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Dynamics of Recursive Sequence of Order Two

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Abstract. In this paper we study some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, ...,$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants with $cx_0 - dx_{-1} \neq 0$.

1. Introduction

In this paper we deal with some properties of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants with $cx_0 - dx_{-1} \neq 0$.

Recently there has been a lot of interest in studying the global attractivity, boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [21-41].

Many researchers have investigated the behavior of the solution of difference equations for example:

In [5] Elabbasy et al. investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}.$$

Elabbasy et al. [6] investigated the global stability, boundedness, periodicity char-

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acter and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [7] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Aloqeili [1] obtained the form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Simsek et al. [32] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Other related results on rational difference equations can be found in refs. [2-20].

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and the function f has continuous partial derivatives on I^{k+1} where $I^{k+1} = I \times I \times ... \times I$ $(k+1-$ times). Then, for initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1(Stability)**.**
(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now assume that the characteristic equation associated with Eq.(3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{4}$$

where $p_i = \dfrac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}$.

Theorem A([28]). *Assume that* $p_i \in R$, $i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1, \tag{5}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \quad n = 0, 1, ...$$

Consider the following equation

$$x_{n+1} = f(x_n, x_{n-1}). \tag{6}$$

The following theorems will be useful for the proof of our main results in this paper.

Theorem B([27]). *Let* $f : [a, b]^2 \to [a, b]$ *be a continuous function, where* a *and* b *are real numbers with* $a < b$. *Suppose that* f *satisfies the following conditions:*

(*a*) $f(x, y)$ *is non-decreasing in* $x \in [a, b]$ *for each fixed* $y \in [a, b]$, *and is non-decreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$.

(*b*) *If* (m, M) *is a solution of the system*

$$m = f(m, m) \quad \textit{and} \quad M = f(M, M),$$

then

$$m = M.$$

Then there exists exactly one equilibrium $\overline{x}$ of Eq. (6), *and every solution of Eq.(6) converges to* $\overline{x}$.

Theorem C([27]). *Let* $f : [a,b]^2 \to [a,b]$ *be a continuous function, where* a *and* b *are real numbers with* $a < b$. *Suppose that* f *satisfies the following conditions:*

(a) $f(x,y)$ *is non-increasing in* $x \in [a,b]$ *for each fixed* $y \in [a,b]$, *and is non-decreasing in* $y \in [a,b]$ *for each* $x \in [a,b]$.

(b) *If* (m,M) *is a solution of the system*

$$m = f(M,m) \quad \textit{and} \quad M = f(m,M),$$

then

$$m = M.$$

Then there exists exactly one equilibrium $\overline{x}$ of Eq. (6), *and every solution of Eq.(6) converges to* $\overline{x}$.

2. Periodic solutions

In this section we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions.

Theorem 1. *Eq.*(1) *has positive prime period two solutions if and only if*

$$(c+d)(a+1) > 4d, \quad d \neq ac. \tag{7}$$

Proof. First suppose that there exists a prime period two solution

$$..., p, q, p, q, ...,$$

of Eq.(1). We will prove that Condition (7) holds.

We see from Eq.(1) that

$$p = aq + \frac{bq}{cq - dp},$$

and

$$q = ap + \frac{bp}{cp - dq}.$$

Then

$$cpq - dp^2 = acq^2 - adpq + bq, \tag{8}$$

and

$$(9) \qquad cpq - dq^2 = acp^2 - adpq + bp.$$

Subtracting (9) from (8) gives

$$d(q^2 - p^2) = ac(q^2 - p^2) + b(q - p).$$

Since $p \neq q$, it follows that

$$(10) \qquad p + q = \frac{b}{d - ac}.$$

Again, adding (8) and (9) yields

$$(11) \qquad 2cpq - d(p^2 + q^2) = ac(p^2 + q^2) - 2adpq + b(p + q).$$

It follows by (10), (11) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \text{for all } p, q \in R,$$

that

$$(12) \qquad pq = \frac{b^2 d}{(d - ac)^2 (c + d)(a + 1)}.$$

Now it is clear from Eq.(10) and Eq.(12) that p and q are the two positive distinct roots of the quadratic equation

$$(13) \qquad (d - ac)t^2 - bt + \frac{b^2 d}{(d - ac)(c + d)(a + 1)} = 0.$$

and so

$$b^2 > \frac{4b^2 d}{(c + d)(a + 1)}.$$

Therefore inequality (7) holds.

