1 into (11.19).

Lemma 11.3 completes the proof. □

Remark. As the above proof shows, it is enough to assume that the maps are combinatorially equivalent up to level $m+N$ and the critical values $g_{n+1}0$ and $\tilde{g}_{n+1}0$ belong to the corresponding pieces Λ and $\tilde{\Lambda}$.

11.7. Proof of the Main Lemma

Let $\{i(k)\}$ be the sequence of non-central levels in the principal nest $V^0 \supset V^1 \supset \ldots$. Let

$i(n-1)+1<m\leqslant i(n)+2$. By Lemma 11.9,

$$\tau_m \leqslant \log K^* + O\left(\sum_{k=0}^{n-1} \exp\left(-\tfrac{1}{4}\mu_{i(k)+2}\right)\right). \tag{11.20}$$

But by Theorem III, the principal moduli $\mu_{i(k)+1}$ grow at linear rate: $\mu_{i(k)+1}\geqslant Bk$, where the constant B depends only on μ_1. Hence the sum (11.20) is bounded by $\log K_* + C(\mu_1)$.

In turn, by Theorem I the modulus μ_1 is bounded by a constant depending only on the selected limbs and a priori bounds. Hence $\tau_n \leqslant \log K_* + A$, where A depends only on the choice of limbs and a priori bounds. The Main Lemma is proved. □

11.8. Last cascade

If the map $F\equiv F_m = R^m f$ is not renormalizable then the principal nest consists of infinitely many central cascades, and the Main Lemma gives a uniform bound on the Teichmüller distance between the corresponding generalized renormalizations.

Otherwise the principal nest ends up with an infinite central cascade $V^{n-1}\supset V^n\supset\ldots$ shrinking to the little Julia set J^{m+1} of the next renormalization $g_n = F_{m+1}\equiv R^{m+1}f$. Recall that the levels $n-1, n, n+1, \ldots$ of this final cascade are called DH levels.

LEMMA 11.10. *Let $n-1$ be a DH level and*

$$H: (V^{n-1}, V^n) \to (\widetilde{V}^{n-1}, \widetilde{V}^n)$$

be a K-qc pseudo-conjugacy between g_n and $\tilde{g}_n$. Then there is a homeomorphism

$$\phi: (V^{n-1}, J^{m+1}) \to (\widetilde{V}^{n-1}, \tilde{J}^{m+1})$$

homotopic to h rel $(J^{m+1}\cup\partial V^{n-1})$, and K-qc on $V^{n-1}\setminus J^{m+1}$.

Proof. Recall that $A^n = V^{n-1}\setminus V^n$. The map $H: A^n\to\tilde{A}^n$ admits a lift to qc maps (with the same dilatation) $H_k: A^{n+k}\to\tilde{A}^{n+k}$ homotopic to h rel the annuli boundary. These maps match to a single qc map $\phi: V^{n-1}\setminus J^{m+1}\to\widetilde{V}^{n-1}\setminus\tilde{J}^{m+1}$ with the same dilatation conjugating F_{m+1} to $\widetilde{F}_{m+1}$. By Corollary 10.5, this map (and the whole homotopy between it and h) matches with h on $J(F_{m+1})$. □

11.9. Spreading around

Let us consider the pieces $P_j\subset Y\equiv Y^{(0)}$ of the initial partition (11.6), and the Markov map $G: \bigcup P_i\to Y$ (see §3.2). Let us consider the first landing map to $V^0\equiv P_0$, $T_0: \bigcup \Omega_i^0\to P_0$.

By Lemmas 11.6 and 11.2, there is a qc pseudo-conjugacy $\phi_0: (Y, \bigcup \Omega_i^0) \to (\widetilde{Y}, \bigcup \widetilde{\Omega}_i^0)$. Let us also consider the following maps:

- the first landing maps to V^n corresponding to the $g_n: \bigcup V_i^n \to V^{n-1}$:

$$T_n: \bigcup \Omega_i^n \to V^n, \quad \Omega_i^n \subset V^{n-1};$$

- the first landing maps to V^n corresponding to G:

$$S_n: \bigcup O_i^n \to V^n, \quad O_i^n \subset Y.$$

Clearly

$$S_0 = T_0 \quad \text{and} \quad S_n = T_n \circ S_{n-1}. \tag{11.21}$$

By the Main Lemma and Lemma 11.2, there is a sequence of qc pseudo-conjugacies

$$\phi_n: (V^{n-1}, \bigcup \Omega_i^n) \to (\widetilde{V}^{n-1}, \bigcup \widetilde{\Omega}_i^n), \quad n < N+1,$$

where N is the first DH level (if F is non-renormalizable then $N=\infty$). Let us turn it inductively into a sequence of pseudo-conjugacies

$$H_n: (Y, \bigcup O_i^n) \to (\widetilde{Y}, \bigcup \widetilde{O}_i^n) \tag{11.22}$$

between S_n and $\widetilde{S}_n$ (with the same dilatation). Indeed, using (11.21), we can define it as

$$H_n | O_i^{n-1} = (\widetilde{S}_{n-1} | \widetilde{O}_i^{n-1})^{-1} \circ (\phi_n | V^{n-1}) \circ S_{n-1} | O_i^{n-1}.$$

As these maps match with H_{n-1} on the boundaries ∂O_i^{n-1}, they glue together into single qc conjugacy (11.22) with the same dilatation.

If F is non-renormalizable then we obtain an infinite sequence of qc pseudo-conjugacies H_n (with uniformly bounded dilatation). As the pieces V_i^n shrink as $n \to \infty$, there is the limit qc map

$$H: (Y, J(F) \cap Y) \to (\widetilde{Y}, \tilde{J}(F) \cap \widetilde{Y}) \tag{11.23}$$

homotopic to $h: J(F) \cap Y \to \tilde{J}(F) \cap \widetilde{Y}$ rel $\partial Y \cup J(F)$.

