

Advances in

Discrete Mathematics and Applications

Volume 5

**Dynamics of
Third-Order
Rational Difference
Equations with
Open Problems
and Conjectures**

**Elias Camouzis
Gerasimos Ladas**



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