Second suppose that inequality (7) is true. We will show that Eq.(1) has a prime period two solution.

Assume that

$$p = \frac{b + \alpha}{2(d - ac)},$$

and

$$q = \frac{b - \alpha}{2(d - ac)},$$

where $\alpha = \sqrt{b^2 - \dfrac{4b^2 d}{(c + d)(a + 1)}}$.

From inequality (7) it follows that α is a real positive numbers, therefore p and q are distinct positive real numbers.

Set

$$x_{-1} = p \text{ and } x_0 = q.$$

We show that $x_1 = x_{-1} = p$ and $x_2 = x_0 = q$.

It follows from Eq.(1) that

$$x_1 = aq + \frac{bq}{cq - dp} = \frac{acq^2 - adpq + bq}{cq - dp}$$

$$= \frac{ac\left[\frac{b-\alpha}{2(d-ac)}\right]^2 - ad\left[\frac{b^2 d}{(d-ac)^2(c+d)(a+1)}\right] + b\left[\frac{b-\alpha}{2(d-ac)}\right]}{c\left[\frac{b-\alpha}{2(d-ac)}\right] - d\left[\frac{b+\alpha}{2(d-ac)}\right]}.$$

Multiplying the denominator and numerator by $4(d-ac)^2$

$$x_1 = \frac{2b^2 d - \frac{4ab^2cd + 4ab^2d^2}{(c+d)(a+1)} - 2bd\alpha}{2(d-ac)\left\{cb - bd - (c+d)\alpha\right\}}.$$

Multiplying the denominator and numerator by $\{cb - bd + (c+d)\alpha\}\{(c+d)(a+1)\}$ we get

$$x_1 = \frac{\left[\left(4b^3d^3 + 4b^3cd^2 - 4ab^3c^2d - 4ab^3cd^2\right) + \left(4b^2cd^2 + 4b^2d^3 - 4ab^2c^2d - 4ab^2cd^2\right)\alpha\right]}{2(d-ac)\left\{4b^2cd^2 + 4b^2d^3 - 4ab^2c^2d - 4ab^2cd^2\right\}}.$$

Dividing the denominator and numerator by $\left\{4b^2cd^2 + 4b^2d^3 - 4ab^2c^2d - 4ab^2cd^2\right\}$ gives

$$x_1 = \frac{b+\alpha}{2(d-ac)} = p.$$

Similarly as before one can easily show that

$$x_2 = q.$$

Then it follows by induction that

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$\ldots, p, q, p, q, \ldots,$$

where p and q are the distinct roots of the quadratic equation (13) and the proof is complete. □

3. Local stability of the equilibrium point

In this section we study the local stability character of the solutions of Eq.(1). The equilibrium points of Eq.(1) are given by the relation

$$\overline{x} = a\overline{x} + \frac{b\overline{x}}{c\overline{x} - d\overline{x}}.$$

If $(c-d)(1-a) > 0$, then the only positive equilibrium point of Eq.(1) is given by

$$\overline{x} = \frac{b}{(c-d)(1-a)}.$$

Let $g : (0,\infty)^2 \longrightarrow (0,\infty)$ be a function defined by

$$g(u,v) = au + \frac{bu}{cu - dv}. \tag{14}$$

Therefore

$$\begin{aligned} \frac{\partial g(u,v)}{\partial u} &= a - \frac{bdv}{(cu-dv)^2}, \\ \frac{\partial g(u,v)}{\partial v} &= \frac{bdu}{(cu-dv)^2}. \end{aligned}$$