Assume that F is renormalizable, but not immediately renormalizable (we leave to the reader an adjustment of the argument to the immediate case). Let $\mathcal{I}$ be the family of little Julia sets J_i^{m+1} contained in Y, $J^{m+1} \equiv J_0^{m+1}$. Let us consider the last pseudo-conjugacy (11.22) on the DH level N. Let us replace it by the pseudo-conjugacy

$$\phi_N: (V^N, J^{m+1}) \to (\widetilde{V}^N, \tilde{J}^{m+1})$$

constructed in Lemma 11.10. Spread it around by the landing map S_N, that is, set

$$H|O_i^N=(\widetilde{S}_N|\widetilde{O}_i^N)^{-1}\circ(\phi_N|V^N)\circ S_N|O_i^N,$$

where $O_i^N\supset J_i^{m+1}$. As these maps match on the ∂O_N with H_N, they glue together into a homeomorphism

$$H\colon (Y, \bigcup_{i\in\mathcal{I}} J_i^{m+1})\to(\widetilde{Y}, \bigcup_{i\in\mathcal{I}} \tilde{J}_i^{m+1}), \tag{11.24}$$

quasi-conformal on $Y\setminus\bigcup_{i\in\mathcal{I}} J_i^{m+1}$ (with dilatation depending only on the choice of limbs and a priori bounds), and homotopic to h rel $\partial Y\cup\bigcup_{i\in\mathcal{I}} J_i^{m+1}$.

Let us consider the backward orbit $Y\equiv Y_0, Y_{-1}, ..., Y_{-r+1}$ of Y under f such that $Y_{-k}\ni f^{r-k}0$, where r is the first return time of the critical orbit to Y. The disks Y_{-k} have disjoint interiors. Let us pull the map H back to these disks, that is, set

$$h_{m+1}|Y_{-k}=(\tilde{f}^k|\widetilde{Y}_{-k})^{-1}\circ H\circ f^k|Y_{-k}.$$

As this map respects the boundary marking of the Y_{-k}, it extends to the whole plane as h_m, which provides the desired next approximation to the Thurston conjugacy (see §10.3).

Theorem VI and hence the Rigidity Theorem are proved. □

12. Rigidity and deformations of real maps

12.1. MLC at real points with big essential periods

Let us go back to the discussion of long central cascades in §7.3 and §8. Consider a map $f\in\mathcal{SL}$ and all its renormalizations $R^m f$ (finite or infinite sequence). Let P_m stand for the straightenings of the $R^m f$. We refer to central cascades of the P_m as the cascades associated to f. They are represented by a compact family of quadratic-like maps.

LEMMA 12.1. *Let f and $\tilde{f}$ be two combinatorially equivalent quadratic-like maps of class $\mathcal{SL}$. Assume that a sequence of central cascades $\mathcal{C}^n$ associated to f converges to a quadratic-like map g. Then all the limits $\tilde{g}$ of the corresponding sequence $\widetilde{\mathcal{C}}^n$ are hybrid equivalent to g.*

Proof. By the Main Lemma for the Rigidity Theorem, the cascades $\mathcal{C}^n$ and $\widetilde{\mathcal{C}}^n$ are K-qc pseudo-conjugate, with a uniform K. Hence the limit maps g and $\tilde{g}$ are qc conjugate. Let P_c and $P_{\tilde{c}}$ be the straightened g and $\tilde{g}$ respectively. These maps are also qc equivalent. But c and $\tilde{c}$ lie on the boundary of the Mandelbrot set and are hence qc rigid (by the standard Beltrami deformation argument). Hence $c=\tilde{c}$. □

In particular, for a map with *real combinatorics* (that is, combinatorially equivalent to a real map) the cascades can still be classified into two types: Ulam–Neumann and saddle-node.

THEOREM VIII. *There exists a p with the following property. Let $f=P_c$ be an infinitely renormalizable real quadratic polynomial. If the essential periods* $\mathrm{per}_e(R^n f)$ *of all the renormalizations are greater than p then MLC holds at c.*

Proof. Let $g=P_a$ be an infinitely renormalizable quadratic map with real combinatorics. If the essential period of $R^m g$ is big then at least one of the four parameters counted in the proof of Lemma 8.7 is big. This implies that $\mathrm{mod}(R^{m+1}g)$ is big (Theorem IV′ takes care of the first three parameters; the last one can be handled easily). By the Rigidity Theorem, $a=c$, and hence MLC holds at c (see §2.5). □

Remark. We have recently proven a stronger version of Theorem VIII, where the essential period p_e is replaced with the period p (in preparation).

12.2. Teichmüller space of quasi-quadratic maps

Real rigidity of quadratic maps (see the Density Theorem stated in the Introduction) yields nice structure on the space of quasi-quadratic maps.

COROLLARY 12.2. *Any two quasi-quadratic maps of Epstein class with the same combinatorics are quasi-symmetrically conjugate.*

Proof. If the map in question is at most finitely renormalizable then by [L4, §4] it admits a polynomial-like generalized renormalization. By the Generalized Straightening Theorem (see §3.7), this renormalization is qc conjugate to a polynomial with one non-escaping critical point. By the Branner–Hubbard–Yoccoz rigidity, such a polynomial is uniquely determined by its combinatorial type (compare [L4, Theorem 5.6]).

If the maps are infinitely renormalizable then by the Complex Bounds Theorem they admit quadratic-like renormalizations, and the result follows from the Straightening and Density Theorems. □

Thus any combinatorial class of Epstein quasi-quadratic maps can be viewed as the space of qc deformations of a reference map. Similar to the quadratic-like situation (see §2.4), the above result allows one to endow this space with the Teichmüller metric

$$\mathrm{dist}_T(f,\tilde f)=\inf_h \mathrm{Dil}(h),$$

where h runs over all local qc extensions of real conjugacies $[-\beta,\beta]\to[-\tilde\beta,\tilde\beta]$ between f and $\tilde f$ (where β stands for the fixed point with positive multiplier).

