Dynamics of SL2(#) Over Moduli Space in Genus Two

Dynamics of $\mathrm{SL}_2(\mathbb{R})$ over moduli space in genus two

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Abstract

This paper classifies orbit closures and invariant measures for the natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{M}_2$, the bundle of holomorphic 1-forms over the moduli space of Riemann surfaces of genus two.

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1 Introduction

Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces of genus g. By Teichmüller theory, every holomorphic 1-form $\omega(z)\,dz$ on a surface $X \in \mathcal{M}_g$ generates a *complex geodesic* $f : \mathbb{H}^2 \to \mathcal{M}_g$, isometrically immersed for the Teichmüller metric.

In this paper we will show:

Theorem 1.1 *Let $f : \mathbb{H}^2 \to \mathcal{M}_2$ be a complex geodesic generated by a holomorphic 1-form. Then $\overline{f(\mathbb{H}^2)}$ is either an isometrically immersed algebraic curve, a Hilbert modular surface, or the full space $\mathcal{M}_2$.*

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In particular, $\overline{f(\mathbb{H}^2)}$ is always an algebraic subvariety of $\mathcal{M}_2$.

Raghunathan's conjectures. For comparison, consider a finite volume hyperbolic manifold M in place of $\mathcal{M}_g$.

While the closure of a geodesic line in M can be rather wild, the closure of a geodesic plane

$$f : \mathbb{H}^2 \to M = \mathbb{H}^n/\Gamma$$

is always an immersed submanifold. Indeed, the image of f can be lifted to an orbit of $U = \mathrm{SL}_2(\mathbb{R})$ on the frame bundle $FM \cong G/\Gamma$, $G = \mathrm{SO}(n,1)$. Raghunathan's conjectures, proved by Ratner, then imply that

$$\overline{Ux} = Hx \subset G/\Gamma$$

for some closed subgroup $H \subset G$ meeting $x\Gamma x^{-1}$ in a lattice. Projecting back to M one finds that $\overline{f(\mathbb{H}^2)} \subset M$ is an immersed hyperbolic k-manifold with $2 \le k \le n$ [Sh].

The study of complex geodesics in $\mathcal{M}_g$ is similarly related to the dynamics of $\mathrm{SL}_2(\mathbb{R})$ on the bundle of holomorphic 1-forms $\Omega\mathcal{M}_g \to \mathcal{M}_g$.

A point $(X,\omega) \in \Omega\mathcal{M}_g$ consists of a compact Riemann surface of genus g equipped with a holomorphic 1-form $\omega \in \Omega(X)$. The Teichmüller geodesic flow, coupled with the rotations $\omega \mapsto e^{i\theta}\omega$, generates an action of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{M}_g$. This action preserves the subspace $\Omega_1\mathcal{M}_g$ of *unit* forms, those satisfying $\int_X |\omega|^2 = 1$.

The complex geodesic generated by $(X,\omega) \in \Omega_1\mathcal{M}_g$ is simply the projection to $\mathcal{M}_g$ of its $\mathrm{SL}_2(\mathbb{R})$-orbit. Our main result is a refinement of Theorem 1.1 which classifies these orbits for genus two.

Theorem 1.2 *Let $Z = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot (X,\omega)}$ be an orbit closure in $\Omega_1\mathcal{M}_2$. Then exactly one of the following holds:*

1. *The stabilizer $\mathrm{SL}(X,\omega)$ of (X,ω) is a lattice, we have*

$$Z = \mathrm{SL}_2(\mathbb{R}) \cdot (X,\omega),$$

and the projection of Z to moduli space is an isometrically immersed Teichmüller curve $V \subset \mathcal{M}_2$.

2. *The Jacobian of X admits real multiplication by a quadratic order of discriminant D, with ω as an eigenform, but $\mathrm{SL}(X,\omega)$ is not a lattice. Then*

$$Z = \Omega_1 E_D$$

coincides with the space of all eigenforms of discriminant D, and its projection to $\mathcal{M}_2$ is a Hilbert modular surface.

3. *The form ω has a double zero, but is not an eigenform for real multiplication. Then*

$$Z = \Omega_1\mathcal{M}_2(2)$$

coincides with the stratum of all forms with double zeros. It projects surjectively to $\mathcal{M}_2$.

4. *The form ω has simple zeros, but is not an eigenform for real multiplication. Then its orbit is dense: we have $Z = \Omega_1\mathcal{M}_2$.*

We note that in case (1) above, ω is also an eigenform (cf. Corollary 5.9).

Corollary 1.3 *The complex geodesic generated by (X, ω) is dense in $\mathcal{M}_2$ iff (X, ω) is not an eigenform for real multiplication.*

Corollary 1.4 *Every orbit closure $\overline{\mathrm{GL}_2^+(\mathbb{R}) \cdot (X, \omega)} \subset \Omega\mathcal{M}_2$ is a complex orbifold, locally defined by linear equations in period coordinates.*

Invariant measures. In the setting of Lie groups and homogeneous spaces, it is also known that every U-invariant measure on G/Γ is algebraic (see §2). Similarly, in §§10—12 we show:

Theorem 1.5 *Each orbit closure Z carries a unique ergodic, $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure μ_Z of full support, and these are all the ergodic probability measures on $\Omega_1\mathcal{M}_2$.*

In terms of local coordinates given by the relative periods of ω, the measure μ_Z is simply Euclidean measure restricted to the 'unit sphere' defined by $\int |\omega|^2 = 1$ (see §3, §8).

Pseudo-Anosov mappings. The classification of orbit closures also sheds light on the topology of complexified loops in $\mathcal{M}_2$.

Let $\phi \in \mathrm{Mod}_2 \cong \pi_1(\mathcal{M}_2)$ be a pseudo-Anosov element of the mapping class group of a surface of genus two. Then there is a real Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{M}_g$ whose image is a closed loop representing $[\phi]$. Complexifying γ, we obtain a totally geodesic immersion

$$f : \mathbb{H}^2 \to \mathcal{M}_g$$

satisfying $\gamma(s) = f(ie^{2s})$. The map f descends to the Riemann surface

$$V_\phi = \mathbb{H}^2/\Gamma_\phi, \qquad \Gamma_\phi = \{A \in \mathrm{Aut}(\mathbb{H}^2) \ : \ f(Az) = f(z)\}.$$

Theorem 1.6 *For any pseudo-Anosov element $\phi \in \pi_1(\mathcal{M}_2)$ with orientable foliations, either*

1. *Γ_ϕ is a lattice, and $f(V_\phi) \subset \mathcal{M}_2$ is a closed algebraic curve, or*
2. *Γ_ϕ is an infinitely generated group, and $\overline{f(V_\phi)}$ is a Hilbert modular surface.*

Proof. The limit set of Γ_ϕ is the full circle S^1_∞ [Mc2], and $f(V_\phi)$ is the projection of the $\mathrm{SL}_2(\mathbb{R})$-orbit of an eigenform (by Theorem 5.8 below). Thus we are in case (1) of Theorem 1.2 if Γ_ϕ is finitely generated, and otherwise in case (2). ■

In particular, the complexification of a closed geodesic as above is never dense in $\mathcal{M}_2$. Explicit examples where (2) holds are given in [Mc2].

Connected sums. A central role in our approach to dynamics on $\Omega\mathcal{M}_2$ is played by the following result (§7):

Theorem 1.7 *Any form (X, ω) of genus two can be written, in infinitely many ways, as a connected sum $(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$ of forms of genus one.*

Here $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz)$ are forms in $\Omega\mathcal{M}_1$, and $I = [0, v]$ is a segment in $\mathbb{R}^2 \cong \mathbb{C}$. The connected sum is defined by slitting each torus E_i open along the image of I in $\mathbb{C}/\Lambda_i$, and gluing corresponding edges to obtain X (Figure 1). The forms ω_i on E_i combine to give a form ω on X with two zeros at the ends of the slits. We also refer to a connected sum decomposition as a *splitting* of (X, ω).

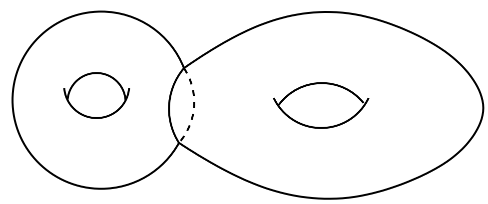

Figure 1. The connected sum of a pair of tori.

Connected sums provide a geometric characterization of eigenforms (§8):

Theorem 1.8 *If $(X, \omega) \in \Omega\mathcal{M}_2$ has two different splittings with isogenous summands, then it is an eigenform for real multiplication. Conversely, any splitting of an eigenform has isogenous summands.*

Here (E_1, ω_1) and (E_2, ω_2) in $\Omega\mathcal{M}_1$ are *isogenous* if there is a surjective holomorphic map $p : E_1 \to E_2$ such that $p^*(\omega_2) = t\omega_1$ for some $t \in \mathbb{R}$.

Connected sums also allow one to relate orbit closures in genus two to those in genus one. We conclude by sketching their use in the proof of Theorem 1.2.

1. Let $Z = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega)}$ be the closure of an orbit in $\Omega_1\mathcal{M}_2$. Choose a splitting
$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2), \tag{1.1}$$
and let $N_I \subset \mathrm{SL}_2(\mathbb{R})$ be the stabilizer of I. Then by $\mathrm{SL}_2(\mathbb{R})$-invariance, Z also contains the connected sums
$$(n \cdot (E_1, \omega_1)) \underset{I}{\#} (n \cdot (E_2, \omega_2))$$
for all $n \in N_I$.

2. Let $N \subset G = \mathrm{SL}_2(\mathbb{R})$ be the parabolic subgroup of upper-triangular matrices, let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let N_Δ and G_Δ be copies of N and G diagonally embedded in $G \times G$. For $u \in \mathbb{R}$ we also consider the twisted diagonals
$$G^u = \{(g, n_u g n_u^{-1}) \,:\, g \in G\} \subset G \times G,$$
where $n_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \in N$.

   The orbit of a pair of forms of genus one under the action of N_I is isomorphic to the orbit of a point $x \in (G \times G)/(\Gamma \times \Gamma)$ under the action of N_Δ. By the classification of unipotent orbits (§2), we have $\overline{Nx} = Hx$ where
$$H = N_\Delta,\ G_\Delta,\ G^u\ (u \neq 0),\ N \times N,\ N \times G,\ G \times N,\ \text{or}\ G \times G.$$

3. For simplicity, assume ω has simple zeros. Then if $H \neq N_\Delta$ and $H \neq G_\Delta$, we can find another point $(X', \omega') \in Z$ for which $H = G \times G$, which implies $Z = \Omega_1 \mathcal{M}_2$ (§11).

4. Otherwise, there are infinitely many splittings with $H = N_\Delta$ or G_Δ.

   The case $H = N_\Delta$ arises when $N_I \cap \mathrm{SL}(X, \omega) \cong \mathbb{Z}$. If this case occurs for two different splittings, then $\mathrm{SL}(X, \omega)$ contains two independent parabolic elements, which implies (X, ω) is an eigenform (§5).

   Similarly, the case $H = G_\Delta$ arises when (E_1, ω_1) and (E_2, ω_2) are isogenous. If this case occurs for two different splittings, then (X, ω) is an eigenform by Theorem 1.8.

5. Thus we may assume $(X, \omega) \in \Omega_1 E_D$ for some D. The summands in (1.1) are then isogenous, and therefore
$$\Gamma_0 = \mathrm{SL}(E_1, \omega_1) \cap \mathrm{SL}(E_2, \omega_2) \subset \mathrm{SL}_2(\mathbb{R})$$
is a lattice. By $\mathrm{SL}_2(\mathbb{R})$-invariance, Z contains the connected sums
$$(E_1, \omega_1) \underset{gI}{\#} (E_2, \omega_2)$$
for all $g \in \Gamma_0$, where $I = [0, v]$. But $\Gamma_0 \cdot v \subset \mathbb{R}^2$ is either discrete or dense. In the discrete case we find $\mathrm{SL}(X, \omega)$ is a lattice, and in the dense case we find $Z = \Omega_1 E_D$, completing the proof (§12).

Invariants of Teichmüller curves. We remark that the orbit closure Z in cases (3) and (4) of Theorem 1.2 is unique, and in case (2) it is uniquely determined by the discriminant D. In the sequels [Mc4], [Mc5], [Mc3] to this paper we obtain corresponding results for case (1); namely, if $\mathrm{SL}(X, \omega)$ is a lattice, then either:

(1a) We have
$$Z \subset \Omega_1 \mathcal{M}_2(2) \cap \Omega E_D,$$
and Z is uniquely determined by the discriminant D and a spin invariant $\epsilon \in \mathbb{Z}/2$; or

(1b) Z is the unique closed orbit in $\Omega_1\mathcal{M}_2(1,1) \cap \Omega E_5$, which is generated by a multiple of the *decagon form* $\omega = dx/y$ on $y^2 = x(x^5-1)$; or

(1c) We have
$$Z \subset \Omega\mathcal{M}_2(1,1) \cap \Omega E_{d^2},$$
and (X,ω) is the pullback of a form of genus one via a degree d covering $\pi : X \to \mathbb{C}/\Lambda$ branched over torsion points.

See e.g. [GJ], [EO], [EMS] for more on case (1c).

It would be interesting to develop similar results for the dynamics of $\mathrm{SL}_2(\mathbb{R})$, and its unipotent subgroups, in higher genus.

Notes and references. There are many parallels between the moduli spaces $\mathcal{M}_g = \mathcal{T}_g/\operatorname{Mod}_g$ and homogeneous spaces G/Γ, beyond those we consider here; for example, [Iv] shows Mod_g exhibits many of the properties of an arithmetic subgroup of a Lie group.

The moduli space of holomorphic 1-forms plays an important role in the dynamics of polygonal billiards [KMS], [V3]. The $\mathrm{SL}_2(\mathbb{R})$-invariance of the eigenform locus ΩE_D was established in [Mc1], and used to give new examples of Teichmüller curves and L-shaped billiard tables with optimal dynamical properties; see also [Mc2], [Ca]. Additional references for dynamics on homogeneous spaces, Teichmüller theory and eigenforms for real multiplication are given in §2, §3 and §4 below.

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2 Dynamics and Lie groups

In this section we recall Ratner's theorems for unipotent dynamics on homogeneous spaces. We then develop their consequences for actions of $\mathrm{SL}_2(\mathbb{R})$ and its unipotent subgroups.

Algebraic sets and measures. Let $\Gamma \subset G$ be a lattice in a connected Lie group. Let $\Gamma_x = x\Gamma x^{-1} \subset G$ denote the stabilizer of $x \in G/\Gamma$ under the left action of G.

A closed subset $X \subset G/\Gamma$ is *algebraic* if there is a closed unimodular subgroup $H \subset G$ such that $X = Hx$ and $H/(H \cap \Gamma_x)$ has finite volume. Then X carries a unique H-invariant probability measure, coming from Haar measure on H. Measures on G/Γ of this form are also called *algebraic.*

Unipotent actions. An element $u \in G$ is *unipotent* if every eigenvalue of $\mathrm{Ad}_u : \mathfrak{g} \to \mathfrak{g}$ is equal to one. A group $U \subset G$ is unipotent if all its elements are.

Theorem 2.1 (Ratner) *Let $U \subset G$ be a closed subgroup generated by unipotent elements. Suppose U is cyclic or connected. Then every orbit closure $\overline{Ux} \subset G/\Gamma$ and every ergodic U-invariant probability measure on G/Γ is algebraic.*

See [Rat, Thms. 2 and 4] and references therein.

Lattices. As a first example, we discuss the discrete horocycle flow on the modular surface.

Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We can regard the G/Γ as the homogeneous space of lattices $\Lambda \subset \mathbb{R}^2$ with $\mathrm{area}(\mathbb{R}^2/\Lambda) = 1$. Let

$$A = \left\{a_t = \left(\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}\right) \,:\, t \in \mathbb{R}_+\right\} \quad \text{and} \quad N = \{n_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \,:\, u \in \mathbb{R}\}$$

denote the diagonal and upper-triangular subgroups of G. Note that G and N are unimodular, but AN is not. In fact we have:

Theorem 2.2 *The only connected unimodular subgroups with $N \subset H \subset G$ are $H = N$ and $H = G$.*

Now consider the unipotent subgroup $N(\mathbb{Z}) = N \cap \mathrm{SL}_2(\mathbb{Z}) \subset G$. Note that N fixes all the horizontal vectors $v = (x, 0) \neq 0$ in $\mathbb{R}^2$. Let $\mathrm{SL}(\Lambda)$ denote the stabilizer of Λ in G. Using the preceding result, Ratner's theorem easily implies:

Theorem 2.3 *Let $\Lambda \in G/\Gamma$ be a lattice, and let $X = \overline{N(\mathbb{Z}) \cdot \Lambda}$. Then exactly one of the following holds.*

1. *There is a horizontal vector $v \in \Lambda$, and $N(\mathbb{Z}) \cap \mathrm{SL}(\Lambda) \cong \mathbb{Z}$. Then $X = N(\mathbb{Z}) \cdot \Lambda$ is a finite set.*
2. *There is a horizontal vector $v \in \Lambda$, and $N(\mathbb{Z}) \cap \mathrm{SL}(\Lambda) \cong (0)$. Then $X = N \cdot \Lambda \cong S^1$.*
3. *There are no horizontal vectors in Λ. Then $X = G \cdot \Lambda = G/\Gamma$.*

Pairs of lattices. Now let G_Δ and N_Δ denote G and N, embedded as diagonal subgroups in $G \times G$. Given $u \in \mathbb{R}$, we can also form the twisted diagonal

$$G^u = \{(g, n_u g n_u^{-1}) \,:\, g \in G\} \subset G \times G,$$

where $n_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \in N$. Note that $G^0 = G_\Delta$.