Then we see that

$$\begin{aligned} \frac{\partial g(\overline{x},\overline{x})}{\partial u} &= a - \frac{d(1-a)}{(c-d)} = p_0, \\ \frac{\partial g(\overline{x},\overline{x})}{\partial v} &= \frac{d(1-a)}{(c-d)} = p_1. \end{aligned}$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} - p_0 y_{n-1} - p_1 y_n = 0. \tag{15}$$

Theorem 2. *Assume that*

$$|ac - d| + d\,|1-a| < |c-d|\,.$$

Then the equilibrium point of Eq.(1) is locally asymptotically stable.
Proof. It is follows by Theorem A that, Eq.(1) is asymptotically stable if

$$|p_1| + |p_0| < 1.$$

That is

$$\left| a - \frac{d(1-a)}{(c-d)} \right| + \left| \frac{d(1-a)}{(c-d)} \right| < 1,$$

then

$$|a(c-d)-d(1-a)|+|d(1-a)|<|c-d|.$$

Thus

$$|ac-d|+d\,|1-a|<|c-d|.$$

The proof is complete. □

4. Global attractor of the equilibrium point of Eq.(1)

In this section we investigate the global attractivity character of solutions of Eq.(1).

Theorem 3. *The equilibrium point $\overline{x}$ of Eq.(1) is a global attractor if $ac>d$.*

Proof. Let p,q are a real numbers and assume that $g:[p,q]^2\longrightarrow[p,q]$ be a function defined by Eq.(14).Therefore

$$\begin{aligned}\frac{\partial g(u,v)}{\partial u} &= a-\frac{bdv}{(cu-dv)^2},\\ \frac{\partial g(u,v)}{\partial v} &= \frac{bdu}{(cu-dv)^2}.\end{aligned}$$

Case (1) If $a-\dfrac{bdv}{(cu-dv)^2}>0$, then we can easily see that the function $g(u,v)$ increasing in u,v.

Suppose that (m,M) is a solution of the system

$$m=g(m,m) \quad and \quad M=g(M,M).$$

Then from Eq.(1), we see that

$$m=am+\frac{bm}{cm-dm},\qquad M=aM+\frac{bM}{cM-dM},$$

then

$$(M-m)=a(M-m),\qquad a\neq 1.$$

Thus

$$M=m.$$

It follows by the Theorem B that $\overline{x}$ is a global attractor of Eq.(1).

Case (2) If $a-\dfrac{bdv}{(cu-dv)^2}<0$, then we can easily see that the function $g(u,v)$ decreasing in u and increasing in v.

Suppose that (m,M) is a solution of the system

$$M=g(m,M) \quad and \quad m=g(M,m).$$

Then from Eq.(1), we see that

$$m = aM + \frac{bM}{cM - dm}, \quad M = am + \frac{bm}{cm - dM},$$

$$cMm - acM^2 - dm^2 + adMm = bM, \quad cMm - acm^2 - dM^2 + adMm = bm,$$

then

$$(M^2 - m^2)(d - ac) = b(M - m), \quad ac > d.$$

Thus

$$M = m.$$

It follows by the Theorem C that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete. □

5. Special case of Eq.(1)

In this section we study the following special case of Eq.(1)

$$x_{n+1} = x_n + \frac{x_n}{x_n - x_{n-1}}, \tag{16}$$

where the initial conditions $x_{-1},\ x_0$ are arbitrary real numbers with $x_{-1},\ x_0 \in R/\{0\}$ and $x_{-1} \neq x_0$.

5.1. The solution form of Eq.(16)

In this section we give a specific form of the solutions of Eq.(16).

Ttheorem 4. *Let $\{x_n\}_{n=-1}^{\infty}$ be the solution of Eq.(16) satisfying $x_{-1} = k,\ x_0 = h$, with $k \neq h,\ k, h \in R/\{0\}$. Then for $n = 0, 1, ...$*

$$\begin{aligned} x_{2n-1} &= k + n\left(h - k + (n-1) + \frac{h}{h-k}\right), \\ x_{2n} &= h + n\left(h - k + n + \frac{h}{h-k}\right). \end{aligned}$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned} x_{2n-3} &= k + (n-1)\left(h - k + (n-2) + \frac{h}{h-k}\right), \\ x_{2n-2} &= h + (n-1)\left(h - k + (n-1) + \frac{h}{h-k}\right). \end{aligned}$$