Let us finish with a few comments on the non-analytic version of Corollary 12.2: *Any two quasi-quadratic maps with the same combinatorics are quasi-symmetrically conjugate.* Under certain combinatorial assumptions (sufficiently slow recurrence of the critical point) this statement was proven in [JS]. In [L5] it is proven for the Fibonacci combinatorics, which is, in a sense, the most recurrent. The general case can be treated as follows:

- The key idea is to consider an asymptotically conformal extension of the maps under consideration [L5].
- If the type of the map is sufficiently big, create (by Lemma 8.6) a generalized quasi-quadratic-like map. Theorem III on the moduli growth is readily extended to this class of maps. (The proof is the same except that all estimates are valid modulo exponentially decaying errors coming from the dilatation of the generalized renormalizations.)
- Then construct a qc pseudo-conjugacy between these quasi-quadratic-like maps as in [L5, §5], and pull it back until the next renormalization level as in §11.
- If the type is essentially bounded, construct the qc pseudo-conjugacy by means of the round discs using essentially bounded geometry (compare [MS, Chapter IV, Theorem 3.1]).
- Passing to the next renormalization level, start the above construction over again interpolating (using complex bounds) between the old and the new maps.

This argument will be elaborated elsewhere.

13. Appendix A: Conformal and quasi-conformal geometry

13.1. Poincaré metric and distortion

A domain $D\subset\mathbf{C}$ is called *hyperbolic* if its universal covering space is conformally equivalent to the unit disc. This happens if and only if $\mathbf{C}\setminus D$ consists of at least two points. Hyperbolic domains possess the *hyperbolic* (*or Poincaré*) *metric* ϱ_D of constant negative curvature. This metric is obtained by pushing down the Poincaré metric $d\varrho_{\mathbf{D}}=|dz/(1-z^2)|$ from the unit disc $\mathbf{D}$.

In the case of a simply-connected hyperbolic domain D ("conformal disk"), $d\varrho_D=p_D(z)\,|dz|$ is the pull-back of the $\varrho_{\mathbf{D}}$ by the Riemann mapping $D\to\mathbf{D}$. In this case its density $p_D(z)$ is comparable with $1/\operatorname{dist}(z,\partial D)$:

$$\tfrac{1}{4}\operatorname{dist}(z,\partial D)^{-1}\leqslant p_D(z)\leqslant \operatorname{dist}(z,\partial D)^{-1}.$$

In the simply-connected case, a set $K\subset D$ has a bounded hyperbolic diameter $\operatorname{diam}_D K$ if and only if there is an annulus $A\subset D$ of definite modulus surrounding K.

More precisely, let $\mu_{\min}(R)$ and $\mu_{\max}(R)$ denote the minimal and maximal possible modulus of an annulus $A \subset D$ surrounding K, where K runs over all subsets of hyperbolic diameter R. Then $0 < \mu_{\min} < \mu_{\max} < \infty$ (all estimates are clearly independent of D). This can be readily seen by passing to the disk model and moving one point of K to the origin. The extremal moduli correspond to the cases of a pair of points and hyperbolic disk of radius R. Moreover, both minimal and maximal moduli behave as $\log(1/R)+O(1)$ (see [Ah, Chapter III]).

Given a univalent holomorphic function $f: D \to \mathbf{C}$, the *distortion* of f on K is defined as

$$\sup_{z,\zeta\in K} \log\left|\frac{f'(z)}{f'(\zeta)}\right|.$$

KOEBE DISTORTION THEOREM. *Let D be a conformal disk, $K \subset D$, and $r = \operatorname{diam}_D K$ be the Poincaré diameter of K in D. Then the distortion of any univalent function f on K is bounded by a constant $C_D(r)$ independent of a particular choice of K. Moreover, $C_D(r) = O(r)$ as $r \to 0$.*

13.2. Moduli defect and capacity

Let D be a topological disk, $\Gamma = \partial D$, $a \in D$, and $\psi: (D, a) \to (\mathbf{D}_r, 0)$ be the Riemann map onto a round disk of radius r with $\psi'(a) = 1$. Then $r \equiv r_a(\Gamma)$ is called the *conformal radius* of Γ about a. The *capacity* of Γ rel a is defined as

$$\operatorname{cap}_a(\Gamma) = \log r_a(\Gamma).$$

LEMMA A.1. *Let $D_0 \supset D_1 \supset K$, where D_i are topological disks and K is a connected compact. Assume that the hyperbolic diameter of K in D_0 and the hyperbolic $\operatorname{dist}(K, \partial D_1)$ are both bounded by an L. Then there is an $\alpha(L) > 0$ such that*

$$\operatorname{mod}(D_1 \setminus K) \leqslant \operatorname{mod}(D_0 \setminus K) - \alpha(L).$$

Proof. Let us take a point $z \in \partial D_1$ whose hyperbolic distance to K is at most L. Then there is an annulus of a definite modulus contained in D_0 and enclosing both K and z.

Let us uniformize $D_0 \setminus K$ by a round annulus $A_r = \{\zeta : r < |\zeta| < 1\}$, and let $\tilde{z}$ correspond to z under this uniformization. Then $\tilde{z}$ stays at a definite Euclidean distance d from the unit circle.

If $R \subset A_r$ is any annulus enclosing the inner boundary of A_r but not enclosing $\tilde{z}$ then by the normality argument $\operatorname{mod}(R) < \operatorname{mod}(A_r) - \alpha_r(d)$ with an $\alpha_r(d) > 0$. (Actually, the extremal annulus is just A_r slit along the radius from $\tilde{z}$ to the unit circle.)