For applications to dynamics over moduli space, it will be important to understand the dynamics of N_Δ on $(G \times G)/(\Gamma \times \Gamma)$. Points in the latter space can be interpreted as pairs of lattices (Λ_1, Λ_2) in $\mathbb{R}^2$ with $\mathrm{area}(\mathbb{R}^2/\Lambda_1) = \mathrm{area}(\mathbb{R}^2/\Lambda_2) = 1$. The action of N_Δ is given by simultaneously shearing these lattices along horizontal lines in $\mathbb{R}^2$.

Theorem 2.4 *All connected N_Δ-invariant algebraic subsets of $(G \times G)/(\Gamma \times \Gamma)$ have the form $X = Hx$, where*

$$H \;=\; N_\Delta,\; G^u,\; N \times N,\; N \times G,\; G \times N,\; \textit{or}\; G \times G.$$

There is one unimodular subgroup between $N \times N$ and $G \times G$ not included in the list above, namely the solvable group $S \cong \mathbb{R}^2 \ltimes \mathbb{R}$ generated by $N \times N$ and $\{(a, a^{-1}) \,:\, a \in A\}$.

Lemma 2.5 *The group S does not meet any conjugate of* $\Gamma \times \Gamma$ *in a lattice.*

Proof. In terms of the standard action of $G \times G$ on the product of two hyperbolic planes, $S \subset AN \times AN$ stabilizes a point $(p, q) \in \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. But the stabilizer of p in Γ is either trivial or isomorphic to $\mathbb{Z}$, as is the stabilizer of q. Thus $S \cap (\Gamma \times \Gamma)$ is no larger than $\mathbb{Z} \oplus \mathbb{Z}$, so it cannot be a lattice in S. The same argument applies to any conjugate. ■

Proof of Theorem 2.4. Let X be a connected, N_Δ-invariant algebraic set. Then $X = Hx$ where H is a closed, connected, unimodular group satisfying

$$N_\Delta \subset H \subset G \times G$$

and meeting the stabilizer of x in a lattice.

It suffices to determine the Lie algebra $\mathfrak{h}$ of H. Let $U \subset G$ be the subgroup of lower-triangular matrices, and let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{u}$ denote the Lie algebras of G, N, A and U respectively. Writing $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{u}$, we have

$$[\mathfrak{n}, \mathfrak{a}] = \mathfrak{n} \text{ and } [\mathfrak{n}, \mathfrak{u}] = \mathfrak{a}.$$

We may similarly express the Lie algebra of $G \times G = G_1 \times G_2$ as

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 \;\;=\;\; (\mathfrak{n}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{u}_1) \oplus (\mathfrak{n}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{u}_2).$$

If H projects faithfully to both factors of $G \times G$, then its image in each factor is N or G by Theorem 2.2. Thus $H = N_\Delta$ or H is the graph of an automorphism $\alpha : G \to G$. In the latter case α must fix N pointwise, since $N_\Delta \subset H$. Then $\alpha(g) = ngn^{-1}$ for some $n = n_u \in N$, and $H = G^u$.

Now assume H does not project faithfully to one of its factors; say H contains $M \times \{\mathrm{id}\}$ where M is a nontrivial connected subgroup of G. Then M is invariant under conjugation by N, which implies $M \supset N$ and thus $N \times N \subset H$.

Assume H is a proper extension of $N \times N$. We claim H is not contained in $AN \times AN$. Indeed, if it were, then (by unimodularity) it would coincide with the solvable subgroup S; but S does not meet any conjugate of $\Gamma \times \Gamma$ in a lattice.

Therefore $\mathfrak{h}$ contains an element of the form $(a_1 + u_1, a_2 + u_2)$ where one of the $u_i \in \mathfrak{u}_i$, say u_1, is nonzero. Bracketing with $(n_1, 0) \in \mathfrak{h}$, we obtain a nonzero vector in $\mathfrak{a}_1 \cap \mathfrak{h}$, so $H \cap G_1$ contains AN. But $H \cap G_1$, like H, is unimodular, so it coincides with G_1. Therefore H contains $G \times N$. Since $H \cap G_2$ is also unimodular, we have $H = G \times N$ or $H = G \times G$. ■

We can now classify orbit closures for N_Δ. We say lattices Λ_1 and Λ_2 are *commensurable* if $\Lambda_1 \cap \Lambda_2$ has finite index in both.

Theorem 2.6 *Let $x = (\Lambda_1, \Lambda_2) \in (G \times G)/(\Gamma \times \Gamma)$ be a pair of lattices, and let $X = \overline{N_\Delta x}$. Then exactly one of the following holds.*

1. *There are horizontal vectors $v_i \in \Lambda_i$ with $|v_1|/|v_2| \in \mathbb{Q}$. Then $X = N_\Delta x \cong S^1$.*

2. *There are horizontal vectors $v_i \in \Lambda_i$ with $|v_1|/|v_2|$ irrational. Then $X = (N \times N)x \cong S^1 \times S^1$.*

3. *One lattice, say Λ_1, contains a horizontal vector but the other does not. Then $X = (N \times G)x \cong S^1 \times (G/\Gamma)$.*

4. *Neither lattice contains a horizontal vector, but Λ_1 is commensurable to $n_u(\Lambda_2)$ for a unique $u \in \mathbb{R}$. Then $X = G^u x \cong G/\Gamma_0$ for some lattice $\Gamma_0 \subset \Gamma$.*

5. *The lattices Λ_1 and $n(\Lambda_2)$ are incommensurable for all $n \in N$, and neither one contains a horizontal vector. Then $X = (G \times G)x = (G \times G)/(\Gamma \times \Gamma)$.*

Proof. Since N_Δ is unipotent, Ratner's theorem implies $X = Hx$ is a connected algebraic set. The result above follows, by considering the list of possible H in Theorem 2.4 and checking when the stabilizer of (Λ_1, Λ_2) in H is a lattice. ■

Locally finite measures. A measure μ is *locally finite* if it assigns finite mass to compact sets. We will show that, for unipotent actions, any locally finite invariant measure is composed of algebraic measures.

Let $\Gamma \subset G$ be a lattice in a connected Lie group. Let $U = (u_t)$ be a 1-parameter unipotent subgroup of G. Then every $x \in G/\Gamma$ naturally determines an algebraic measure ν_x recording the distribution of the orbit Ux. More precisely, by [Rat, Thm. 6] we have:

Theorem 2.7 (Ratner) *For every $x \in G/\Gamma$, there is an ergodic, algebraic, U-invariant probability measure ν_x such that* $\operatorname{supp}(\nu_x) = \overline{U \cdot x}$ *and*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t \cdot x)dt \;=\; \int_{G/\Gamma} f(y)\nu_x(y)$$

for every $f \in C_0(G/\Gamma)$.

Here $C_0(G/\Gamma)$ denotes the space of compactly supported continuous functions. Applying this result, we obtain:

Theorem 2.8 *Let μ be a locally finite U-invariant measure on G/Γ. Then for any $f \in C_0(G/\Gamma)$, we have*

$$\int f\mu = \int \left(\int f(y)\nu_x(y) \right) \mu(x). \tag{2.1}$$

In other words, μ can be expressed as the convolution $\mu(x) * \nu_x$.

Proof. The result is immediate if $\mu(G/\Gamma) < \infty$. Indeed, in this case we can consider the family of uniformly bounded averages $f_T(x) = (1/T)\int_0^T f(u_t \cdot x)dt$, which converge pointwise to $F(x) = \int f\nu_x$ as $T \to \infty$. Then by U-invariance of μ and dominated convergence, we have:

$$\int f\mu = \lim_{T\to\infty} \int f_T\mu = \int \left(\lim_{T\to\infty} f_T\right)\mu = \int F\mu,$$

which is (2.1) .

For the general case, let $K_1 \subset K_2 \subset K_3 \subset \cdots$ be an exhaustion of G/Γ by compact sets, and let

$$E_n = \{x \,:\, \nu_x(K_n) > 1/n\}.$$

Clearly E_n is U-invariant and $\bigcup E_n = G/\Gamma$. Moreover, if we take $f \in C_0(X)$ with $f \geq 0$ and $f = 1$ on E_n, then its averages satisfy

$$F(x) = \lim_{T\to\infty} f_T(x) = \int f\nu_x \geq \nu_x(K_n) > 1/n$$

for all $x \in E_n$. Thus by Fatou's lemma we obtain:

$$(1/n)\mu(E_n) \leq \int F\mu = \int (\lim_{T\to\infty} f_T)\mu \leq \lim \int f_T\mu = \int f\mu < \infty,$$

and therefore $\mu(E_n)$ is finite. Applying the first argument to the finite invariant measure $\mu|E_n$, and letting $n \to \infty$, we obtain the theorem. ∎

Invariant measures on G/Γ. Analogous results hold when $U = (u^n)$ is a cyclic unipotent subgroup. For example, let

$$X = G/\Gamma = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SL}_2(\mathbb{Z}),$$

be the space of lattices again, and let

$$X_H = \{x \in X \,:\, \overline{N(\mathbb{Z})\cdot x} = Hx\}.$$

Then we have a partition of X into three sets,

$$X = X_{N(\mathbb{Z})} \sqcup X_N \sqcup X_G,$$

corresponding exactly to the three alternatives in Theorem 2.3. By Theorem 2.7, the measure ν_x is H-invariant when x is in X_H. Thus the preceding Theorem implies:

Corollary 2.9 *Let μ be a locally finite $N(\mathbb{Z})$-invariant measure on $X = G/\Gamma$. Then $\mu|X_H$ is H-invariant, for $H = N(\mathbb{Z})$, N, and G.*

Dynamics on $\mathbb{R}^2$. The same methods permit an analysis of the action of a lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$.

Theorem 2.10 *Let Γ be a general lattice in $\mathrm{SL}_2(\mathbb{R})$. Then:*

1. *For $v \neq 0$, the orbit $\Gamma v \subset \mathbb{R}^2$ is either dense or discrete, depending on whether the stabilizer of v in Γ is trivial or $\mathbb{Z}$.*
2. *Any Γ-invariant locally finite measure α on $\mathbb{R}^2$ has the form $\alpha = \alpha^a + \alpha^s$, where α^a is a constant multiple of the standard area measure, and α^s assigns full mass to $\{v : \Gamma v \text{ is discrete.}\}$.*
3. *For any discrete orbit, we have $t\Gamma v \to \mathbb{R}^2$ in the Hausdorff topology as $t \to 0$.*
4. *If v_n is a bounded sequence of vectors with infinitely many different slopes, then $\bigcup \Gamma \cdot v_n$ is dense in $\mathbb{R}^2$.*

Proof. We can regard $\mathbb{R}^2 - \{0\}$ as the homogeneous space G/N. Thus the first two statements follow from Ratner's theorems, by relating the action of Γ on G/N to the action of N on $\Gamma\backslash G$. (For (1) use the fact that when Γv is discrete in $\mathbb{R}^2 - \{0\}$ it is also discrete in $\mathbb{R}^2$; this follows from discreteness of Γ.)

To prove (3) and (4), we interpret G/N as the space of horocycles in the hyperbolic plane $\mathbb{H}^2$. Then discrete orbits Γv correspond to preimages of closed horocycles around the finitely many cusps of the surface $X = \Gamma\backslash\mathbb{H}^2$. Thus we can choose nonzero vectors $c_i \in \mathbb{R}^2$, one for each cusp, such that any discrete orbit in $\mathbb{R}^2 - \{0\}$ has the form

$$\Gamma v = t\Gamma c_i$$

for some $t > 0$ and $1 \leq i \leq m$. As $t \to 0$, the length of the corresponding closed horocycle $H_i(t)$ tends to infinity. Thus $H_i(t)$ becomes equidistributed in $T_1(X)$ [EsM, Thm. 7.1], and therefore $t\Gamma v \to \mathbb{R}^2$ in the Hausdorff topology.

To establish (4), we may assume each individual orbit Γv_n is discrete, since otherwise it is already dense by (1). Passing to a subsequence, we can write $\Gamma v_n = t_n \Gamma c_i$ for a fixed value of i. Since Γc_i is discrete in $\mathbb{R}^2$, for the bounded vectors v_n to take on infinitely many slopes, we must have $\liminf t_n = 0$; thus $\bigcup \Gamma v_n$ is dense by (3). ∎

Deserts. A typical example of a discrete Γ-orbit in $\mathbb{R}^2$ is the set of relatively prime integral points,

$$\mathbb{Z}^2_{\mathrm{rp}} = \mathrm{SL}_2(\mathbb{Z}) \cdot (1, 0) = \{(p, q) \in \mathbb{Z}^2 \ : \ \gcd(p, q) = 1\}.$$

We remark that this orbit is *not* uniformly dense in $\mathbb{R}^2$: there exist arbitrarily large 'deserts' in its complement. To see this, fix distinct primes (p_{ij}) indexed by $0 \leq i, j \leq n$. Then the Chinese remainder theorem provides integers $a, b > 0$ such that $(a, b) = (-i, -j) \bmod p_{ij}$ for all (i, j). Thus p_{ij} divides $(a + i, b + j)$, and therefore we have

$$\mathbb{Z}^2_{\mathrm{rp}} \cap [a, a + n] \times [b, b + n] = \emptyset.$$

However a and b are much greater than n, and thus $(1/\max(a, b))\mathbb{Z}^2_{\mathrm{rp}}$ is still very dense, consistent with (3) above.

3 Riemann surfaces and holomorphic 1-forms

In this section we recall the metric and affine geometry of a compact Riemann surface equipped with a holomorphic 1-form. We then summarize results on the moduli space $\Omega\mathcal{M}_g$ of all such forms, its stratification and the action of $\mathrm{SL}_2(\mathbb{R})$ upon it.

Geometry of holomorphic 1-forms. Let ω be a holomorphic 1-form on a compact Riemann surface X of genus g. The g-dimensional vector space of all such 1-forms will be denoted by $\Omega(X)$. Assume $\omega \neq 0$, and let $Z(\omega) \subset X$ be its zero set. We have $|Z(\omega)| \leq 2g - 2$.

The form ω determines a conformal metric $|\omega|$ on X, with concentrated negative curvature at the zeros of ω and otherwise flat. Any two points of $(X, |\omega|)$ are joined by a unique geodesic in each homotopy class. A geodesic is *straight* if its interior is disjoint from $Z(\omega)$. Since a straight geodesic does not change direction, its length satisfies $\int_\gamma |\omega| = |\int_\gamma \omega|$.

Saddle connections. A *saddle connection* is a straight geodesic (of positive length) that begins and ends at a zero of ω (a saddle). When X has genus $g \geq 2$, every essential loop on X is homotopic to a chain of saddle connections.

Affine structure. The form ω also determines a branched complex affine structure on X, with local charts $\phi : U \to \mathbb{C}$ satisfying $d\phi = \omega$. These charts are well-defined up to translation, injective away from the zeros of ω, and of the form $\phi(z) = z^{p+1}$ near a zero of order p.

Foliations. The harmonic form $\rho = \operatorname{Re} \omega$ determines a *measured foliation* $\mathcal{F}_\rho$ on X. Two points $x, y \in X$ lie on the same leaf of $\mathcal{F}_\rho$ iff they are joined by a path satisfying $\rho(\gamma'(t)) = 0$. The leaves are locally smooth 1-manifolds tangent to $\operatorname{Ker} \rho$, coming together in groups of $2p$ at the zeros of ω of order p. The leaves are oriented by the condition $\operatorname{Im} \rho > 0$. In a complex affine chart, we have $\rho = dx$ and the leaves of $\mathcal{F}_\rho$ are the vertical lines in $\mathbb{C}$. The measure of a transverse arc is given by $\mu(\tau) = |\int_\tau \rho|$.

Slopes. Straight geodesics on $(X, |\omega|)$ become straight lines in $\mathbb{C}$ in the affine charts determined by ω. Thus $\mathcal{F}_\rho$ can alternatively be described as the foliation of X by parallel geodesics of constant slope ∞. Similarly, $\mathcal{F}_{\operatorname{Re}(x+iy)\omega}$ gives the foliation of X by geodesics of slope x/y.

The spine. The union of all saddle connections running along leaves of $\mathcal{F}_\rho$ is the *spine* of the foliation. The spine is a finite graph embedded in X.

Cylinders. A *cylinder* $A \subset X$ is a maximal open region swept out by circular leaves of $\mathcal{F}_\rho$. The subsurface $(A, |\omega|)$ is isometric to a right circular cylinder of height $h(A)$ and circumference $c(A)$; its *modulus* $\operatorname{mod}(A) = h(A)/c(A)$ is a conformal invariant. Provided X is not a torus, ∂A is a union of saddle connections.

Periodicity. The foliation $\mathcal{F}_\rho$ is *periodic* if all its leaves are compact. In this case, either X is a torus foliated by circles, or the complement of the spine of $\mathcal{F}_\rho$ in X is a finite union of cylinders $A_1, \ldots, A_n$.

Moduli space. Let $\mathcal{M}_g = \mathcal{T}_g / \operatorname{Mod}_g$ denote the moduli space of compact Riemann surfaces X of genus g, presented as the quotient of Teichmüller space by the action of the mapping class group. Let

$$\Omega\mathcal{T}_g \to \mathcal{T}_g$$

denote the bundle whose fiber over X is $\Omega(X) - \{0\}$. The space $\Omega\mathcal{T}_g$ is the complement of the zero-section of a holomorphic line bundle over $\mathcal{T}_g$. The mapping class group has a natural action on this bundle as well, and we define the *moduli space of holomorphic 1-forms of genus g* by

$$\Omega\mathcal{M}_g = \Omega\mathcal{T}_g / \operatorname{Mod}_g .$$

The projection $\Omega\mathcal{M}_g \to \mathcal{M}_g$ is a holomorphic bundle map in the category of orbifolds; the fiber over $X \in \operatorname{Mod}_g$ is the space $\Omega(X) - \{0\}/\operatorname{Aut}(X)$.