Now, it follows from Eq.(16) that

$$x_{2n-1} = x_{2n-2} + \frac{x_{2n-2}}{x_{2n-2} - x_{2n-3}} = h + (n-1)\left(h - k + (n-1) + \frac{h}{h-k}\right)$$

$$+\frac{h+(n-1)\left(h-k+(n-1)+\frac{h}{h-k}\right)}{\left(h+(n-1)\left(h-k+(n-1)+\frac{h}{h-k}\right)\right)-\left(k+(n-1)\left(h-k+(n-2)+\frac{h}{h-k}\right)\right)}$$

$$=h+(n-1)\left(h-k+(n-1)+\frac{h}{h-k}\right)+\frac{h+(n-1)\left(h-k+(n-1)+\frac{h}{h-k}\right)}{h-k+(n-1)}.$$

Multiplying the denominator and numerator by $(h-k)$ we get

$$\begin{aligned} x_{2n-1} &= k+(n-1)\left(h-k+(n-1)+\frac{h}{h-k}\right)+\frac{(h+(n-1)(h-k))}{(h-k)}+(h-k) \\ &= k+(n-1)\left(h-k+(n-1)+\frac{h}{h-k}\right)+(h-k)+(n-1)+\frac{h}{(h-k)}, \end{aligned}$$

then we have

$$x_{2n-1}=k+n\left(h-k+(n-1)+\frac{h}{h-k}\right).$$

Also, we get from Eq.(16)

$$\begin{aligned} x_{2n} &= x_{2n-1}+\frac{x_{2n-1}}{x_{2n-1}-x_{2n-2}} \\ &= k+n\left(h-k+(n-1)+\frac{h}{h-k}\right)+\frac{k+n\left(h-k+(n-1)+\frac{h}{h-k}\right)}{(n-1)+\frac{h}{(h-k)}}. \end{aligned}$$

Multiplying the denominator and numerator by $(h-k)$ we get

$$\begin{aligned} x_{2n} &= k+n\left(h-k+(n-1)+\frac{h}{h-k}\right) \\ &\quad +\frac{\left(k(h-k)+n(h-k)^2\right)+(n(n-1)(h-k)+nh)}{(n-1)(h-k)+h}, \end{aligned}$$

or

$$x_{2n}=k+n\left(h-k+(n-1)+\frac{h}{h-k}\right)+(h-k+n).$$

Thus we obtain

$$x_{2n}=h+n\left(h-k+n+\frac{h}{h-k}\right).$$

Hence, the proof is completed. □

Remark 2. It is easy to see that every solution of Eq.(16) is unbounded.

Numerical examples

Here we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. Consider $x_{-1} = 29,\ x_0 = 8$ and $a = b = c = d = 1$. See Fig. 1.

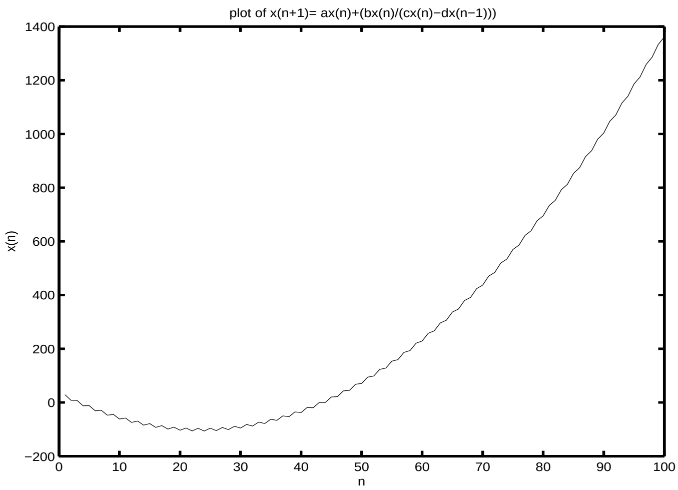

Fig. 1

Example 2. See Fig. 2, since $x_{-1} = 9,\ x_0 = 6$ and $a = 0.8,\ b = 3,\ c = 5,\ d = 2$.