We have to check that $\alpha_r(d)$ is not vanishing as $r\to 0$. Let us fix an outer boundary Γ of R (the unit circle plus the slit in the extremal case). We may certainly assume that the inner boundary coincides with the r-circle. Then the defect $\mathrm{mod}(R)-\log(1/r)$ monotonically increases to $\mathrm{cap}_0(\Gamma)$. By normality this capacity is bounded above by a $-\alpha(d)<0$, and we are done. □

Let A be a standard cylinder of finite modulus, $K\subset A$. Let us define $\mathrm{width}(K)\equiv \mathrm{width}(K|A)$ as the modulus of the smallest concentric subcylinder $A'\subset A$ containing K.

DEFINITE GRÖTZSCH INEQUALITY. *Let A_1 and A_2 be homotopically non-trivial disjoint topological annuli in A. Let K be the set of points in their complement which are separated by $A_1\cup A_2$ from the boundary of A. Then there is a function $\beta(x)>0$ $(x>0)$ such that*

$$\mathrm{mod}(A)\geqslant \mathrm{mod}(A_1)+\mathrm{mod}(A_2)+\beta(\mathrm{width}(K)).$$

Proof. For a given cylinder this follows from the usual Grötzsch inequality and the normality argument. Let us fix a K, and let $\mathrm{mod}(A)\to\infty$. We can assume that A_i are lower and upper components of $A\setminus K$. Then the modulus defect

$$\mathrm{mod}(A)-\mathrm{mod}(A_1)-\mathrm{mod}(A_2)$$

decreases by the usual Grötzsch inequality. At limit the cylinder becomes the punctured plane, and the modulus defect converges to $-(\mathrm{cap}_0(K)+\mathrm{cap}_\infty(K))$.

It follows from the area inequality that this sum of capacities is negative, unless K is a circle centered at the origin. Moreover, the estimate depends only on $\mathrm{width}(K)$. Indeed, let D_0 and D_∞ be the components of $\mathbf{C}\setminus K$ containing 0 and ∞ respectively. Let $\phi_0\colon B(0,R_0)\to D_0$ and $\phi_\infty\colon \mathbf{C}\setminus B(0,R_\infty)\to D_\infty$ be the Riemann mappings normalized by $\phi_0(z)\sim z$ as $z\to 0$, and $\phi_\infty(z)\sim z$ as $z\to\infty$. Then $\mathrm{cap}_0 K=\log R_0$ and $\mathrm{cap}_\infty K=\log(1/R_\infty)$.

As scaling does not change $\mathrm{cap}_0(K)+\mathrm{cap}_\infty(K)$, we can assume that $R_\infty(K)=1$. Let

$$\phi_\infty(z)=z-\sum_{k=1}^{\infty}\frac{a_k}{z^k}.$$

Then

$$\mathrm{area}(\mathbf{C}\setminus D_\infty)=\tfrac{1}{2}i\int_{|z|=1}\phi_\infty\, d\bar{\phi}_\infty=\pi\left(1-\sum_{k=1}^{\infty}k|a_k|^2\right)\leqslant\pi,$$

with equality only in the case when $\phi_\infty=\mathrm{id}$. Hence $\mathrm{area}(D_0)\leqslant\pi$ with equality only in the case when K is the unit circle S^1.

As $|\phi_0|^2$ is a subharmonic function,

$$1=|\phi_0(0)|^2 \leqslant \frac{\operatorname{area}(D_0)}{\pi R_0^2} \leqslant \frac{1}{R_0^2},$$

with equality only in the case when $K=S^1$. Hence $\operatorname{cap}_0(K)<0$ unless $K=S^1$. Moreover, by a normality argument $\operatorname{cap}_0(K)\leqslant c(\operatorname{width}(K))<0$. (Indeed, otherwise there would be a sequence of domains D_0^m as above converging to a domain Ω different from the unit disk, with $\operatorname{cap}_0(\Omega)=0$.) The lemma is proved. □

13.3. Eccentricity and pinching

Let Γ be a Jordan curve surrounding a point a. Let $d_a(\Gamma)$ and $r_a(\Gamma)$ be the Euclidean radii of the inscribed and circumscribed circles about Γ centered at a. Then let us define the eccentricity of Γ about a as

$$e_a(\Gamma)=\log\frac{r_a(\Gamma)}{d_a(\Gamma)}.$$

LEMMA A.2. *Let $A\subset \mathbf{C}\setminus\{0\}$ be an annulus homotopically non-trivially embedded in the punctured plane, $\Gamma\subset A$ be a homotopically non-trivial Jordan curve, and A_i be the components of $A\setminus\Gamma$. Assume that* $\operatorname{mod}(A_i)\geqslant\mu>0$. *If* $e_0(\Gamma)\geqslant e$ *then* $\operatorname{width}(\Gamma|A)\geqslant w(e)$, *where $w(e)\to\infty$ as $e\to\infty$.*

Proof. Assume that there is a sequence of annuli A^n and curves $\Gamma^m\subset A^m$ satisfying the assumptions of the lemma, such that $\operatorname{width}(\Gamma^m|A^m)\leqslant w$, while $e_0(\Gamma^m)\to\infty$. Let us consider the uniformization $\phi_m\colon \bar A^m\to A^m$ of the A^m by round annuli centered at 0. Let $\bar\Gamma^m=\phi_m^{-1}\Gamma^m$. Then $\bar\Gamma^m$ is contained in a round annulus $\bar R^m$ of modulus $\leqslant w$ concentric with $\bar A^m$. Let $R^m=\phi_m\bar R^m$.

Let us normalize the annuli A^m and $\bar A^m$ (by scaling and rotation) so that the inner radii of R^m and $\bar R^m$ are equal to 1, and $\phi_m(1)=1$. Passing to a subsequence (without change of notations) we can find concentric annuli $\bar R\subset\bar A$ such that the inner radius of $\bar R$ is equal to 1, $\operatorname{mod}(\bar R)\leqslant w$, both components of $\bar A\setminus\bar R$ have moduli at least $\alpha(\mu,w)$, and $\bar R^m\subset\bar R$, $\bar A^m\supset\bar A$.