For brevity, we refer $(X, \omega) \in \Omega\mathcal{M}_g$ as *a form of genus g.*

Action of $\mathbf{GL}_2^+(\mathbb{R})$. The group $\mathrm{GL}_2^+(\mathbb{R})$ of automorphism of $\mathbb{R}^2$ with $\det(A) > 0$ has a natural action on $\Omega\mathcal{M}_g$. To define $A \cdot (X, \omega)$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$, consider the harmonic 1-form

$$\omega' = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \operatorname{Re}\omega \\ \operatorname{Im}\omega \end{pmatrix} \tag{3.1}$$

on X. Then there is a unique complex structure with respect to which ω' is holomorphic; its charts yield a new Riemann surface X', and we define $A \cdot (X, \omega) = (X', \omega')$

Periods. Given $(X, \omega) \in \Omega\mathcal{M}_g$, the *relative period map*

$$I_\omega : H_1(X, Z(\omega); \mathbb{Z}) \to \mathbb{C}$$

is defined by $I_\omega(C) = \int_C \omega$. Its restriction

$$I_\omega : H_1(X, \mathbb{Z}) \to \mathbb{C}$$

to chains with $\partial C = 0$ is the *absolute period map*, whose image $\operatorname{Per}(\omega) \subset \mathbb{C}$ is the group of *absolute periods* of ω.

Strata. Let $(p_1, \ldots, p_n)$ be an unordered partition of $2g-2 = \sum p_i$. The space $\Omega\mathcal{T}_g$ breaks up into *strata* $\Omega\mathcal{T}_g(p_1, \ldots, p_n)$ consisting of those forms (X, ω) whose n zeros have multiplicities $p_1, \ldots, p_n$. Clearly this stratification is preserved by the action of $\mathrm{GL}_2^+(\mathbb{R})$.

The bundle of groups $H_1(X, Z(\omega); \mathbb{Z})$ is locally trivial over a stratum. Thus on a neighborhood U of (X_0, ω_0) in $\Omega\mathcal{T}_g$ we can define *period coordinates*

$$p : U \to H^1(X_0, Z(\omega_0); \mathbb{C})$$

sending $(X, \omega) \in U$ to the cohomology class of ω.

Theorem 3.1 *The period coordinate charts are local homeomorphisms, giving $\Omega\mathcal{T}_g(p_1, \ldots, p_n)$ the structure of a complex manifold of dimension $2g + n - 1$.*

Measures. The transition functions between period charts are integral linear maps induced by homeomorphisms of (X, Z). Thus a stratum carries a natural volume element, a local linear integral structure and a global quadratic function $q(X,\omega) = \int_X |\omega|^2$, all inherited from $H^1(X, Z; \mathbb{C})$.

These structures descend to a stratification of the moduli space $\Omega\mathcal{M}_g$ by the orbifolds

$$\Omega\mathcal{M}_g(p_1, \ldots, p_n) = \Omega\mathcal{T}_g(p_1, \ldots, p_n)/\operatorname{Mod}_g .$$

Unit area bundle. For $t > 0$, let $\Omega_t\mathcal{M}_g$ denote the bundle of forms (X, ω) with total area $\int_X |\omega|^2 = t$. Each stratum of this space also carries a natural measure, defined on $U \subset \Omega_t\mathcal{M}_g(p_1, \ldots, p_n)$ to be proportional to the measure of the cone $(0, 1) \cdot U \subset \Omega\mathcal{M}_g(p_1, \ldots, p_n)$.

Theorem 3.2 *Each stratum of $\Omega_1\mathcal{M}_g$ has finite measure and finitely many components.*

Theorem 3.3 *The action of $\mathrm{SL}_2(\mathbb{R})$ is volume-preserving and ergodic on each component of each stratum of $\Omega_1\mathcal{M}_g$.*

Real-affine maps and $\mathbf{SL}(X,\omega)$. The stabilizer of $(X, \omega) \in \Omega\mathcal{M}_g$ is the discrete group $\mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(\mathbb{R})$.

Here is an intrinsic definition of $\mathrm{SL}(X, \omega)$. A map $\phi : X \to X$ is *real-affine* with respect to ω if, after passing to the universal cover, there is an $A \in \mathrm{GL}_2(\mathbb{R})$ and $b \in \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\phi}} & \widetilde{X} \\ {\scriptstyle D_\omega}\downarrow & & {\scriptstyle D_\omega}\downarrow \\ \mathbb{C} & \xrightarrow{Av+b} & \mathbb{C} \end{array}$$

commutes. Here the *developing map* $D_\omega(q) = \int_p^q \omega$ is obtained by integrating the lift of ω.

We denote the linear part of ϕ by $D\phi = A \in \mathrm{SL}_2(\mathbb{R})$. Then $\mathrm{SL}(X, \omega)$ is the image of the group $\mathrm{Aff}^+(X, \omega)$ of orientation preserving real-affine automorphisms of (X, ω) under $\phi \mapsto D\phi$.

Teichmüller curves. A *Teichmüller curve* $f : V \to \mathcal{M}_g$ is finite volume hyperbolic Riemann surface V equipped with a holomorphic, totally geodesic, generically 1-1 immersion into moduli space. We also refer to $f(V)$ as a Teichmüller curve; it is an irreducible algebraic curve on $\mathcal{M}_g$ whose normalization is V.

Theorem 3.4 *The following are equivalent.*

1. *The group $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$.*
2. *The orbit $\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega)$ is closed in $\Omega\mathcal{M}_g$.*
3. *The projection of the orbit to $\mathcal{M}_g$ is a Teichmüller curve.*

In this case we say (X,ω) *generates* the Teichmüller curve $V \to \mathcal{M}_g$. Assuming ω is normalized so its area is one, the orbit

$$\Omega_1 V = \mathrm{SL}_2(\mathbb{R}) \cdot (X,\omega) \subset \Omega_1 \mathcal{M}_g$$

can be regarded as a circle bundle over V.

Genus one and two. The space $\Omega\mathcal{M}_1$ is canonically identified with the space of lattices $\Lambda \subset \mathbb{C}$, via the correspondence $(E,\omega) = (\mathbb{C}/\Lambda, dz)$. Moreover the action of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega_1\mathcal{M}_1$ is transitive; if we take the square lattice $\mathbb{Z} \oplus \mathbb{Z}i$ as a basepoint, we then obtain an $SL_2(\mathbb{R})$-equivariant isomorphism

$$\Omega_1\mathcal{M}_1 \cong \mathrm{SL}_2(\mathbb{R}) / \mathrm{SL}_2(\mathbb{Z}).$$

In particular $\mathrm{SL}(E,\omega)$ is conjugate to $\mathrm{SL}_2(\mathbb{Z})$ for any form of genus one.

In higher genus the action of $\mathrm{SL}_2(\mathbb{R})$ is not transitive. However we do have:

Theorem 3.5 *In genus two the strata are connected, and* $\mathrm{SL}_2(\mathbb{R})$ *acts ergodically on* $\Omega_1\mathcal{M}_2(2)$ *and* $\Omega_1\mathcal{M}_2(1,1)$.

References. The metric geometry of holomorphic 1-forms, and more generally of quadratic differentials, is developed in [Str] and [Gd]. Period coordinates and strata are discussed in [V4], [MS, Lemma 1.1] and [KZ]. Finiteness of the measure of $\Omega_1\mathcal{M}_g(p_1,\ldots,p_n)$ is proved in [V2]; see also [MS, Thm. 10.6]. The ergodicity of $\mathrm{SL}_2(\mathbb{R})$ is shown in [Mas], [V1] for the principle stratum $\Omega\mathcal{M}_g(1,\ldots,1)$, and in [V2, Theorem 6.14] for general strata. The equivalence of (1) and (2) in Theorem 3.4 is due to Smillie; see [V5, p.226]. For more on Teichmüller curves, see [V3] and [Mc1]. The classification of the components of strata is given in [KZ]. For related results, see [Ko], [EO], and [EMZ].

4 Abelian varieties with real multiplication

In this section we review the theory of real multiplication, and classify eigenforms for Riemann surfaces of genus two.

An Abelian variety $A \in \mathcal{A}_g$ admits real multiplication if its endomorphism ring contains a self-adjoint order $\mathfrak{o}$ of rank g in a product of totally real fields. Let

$$\begin{aligned}\mathcal{E}_2 \;=\; \{&(X,\omega) \in \Omega\mathcal{M}_2 \,:\, \mathrm{Jac}(X) \text{ admits real multiplication with} \\ &\omega \text{ as an eigenform}\},\end{aligned}$$

let $\mathfrak{o}_D$ denote the real quadratic order of discriminant D, and let

$$\Omega E_D = \{(X,\omega) \in \Omega\mathcal{M}_2 \,:\, \omega \text{ is an eigenform for real multiplication by } \mathfrak{o}_D\}.$$

We will show:

Theorem 4.1 *The eigenform locus in genus two is a disjoint union*

$$\mathcal{E}_2 = \bigcup \Omega E_D$$

of closed, connected, complex orbifolds, indexed by the integers $D \geq 4$, $D = 0$ or $1 \bmod 4$.

The discussion will also yield a description of ΩE_D as a $\mathbb{C}^*$-bundle over a Zariski open subset E_D of the Hilbert modular surface $X_D = (\mathbb{H}^2 \times -\mathbb{H}^2)/\operatorname{SL}_2(\mathfrak{o}_D)$, and an identification of ΩE_{d^2} with the space of elliptic differentials of degree d.

Abelian varieties. Let $\mathcal{A}_g$ denote the moduli space of principally polarized Abelian varieties of dimension g. The space $\mathcal{A}_g$ is isomorphic to the quotient $\mathcal{H}^g/\operatorname{Sp}_{2g}(\mathbb{Z})$ of the Siegel upper halfspace by the action of the integral symplectic group.

Let $\Omega(A)$ denote the g-dimensional space of holomorphic 1-forms on A, and let $\Omega\mathcal{A}_g \to \mathcal{A}_g$ be the bundle of pairs (A, ω) with $\omega \neq 0$ in $\Omega(A)$.

Endomorphisms. Let $\operatorname{End}(A)$ denote the ring of endomorphisms of A as a complex Lie group. The endomorphism ring is canonically isomorphic to the ring of homomorphisms

$$T : H_1(A, \mathbb{Z}) \to H_1(A, \mathbb{Z})$$

that preserve the Hodge decomposition $H^1(A, \mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A)$.

The polarization of A provides $H_1(A, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ with a unimodular symplectic form $\langle v_1, v_2 \rangle$, isomorphic to the intersection form $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ on the homology of a Riemann surface of genus g. The polarization is compatible with the Hodge structure and makes A into a Kähler manifold.

Each $T \in \operatorname{End}(A)$ has an associated adjoint operator T^*, characterized by $\langle Tv_1, v_2 \rangle = \langle v_1, T^* v_2 \rangle$. If $T = T^*$ we say T is *self-adjoint.* (The map $T \mapsto T^*$ is known as the *Rosati involution.*)

Real multiplication. Let K be a totally real field of degree g over $\mathbb{Q}$, or more generally a product $K = K_1 \times K_2 \times \cdots \times K_n$ of such fields with $\sum \deg(K_i/\mathbb{Q}) = g$.

We say $A \in \mathcal{A}_g$ admits *real multiplication* by K if there is a faithful representation

$$K \hookrightarrow \operatorname{End}(A) \otimes \mathbb{Q}$$

satisfying $\langle kv_1, v_2 \rangle = \langle v_1, kv_2 \rangle$. The breadth of this definition permits a uniform treatment of Humbert surfaces and elliptic differentials, in addition to the traditional Hilbert modular varieties $(\mathbb{H}^2)^g/\operatorname{SL}_2(\mathcal{O}_K) \to \mathcal{A}_g$.

The map $K \to \operatorname{End}(A)$ makes $H^1(A, \mathbb{Q})$ into a free K-module of rank 2. Since $\operatorname{End}(A)$ respects the Hodge decomposition, we have a complex-linear action of K on $\Omega(A) \cong H^{1,0}(A)$. Choosing a basis of eigenforms, we obtain a direct sum decomposition

$$\Omega(A) = \bigoplus_1^g \mathbb{C}\omega_i$$

diagonalizing the action of K.

Orders. The integral points $\mathfrak{o} = K \cap \operatorname{End}(A)$ are an order[1] in K acting by self-adjoint endomorphisms of A. This action is *proper* in the sense that no larger order in K acts on A; equivalently, $(\mathbb{Q} \cdot \mathfrak{o}) \cap \operatorname{End}(A) = \mathfrak{o}$.

Now let $L \cong (\mathbb{Z}^{2g}, \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right))$ be a unimodular symplectic lattice of rank $2g$, and let $\mathfrak{o}$ be an order in K. Fix a representation

$$\rho : \mathfrak{o} \hookrightarrow \operatorname{End}(L) \cong \mathrm{M}_{2g}(\mathbb{Z})$$

giving a proper, faithful action of $\mathfrak{o}$ on L by self-adjoint endomorphisms. This representation makes L into an $\mathfrak{o}$-module of rank 2.

We say $A \in \mathcal{A}_g$ admits *real multiplication by* $(\mathfrak{o}, \rho)$ if there is a symplectic isomorphism $L \cong H_1(A, \mathbb{Z})$ sending $\rho(\mathfrak{o})$ into $\operatorname{End}(A)$. This definition refines the notion of real multiplication by $K = \mathfrak{o} \otimes \mathbb{Q}$.

Synthesis. To construct all Abelian varieties with real multiplication by $(\mathfrak{o}, \rho)$, we begin by diagonalizing the action of $\mathfrak{o}$ on L; the result is a splitting

$$L \otimes \mathbb{R} = \bigoplus_1^g S_i \tag{4.1}$$

into orthogonal, symplectic eigenspaces $S_i \cong \mathbb{R}^2$. The set of $\mathfrak{o}$-invariant complex structures on $L \otimes \mathbb{R}$, positive with respect to the symplectic form, is parameterized by a product of upper halfplanes $(\mathbb{H}^2)^g$, one for each S_i. Given $\tau \in (\mathbb{H}^2)^g$, we obtain an $\mathfrak{o}$-invariant complex structure on $L \otimes \mathbb{R}$ and hence an Abelian variety

$$A_\tau = (L \otimes \mathbb{R})_\tau / L \cong \mathbb{C}^g / L_\tau,$$

equipped with real multiplication by $(\mathfrak{o}, \rho)$. Conversely, an Abelian variety A with real multiplication by $(\mathfrak{o}, \rho)$ determines an $\mathfrak{o}$-invariant complex structure on $\widetilde{A} \cong L \otimes \mathbb{R}$, and hence $A = A_\tau$ for some τ.

Hilbert modular varieties. The symplectic automorphisms of $L \otimes \mathbb{R}$ commuting with $\mathfrak{o}$ preserve the splitting $\oplus_1^g S_i$, and hence form a subgroup isomorphic to $\mathrm{SL}_2(\mathbb{R})^g$ inside $\operatorname{Sp}(L \otimes \mathbb{R}) \cong \operatorname{Sp}_{2g}(\mathbb{R})$. The automorphisms of L itself, as a symplectic $\mathfrak{o}$-module, are given by the integral points of this subgroup:

$$\Gamma(\mathfrak{o}, \rho) = \mathrm{SL}_2(\mathbb{R})^g \cap \operatorname{Sp}_{2g}(\mathbb{Z}).$$

Via its embedding in $\mathrm{SL}_2(\mathbb{R})^g$, the discrete group $\Gamma(\mathfrak{o}, \rho)$ acts isometrically on $(\mathbb{H}^2)^g$, with finite volume quotient the *Hilbert modular variety*

$$X(\mathfrak{o}, \rho) = (\mathbb{H}^2)^g / \Gamma(\mathfrak{o}, \rho).$$

The variety $X(\mathfrak{o}, \rho)$ is the moduli space of pairs $(A, \mathfrak{o} \to \operatorname{End}(A))$ compatible with ρ. By forgetting the action of $\mathfrak{o}$, the map $\tau \mapsto A_\tau$ yields a commutative diagram

$$\begin{array}{ccc} (\mathbb{H}^2)^g & \longrightarrow & \mathcal{H}_g \\ \downarrow & & \downarrow \\ X(\mathfrak{o}, \rho) & \longrightarrow & \mathcal{A}_g. \end{array}$$

[1] An order is a subring of finite index in the full ring of integers $\mathcal{O}_K = \mathcal{O}_{K_1} \times \cdots \times \mathcal{O}_{K_n}$.

Here $X(\mathfrak{o},\rho) \rightarrow \mathcal{A}_g$ is a proper finite morphism of algebraic varieties.

Theorem 4.2 *The image of $X(\mathfrak{o},\rho)$ consists exactly of those $A \in \mathcal{A}_g$ admitting real multiplication by $(\mathfrak{o},\rho)$.*

Corollary 4.3 *The locus of real multiplication in $\mathcal{A}_g$ is covered by a countable union of Hilbert modular varieties.*

Genus two. We now specialize to the case of genus $g = 2$. This case is simplified by the fact that every quadratic order

$$\mathfrak{o}_D = \mathbb{Z}[x]/(x^2 + bx + c)$$

is determined up to isomorphism by its discriminant $D = b^2 - 4c$. Thus we may assume $\mathfrak{o} = \mathfrak{o}_D$ for some $D > 0$, $D = 0$ or $1 \bmod 4$.

The classification is further simplified by the fact that there is an essentially unique representation $\rho : \mathfrak{o}_D \rightarrow \mathrm{End}(L)$, where $L = (\mathbb{Z}^4, \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right))$.

Theorem 4.4 *There is a unique proper, faithful, self-adjoint representation*

$$\rho_D : \mathfrak{o}_D \rightarrow \mathrm{End}(L) \cong \mathrm{M}_4(\mathbb{Z})$$

up to conjugation by elements of $\mathrm{Sp}(L) \cong \mathrm{Sp}_4(\mathbb{Z})$ *.*

Because of this uniqueness, we will write X_D and Γ_D for $X(\mathfrak{o}_D,\rho_D)$ and $\Gamma(\mathfrak{o}_D,\rho_D)$.