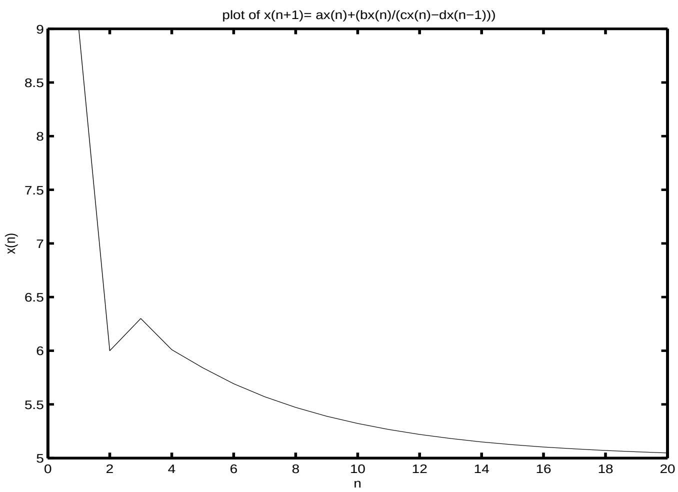

Fig. 2

Example 3. See Fig. 3, since $x_{-1} = 7,\ x_0 = 9$ and $a = 0.5,\ b = 13,\ c = 6,\ d = 3$.

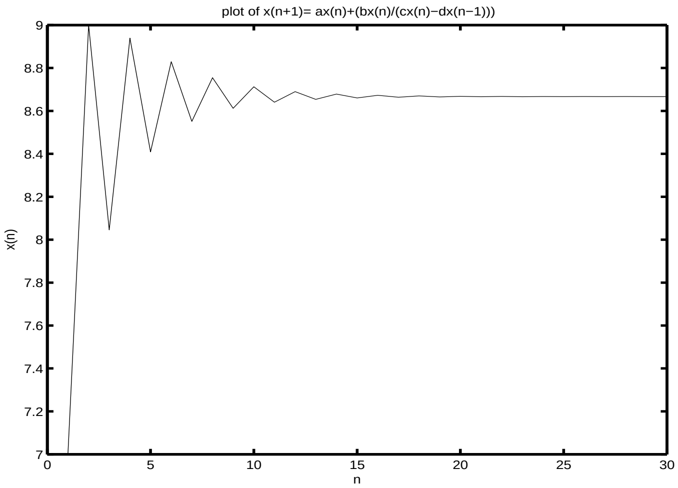

Fig. 3

Example 4. See Fig. 4, since $x_{-1} = 1.928 \left(= p = \dfrac{b+\alpha}{2(d-ac)}\right),\ x_0 = 1.071 \left(= q = \dfrac{b-\alpha}{2(d-ac)}\right)$ and $a = 2,\ b = 6,\ c = 4,\ d = 10$.

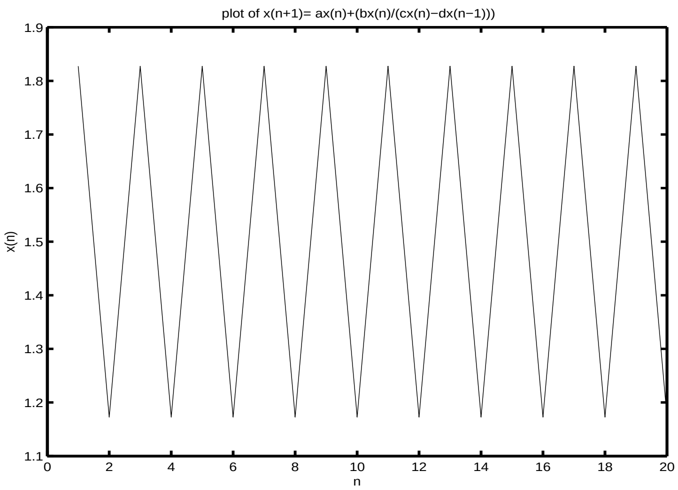

Fig. 4

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