By the Koebe Theorem, the family of functions ϕ_m is normal in A. Hence these functions are uniformly bounded on $\bar R$ contradicting the assumption that the eccentricities of Γ^m about 0 go to ∞. □

"Pinching" of a Jordan curve means creation of a narrow region which in limit makes the curve non-simple. Below we will quantify this process.

Let us take a number $0<k<1$ called the "pinching parameter". Let us define the *k-pinching* of a Jordan curve Γ as $\xi_k(\Gamma)=\inf \operatorname{dist}(z_1, z_2)$, where the infimum is taken over all pairs of points $z_i\in\Gamma$ such that both components Γ_1 and Γ_2 of $\Gamma\setminus\{z_1, z_2\}$ have diameter at least $k \operatorname{diam}\Gamma$.

We say that a curve Γ is (k,ε)*-pinched* if $\xi_k(\Gamma)<\varepsilon$. Note that if the curve is symmetric about 0 and $e_0(\Gamma)\geqslant e$, then it is $(0.5-e^{-1}, e^{-1})$-pinched.

The following lemma shows that a sufficiently pinched curve has a definite width:

LEMMA A.3. *Let Γ, A and A_i be the same objects as in Lemma* A.2. *Let also* $\operatorname{mod}(A_i)\geqslant\mu>0$. *If Γ is (k,ε)-pinched, then there exists a $w=w(\mu,k)>0$ such that* $\operatorname{width}(\Gamma|A)\geqslant w>0$ *for all sufficiently small $\varepsilon>0$.*

Proof. Otherwise we can find a sequence A^m of annuli as above with

$$\operatorname{width}(\Gamma^m|A^m)\to 0 \quad \text{as } m\to\infty,$$

and Γ^m is $(k, 1/m)$-pinched. As in the previous lemma, let $\phi_m\colon \bar{A}^m\to A^m$ be the uniformizations by round annuli normalized in such a way that Γ and $\bar{\Gamma}=\phi_m^{-1}\Gamma$ pass through 1.

Then the curves $\bar{\Gamma}^m$ should converge to the unit circle in the Hausdorff metric. Moreover, the family ϕ_m is well-defined and normal on an appropriate concentric annulus $\bar{A}$. Hence any Hausdorff limit of the family of curves Γ^m is an analytic Jordan curve. On the other hand, these curves should be $(k,0)$-pinched (that is, they are non-simple). Contradiction. ☐

LEMMA A.4. *Let Γ, A and A_i be the same objects as in Lemma* A.2, *and* $\operatorname{mod}(A_i)\geqslant$ $\mu>0$. *Let D be a topological disk bounded by Γ, and $b\in D$. Then there is a function $\delta(w,L)\to 0$ as $w\to 0$, $L\to\infty$ such that*

$$\operatorname{width}(\Gamma|A)<w \quad \textit{and} \quad \operatorname{dist}(b,\partial\Gamma)\leqslant\delta(w,L)\operatorname{diam}\Gamma,$$

provided $\varrho_D(0,b)\geqslant L$.

(Thus, if b is hyperbolically far away from 0 then it is close to ∂D in the Euclidean metric, in the scale of D.)

Proof. Otherwise there is a sequence of the curves $\Gamma^m=\partial D^m$ as above, and points $b^m\in D^m$ such that $\operatorname{diam} D^m=1$,

$$\operatorname{width}(\Gamma^m|A^m)\to 0, \quad \varrho_{D^m}(0,b^m)\to\infty, \tag{13.1}$$

and

$$\operatorname{dist}(b^m,\Gamma^m)\geqslant\delta. \tag{13.2}$$

Passing to a Carathéodory limit along some subsequence (without change of notations), we have $(D^m, 0, b^m) \to (D, 0, b)$, where D is a topological disk bounded by an analytic Jordan curve. But then $\varrho_{D^m}(0, b^m) \to \varrho_D(0, b) < \infty$, contradicting (13.1). □

LEMMA A.5. *Let Γ be a Jordan curve which does not pass through* 0. *If it is (k, ε)-pinched then its pull-back under the quadratic map $\Phi: z \mapsto z^2$ is $(C^{-1}k, C\sqrt{\varepsilon})$-pinched, where $C>1$ is an absolute constant.*

Proof. Let z_1 and z_2 be two points on Γ such that $\operatorname{dist}(z_1, z_2) < \varepsilon$, while $\operatorname{diam} \Gamma_i > k$, where Γ_i are complementary components of $\Gamma \setminus \{z_1, z_2\}$. Let us mark the Φ-preimages of the corresponding objects with tilde (select the closest preimages of the points z_i).

We can assume that $\operatorname{diam} \Gamma = \frac{1}{4}$. If $\operatorname{dist}(\Gamma, 0) > \frac{1}{4}$ then the distortion of the quadratic map on Γ is bounded by an absolute constant, and the conclusion follows. Otherwise Γ is contained in the unit disk. Hence $\frac{1}{2} \geqslant \operatorname{diam} \widetilde{\Gamma} \geqslant \frac{1}{8}$ and $\operatorname{diam} \widetilde{\Gamma}_i \geqslant \frac{1}{2} \operatorname{diam} \Gamma_i \geqslant \frac{1}{8} k$. Moreover, $\operatorname{dist}(\tilde{z}_1, \tilde{z}_2) = O(\sqrt{\varepsilon})$, and we are done. □

13.4. Quasi-conformal maps

There are a few Russian and English sources on quasi-conformal maps: [Ah], [Bel], [Kr], [LV], [Vo]. We assume that the reader is familiar with the basics of the theory. Below we will quote only a couple of facts especially important for this paper, or whose search in the literature may need some effort.

In what follows by a conformal structure we will mean a structure associated to a measurable Beltrami differential μ with $\|\mu\|_\infty < 1$. We will denote by σ the standard structure corresponding to zero Beltrami differential.