Sketch of the proof. We briefly describe three approaches.

1. The classical proof is by a direct matrix calculation [Hu, pp. 301–308], succinctly presented in [Ru, Thm. 2].

2. A second approach is by the classification of $\mathfrak{o}_D$-modules. Although $\mathfrak{o}_D$ need not be a Dedekind domain, it admits a similar module theory, as shown in [Ba] and [BF] (for orders in quadratic fields). In particular, we have $L \cong \mathfrak{o}_D \oplus \mathfrak{a}$ for some ideal $\mathfrak{a} \subset \mathfrak{o}_D$. Then unimodularity of the symplectic form implies $L \cong \mathfrak{o}_D \oplus \mathfrak{o}_D^\vee$, with $\langle v_1, v_2\rangle = \mathrm{Tr}_{\mathbb{Q}}^K(v_1 \wedge v_2)$.

3. A third approach is based on the fact that the self-adjoint elements $T \in \mathrm{End}(L)$ correspond bijectively to $\wedge^2 L \cong H^2(A,\mathbb{Z})$ via $T \mapsto Z_T = \langle Tv_1, v_2\rangle$. The identity element gives the original symplectic form on L, which corresponds to the theta-divisor $Z_I = \Theta$. The intersection form $Z \cdot W$ makes $\wedge^2 L$ into a lattice of signature $(3,3)$, with $\Theta^2 = 2$, from which we obtain a quadratic form

$$q(Z) = (Z \cdot \Theta)^2 - 2Z^2$$

of signature $(3,2)$ on the quotient space $M = (\wedge^2 L)/(\mathbb{Z}\Theta)$. The proper, self-adjoint orders $\mathfrak{o} = \mathbb{Z}[T] \subset \mathrm{End}(L)$ of discriminant D correspond bijectively to primitive vectors $Z \in M$ with $q(Z_T) = D$; compare [Ka1], [GH, §3]. The uniqueness of the order of discriminant D in $\mathrm{End}(L)$ then reduces to transitivity of the action of $\mathrm{Sp}(L)$ on the primitive elements of norm D in M, which in turn follows from general results on lattices containing a sum of hyperbolic planes [Sc, Prop. 3.7.3]. ∎

Galois involution. Note we have

$$K = \mathfrak{o}_D \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} \times \mathbb{Q} & \text{if } D = d^2 \text{ is a square, and} \\ \mathbb{Q}(\sqrt{D}) & \text{otherwise.} \end{cases}$$

We let $k \mapsto k'$ denote the Galois involution of $K/\mathbb{Q}$, with $(x,y)' = (y,x)$ when $K = \mathbb{Q} \times \mathbb{Q}$. As usual, the trace and norm of $k \in K$ are defined by $k + k'$ and kk' respectively.

Explicit models. A model for the unique action of $\mathfrak{o}_D$ by real multiplication on L is obtained by taking $L \cong \mathfrak{o}_D \times \mathfrak{o}_D$, with the symplectic form

$$\langle v_1, v_2 \rangle = \mathrm{Tr}_{\mathbb{Q}}^K (D^{-1/2} v_1 \wedge v_2).$$

This form is clearly alternating, and it is easily shown to be unimodular (since $D^{-1/2}\mathfrak{o}_D$ is the inverse different of $\mathfrak{o}_D$).

The automorphism group of $L \cong \mathfrak{o}_D \oplus \mathfrak{o}_D$ as a symplectic $\mathfrak{o}_D$-module is given simply by $\Gamma_D = \mathrm{SL}_2(\mathfrak{o}_D)$. Thus the Hilbert modular surface for real multiplication by $\mathfrak{o}_D$ is given by

$$X_D = (\mathbb{H}^2 \times -\mathbb{H}^2) / \mathrm{SL}_2(\mathfrak{o}_D),$$

where the embedding $\mathrm{SL}_2(\mathfrak{o}_D) \to \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ comes from the two real places of K. (The $-\mathbb{H}^2$ factor arises because $\sqrt{D}' = -\sqrt{D} < 0$.)

Humbert surfaces. The image of X_D in $\mathcal{A}_2$ is the *Humbert surface*

$$H_D = \{A \in \mathcal{A}_2 \,:\, A \text{ admits real multiplication by } \mathfrak{o}_D\}.$$

The map $X_D \to H_D$ is generically two-to-one. In fact, the normalization of H_D is X_D/ι, where ι changes the inclusion $\mathfrak{o}_D \hookrightarrow \mathrm{End}(A)$ by precomposition with the Galois involution $k \mapsto k'$. The map ι interchanges the factors of $\widetilde{X_D} = \mathbb{H}^2 \times \mathbb{H}^2$ when lifted to the universal cover. Thus the normalization of H_D is a symmetric Hilbert modular surface.

Theorem 4.5 *Each Humbert surface H_D is irreducible, and the locus of real multiplication in $\mathcal{A}_2$ coincides with $\bigcup H_D$.*

Proof. Since X_D is connected, so is H_D; and if A admits real multiplication by $\mathfrak{o}$, then $\mathfrak{o} \cong \mathfrak{o}_D$ for some D. ∎

Eigenforms. For a more precise connectedness result, consider the eigenform bundle $\Omega H_D \to H_D$ defined by

$$\Omega H_D = \{(A, \omega) \in \Omega\mathcal{A}_2 \,:\, \omega \text{ is an eigenform for real multiplication by } \mathfrak{o}_D\}.$$

The fiber of ΩH_D over a generic point $A \in H_D$ has two components, one for each of the eigenspaces of $\mathfrak{o}_D$ acting on $\Omega(A)$. Nevertheless we have:

Theorem 4.6 *The eigenform bundle $\Omega H_D \to H_D$ is connected.*

For the proof, consider the bundle $\Omega X_D \to X_D$ whose fiber over τ consists of the nonzero eigenforms in S_1^*, where

$$\Omega(A_\tau) \cong (L \otimes \mathbb{R})^*_\tau \cong S_1^* \oplus S_2^*.$$

(This bundle depends on the chosen ordering of (S_1, S_2), or equivalently on the choice of a real place $\nu : K \to \mathbb{R}$.) Clearly ΩX_D is connected; thus the connectedness of ΩH_D follows from:

Theorem 4.7 *The natural map $\Omega X_D \to \Omega H_D$ is an isomorphism.*

Proof. Given $(A, \omega) \in \Omega H_D$, there is a unique order $\mathfrak{o} \subset \mathrm{End}(A)$ with ω as an eigenvector. Of the two Galois conjugate isomorphisms $\mathfrak{o}_D \cong \mathfrak{o}$, there is a unique one such that ω lies in the eigenspace S_1^*. The resulting inclusion $\mathfrak{o}_D \hookrightarrow \mathrm{End}(A)$ uniquely determines the point in ΩX_D corresponding to (A, ω). ∎

Jacobians. By the Torelli theorem, the map $X \mapsto \mathrm{Jac}(X)$ from $\mathcal{M}_g$ to $\mathcal{A}_g$ is injective. In the case of genus $g = 2$, the image is open, and in fact we have

$$\mathcal{A}_2 = \mathcal{M}_2 \sqcup H_1.$$

The Humbert surface $H_1 \cong \mathcal{A}_1 \times \mathcal{A}_1/(\mathbb{Z}/2)$ consists of products of polarized elliptic curves. The intersection $H_D \cap H_1$ is a finite collection of curves.

Let $E_D \subset X_D$ be the Zariski open subset lying over $H_D - H_1$; it consists of the Jacobians in X_D. Using the embedding $\Omega\mathcal{M}_2 \subset \Omega\mathcal{A}_2$, we can regard

$$\Omega E_D = \{(X, \omega) \in \Omega\mathcal{M}_2 \, : \, \omega \text{ is an eigenform for real multiplication by } \mathfrak{o}_D\}$$

as the set of pairs $(\mathrm{Jac}(X), \omega) \in \Omega H_D$. By Theorem 4.7, we have

$$\Omega E_D = \Omega H_D | (H_D - H_1) = \Omega X_D | E_D.$$

Theorem 4.8 *The locus $\Omega H_D \subset \Omega\mathcal{A}_2$ is closed.*

Proof. Consider a sequence in ΩH_D such that $(A_n, \omega_n) \to (A, \omega) \in \Omega\mathcal{A}_2$. Write $\mathfrak{o}_D = \mathbb{Z}[t]/p(t)$ with

$$p(t) = t^2 + bt + c = (t - k)(t - k').$$

Then for each n, there is a self-adjoint holomorphic endomorphism

$$T_n : A_n \to A_n,$$

with ω_n as an eigenform, satisfying $p(T_n) = 0$. Since T_n is self-adjoint, we have

$$\|DT_n\| \leq \max(|k|, |k'|)$$

with respect to the Kähler metric on A_n coming from its polarization. By equicontinuity, there is a subsequence along which T_n converges to a self-adjoint holomorphic endomorphism $T : A \to A$. In the limit ω is an eigenform for T and $p(T) = 0$; therefore $(A, \omega) \in \Omega H_D$. ∎

Corollary 4.9 *The locus $\Omega E_D \subset \Omega\mathcal{M}_2$ is closed.*

Proof of Theorem 4.1. Whenever X admits real multiplication, we have $\mathrm{Jac}(X) \in H_D - H_1$ for some D, and thus $\mathcal{E}_2 = \bigcup \Omega E_D$. (We can take $D \geq 4$ since E_1 is empty.) Because X_D is connected, so is E_D, and therefore so is the $\mathbb{C}^*$-bundle $\Omega E_D \to E_D$. ■

Elliptic differentials. A holomorphic 1-form $(X, \omega) \in \Omega\mathcal{M}_g$ is an *elliptic differential* if there is an elliptic curve $E = \mathbb{C}/\Lambda$ and a holomorphic map $p : X \to E$ such that $p^*(dz) = \omega$. By passing to a covering space of E if necessary, we can assume $p_*(H_1(X, \mathbb{Z})) = H_1(E, \mathbb{Z})$; then the *degree* of ω is the degree of p.

Theorem 4.10 *The locus $\Omega E_{d^2} \subset \Omega\mathcal{M}_2$ coincides with the set of elliptic differentials of degree d.*

Proof. An elliptic differential (X, ω) of degree d, pulled back via a degree d map $p : X \to E$, determines a splitting

$$H^1(A, \mathbb{Q}) = H^1(E, \mathbb{Q}) \oplus H^1(E, \mathbb{Q})^\perp$$

where $A = \mathrm{Jac}(X)$. Let $\pi \in \mathrm{End}(A) \otimes \mathbb{Q}$ denote projection to $H^1(E)$, and let $T(v) = \pi(d \cdot v)$. Then T is a primitive, self-adjoint element of $\mathrm{End}(A)$, and $T^2 = dT$; thus A admits real multiplication by $\mathbb{Z}[T] \cong \mathfrak{o}_{d^2}$, and we have $(X, \omega) \in \Omega E_{d^2}$.

Conversely, given $(X, \omega) \in \Omega E_{d^2}$, the eigenspaces for the action of $\mathfrak{o}_{d^2}$ determine a splitting of $H^1(\mathrm{Jac}(X), \mathbb{Q})$ as above, which in turn yields a degree d map $X \to E$ that exhibits ω as an elliptic differential. Compare [Ka1, §4]. ■

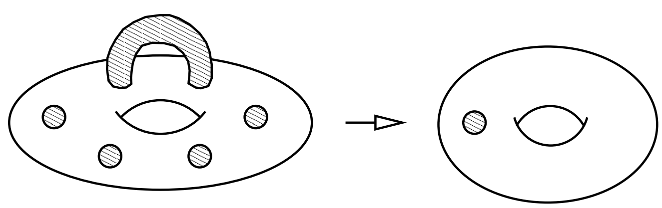

Figure 2. A degree 6 branched covering. The shaded handle maps to the shaded disk with 2 branch points.

Letting Σ_g denote a smooth oriented surface of genus g, we have the following purely topological result.

Corollary 4.11 *Up to the action of $\mathrm{Diff}^+(\Sigma_2) \times \mathrm{Diff}^+(\Sigma_1)$, there is a unique degree d covering map*

$$p : \Sigma_2 \to \Sigma_1,$$

branched over two points, such that $p_* : H_1(\Sigma_2, \mathbb{Z}) \to H_1(\Sigma_1, \mathbb{Z})$ is surjective.

Proof. Maps p as above are classified by components of the locus $U \subset \Omega E_{d^2}$ where the relative and absolute periods of ω are distinct. Since U is Zariski open and ΩE_{d^2} is connected, p is unique up to diffeomorphism. ∎

The unique branched covering of degree 6 is shown in Figure 2.

Theorem 4.12 *The singular locus of the orbifold $\Omega\mathcal{M}_2$ coincides with ΩE_4, with local fundamental group $\mathbb{Z}/2$.*

Proof. A point (X, ω) in the manifold cover $\Omega\mathcal{T}_2 \to \Omega\mathcal{M}_2$ has a nontrivial stabilizer in Mod_2 iff ω descends to a holomorphic 1-form on $E = X/\Gamma$, where Γ is a nontrivial subgroup of $\mathrm{Aut}(X)$. But then E must be an elliptic curve, and $X \to E$ must be a regular degree 2 branched cover. ∎

Notes. The above results on real multiplication in genus two originate in the work of Humbert in the late 1890s [Hu]; see also [vG, Ch. IX] and [Ru, §4]. Additional material on real multiplication and Hilbert modular varieties can be found in [HG], [vG] and [BL]. Elliptic differentials and the Humbert surfaces H_{d^2} are discussed in detail in [Ka1]; see also [Her], [KS1], [Ka2]. We remark that our notation here differs slightly from [Mc1], where eigenforms for $\mathbb{Q} \times \mathbb{Q}$ were excluded from $\mathcal{E}_2$.

5 Recognizing eigenforms

In this section we introduce a homological version of the group $\mathrm{SL}(X, \omega)$, and recall the notion of complex flux. We then establish the following characterization of eigenforms for real multiplication.

Theorem 5.1 *Let $K \subset \mathbb{R}$ be a real quadratic field, and let (X, ω) belong to $\Omega\mathcal{M}_2$. Then the following conditions are equivalent.*

1. $\mathrm{Jac}(X)$ *admits real multiplication by K with ω as an eigenform.*
2. *The trace field of* $\mathrm{SL}(H_1(X, \mathbb{Q}), \omega)$ *is K.*
3. *The span $S(\omega) \subset H^1(X, \mathbb{R})$ of* $(\mathrm{Re}\,\omega, \mathrm{Im}\,\omega)$ *is defined over K, and satisfies $S(\omega)' = S(\omega)^\perp$.*
4. *After replacing (X, ω) by $g \cdot (X, \omega)$ for suitable $g \in \mathrm{GL}_2^+(\mathbb{R})$, the form ω has absolute periods in $K(i)$ and zero complex flux.*

We also give similar results for $K = \mathbb{Q} \times \mathbb{Q}$, and applications to Teichmüller curves.

The homological affine group. We begin by explaining condition (2). Let $F \subset \mathbb{R}$ be a subring (such as $\mathbb{Z}$ or $\mathbb{Q}$). Given $(X, \omega) \in \Omega\mathcal{M}_g$, consider the absolute period map

$$I_\omega : H_1(X, F) \to \mathbb{C} \cong \mathbb{R}^2$$

defined by $I_\omega(C) = \int_C \omega$. We say a symplectic automorphism Φ of $H_1(X,F)$ is *affine* with respect to ω if there is a real-linear map $D\Phi : \mathbb{C} \to \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} H_1(X,F) & \xrightarrow{\Phi} & H_1(X,F) \\ {\scriptstyle I_\omega}\big\downarrow & & {\scriptstyle I_\omega}\big\downarrow \\ \mathbb{C} & \xrightarrow{D\Phi} & \mathbb{C} \end{array} \tag{5.1}$$

commutes. Equivalently, let $S(\omega) \subset H^1(X,\mathbb{R})$ denote the span of $(\operatorname{Re}\omega, \operatorname{Im}\omega)$; then Φ is affine iff $\Phi^*(S(\omega)) = S(\omega)$.

The group of all Φ as above will be denoted by $\operatorname{Aff}^+(H_1(X,F),\omega)$, and its image under $\Phi \mapsto D\Phi$ by

$$\mathrm{SL}(H_1(X,F),\omega) \ \subset \ \mathrm{SL}_2(\mathbb{R}).$$

This group records all the affine symmetries of (X,ω) that are feasible on a homological level. Note that $\mathrm{SL}(H_1(X,F),\omega)$ changes by conjugacy in $\mathrm{SL}_2(\mathbb{R})$ if we replace (X,ω) by $g\cdot(X,\omega)$, $g \in \mathrm{SL}_2(\mathbb{R})$.

Properties. Any affine homeomorphism of (X,ω) induces an affine map on homology, so we have:

Theorem 5.2 *The group* $\mathrm{SL}(X,\omega)$ *is a subgroup of* $\mathrm{SL}(H_1(X,F),\omega)$.

The *trace field* of $\mathrm{SL}(H_1(X,\mathbb{Q}),\omega)$ is the field generated over $\mathbb{Q}$ by $\{\operatorname{Tr} D\Phi\}$; it is constant along the orbit $\mathrm{SL}_2(\mathbb{R})\cdot(X,\omega)$.