Two fundamental properties of qc maps exploited in this paper are:

COMPACTNESS LEMMA. *The space of K-qc maps $h: \mathbf{C} \to \mathbf{C}$ normalized by $h(0)=0$ and $h(1)=1$ is compact in the uniform topology on the Riemann sphere.*

GLUING LEMMA. *Let us have two disjoint domains D_1 and D_2 with a piecewise smooth arc γ of their common boundary. Let $D = D_1 \cup D_2 \cup \gamma$. If $h: D \to \mathbf{C}$ is a homeomorphism such that $h|D_i$ is K-qc, then h is K-qc.*

Note that the Gluing Lemma makes a difference between complex qc and real qs maps which is crucial for the pull-back argument.

Let D be a simply-connected domain conformally equivalent to the hyperbolic plane $\mathbf{H}^2$. Given a family of subsets $\{S_k\}_{k=1}^n$ in D, let us say that a family of disjoint annuli $A_k \subset D \setminus \bigcup S_i$ is *separating* if A_k surrounds S_k but does not surround the S_i, $i \neq k$. The following lemma is used in the present paper uncountably many times:

MOVING LEMMA. • *Let $a, b \in D$ be two points on hyperbolic distance $\varrho \leqslant \bar{\varrho}$. Then there is a diffeomorphism $\phi: (D, a) \to (D, b)$, identical near ∂D, with dilatation* $\mathrm{Dil}(\phi) = 1 + O(\varrho)$, *where the constant depends only on $\bar{\varrho}$.*

• *Let $\{(a_k, b_k)\}_{k=1}^n$ be a family of pairs of points which admits a family of separating annuli A_k with* $\mathrm{mod}(A_k) \geqslant \mu$. *Then there is a diffeomorphism $\phi: (D, a_1, \dots, a_n) \to (D, b_1, \dots, b_n)$, identical near ∂D, with dilatation* $\mathrm{Dil}(\phi) = 1 + O(e^{-\mu})$.

Proof. As the statement is conformally equivalent, we can work with the unit disk model of the hyperbolic plane, and can also assume that $a=0$, $b>0$. Also, it is enough to prove the statement for sufficiently small ϱ.

There is a smooth function $\psi: [0,1] \to [b,1]$ such that $\psi(x) \equiv b$ near 0, $\psi(x) \equiv 0$ near 1, and $\psi'(x) = O(\varrho)$, with a constant depending only on $\bar{\varrho}$.

Let us define a smooth map $\phi: (\mathbf{D}, 0) \to (\mathbf{D}, b)$ as $z \mapsto z + \psi(|z|)$. Then

$$\partial\phi(z) = 1 + \psi'(|z|)\frac{\bar{z}}{2|z|} = 1 + O(\varrho), \quad \bar{\partial}\phi(z) = \psi'(|z|)\frac{z}{2|z|} = O(\varrho). \tag{13.3}$$

Thus

$$\mathrm{Jac}(f) = |\partial\phi(z)|^2 - |\bar{\partial}\phi(z)|^2 = 1 + O(\varrho).$$

Hence for sufficiently small $\varrho > 0$, f is a local orientation preserving diffeomorphism. As $f: \partial\mathbf{D} \to \partial\mathbf{D}$, f is a proper map. Hence it is a diffeomorphism.

Finally, (13.3) yields that the Beltrami coefficient $\mu_f = O(\varrho)$, so that the dilatation $\mathrm{Dil}(f) = 1 + O(\varrho)$. □

Let $Q \subset \mathbf{C}$, $h: Q \to \mathbf{C}$ be a homeomorphism onto its image. It is called *quasi-symmetric* (qs) if for any three points $a, b, c \in Q$ such that $q^{-1} \leqslant |a-b|/|b-c| \leqslant q$, we have: $\varkappa(q)^{-1} \leqslant |h(a)-h(b)|/|h(b)-h(c)| \leqslant \varkappa(q)$. It is called $\varkappa$-quasi-symmetric if $\varkappa(1) \leqslant \varkappa$. It follows from the Compactness Lemma that any K-qc map is $\varkappa$-quasi-symmetric, with a $\varkappa$ depending only on K.

13.5. Removability

A compact set $X \subset \mathbf{C}$ is called *removable* if for any neighborhood $U \supset X$, any conformal map $h: U \setminus X \to \mathbf{C}$ admits a conformal extension across X.

Remark. A more common notion of removability includes the assumption that the map h is continuous accross X. The above stronger notion better suits our purposes.

Let us show that removability is quasi-conformally invariant.

LEMMA A.6. *Let $\phi\colon(\mathbf{C},X)\to(\mathbf{C},\widetilde{X})$ be a qc map. If the set X is removable then $\widetilde{X}$ is removable as well.*

Proof. Let σ be the standard conformal structure on $\mathbf{C}$. Let $\widetilde{U}\supset\widetilde{X}$ be a neighborhood of $\widetilde{X}$, and let $\tilde{h}\colon\widetilde{U}\setminus\widetilde{X}\to\mathbf{C}$ be a conformal map. Let us consider a conformal structure $\tilde{\mu}$ on $\mathbf{C}$ which is equal to $(\tilde{h}\circ\phi)_*(\sigma)$ on $\tilde{h}(\widetilde{U}\setminus\widetilde{X})$, and is equal to σ outside. By the Measurable Riemann Mapping Theorem, there is a qc map $\psi\colon\mathbf{C}\to\mathbf{C}$ such that $\tilde{\mu}=\psi_*(\sigma)$.

Let $U=\phi^{-1}\widetilde{U}$. Then the function $h=\psi^{-1}\circ\tilde{h}\circ\phi\colon U\setminus X\to\mathbf{C}$ is conformal. As X is removable, it admits a conformal extension across X. We will use the same notation h for the extended function. Then the formula $\tilde{h}=\psi\circ h\circ\phi^{-1}$ provides us with a conformal extension of $\tilde{h}$ across $\widetilde{X}$. □

Let us now show that removable sets are also qc-removable.