Theorem 5.3 *The trace field of* $\mathrm{SL}(H_1(X,\mathbb{Q}),\omega)$ *has degree at most g over $\mathbb{Q}$, where g is the genus of X.*

Proof. Using the identity $\operatorname{tr}(A)\operatorname{tr}(B) = \operatorname{tr}(AB) + \operatorname{tr}(AB^{-1})$, any element of the trace field can be expressed in the form $t = \sum a_i \operatorname{tr}(D\Phi_i)$, $a_i \in \mathbb{Q}$. Since the endomorphism

$$T = \sum a_i(\Phi_i + \Phi_i^{-1})$$

of $H^1(X,\mathbb{Q})$ satisfies $T|S(\omega) = tI$, t is an eigenvalue of multiplicity at least two. Since $\dim H^1(X) = 2g$, we have $\deg(\mathbb{Q}(t)/\mathbb{Q}) \le g$. Compare [Mc1, Thm. 5.1], [KS2]. ■

Theorem 5.4 *Let $(X,\omega) \in \Omega E_D$ be an eigenform of genus two. Then we have:*

$$\begin{aligned} \mathrm{SL}(H_1(X,\mathbb{Q}),\omega) &\cong \mathrm{SL}_2(\mathbb{Q}(\sqrt{D})), \quad \textit{and} \\ \mathrm{SL}(H_1(X,\mathbb{Z}),\omega) &\cong \begin{cases} \mathrm{SL}_2(\mathbb{Z}) & \textit{if } D \textit{ is a square,} \\ \mathrm{SL}_2(\mathfrak{o}_D) & \textit{otherwise.} \end{cases} \end{aligned}$$

Proof. The condition $\Phi^*(S(\omega)) = S(\omega)$ is equivalent to the condition that Φ^* commutes with $\mathfrak{o}_D$ acting on $H^1(X, \mathbb{R})$. Thus we have

$$\mathrm{Aff}^+(H_1(X,\mathbb{Z}),\omega) \cong \Gamma_D \cong \mathrm{SL}_2(\mathfrak{o}_D)$$

(see §4). The action of affine automorphisms on $S(\omega)$ gives a map $\mathrm{SL}_2(\mathfrak{o}_D) \to \mathrm{SL}_2(\mathbb{R})$ which is an isomorphism when D is not a square; otherwise, its image is $\mathrm{SL}_2(\mathbb{Z})$, and the result follows. ∎

Cohomology over $\boldsymbol{K}$. Next we explain condition (3). The subspace $S(\omega) \subset H^1(X,\mathbb{R})$ is *defined over* K if $S(\omega) = V \otimes_K \mathbb{R}$ for some $V \subset H^1(X, K)$. The Galois involution of $K/\mathbb{Q}$ determines a second subspace $S(\omega)' = V' \otimes \mathbb{R}$, and (3) says the cohomology of X decomposes as an orthogonal direct sum

$$H^1(X,\mathbb{R}) = S(\omega) \oplus S(\omega)'$$

with respect to the symplectic form.

Complex flux. Finally we explain condition (4). Consider the complex extension field $K(i) \subset \mathbb{C}$. Let $(k_1 + ik_2)' = k_1' + ik_2'$ and $\overline{k_1 + ik_2} = k_1 - ik_2$. These involutions generate the Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$ of $K(i)/\mathbb{Q}$.

Now suppose that $(X, \omega) \in \mathcal{M}_2$ has absolute periods in $K(i)$. Then ω determines both a cohomology class $[\omega]$ in $H^1(X, K(i))$, and its Galois conjugate classes $[\overline{\omega}]$, $[\omega']$ and $[\overline{\omega}']$. We say ω has *zero complex flux* [Mc1] if

$$\int \omega \wedge \omega' = \int \omega \wedge \overline{\omega}' = 0.$$

Remark. If ω has *relative* periods in $K(i)$ and zero complex flux, then the group $\mathrm{SL}(X,\omega)$ is large — its limit set is the full circle at infinity [Mc1, Thm. 8.4].

Proof of Theorem 5.1. By Theorem 5.4, (1) $\implies$ (2). To see (2) $\implies$ (3), choose $\Phi \in \mathrm{Aff}^+(H_1(X,\mathbb{Q}),\omega)$ such that $\mathrm{Tr}(D\Phi) = k$ and $K = \mathbb{Q}(k)$. Let $T = \Phi^* + (\Phi^*)^{-1}$ acting on $H^1(X,\mathbb{Q})$. Since Φ^* preserves the splitting

$$H^1(X,\mathbb{R}) = S(\omega) \oplus (S\omega)^\perp,$$

we have $T = \left(\begin{smallmatrix} kI & 0 \\ 0 & k'I \end{smallmatrix}\right)$ with respect to this decomposition. Consequently the subspace $S(\omega) = \mathrm{Ker}(T - kI)$ is defined over K, and $S(\omega)^\perp = \mathrm{Ker}(T - k'I)$ is its Galois conjugate.

The reverse argument shows (3) $\implies$ (1). Indeed, suppose the two factors in the direct sum decomposition

$$H^1(X,\mathbb{R}) = S(\omega) \oplus S(\omega)^\perp$$

are defined over K and Galois conjugate, and write $K = \mathbb{Q}(k)$. Then $T = \left(\begin{smallmatrix} kI & 0 \\ 0 & k'I \end{smallmatrix}\right)$ preserves $H^1(X,\mathbb{Q})$. Orthogonality of the factors implies T is self-adjoint with respect to the symplectic form. The first factor maps to a complex line under the isomorphism $H^1(X,\mathbb{R}) \cong \Omega(X)$, so the second does as well. Thus T defines an element of $\mathrm{End}(\mathrm{Jac}(X)) \otimes \mathbb{Q}$, and consequently $\mathrm{Jac}(X)$ admits real multiplication by $\mathbb{Q}(T) \cong \mathbb{Q}(k) \cong K$.

The equivalence of (3) and (4) follows easily by choosing a suitable basis for $S(\omega)$. ■

A similar argument shows:

Theorem 5.5 *For $K = \mathbb{Q} \times \mathbb{Q}$, the following conditions are equivalent.*

1. *The Jacobian of X admits real multiplication by K with ω as an eigenform.*
2. *The absolute periods of ω form a lattice in $\mathbb{C}$.*
3. *The span $S(\omega) \subset H^1(X,\mathbb{R})$ of $(\mathrm{Re}\,\omega, \mathrm{Im}\,\omega)$ is defined over $\mathbb{Q}$.*

From condition (2) in Theorems 5.1 and 5.5 we have:

Corollary 5.6 *Membership in the eigenform locus $\mathcal{E}_2$ depends only on the absolute period map $I_\omega : H_1(X,\mathbb{Z}) \to \mathbb{C}$.*

Corollary 5.7 *The eigenform locus $\mathcal{E}_2 \subset \mathcal{M}_2$ is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$, as is each of its connected components ΩE_D.*

Compare [Mc1, Thm. 7.2].

Theorem 5.8 *If $\mathrm{SL}(X,\omega)$ contains a hyperbolic element, then $(X,\omega) \in \mathcal{E}_2$.*

Proof. In fact, if $A \in \mathrm{SL}(X,\omega)$ is hyperbolic with irrational trace, then (X,ω) is an eigenform for real multiplication by $\mathbb{Q}(\mathrm{tr}(A))$ (Theorem 5.1). Otherwise, ω is an elliptic differential [Mc2, Thm. 9.5]. ■

Corollary 5.9 *If (X,ω) generates a Teichmüller curve, then (X,ω) is an eigenform. In particular, every Teichmüller curve in $\mathcal{M}_2$ lies on a Hilbert modular surface.*

Theorem 5.10 *A form (X,ω) in $\Omega\mathcal{M}_2(2)$ generates a Teichmüller curve iff $(X,\omega) \in \mathcal{E}_2$.*

Proof. Let $(X,\omega) \in \Omega\mathcal{M}_2(2)$ be an eigenform for real multiplication by K. If K is a quadratic field, then $\mathrm{SL}(X,\omega)$ is a lattice by [Mc1, Thm 1.3]. Otherwise, $\omega = \pi^*(dz)$ is an elliptic differential, pulled back via a map $\pi : X \to E = \mathbb{C}/\Lambda$. Since ω has a double zero, π is branched over only one point, and hence a subgroup of finite index in $\mathrm{SL}(E,dz) \cong \mathrm{SL}_2(\mathbb{Z})$ lifts to $\mathrm{SL}(X,\omega)$ [GJ]; so $\mathrm{SL}(X,\omega)$ is a lattice in this case as well.

The converse is immediate from Theorem 5.8. ■

Corollary 5.11 *For each discriminant D, there is a finite collection of Teichmüller curves such that $\Omega E_D \cap \Omega\mathcal{M}_2(2) = \bigcup \Omega V_i$.*

Proof. Each connected component of $\Omega E_D \cap \Omega\mathcal{M}_2(2)$ lies over a Teichmüller curve by the preceding result, and the number of components is finite because the intersection is an algebraic variety. ∎

6 Algebraic sums of 1-forms

In this section we introduce the notion of algebraic sums of 1-forms. We find that whenever an eigenform in genus 2 is presented as a sum of forms of genus 1, the corresponding elliptic curves are isogenous. Conversely, we will show:

Theorem 6.1 *Let (X, ω) be a holomorphic 1-form of genus 2 that can be presented, in more than one way, as an algebraic sum*

$$(X, \omega) \cong (E_1, \omega_1) + (E_2, \omega_2)$$

of isogenous forms of genus 1. Then $\mathrm{Jac}(X)$ *admits real multiplication with ω as an eigenform.*

Isogeny. An *isogeny* between a pair of elliptic curves is a surjective holomorphic map $p : E_1 \to E_2$. We say a pair of 1-forms $(E_i, \omega_i) \in \Omega\mathcal{M}_1$ are isogenous if there is a $t > 0$ and an isogeny $p : E_1 \to E_2$ such that $p^*(\omega_2) = t\omega_1$. This is equivalent to the condition that $t\Lambda_1 \subset \Lambda_2$, where $\Lambda_i \subset \mathbb{C}$ is the period lattice of (E_i, ω_i).

Algebraic sum. Let (X, ω) be a form of genus g, and let (Y_i, ω_i) be forms of genus g_i, $g = g_1 + g_2$. A symplectic isomorphism

$$H_1(X, \mathbb{Z}) \cong H_1(Y_1, \mathbb{Z}) \oplus H_1(Y_2, \mathbb{Z})$$

presents (X, ω) as an *algebraic sum*,

$$(X, \omega) \cong (Y_1, \omega_1) + (Y_2, \omega_2),$$

if we have

$$[\omega] = [\omega_1] + [\omega_2]$$

upon passing to cohomology with coefficient in $\mathbb{C}$. Two algebraic splittings of (X, ω) are *equivalent* if they come from the same unordered splitting $H_1(X) = H_1(Y_1) \oplus H_1(Y_2)$.

We emphasize that the sublattices $H_1(Y_1, \mathbb{Z})$ and $H_1(Y_2, \mathbb{Z})$ of $H_1(X, \mathbb{Z})$ are orthogonal and of determinant 1.

Genus two. We will be interested in expressing forms of genus 2 as algebraic sums of forms of genus 1,

$$(X, \omega) \cong (E_1, \omega_1) + (E_2, \omega_2).$$

In this case the forms $(E_i, \omega_i) \in \Omega\mathcal{M}_1$ are uniquely determined by the splitting

$$H_1(X, \mathbb{Z}) = S_1 \oplus S_2, \tag{6.1}$$

$S_i = H_1(E_i, \mathbb{Z})$; indeed, we have $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz)$ where $\Lambda_i = I_\omega(S_i)$.

On the other hand, not every symplectic splitting as in (6.1) determines an algebraic sum; the lattices $\Lambda_i = I_\omega(S_i)$ must be nondegenerate, and each map $S_i \to \Lambda_i \subset \mathbb{C}$ must preserve the orientation of S_i as a symplectic subspace of $H^1(X, \mathbb{Z})$.

Real multiplication. To motivate Theorem 6.1, we first prove:

Theorem 6.2 *Let (X, ω) be an eigenform for real multiplication, expressed as an algebraic sum*

$$(X, \omega) \cong (E_1, \omega_1) + (E_2, \omega_2)$$

of forms of genus 1. Then (E_1, ω_1) and (E_2, ω_2) are isogenous.

Lemma 6.3 *Suppose $A \in \mathcal{A}_2$ admits real multiplication by a field K, and $H_1(A, \mathbb{Q}) = S \oplus S^\perp$, $\dim_{\mathbb{Q}} S = 2$. Then there is a $k \in K$ such that $kS = S^\perp$.*

Proof. Let (a, b) be a basis for S over $\mathbb{Q}$, with $\langle a, b\rangle = 1$. Let $k \neq 0$ be an element in the kernel of the map $K \to \mathbb{Q}$ given by $k \mapsto \langle ka, b\rangle$. By self-adjointness of the action of K, we have $\langle ka, b\rangle = \langle a, kb\rangle = 0$, as well as $\langle ka, a\rangle = \langle kb, b\rangle = 0$. Thus $kS = S^\perp$. ∎

Proof of Theorem 6.2. Let

$$H_1(X, \mathbb{Z}) = L_1 \oplus L_2 \cong H_1(E_1, \mathbb{Z}) \oplus H_1(E_2, \mathbb{Z})$$

be the isomorphism underlying the algebraic sum, and let $\Lambda_i = I_\omega(L_i) = \mathrm{Per}(\omega_i)$.

Assume (X, ω) is an eigenform for real multiplication by K. If $K = \mathbb{Q} \times \mathbb{Q}$, then $\Lambda = I_\omega(H_1(X, \mathbb{Z})) \cong \mathbb{Z}^2$ is a lattice in $\mathbb{C}$ containing Λ_1 and Λ_2 with finite index, so (E_1, ω_1) and (E_2, ω_2) are isogenous.

Otherwise K is a field. The eigenform ω determines an embedding $K \subset \mathbb{R}$ such that $I_\omega(k \cdot v) = kI_\omega(v)$ for all v in $H_1(X, \mathbb{Q})$. Let $S_i = L_i \otimes \mathbb{Q} \subset H_1(X, \mathbb{Q})$. By the preceding Lemma, we have $S_1^\perp = S_2 = kS_1$ for some $k \in K$; therefore

$$\Lambda_1 \otimes \mathbb{Q} = I_\omega(S_1) = I_\omega(kS_2) = k\Lambda_2 \otimes \mathbb{Q},$$

which again implies isogeny. ∎

Isogeny and $\mathbf{SL_2(\mathbb{Q})}$. To prove Theorem 6.1, we will show that multiple splittings of (X, ω) with isogenous summands make the group $\mathrm{SL}(H_1(X, \mathbb{Q}), \omega)$ large.

Lemma 6.4 *Let $(X,\omega) = (E_1,\omega_1) + (E_2,\omega_2)$ be an algebraic sum of isogenous forms. Then the largest subgroup*

$$H \subset \mathrm{Aff}^+(H_1(X,\mathbb{Q}),\omega)$$

that preserves each summand in the splitting

$$H_1(X,\mathbb{Q}) = H_1(E_1,\mathbb{Q}) \oplus H_1(E_2,\mathbb{Q})$$

is isomorphic to $\mathrm{SL}_2(\mathbb{Q})$*.*

Proof. Let $p : E_1 \to E_2$ be an isogeny satisfying $p^*(\omega_2) = t\omega_1$, and let

$$P = p_* : H_1(E_1,\mathbb{Q}) \to H_1(E_2,\mathbb{Q}).$$

Then $tI_\omega = I_\omega P$ on $H_1(E_1) \subset H_1(X)$. It follows that any symplectic automorphism A of $H_1(E_1,\mathbb{Q})$ has a unique extension

$$\Phi = A \oplus PAP^{-1}$$

to an ω-affine automorphism of $H_1(E_1) \oplus H_1(E_2) \cong H_1(X)$. Thus H is isomorphic to $\mathrm{Sp}(H_1(E_1,\mathbb{Q})) \cong \mathrm{SL}_2(\mathbb{Q})$. ∎

Proof of Theorem 6.1. Suppose (X,ω) can be presented as an algebraic sum of isogenous forms in two different ways,

$$(X,\omega) \cong (E_1,\omega_1) + (E_2,\omega_2) \cong (F_1,\eta_1) + (F_2,\eta_2).$$

Then we have

$$\mathrm{Per}(\omega) = \mathrm{Per}(\omega_1) + \mathrm{Per}(\omega_2) = \Lambda_1 + t\Lambda_2,$$

where Λ_1 and Λ_2 are commensurable lattices. If $t > 0$ is rational then the periods of ω also form a lattice, in which case (X,ω) is an eigenform for real multiplication by $\mathbb{Q} \times \mathbb{Q}$ by Theorem 5.5.

Now assume t is irrational. Then the period map $I_\omega : H_1(X,\mathbb{Q}) \to \mathbb{C}$ is an embedding. This implies the map $\Phi \mapsto D\Phi$ gives an isomorphism

$$G = \mathrm{Aff}^+(H_1(X,\mathbb{Q}),\omega) \cong \mathrm{SL}(H_1(X,\mathbb{Q}),\omega) = \Gamma.$$

We may assume the trace field K of Γ is $\mathbb{Q}$, since otherwise (X,ω) is an eigenform for real multiplication by K (Theorem 5.1).

Let $H \subset G$ be the subgroup stabilizing the splitting

$$H_1(X) = H_1(E_1) \oplus H_1(E_2).$$

By the preceding Lemma, H isomorphic to $\mathrm{SL}_2(\mathbb{Q})$. But any proper extension of $DH \cong \mathrm{SL}_2(\mathbb{Q}) \subset \Gamma$ has irrational trace field, so $DH = \Gamma$ and $H = G$. Thus the isogeny from E_1 to E_2 determines a map $P : H_1(E_1, \mathbb{Q}) \to H_1(E_2, \mathbb{Q})$ satisfying $\det(P) > 0$ and $P(\Phi x) = \Phi(P(x))$ for all $\Phi \in G$.

Now consider the second splitting, $H_1(X) = H_1(F_1) \oplus H_1(F_2)$. Let $A_1 \oplus A_2$ denote two factors of the map

$$H_1(F_1, \mathbb{Z}) \hookrightarrow H_1(E_1, \mathbb{Z}) \oplus H_1(E_2, \mathbb{Z}).$$

Then $\det(A_1)+\det(A_2) = 1$, since the oriented sublattice $H_1(F_1, \mathbb{Z}) \subset H_1(X, \mathbb{Z})$ has determinant one.

By the same reasoning as above, G stabilizes the splitting $H_1(F_1) \oplus H_1(F_2)$. But if the second splitting is different from the first, then the subspace $H_1(F_1, \mathbb{Q})$ is the graph of a map $Q : H_1(E_1, \mathbb{Q}) \to H_1(E_2, \mathbb{Q})$, $Q = A_2 \circ A_1^{-1}$, that also respects the action of G. Since the action of $G \cong \mathrm{SL}_2(\mathbb{Q})$ on $H_1(E_i, \mathbb{Q})$ is irreducible, we have $Q = A_2 \circ A^{-1} = cP$ for some $c \in \mathbb{Q}^*$. But this implies $\det(A_1)\det(A_2) > 0$, which is impossible for two integers satisfying $\det(A_1) + \det(A_2) = 1$. ■

7 Connected sums of 1-forms

In this section we introduce the connected sum construction and establish:

Theorem 7.1 *Any holomorphic 1-form of genus 2 can be expressed in infinitely many ways as a connected sum* $(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$ *of forms of genus 1.*

We also discuss the relation to algebraic sum.

Connected sum. Let $I = [0, v] = [0, 1] \cdot v$ be the segment from 0 to $v \neq 0$ in $\mathbb{C}$. Let (Y_i, ω_i) be holomorphic 1-forms of genus g_i, $i = 1, 2$, and let

$$\gamma_i : I \to Y_i$$

be smooth embeddings such that $\gamma_i^*(\omega_i) = dz$. The arcs $J_i = \gamma_i(I)$ are straight geodesics on $(Y_i, |\omega_i|)$, possibly with zeros of ω at their endpoints.

Slitting the two surfaces open along these arcs, and gluing corresponding edges using γ_i, we obtain the *connected sum*

$$(X, \omega) = (Y_1, \omega_1) \underset{I}{\#} (Y_2, \omega_2).$$

Here X has genus $g_1 + g_2$. If J_i joins zeros of orders a_i and b_i on Y_i, then ω has zeros of order $a_1 + a_2 + 1$ and $b_1 + b_2 + 1$ on X. The simplest case, in which there are no zeros on the arcs J_i, results in a pair of simple zeros on X; see Figure 3.

It is straightforward to check that the connected sum operation commutes with the action of $\mathrm{GL}_2^+(\mathbb{R})$: we have

$$g \cdot ((Y_1, \omega_1) \underset{I}{\#} (Y_2, \omega_2)) \quad = \quad g \cdot (Y_1, \omega_1) \underset{g \cdot I}{\#} g \cdot (Y_2, \omega_2) \qquad (7.1)$$

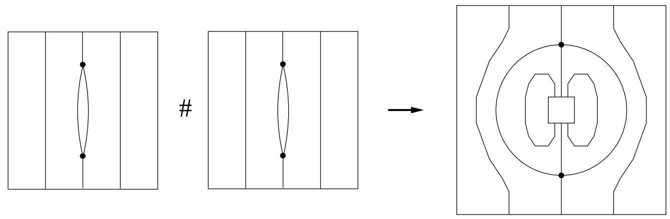

Figure 3. Local picture of a connected sum.

for all $g \in \mathrm{GL}_2^+(\mathbb{R})$.

Splitting. To present (X, ω) as a connected sum, it suffices to find a loop $L \subset X$ consisting of a pair of homologous saddle connections. Then L *splits* (X, ω) into two slit surfaces; regluing the slits, we obtain a pair (Y_1, ω_1), (Y_2, ω_2) whose connected sum is the original surface.

Genus two. We will be interested in presenting forms of genus two as connected sums of forms of genus one,

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2).$$

It is convenient to identify elements of $\Omega\mathcal{M}_1$ with lattices $\Lambda \subset \mathbb{C}$, via the correspondence $(E, \omega) = (\mathbb{C}/\Lambda, dz)$. The gluing data can always be normalized so that $\gamma_i(z) = z$.

Thus a connected sum of tori is completely determined by a pair of lattices $\Lambda_1, \Lambda_2 \subset \mathbb{C}$, and a vector $v \in \mathbb{C}^*$, such that

$$[0, v] \cap \Lambda_1 = [0, v] \cap \Lambda_2 = \{0\}. \tag{7.2}$$

The last condition guarantees that $I = [0, v]$ embeds in $E_i = \mathbb{C}/\Lambda_i$.

We extend the operation of connected sum in a natural way to include the case where

$$[0, v] \cap \Lambda_1 = \{0, v\}, \quad [0, v] \cap \Lambda_2 = \{0\} \tag{7.3}$$

or vice versa. In this case I maps to a loop in E_1 and remains embedded in E_2. The connected sum then results in a double zero for ω, lying on a figure-eight $L \subset X$ coming from the slits on E_1 and E_2.

Local homeomorphism. Let $D(1,1)$ and $D(2)$ denote the set of triples

$$(\Lambda_1, \Lambda_2, v) \in \Omega\mathcal{M}_1 \times \Omega\mathcal{M}_1 \times \mathbb{C}^*$$

satisfying (7.2) and (7.3) respectively. (In $D(2)$ we allow I to map to a loop in *either* E_1 or E_2.) The domain $D(1,1)$ is open and dense, while $D(2)$ is open and dense in a codimension one hypersurface. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on the space of triples $(\Lambda_1, \Lambda_2, v)$, leaving $D(1,1)$ and $D(2)$ invariant.

By sending the data $(\Lambda_1, \Lambda_2, v)$ to the connected sum it defines, we obtain natural maps $\Phi(1,1) : D(1,1) \to \Omega\mathcal{M}_2(1,1)$ and $\Phi(2) : D(2) \to \Omega\mathcal{M}_2(2)$.

Theorem 7.2 *The connected sum mappings* $\Phi(1,1)$ *and* $\Phi(2)$ *are surjective,* $\mathrm{GL}_2^+(\mathbb{R})$*-equivariant local covering maps.*

Proof. Surjectivity is Theorem 7.1, and equivariance follows from equation (7.1). To see these maps are locally coverings (in the category of orbifolds), recall that the relative periods of (X,ω) provide local coordinates on the universal cover of each stratum (§3). The absolute periods determine Λ_1 and Λ_2, while v is a relative period of ω. Thus $\Phi(1,1)$ and $\Phi(2)$ have continuous local inverses, after passing to a manifold local cover in $\Omega\mathcal{M}_2(1,1)$ and $\Omega\mathcal{M}_2(2)$. ∎

Note. Since the result of a connected sum is independent of the order of its summands, the map $\Phi(1,1)$ is locally 2-to-1 along the diagonal $\Lambda_1 = \Lambda_2$ in $D(1,1)$. The image of the diagonal is the singular locus ΩE_4 of $\Omega\mathcal{M}_2$, discussed in Theorem 4.12.

Saddle connections. Now fix a surface (X,ω) of genus two, and let $\eta : X \to X$ be the hyperelliptic involution.

Theorem 7.3 *Let* $J \supset Z(\omega)$ *be a saddle connection such that* $J \neq \eta(J)$*. Then* (X,ω) *splits along* $L = J \cup \eta(J)$ *as a connected sum of tori.*

The proof is straightforward. Note that L is a loop when ω has simple zeros, and a figure eight otherwise; and $J = \eta(J)$ iff J contains a Weierstrass point in its interior.

To see that many splittings exist, we next establish:

Theorem 7.4 *Let* (X,ω) *be a form of genus two, and let* C *be a cylinder on* X*. Then there is a saddle connection* $J \supset Z(\omega)$ *such that* $J \neq \eta(J)$ *and* J *does not cross* ∂C*.*

(Note: We allow J to have endpoints on ∂C or to be contained in ∂C.)

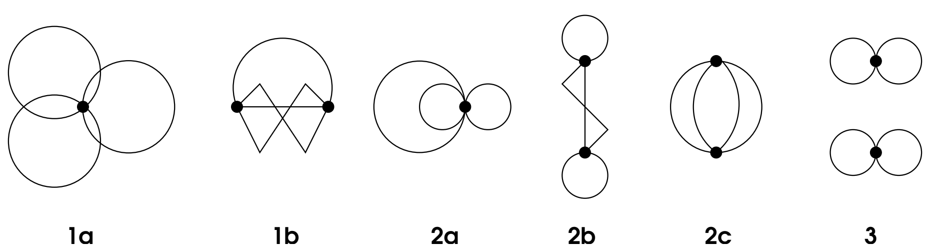

Figure 4. The 6 possible ribbon graphs for periodic 1-forms of genus two.

Let $\mathcal{F}$ be the foliation of $(X,|\omega|)$ by geodesics parallel to ∂C.

Lemma 7.5 *The required saddle connection exists if the foliation* $\mathcal{F}$ *is periodic.*

Proof. In this case the complement of the spine S of $\mathcal{F}$ is a union of $n \leq 3$ cylinders, one of which is C. Each cylinder contains two Weierstrass points, so $\eta|S$ has exactly $6 - 2n$ fixed-points.

The spine itself is a ribbon graph isomorphic to one of the six examples shown in Figure 4 (compare [Z]).

In this Figure the round dots represent zeros of ω and the edges represent saddle connections. The local planar embedding at each zero is essential, while the internal crossing of edges should be ignored. The graphs are labeled according to the number of cylinders of $\mathcal{F}$. Note that ω has a double zero in cases (1a) and (2a), and otherwise a pair of simple zeros.

It is straightforward to construct J in each case. In cases (2a) and (2b) we can take J to be one of the edges of S whose interior is disjoint from the two fixed-points of $\eta|S$. In the remaining cases, one of the cylinders $C' \subset X - S$ contains the interiors of three disjoint saddle connections joining the zeros of ω. Since C' contains only two Weierstrass points, one of these saddle connections satisfies $\eta(J) \neq J$. ∎

Proof of Theorem 7.4. Consider the subgraph of the spine of $\mathcal{F}$ given by $L = \partial\overline{C}$.

If L is empty, then $\mathcal{F}$ is periodic with one cylinder and the preceding Lemma applies.

If L is connected, then it is either a loop or a figure eight containing $Z(\omega)$ and invariant under the hyperelliptic involution; thus it has the form $L = J \cup \eta(J)$, where J is a saddle connection, and we are done.

Now suppose L is disconnected. Then $L = L_1 \cup L_2$ has two components, and ω has a pair of simple zeros z_1, z_2 with $z_i \in L_i$. If L is a pair of figure-eights, then $\mathcal{F}$ is periodic with 3 cylinders and again the Lemma applies.

Thus we are left with the case where L is a pair of topological circles. In this case $\overline{C}$ is an embedded annulus in X, and so its complement $F = X - C$ is a torus with 2 boundary components. Note that $(F, |\omega|)$ has interior angle 3π at each zero $z_i \in \partial F$.

We can assume that L coincides with the spine S, since otherwise $\mathcal{F}$ is periodic. Let $A \in \mathrm{GL}_2^+(\mathbb{R})$ be a linear transformation that acts by isometry on the leaves of $\mathcal{F}$ and stretches in the orthogonal direction. Then, after replacing (X,ω) with $A^n \cdot (X,\omega)$ for $n \gg 0$, we can also assume that L_1 and L_2 are the two *shortest* saddle connections on X.

Now let $K \subset F$ be the shortest geodesic joining z_1 to z_2. If $\eta(K) \neq K$ we can take $J = K$ to complete the proof; so assume that $\eta(K) = K$. Let $\alpha + \beta = 3\pi$ be the two interior angles formed by K and ∂F at z_2.

If $\min(\alpha,\beta) < \pi$, we can construct a broken geodesic $M' = K * L_2$ also joining z_1 to z_2, with a bending angle of less than π at the break. Let $M \subset F$ be the geodesic in the same homotopy class as M' relative to its endpoints. Because of the bend in M', we have

$$|M| < |M'| = |K| + |L_2|.$$

Note that $M \neq \eta(M)$ because these geodesics are in different homotopy classes relative to their endpoints. Thus if M is a saddle connection from z_1 to z_2, we can take $J = M$ to complete the proof. Otherwise, M contains both a saddle connection N from z_1 to z_2, and at least one other saddle connection, necessarily of length $\geq |L_2|$. But then $|N| \leq |M| - |L_2| < |K|$, contrary to our assumption that K is the shortest geodesic joining z_1 to z_2.

Finally suppose $\alpha, \beta \geq \pi$. Using the fact that $\eta(K) = K$, it is not hard to see that the metric completion of $(F - K, |\omega|)$ is isometric to $(E - P, |dz|)$, the complement of an open parallelogram P in a flat torus E. (When α or β is equal to π, P degenerates to a slit.) There is a natural collapsing map

$$\pi : (E - P, \partial P) \to (\overline{F}, L_1 \cup K \cup L_2)$$

that sends the cyclically ordered vertices (a, b, b', a') of ∂P to (z_1, z_2, z_2, z_1) (Figure 5). The opposite edges ab and $a'b'$ are identified to form K. Note that the hyperelliptic involution η of $E - P$ sends a to b' and b to a'.

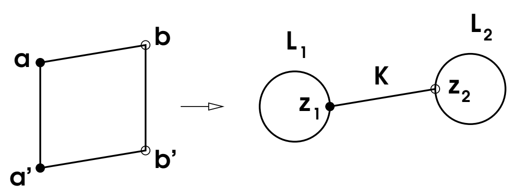

Figure 5. The collapsing map $P \to L_1 \cup K \cup L_2$.

Next we construct a straight geodesic I joining a to b through the interior of $E - P$. Let $R \subset E - P$ be a maximal parallelogram with sides parallel to those of P, and sharing the edge bb' with P. By maximality, ∂R contains a or a'. If $a \in \partial R$, let I be straight geodesic from a to b inside R. If $a' \in \partial R$, let I' join a' to b' through R and let $I = \eta(I')$.

Now consider the saddle connection $J = \pi(I)$ from z_1 to z_2. We have $I \neq \eta(I)$ since the latter segment joins a' to b'. Since I runs through the interior of $E - P$, we have $\eta(J) = \pi(\eta(I)) \neq J$. By construction J is disjoint from C, so the proof is complete. ∎

Corollary 7.6 *The slopes of saddle connections $J \supset Z(\omega)$ with $J \neq \eta(J)$ are dense in $\mathbb{R} \cup \{\infty\}$.*

Proof. Let $U \subset \mathbb{P}^1(\mathbb{R})$ be an open interval. We will show there is a saddle connection J as above with slope in U.

It is known that the uniquely ergodic slopes are dense, as are the slopes of cylinders [MT, §3, §4]. Thus we can find a long, thin cylinder C on X such that every saddle connection with slope outside U crosses ∂C. By the preceding result, we can find a J that does not cross ∂C, and hence its slope is in U. ∎

Proof. of Theorem 7.1. The saddle connections J provided by the Corollary give, by Theorem 7.3, infinitely many distinct splittings of (X, ω). ∎

Algebraic sum. By Mayer–Vietoris, any geometric splitting of (X, ω) gives rise to a natural isomorphism $H_1(X, \mathbb{Z}) \cong H_1(Y_1, \mathbb{Z}) \oplus H_1(Y_2, \mathbb{Z})$, and hence an algebraic splitting $(X, \omega) = (Y_1, \omega_1) + (Y_2, \omega_2)$. Thus Theorem 7.1 gives immediately:

Corollary 7.7 *Any form of genus two can be expressed in infinitely many ways as an algebraic sum of forms of genus one.*

Remarks. Variants of the connected sum construction are discussed in [Mc2, §7] and [EMZ, §7]. The argument above also uses the 'parallelogram construction', developed in more detail in [EMZ, §10]. We remark that the connected sum operation can be extended to yield points in $\overline{\mathcal{M}}_2$; for example, if I maps to a loop in both tori, then $(E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$ naturally defines a form on a stable curve $X \in \mathcal{M}_{1,2} \subset \partial \mathcal{M}_2$.

8 Eigenforms as connected sums

In this section we discuss the special properties of splittings of eigenforms $(X, \omega) \in \Omega E_D$, and the natural measure on $\Omega E_D(1, 1)$.

Isogeny. We begin by observing that eigenforms are characterized by isogeny of their summands. More precisely, since any geometric sum gives rise to an algebraic sum, Theorems 6.1 and 6.2 imply:

Theorem 8.1 *Any splitting $(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$ of an eigenform for real multiplication has isogenous summands.*

Theorem 8.2 *If $(X, \omega) \in \Omega \mathcal{M}_2$ has splits as a connected sum of isogenous forms in two different ways, then (X, ω) is an eigenform for real multiplication.*

Finiteness. Next we show that the discriminant D controls the shape of the isogeny.

Theorem 8.3 *An algebraic sum $(X, \omega) \cong (E_1, \omega_1) + (E_2, \omega_2)$ lies in ΩE_D if and only if there are integers satisfying $e^2 + 4d = D$, and an isogeny $p : E_1 \to E_2$, such that:*

- $\deg(p) = d \geq 1$, $\gcd(p, e) = 1$, *and*
- $p^*(\omega_2) = \lambda \omega_1$, *where $\lambda^2 = e\lambda + d$.*

In particular, D determines $\deg(p)$ *and λ up to finitely many choices.*

Here $\gcd(p, e) = \gcd(p_{11}, p_{12}, p_{21}, p_{22}, e)$, where $(p_{ij}) \in \mathrm{M}_2(\mathbb{Z})$ is a matrix for the map $p_* : H_1(E_1, \mathbb{Z}) \to H_1(E_2, \mathbb{Z})$.

Proof. Since (X, ω) is an eigenform, there is a primitive self-adjoint endomorphism

$$T : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$

such that $\mathbb{Z}[T] \cong \mathfrak{o}_D$, and $T^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}$. Equivalently, we have

$$I_\omega(T(x)) = \int_{T(x)} \omega = \lambda \cdot I_\omega(x)$$

for all cycles $x \in H_1(X, \mathbb{Z})$, and $\gcd(T_{ij}) = 1$. Using the algebraic splitting

$$H_1(X, \mathbb{Z}) = H_1(E_1, \mathbb{Z}) \oplus H_1(E_2, \mathbb{Z}),$$

we can write $x = x_1 + x_2$ with $x_i \in H_1(E_i, \mathbb{Z})$; then $I_\omega(x) = I_{\omega_1}(x_1) + I_{\omega_2}(x_2)$.