LEMMA A.7. *Let X be a removable set and $U\supset X$ be its neighborhood. Then any qc map h on $U\setminus X$ admits a qc extension across X.*

Proof. Let us consider a conformal structure μ equal to $h^*(\sigma)$ on $U\setminus X$, and equal to σ on the rest of $\mathbf{C}$. By the Measurable Riemann Mapping Theorem, there exists a qc map $\phi\colon\mathbf{C}\to\mathbf{C}$ such that $\mu=\phi^*(\sigma)$ (the solution of the Beltrami equation). Then the function $\tilde{h}=h\circ\phi^{-1}$ is univalent on $\widetilde{U}\setminus\widetilde{X}\equiv\phi U\setminus\phi X$.

By Lemma A.6, the set $\widetilde{X}$ is removable. Hence $\tilde{h}$ admits a conformal extension across $\widetilde{X}$. Then the formula $h=\tilde{h}\circ\phi$ provides us with a qc extension of h across X. □

Let us finally state a useful criterion for removability (see, e.g., [SaN]):

REMOVABILITY CRITERION. *Let X be a Cantor set satisfying the following property. There is an $\eta>0$ such that for any point $z\in X$ there is a nest of disjoint annuli $A_i(z)\subset\mathbf{C}\setminus X$ surrounding z with $\operatorname{mod}(A_i(z))\geqslant\eta$. Then X is removable.*

14. Appendix B: Reference notes

— For recent surveys on renormalization, rigidity, puzzle and related topics see [L6], [L7], [Mc2], [Mc4], [Sh2].

— The first applications of the puzzle which appeared after Yoccoz' work were concerned with the problem of the Lebesgue measure of the quadratic Julia sets (Lyubich [L2], [L3] and Shishikura [Sh2]), and to its real counterpart, Milnor's problem of attractors [LM], [L3], [L4].

— The results and main ideas of this paper were first presented at the Warwick and Durham Workshops on Hyperbolic Geometry and Holomorphic Dynamics in June–July

1993 (see the preprint IMS at Stony Brook, 1993/9). Since then they have been presented at many different meetings around the world including the ICM-94 in Zürich [L6]. A more detailed account of the proof appeared in the preprint MSRI, 026-95 and the preprint IMS at Stony Brook, 1995/14.

The only essential modification of the 1993 paper is that the Rigidity Theorem is now stated under the assumption of a priori bounds, while earlier it was stated under the assumption of big type yielding a priori bounds (see Corollary 1.1). In this formulation, the proof requires an adjustment of the fundamental annuli, which is done in §9. As the complex bounds have recently become available for all real maps, the new version is directly applied to the real maps (in the early version essentially bounded and high combinatorics were treated in a different manner, see [L7]).

— The main geometric result of this paper, Theorem III, §6, is a complex analogue of Theorem II of [L4] on the exponential decay of the scaling factors of S-unimodal maps. The latter result, in turn, extends to arbitrary combinatorics the corresponding theorem for real Fibonacci maps, obtained jointly with Milnor [LM]. This development began in the fall of 1990 with Milnor's computer experiment which showed the exponential decay of the scaling factors for the Fibonacci map.

The statement of Theorem II (§5 of this paper) is analogous to Martens' Theorem 8.1 in real dynamics [Mar]. However, the proofs of these results are totally different (Martens exploits the "minimal interval argument" which is at present not available in the complex dynamics). It is crucial for our proof of Theorem II (and Theorem III as well) to have a geometric parameter which behaves monotonically under generalized renormalization. Such a parameter, "the asymmetric Poincaré length", was defined in [L4] for real unimodal maps. Its complex counterpart was suggested, for the Fibonacci combinatorics, by Jeremy Kahn (IHES, April 1992). The general definition of the asymmetric modulus involving admissible families and isles was given by the author.

— The first advance in the problem of a priori bounds was achieved in the work of Sullivan (see [MS], [S2]) where it was resolved for real quadratics of bounded type. The idea to use hyperbolic neighborhoods in the slit planes came from that work.

The gap between that result and our Theorem V of §8.2 consists of maps with essentially bounded but unbounded combinatorics. In a joint work with Yampolsky [LY] we gave an appropriate modification of Sullivan's method to cover essentially bounded combinatorics as well. This yields the Complex Bounds Theorem.

This result was independently proven by Levin and van Strien [LvS]. The method of [LvS] is quite different: it does not need a detailed combinatorial analysis but rather needs specific numerical estimates for real geometry. It does not tackle the phenomenon of big space.

Also, the gap between [S2] and Theorem V was independently filled by Graczyk and Świątek [GS2]. The method of the latter work is specifically adopted to essentially bounded but *unbounded* combinatorics.

The creation of a generalized quadratic-like map by pulling back the Euclidean disc (see §8) is crucial for our treatment of real dynamics. The key estimate, for the Fibonacci combinatorics, appeared in [LM, Lemma 8.2]. It was extended in [L4, §4] onto arbitrary combinatorics of sufficiently big type.

Theorems IV and IV′ of §7 are the first advances in the problem of a priori bounds outside the real line (compare Rees [R]).

— Local connectivity is a nice quality of a Julia set, since such a set admits an explicit topological model (see Douady [D2]). By now there is a large pool of quadratics whose Julia sets are known to be locally connected:

- Parabolic and Misiurewicz points (Douady–Hubbard [DH1]);
- At most finitely renormalizable points (Yoccoz, see [Hu], [Mi2]);
- Diophantine Siegel disks (Petersen [Pe], see also Yampolsky [Yam]);
- Real infinitely renormalizable maps of bounded type (Hu–Jiang [HJ]);
- Maps of class $\mathcal{SL}$ of sufficiently big type; in particular, real maps with essentially big combinatorics (this work);
- All real maps (Levin–van Strien [LvS], Lyubich–Yampolsky [LY]).

However, there are also counter-examples: Cremer and some infinitely renormalizable quadratics have non-locally connected Julia sets (see [L1], [Mi2]).

— Prior to this work the MLC was established in the following cases:

- Parabolic points (Douady–Hubbard [DH1]);
- Boundaries of the hyperbolic components (Yoccoz, see Hubbard [Hu]);
- At most finitely renormalizable maps (Yoccoz, see Hubbard [Hu]). Then Kahn [Ka] proved a stronger result that the Julia set of a quadratic map under consideration is removable.

Note that it is easy to construct some infinitely renormalizable parameter values of unbounded type where MLC holds (oral communications by A. Douady and J.-C. Yoccoz, Durham 93). For example, first find arbitrary small copies M_n of the Mandelbrot set near $c{=}-2$. Then for an appropriate subsequence $n(k)$, the tuned Mandelbrot copies $M_{n(1)} * M_{n(2)} * \ldots * M_{n(l)}$ shrink to a single point.

Let us also note that the MLC problem is closely related to the problem of landing of parameter rays at points $c \in \partial M$. MLC certainly yields landing of all rays, but, on the other hand, landing of some special rays has been a basis for progress in the MLC problem. The first results in this direction (landing at parabolic and Misiurewicz points) were obtained by Douady and Hubbard (see [DH1], [Mi4], [Sc2]). Recently Manning [Man] has estimated the Hausdorff dimension of the set of rays landing at infinitely renormalizable points.

— The origin of our approach to the rigidity problem can be traced back to the proof of Mostow rigidity: from topological to quasi-conformal equivalence, and then (by means of ergodic theory) to conformal equivalence. This set of ideas was brought to the iteration theory by Sullivan and Thurston.

The passage from quasi-conformal to conformal equivalence in our setting is settled by McMullen's Rigidity Theorem [Mc2]. Our main task was to pass from topological to quasi-conformal equivalence. A way to do this, called "pull-back argument", is to start with a quasi-conformal map respecting some dynamical data, and to pull it back so that it will respect more and more data on every step. At the end it will become (with some luck) a quasi-conformal conjugacy. This method was introduced by Thurston (see [DH3] and also [Mc1]) for postcritically finite maps, and exploited by Sullivan ([S2], [MS]) for real infinitely renormalizable maps of bounded type. These first applications dealt with maps with rather simple combinatorics.

For more complicated combinatorics, a certain real version of this method based on the so called "inducing" was suggested by Jacobson and Świątek [JS]. (Roughly speaking, "inducing" means building out of f an expanding map with a definite range.) On the other hand, by means of a purely complex pull-back argument in the puzzle framework, Jeremy Kahn [Ka] proved his removability result.

Our way is different from all the above, though it has some common features with them. We believe that holomorphic dynamics is the right framework for the rigidity problem, and our method is purely complex. Rather than building an induced expanding map, we pass consecutively from bigger to smaller scales by means of generalized renormalization, and carry out the pull-back using growth of *complex* moduli and *complex a priori* bounds.

Let us note that there is a different approach to rigidity problems, by comparing the dynamical and parameter planes. This method was used by Branner and Hubbard [BH] and Yoccoz to prove their rigidity results. In this spirit, we can give an alternative proof of Corollary 1.1. Namely, in [L8] we transfer the geometric result of Theorem III to the parameter plane. This yields disjoint definite collars around little Mandelbrot copies containing a point c satisfying the assumptions of Corollary 1.1. Hence these copies shrink to c.

— Real rigidity of infinitely renormalizable maps of bounded type was proven by Sullivan (see [S2], [MS], [L7]). The proof is based upon real a priori bounds and the pull-back argument.

Density of hyperbolic maps in the real quadratic family was first announced by

Świątek (see the preprint IMS at Stony Brook, 1992/10). That work did not have results on the moduli growth and complex a priori bounds, which play a key role in our argument.

A later work of Graczyk and Świątek [GS1] announced at the Durham Workshop (July 1993) already contained a result on the growth of moduli (within a certain nest of domains different from the principal nest, for *real* quadratics). However, it was not incorporated in [GS1] into the pull-back argument. Note also that both the combinatorial and analytic parts of [GS1] essentially rely on the real line.

— The combinatorial analysis of §3 based on the nest of the first return maps is analogous to the corresponding study of S-unimodal maps (compare Lyubich [L4], Martens [Mar]). The return graph was suggested by Martens (in the abstract theory of minimal dynamical systems such a graph is known under the name "Bratteli–Vershik", see [Ve], [HPS]). The generalized renormalization was first introduced and exploited in [LM], [L2].

— In this paper we have concentrated on the quadratic case. The higher degree unimodal case $z \mapsto z^d + c$ has some essentially different features. The basic difference is that the geometric Theorem III is not true any more for $d>2$ (as the Fibonacci examples show). However, there are many ingredients which are still valid (see the remarks throughout the paper). In particular, our proof of Theorem V (and hence the Complex Bounds Theorem) comes through for $d>2$ as well (see the remark at the end of §8). The author has noticed this after he received the paper [LvS], where the Complex Bounds Theorem was proven for all degrees in an essentially different manner.

A geometric and measure-theoretic analysis of higher degree Fibonacci maps has been carried out by Bruin, Keller, Nowicki, van Strien [KN], [BKNS], [SN] and Buff [Bu]. The goal of these works is to prove existence of "wild" attractors and Julia sets of positive measure (in drastic contrast with the corresponding quadratic results discussed above). Numerically these phenomena had been studied by the author jointly with F. Tangerman and S. Sutherland (see [L6]).

— For further applications of the results of this paper see [L8], [L9], [L10], [MN], [Pra1], [Pra2], [Prz], [W], [Yar].

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MIKHAIL LYUBICH
Department of Mathematics
SUNY
Stony Brook, NY 11794-3660
U.S.A.
mlyubich@math.sunysb.edu

Received March 2, 1995
Received in revised form November 24, 1995