Now let (a_i, b_i) be a symplectic basis for $H_1(E_i, \mathbb{Z})$, satisfying $\langle a_i, b_i \rangle = 1$. Then with respect to the basis (a_1, b_1, a_2, b_2) for $H_1(X, \mathbb{Z})$, the symplectic form is given by the block matrix $S = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The self-adjointness condition $\langle Tx, y \rangle = \langle x, Ty \rangle$ is equivalent to $ST^t = TS$. Equating corresponding blocks, we find that T has the form

$$T = \begin{pmatrix} fI & B \\ C & eI \end{pmatrix}$$

with $JB^t = CJ$. In particular B^t and C are conjugate.

After replacing T with $T - fI$ (which still generates $\mathfrak{o}_D$), we can assume $f = 0$. Then for all $x \in H_1(E_1, \mathbb{Z})$, we have $T(x) = C(x) \in H_1(E_2, \mathbb{Z})$. Moreover, we have

$$I_{\omega_2}(C(x)) = I_\omega(T(x)) = \lambda \cdot I_\omega(x) = \lambda \cdot I_{\omega_1}(x).$$

Thus $C : H_1(E_1, \mathbb{Z}) \to H_1(E_2, \mathbb{Z})$ determines an isogeny $p : E_1 \to E_2$ of degree $d = \det(C) > 0$, satisfying $p^*(\omega_2) = \lambda\omega_1$.

We have $\gcd(T_{ij}) = \gcd(p, e) = 1$, since C is a matrix for p. Moreover T generates $\mathfrak{o}_D$, so it satisfies $T^2 = e'T + d'$ with $(e')^2 + 4d' = D$.

We claim that $(d', e') = (d, e)$. To see this, first note that

$$T^2 = \begin{pmatrix} BC & eB \\ eC & e^2 I \end{pmatrix} = e'T + d' = \begin{pmatrix} d'I & e'B \\ e'C & (e')^2 I \end{pmatrix}.$$

This equation implies $e = e'$ and $d^2 = \det(BC) = (d')^2$. We also have $\mathrm{tr}(B) = \mathrm{tr}(C) = \mathrm{tr}(d'C^{-1}) = (d'/d)\,\mathrm{tr}(C)$, and thus $d' = d$ provided $\mathrm{tr}(B) \neq 0$. But the same conclusion holds when $\mathrm{tr}(B) = 0$. For if $d' = -d$ and $\mathrm{tr}(B) = 0$, then $C = -dB^{-1} = B$ and thus $JB^t = BJ$, which implies $B = 0$, contrary to the fact that $\det(B) = d > 0$.

Since $T^2 = eT + d$, the same relation holds for its eigenvalue λ.

The converse follows similar lines. ∎

Corollary 8.4 *Fix a discriminant D. Then there is a finite collection of pairs of forms of genus 1 such that every splitting of an eigenform in $\Omega_1 E_D$ is given by*

$$(X, \xi) = (g \cdot (E_i, \omega_i)) \underset{I}{\#} (g \cdot (F_i, \eta_i)), \tag{8.1}$$

for some $I \subset \mathbb{C}$, $g \in \mathrm{SL}_2(\mathbb{R})$, and $i \in \{1, \ldots, n_D\}$.

Proof. Let $(X, \xi) = (E, \omega) \underset{I}{\#} (F, \eta)$ be a splitting of a form in $\Omega_1 E_D$. By the preceding theorem, there is an isogeny $p : E \to F$ of degree d satisfying $p^*(\eta) = \lambda\omega$, where D determines the pair (d, λ) up to finitely many choices. Adjusting by the action of $\mathrm{SL}_2(\mathbb{R})$, we can assume $\mathrm{Per}(\eta) = \mathbb{Z}s \oplus \mathbb{Z}is$ for some $s > 0$. Since $\mathrm{area}(X, \xi) = 1$, we have $s^2(1 + d/\lambda^2) = 1$, so the data (d, λ) determines the square torus (F, η). But $d = \deg(p)$ also determines E and p up to finitely many choices, and for each of these we have $\omega = \lambda^{-1} p^*(\eta)$. ■

Volume. Recall that the relative periods of (X, ω) provide local (orbifold) coordinates on $\Omega\mathcal{M}_2(1, 1)$. In these charts $\Omega E_D(1, 1) = \Omega E_D \cap \Omega\mathcal{M}_2(1, 1)$ is locally a linear subspace of complex codimension 2, determined by constraints that hold

$$\mathrm{Ker}(I_\omega) \subset H_1(X, \mathbb{C})$$

fixed as ω varies. (Cf. Theorem 5.1(3)). Thus the Euclidean volume element in period coordinates determines a natural $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\Omega E_D(1, 1)$ and $\Omega_1 E_D(1, 1)$.

Theorem 8.5 *The space of eigenforms $\Omega_1 E_D(1, 1)$ has finite volume.*

Proof. The proof follows the same lines as the proof of finiteness of the volume of $\Omega_1\mathcal{M}_2(1, 1)$, using the fact that $\partial\Omega_1 E_D(1, 1)$ lies over a finite set of curves in $\overline{\mathcal{M}_2}$. ■

Remark. Corollary 8.4 can also be proved by starting with a splitting $(X, \xi) = (E, \omega) \underset{I}{\#} (F, \eta)$ in $\Omega_1 E_D$, letting $I \to 0$ to obtain a stable curve $Y \in \partial\mathcal{M}_2$, noting that $\mathrm{Jac}(Y) \cong E \times F$ belongs to the Humbert surface $H_1 \in \mathcal{A}_2$, and appealing to the fact that $H_1 \cap H_D$ is a finite union of irreducible curves.

9 Pairs of splittings

Since any $(X, \omega) \in \Omega\mathcal{M}_2$ splits as a connected sum in many ways, it is useful to understand how various splittings are related to one another. In this section we classify splittings according to the cylinders they produce, and explore how these cylinders behave when the splitting is changed or deformed. We also show that, after varying the factors in a connected sum, there is almost surely a second splitting which is generic.

Cylinders and splittings. Consider a 1-form on a surface of genus two, presented as a connected sum

$$(X, \omega) = (E_1, \omega_1) \mathop{\#}_I (E_2, \omega_2) \tag{9.1}$$

of a pair of tori. Let $N_I \subset \mathrm{SL}_2(\mathbb{R})$ denote the stabilizer of I.

The foliation $\mathcal{F}$ of (X, ω) by geodesics parallel to I has at most two cylinders, one for each subtorus E_i. It is easy to see that the foliation $\mathcal{F}$

- *has a cylinder C_i on E_i* iff $N_I \cap \mathrm{SL}_2(E_i, \omega_i) \cong \mathbb{Z}$, and
- *has two cylinders, with a rational ratio of moduli* iff $N_I \cap \mathrm{SL}_2(X, \omega) \cong \mathbb{Z}$.

In the latter case, a suitable product of Dehn twists around C_1 and C_2 gives an affine automorphism of (X, ω) generating $N_I \cap \mathrm{SL}(X, \omega) \cong \mathbb{Z}$. For brevity, in this case we say the foliation $\mathcal{F}$ is *rational*.

When there are two cylinders, $\mathcal{F}$ is *periodic*, because all its leaves are closed.

When there are no cylinders at all, the splitting presents $(X, \mathcal{F})$ as a connected sum of tori with irrational foliations.

We say the splitting (9.1) is *rational*, *periodic*, *has no cylinders*, etc., if the corresponding property holds for $\mathcal{F}$.

Theorem 9.1 *Any splitting of an eigenform $(X, \omega) \in \mathcal{E}_2$ either has two cylinders with a rational ratio of moduli, or no cylinders at all.*

Proof. By Theorem 5.1, after normalizing (X, ω) by the action of $\mathrm{GL}_2^+(\mathbb{R})$, we can assume ω has absolute periods in $K(i)$ and zero complex flux. Let $\mathcal{F}$ be the foliation of (X, ω) by geodesics parallel to I. If $\mathcal{F}$ has a cylinder on E_1 or E_2, then its leaves have the same slope as an absolute period of ω, and therefore $(X, \mathcal{F})$ decomposes into a pair of cylinders with a rational ratio of moduli by [Mc2, Thm 8.3]. ■

Rectangles. The next result gives an explicit description of two different splittings of certain surfaces. Let us say a based lattice in $\mathbb{C} \cong \mathbb{R}^2$ is *rectangular* if it has the form $\Lambda = \mathbb{Z}(x, 0) \oplus \mathbb{Z}(0, y)$, $x, y > 0$. Similarly, we say an element of $\Omega\mathcal{M}_1$ is rectangular if it has the form $(R, \rho) = (\mathbb{C}/\Lambda, dz)$, where Λ is a rectangular lattice.

Theorem 9.2 *Let $I = [0, v] \subset \mathbb{R}$, and let $(X, \omega) = (R_1, \rho_1) \mathop{\#}_I (R_2, \rho_2)$ be a connected sum of tori with rectangular period lattices*

$$\mathrm{Per}(\rho_i) = \mathbb{Z}a_i + \mathbb{Z}b_i.$$

Then we also have $(X, \omega) = (F_1, \eta_1) \mathop{\#}_K (F_2, \eta_2)$, where

$$\begin{aligned} \mathrm{Per}(\eta_1) &= \mathbb{Z}(a_1 + b_2) \oplus \mathbb{Z}b_1, \\ \mathrm{Per}(\eta_2) &= \mathbb{Z}(a_2 + b_1) \oplus \mathbb{Z}b_2, \end{aligned}$$

and $K = [0, v + b_1 + b_2]$.

Proof. Let $P \subset \mathbb{C}$ be the polygon consisting of an $a_2 \times b_2$ rectangle resting atop an $a_1 \times b_1$ rectangle, overlapping along $I = [0, v]$ as shown in Figure 6. By assumption, (X, ω) is isomorphic to the quotient space $(P, dz)/\sim$ obtained by gluing opposite edges of P using horizontal and vertical translations. In the Figure, ω has two simple zeros, labeled by white and black dots along ∂P.

Now let $J = [-b_1, v + b_2]$ be the saddle connection crossing I as shown. By splitting (X, ω) along the loop $L = J \cup \eta(J)$ (represented by the three slanting lines in P), we obtain a second connected sum decomposition $(X, \omega) = (F_1, \eta_1) \underset{K}{\#} (F_2, \eta_2)$. To check its periods, note that the new and old splittings are related by a vertical Dehn twist. ∎

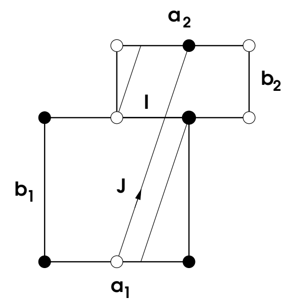

Figure 6. Resplitting a sum of rectangles.

Since the saddle connections used above persist throughout a neighborhood of (X, ω) in $\Omega\mathcal{M}_2(1,1)$ or $\Omega\mathcal{M}_2(2)$, we have:

Theorem 9.3 *Corresponding pairs of splittings exist for all (X', ω') close enough to (X, ω) in its stratum.*

Shearing. We now study how the splittings above change when one of the original rectangles is sheared along horizontal lines.

By reordering the summands, we can assume that I embeds in R_1. Let $n_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, and let

$$(X(u), \omega(u)) = (n_u \cdot (R_1, \rho_1)) \underset{I}{\#} (R_2, \rho_2).$$

For all u sufficiently small, the second splitting $(X, \omega) = (F_1, \eta_1) \underset{K}{\#} (F_2, \eta_2)$ deforms to give a second splitting

$$(X(u), \omega(u)) = (F_1(u), \eta_1(u)) \underset{K(u)}{\#} (F_2(u), \eta_2(u)). \tag{9.2}$$

Theorem 9.4 *For all but countably many u, the splitting (9.2) has no cylinder on $F_1(u)$. In addition,* $\operatorname{area}(F_1(u), \eta_1(u)) = Au + B$ *with $A \neq 0$.*

Proof. We have $n_u \cdot \mathrm{Per}(\rho_1) = \mathbb{Z}a_1 \oplus \mathbb{Z}(ua_1 + b_1)$, and thus by continuity

$$\mathrm{Per}(\eta_1(u)) = \mathbb{Z}(a_1 + b_2) \oplus \mathbb{Z}(ua_1 + b_1). \tag{9.3}$$

Similarly $K(u) = [0, v(u)]$ with $v(u) = v + ua_1 + b_1 + b_2$.

We claim that, for all but countably many u, we have

$$\mathrm{Per}(\eta_1(u)) \cap \mathbb{R} \cdot K(u) = \{0\}. \tag{9.4}$$

Otherwise, there are integers $(n, m) \neq (0, 0)$ such that the vector

$$\lambda(u) = n(a_1 + b_2) + m(ua_1 + b_1) \in \mathrm{Per}(\eta_1(u))$$

has the same slope as $v(u)$ for all u. But since $\lambda(u)$ and $v(u)$ move along straight lines in the plane, and $v(u)$ never passes through the origin, for the slopes always to agree we must have $\lambda(u) = mv(u)$. This easily implies $v = a_1$, contrary to the fact that $|v| < |a_1|$ (since I embeds in (R_1, ρ_1)).

Thus the second splitting almost never has a cylinder on $F_1(u)$. Finally (9.3) implies

$$\mathrm{area}(F_1(u), \eta_1(u)) = |a_1||b_1| - u|a_1||b_2|$$

is a nonconstant linear function of u. ∎

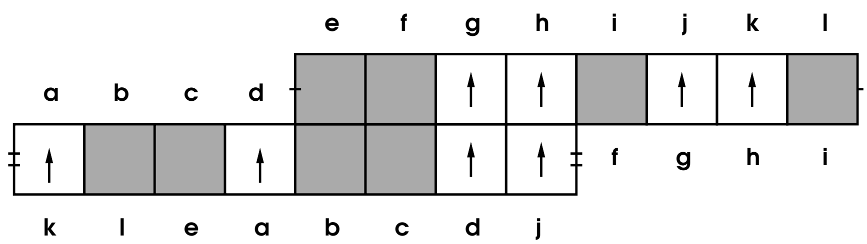

Figure 7. A pair of two-cylinder splittings.

Persistent cylinders. Starting with a 2-cylinder splitting along $I \subset \mathbb{R}$, one might expect that the sheared surfaces

$$(X(u), \omega(u)) = (n_u \cdot (E_1, \omega_1)) \underset{I}{\#} (E_2, \omega_2)$$

have a *unique* 2-cylinder splitting for almost every u (namely the splitting above). But this need not be the case; there may be a second 2-cylinder splitting that persists under shearing.

An example is shown in Figure 7. The result of gluing the edges of the parallelogram in the indicated pattern is a surface (X, ω) that splits as a connected sum of tori in both the horizontal and vertical directions. The two horizontal tori correspond to the 1×8 rectangles in the figure; the two vertical tori are distinguished by shading.

The key property of this example is that the intersection matrix for the horizontal and vertical cylinders $(C_1, C_2), (D_1, D_2)$ is singular; we have $C_i \cdot D_j = 4$ for all i and j. Because of this, the cylinders D_1 and D_2 remain parallel as we shear along C_1, and therefore $(X(u), \omega(u))$ has *two* distinct 2-cylinder splittings for all u sufficiently small.

On the other hand, it is easy to check:

Theorem 9.5 *Let* $(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$*. Then for almost every* $g \in \mathrm{SL}_2(\mathbb{R})$*, the deformed sum*

$$(X', \omega') = (g \cdot (E_1, \omega_1)) \underset{I}{\#} (E_2, \omega_2)$$

has no 2-cylinder splittings.

Proof. Almost every choice of g puts the lattice $g \cdot \mathrm{Per}(\omega_1)$ into general position with respect to I and $\mathrm{Per}(\omega_2)$. Thus I embeds into (E_1, ω_1) and the connected sum is well-defined. Similarly, the periods $\int_{C_i} \omega'$ of any linearly independent homology classes $C_1, C_2 \in H_1(X', \mathbb{Z})$ almost surely have different slopes, so (X', ω') cannot carry a pair of parallel cylinders. ■

Generic splittings. We say a splitting

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$$

is *generic* if it has no cylinders and if (E_1, ω_1) is *not* isogenous to $n \cdot (E_2, \omega_2)$ for any $n \in N_I$. By Theorem 2.6, this is equivalent to the condition

$$\overline{N_I \cdot (\Lambda_1, \Lambda_2)} = (G \times G)/(\Gamma \times \Gamma),$$

where Λ_i is the period lattice $\mathrm{Per}(\omega_i)$ rescaled to have determinant one. The existence of a generic splitting is useful for showing $G \cdot (X, \omega)$ is large.

To treat the case of the twisted diagonals G^u, $u \neq 0$ that arise in Theorem 2.6, we will use:

Theorem 9.6 *If* $h \neq \mathrm{id}$ *in* G*, then*

$$(X_g, \omega_g) = (g \cdot (E_1, \omega_2)) \underset{I}{\#} (hg \cdot (E_2, \omega_2)) \tag{9.5}$$

admits a generic splitting for almost every $g \in G$*.*

Lemma 9.7 *Let* $L_1, L_2 : M_2(\mathbb{R}) \to \mathbb{R}^2$ *be linear maps, and let* $V \subset \mathrm{M}_2(\mathbb{R})$ *be the subvariety defined by*

$$((1, 0) + L_1(g)) \wedge L_2(g) = 0. \tag{9.6}$$

Then either V *meets* $G = \mathrm{SL}_2(\mathbb{R}) \subset \mathrm{M}_2(\mathbb{R})$ *in a proper subvariety, or*

$$(1, 0) \wedge L_2(g) = L_1(g) \wedge L_2(g) = 0$$

for all $g \in \mathrm{M}_2(\mathbb{R})$*.*

Proof. The defining equation for V has the form

$$f(g) = (1,0) \wedge L_2(g) + L_1(g) \wedge L_2(g) = f_1(g) + f_2(g),$$

where $f_i(g)$ is homogeneous of degree i in the entries of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Since G is the irreducible smooth subvariety of $\mathrm{M}_2(\mathbb{R})$ defined by the equation $ad - bc = 1$, if V contains G then $f(g) = h(g)(ad - cb - 1)$ for some polynomial $h(g)$. But f is quadratic with no constant term, so we must have $h = f = 0$ and thus $f_1 = f_2 = 0$. ∎

Proof of Theorem 9.6. Given $g \in G$, pick a pair of splittings for (X_g, ω_g) other than the given one. Let U be an open neighborhood of g on which these splittings persist, let $V_i \subset U$, $i = 1, 2$ be the locus where the ith splitting has isogenous summands, and let $V = V_1 \cap V_2$. Then by Theorem 8.2, (X_g, ω_g) is an eigenform for all $g \in V$, and thus its original splitting (9.5) also has isogenous summands. Consequently (E_1, ω_1) is isogenous to $g^{-1}hg \cdot (E_2, \omega_2)$ for all $g \in V$, which implies the set $\{g^{-1}hg \ : \ g \in V\}$ is countable. Since $h \neq \mathrm{id}$, this in turn implies $V \subset G$ has measure zero. But V_1 and V_2 are countable unions of algebraic subsets of G, so one of these has measure zero as well.

Thus for all $g \in U$ we can choose a new, continuously varying splitting

$$(X_g, \omega_g) = (F_1(g), \eta_1(g)) \underset{J(g)}{\#} (F_2(g), \eta_2(g)) \tag{9.7}$$

whose summands are almost surely not isogenous. Since this splitting is distinct from (9.5), the interval $J(g) = [0, v(g)]$ is not parallel to I. To complete the proof, we will show this new splitting is generic for almost every $g \in U$.

By symmetry we can assume $I = [0, 1]$. Note that

$$\mathrm{Per}(\omega_g) = g \cdot \mathrm{Per}(\omega_1) \oplus hg \cdot \mathrm{Per}(\omega_2),$$

and thus any period of ω_g has the form

$$w(g) = L_2(g) = g(w_1) + hg(w_2)$$

with $w_i \in \mathrm{Per}(\omega_i)$. Similarly, since $J(g)$ and I are homologous *rel* their endpoints, the difference $v(g) - (1, 0)$ is a period of ω_g and hence

$$v(g) = (1, 0) + L_1(g) = (1, 0) + g(v_1) + hg(v_2)$$

for fixed $v_i \in \mathrm{Per}(\omega_i)$. Note that L_1 and L_2 depend linearly on the coefficients of g, and thus they extend to linear maps $L_1, L_2 : \mathrm{M}_2(\mathbb{R}) \to \mathbb{R}^2$.

We begin by ruling out cylinders. If the splitting (9.7) has a cylinder, then there is a vector $w(g) \in \mathrm{Per}(\omega_g)$ that is parallel to $J(g) = [0, v(g)]$. In other words, we have

$$v(g) \wedge w(g) = ((1, 0) + L_1(g)) \wedge L_2(g) = 0. \tag{9.8}$$

Since $w(g) = L_2(g)$ is parallel to $J(g)$, it is not parallel to $(1, 0)$, and thus $(1, 0) \wedge L_2(g) \neq 0$. But then by the preceding Lemma, the relation (9.8) (for fixed w_1 and w_2) can only hold on a proper subvariety of G. Since there are only countably many possibilities for w_1 and w_2, the locus where the splitting (9.7) has a cylinder is a countable union of proper subvarieties, and hence it has measure zero.

We conclude by ruling out commensurability relations. Suppose $g \in U$ and $(F_1(g), \eta_1(g))$ is isogenous to $n(g) \cdot (F_2(g), \eta_2(g))$ for some (almost surely unique) $n(g) \in N_{J(g)}$. We have already arranged that the summands of (9.7) are almost surely not commensurable, so we can assume $n(g) \neq \mathrm{id}$. Then we can find $n_i(g) \in \mathrm{Per}(\eta_i(g))$ such that $n_1(g) = n(g) \cdot n_2(g) \neq n_2(g)$ and thus

$$0 \neq w(g) = n_1(g) - n_2(g) \in \mathrm{Per}(\omega_g)$$

is parallel to $J(g)$; that is, $v(g) \wedge w(g) = 0$. But again, the relation $v(g) \wedge w(g) = 0$ (for fixed w_1 and w_2) can only hold on a set of measure zero in U. Thus the summands of (9.7) almost surely satisfy no commensurability relation, and so the new splitting is generic for almost every $g \in U$. ■

Similar arguments yield:

Theorem 9.8 *For almost every* $(g_1, g_2) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, *every splitting of the form*

$$(X, \omega) = (g_1 \cdot (E_1, \omega_1)) \underset{I}{\#} (g_2 \cdot (E_2, \omega_2))$$

is generic.

10 Dynamics on $\Omega\mathcal{M}_2(2)$

In this section we begin the proof of our main results, by analyzing orbit closures and invariant measures for forms with double zeros. We will show:

Theorem 10.1 *Let Z be the closure of the* $\mathrm{SL}_2(\mathbb{R})$*-orbit of* $(X, \omega) \in \Omega_1\mathcal{M}_2(2)$. *Then either:*

- *$Z = \Omega_1 V$ for some Teichmüller curve $V \to \mathcal{M}_2$, or*
- *$Z = \Omega_1\mathcal{M}_2(2)$.*

In the first case (X, ω) is an eigenform for real multiplication, while in the second case it is not.

Theorem 10.2 *Any ergodic* $\mathrm{SL}_2(\mathbb{R})$*-invariant probability measure α on* $\Omega_1\mathcal{M}_2(2)$ *is either the standard measure on the full space, or the standard measure on*

$$\Omega_1 V \subset \Omega_1\mathcal{M}_2(2)$$

for some Teichmüller curve V.

The Weierstrass foliation. The orbits of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega_1\mathcal{M}_2(2)$ project to the leaves of the *Weierstrass foliation* $\mathcal{F}$ of $\mathcal{M}_2$. This "foliation" has six leaves passing through each $X \in \mathcal{M}_2$, corresponding to the six Weierstrass points on X and the associated forms with double zeros. Each leaf is an isometrically immersed Riemann surface. Theorem 10.1 implies:

Corollary 10.3 *Each leaf of the Weierstrass foliation is either an isometrically immersed algebraic curve, or a dense subset of $\mathcal{M}_2$.*

Orbit closures. We proceed to the proof of Theorem 10.1. Let

$$Z = \overline{\mathrm{SL}_2(\mathbb{R}) \cdot (X,\omega)} \subset \Omega_1\mathcal{M}_2(2).$$

Choose a splitting

$$(X,\omega) = (E_1,\omega_1) \underset{I}{\#} (E_2,\omega_2). \tag{10.1}$$

Order the summands so that I embeds in E_1 and projects to a loop in E_2. The splitting always has a cylinder on E_2.

Let $N \subset G = \mathrm{SL}_2(\mathbb{R})$ be the upper-triangular subgroup as usual. Let N_I denote the conjugate of N stabilizing I, and let $N_I(\mathbb{Z}) = N_I \cap \mathrm{SL}(E_2,\omega_2) \cong \mathbb{Z}$. Since $N_I(\mathbb{Z})$ leaves both (E_2,ω_2) and I invariant, we have:

$$n \cdot (X,\omega) = (n \cdot (E_1,\omega_1)) \underset{I}{\#} (E_2,\omega_2) \in Z \tag{10.2}$$

for all $n \in N_I(\mathbb{Z})$. Applying the results of §2 to the unipotent orbit $N_I(\mathbb{Z}) \cdot (E_1,\omega_1)$, we obtain:

Lemma 10.4 *At least one of the following alternatives holds.*

1. *The splitting (10.1) has two cylinders, and $N_I \cap \mathrm{SL}(X,\omega) \cong \mathbb{Z}$.*
2. *The splitting has two cylinders, and there are a pair of rectangular tori such that $(n \cdot (R_1,\rho_1)) \underset{[0,1]}{\#} (R_2,\rho_2) \in Z$ for all $n \in N$.*
3. *The splitting has only one cylinder, and we have $(F_1,\eta_1)\underset{K}{\#}(F_2,\eta_2) \in Z$ whenever $\mathrm{area}(F_i,\eta_i) = \mathrm{area}(E_i,\omega_i)$ and K projects to a loop in F_2.*

Proof. Since the discussion is $\mathrm{SL}_2(\mathbb{R})$-invariant, after replacing (X,ω) with $g \cdot (X,\omega)$ we can assume $I = [0,t]$ is horizontal, (E_2,ω_2) is a square torus, and $N_I(\mathbb{Z}) = N(\mathbb{Z})$.

Now let

$$Z_1 = \overline{N(\mathbb{Z}) \cdot (X_1,\omega_1)} \subset \Omega\mathcal{M}_1 \cong G/\Gamma.$$

By the classification of $N(\mathbb{Z})$ orbits on G/Γ, we have $Z_1 = H \cdot (X_1,\omega_1)$ where $H = N(\mathbb{Z})$, N or G.

These three alternative correspond to the three cases in the Lemma above. Indeed, by Theorem 2.3, exactly one of the following holds.

1. $H = N(\mathbb{Z})$, Z_1 is finite and there is a horizontal vector in $\Lambda = \mathrm{Per}(\omega_1)$. Then the splitting has two cylinders because of the horizontal vector, and $N_I \cap \mathrm{SL}(X, \omega) = \mathbb{Z}$ because the stabilizer of (X_1, ω_1) has finite index in $N(\mathbb{Z})$. Thus we are in case (1).

2. $H = N$ and Λ has a horizontal vector. Then the splitting again has two cylinders, and by $\mathrm{SL}_2(\mathbb{R})$-invariance of Z and (10.2), we find that

$$(g_1 \cdot (E_1, \omega_1)) \underset{I}{\#} (g_2 \cdot (E_2, \omega_2)) \in Z \tag{10.3}$$

   for all $g_1, g_2 \in N$. The action of N can be used to make these tori rectangular, so we are in case (2).

3. $H = G$. Then (10.3) holds for all $g_1, g_2 \in G$, and we are in case (3).

■

Proof of Theorem 10.1. Recall that (X, ω) splits as a connected sum in infinitely many ways. If case (1) of the Lemma holds for two different splittings, then $\mathrm{SL}(X, \omega)$ contains a pair of independent parabolic subgroups. Therefore it also contains a hyperbolic element, and consequently $Z = \Omega_1 V$ is a circle bundle over a Teichmüller curve, by Theorems 5.8 and 5.10.

Now assume case (2) of the Lemma holds, and let $n_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$. By Theorem 9.4, for almost all sufficiently small u, the form

$$(n_u \cdot (R_1, \rho_1)) \underset{[0,1]}{\#} (R_2, \rho_2) \in Z$$

admits a second splitting

$$(F_1(u), \eta_1(u)) \underset{K(u)}{\#} (F_2(u), \eta_2(u))$$

with no cylinder on F_1. Moreover $A(u) = \mathrm{area}(F_1(u), \eta_1(u))$ is a linear function of u.

By case (3) of the Lemma, when there is no cylinder on F_1, Z contains all connected sums in $\Omega_1\mathcal{M}_2(2)$ whose first summand has area $A(u)$. Since $A(u)$ ranges through a set of positive measure, Z has positive measure. Thus Z coincides with the full stratum $\Omega_1\mathcal{M}_2(2)$, by ergodicity of the action of $\mathrm{SL}_2(\mathbb{R})$.

The same argument shows $Z = \Omega_1\mathcal{M}_2(2)$ when the initial splitting satisfies case (3) of the Lemma, since Z contains a connected sum of rectangles in this case as well. ■

The classification of ergodic invariant measures follows similar lines, using conditional measures.

The space of splittings. To formalize the discussion, we will use a local covering map

$$f : G \times T \to \Omega_1\mathcal{M}_2(2)$$

to pass to a space $G \times T$ which parameterizes points $(X, \omega) \in \Omega_1\mathcal{M}_2(2)$ equipped with a choice of splitting. We refer to $G \times T$ as the *splitting space* for $\Omega_1\mathcal{M}_2(2)$.

To define f, let $0 < s, t < 1$ satisfy $s^2 + t^2 = 1$. Let $\Lambda_2 = \mathbb{Z}t \oplus \mathbb{Z}it$, so that $(R_2, \rho_2) = (\mathbb{C}/\Lambda_2, dz)$ is the square torus of area t^2, and let $I = [0, t]$. Define $T_s \subset \Omega_{s^2}\mathcal{M}_1$ by

$$T_s = \{(E_1, \omega_1) \ : \ \mathrm{area}(E_1, \omega_1) = s^2 \text{ and } \mathrm{Per}(\omega_1) \cap I = \{0\}\},$$

and define

$$\Phi_s : T_s \to \Omega_1\mathcal{M}_2(2)$$

by

$$\Phi_s(E_1, \omega_1) = (E_1, \omega_1) \underset{I}{\#} (R_2, \rho_2). \tag{10.4}$$

This map slits open a torus of area s^2 and inserts a square cylinder of area t^2, to yield a surface of genus two with area one.

Let $N \subset G \cong \mathrm{SL}_2(\mathbb{R})$ be the upper-triangular subgroup as usual, and let $N(\mathbb{Z}) = N \cap \mathrm{SL}_2(\mathbb{Z})$. Since $N(\mathbb{Z})$ stabilizes (Λ_2, I), we have

$$\Phi_s(nx) = n \cdot \Phi_s(x)$$

for all $n \in N(\mathbb{Z})$.

Now let $T = \bigcup T_s \subset \Omega\mathcal{M}_1$, equipped with the natural projection $\pi : T \to (0, 1)$ sending T_s to $\{s\}$. The union of the mappings Φ_s gives a map

$$\Phi : T \to \Omega_1\mathcal{M}_2(2)$$

satisfying $\Phi|T_s = \Phi_s$, and we define $f : G \times T \to \Omega_1\mathcal{M}_2(2)$ by

$$f(g, (E, \omega)) = g \cdot \Phi(E, \omega).$$

It is easy to see that f is a surjective local covering map (cf. Theorem 7.2). Note that

$$g \cdot f(h, (E, \omega)) = f(gh, (E, \omega)) \tag{10.5}$$

for all $g, h \in G$, and

$$f(gn, (E, \omega)) = f(g, n \cdot (E, \omega)) \tag{10.6}$$

for all $n \in N(\mathbb{Z})$.

Types of splittings. It is useful to classify the elements of $T \subset \Omega\mathcal{M}_1$ according to the types of splittings they determine. Let

$$T_H = \{x \in T \ : \ \overline{N(\mathbb{Z}) \cdot x} = H \cdot x \subset \Omega\mathcal{M}_1\}.$$

Then by the results of §2, we have:

$$T = T_{N(\mathbb{Z})} \sqcup T_N \sqcup T_G.$$

Moreover, the splitting (10.4) has:

- two cylinders, with a rational ratio of moduli, when $(E_1, \omega_1) \in T_{N(\mathbb{Z})}$;
- two cylinders, with an irrational ratio of moduli, when $(E_1, \omega_1) \in T_N$; and
- only one cylinder, when $(E_1, \omega_1) \in T_G$.

Pullback of measures. Now let α be an ergodic, $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on $\Omega_1\mathcal{M}_2(2)$. Since f is a local covering map, we can pull back α to a *locally* finite measure $f^*(\alpha)$ on $G \times T$. Equation (10.5) implies

$$f^*(\alpha) = dg \times \mu$$

where dg is Haar measure on G, and μ is a locally finite measure on T; and equation (10.6) implies μ is $N(\mathbb{Z})$-invariant.

By standard results on disintegration of measures [Bou, §2.7], there is a family μ_s of locally finite, $N(\mathbb{Z})$-invariant *conditional measures* on T_s, and a locally finite measure ξ on $(0,1)$, such that

$$\int_T \phi\mu = \int_{(0,1)} \left(\int_{T_s} \phi\mu_s \right) \xi(s) \tag{10.7}$$

for all $\phi \in C_0(T)$. In other words, μ can be expressed as the convolution $\mu_s * \xi(s)$.

This construction reduces the study of α to the study of the $N(\mathbb{Z})$-invariant measures μ_s on the homogeneous spaces $\Omega_{s^2}\mathcal{M}_1 \supset T_s$. By applying Corollary 2.9 to these measures, we obtain:

Lemma 10.5 *The restriction of μ to $T_H \subset T$ is H-invariant, for $H = N(\mathbb{Z})$, N, or G.*

Proof of Theorem 10.2. We consider how the mass of the measure μ on T is distributed among the blocks of the partition $T = T_{N(\mathbb{Z})} \sqcup T_N \sqcup T_G$.

First suppose μ assigns full measure to $T_{N(\mathbb{Z})}$. Then α assigns full measure to the set of $(X,\omega) \in \Omega\mathcal{M}_2(2)$ such that every splitting of (X,ω) is rational. But every such (X,ω) generates a Teichmüller curve V (by Theorems 5.8 and 5.10). Since the collection of Teichmüller curves is countable, we have $\alpha(\Omega_1 V) > 0$ for some V, and by ergodicity of α, V is unique. Since α is $\mathrm{SL}_2(\mathbb{R})$-invariant, it coincides with the standard probability measure on

$$\Omega_1(V) \cong \mathrm{SL}_2(\mathbb{R}) / \mathrm{SL}(X,\omega),$$

completing the proof in the case $\mu(T_N) = \mu(T_G) = 0$.

Now suppose μ assigns positive mass to T_N or T_G. We claim $\mu(T_G) > 0$. Indeed, $\nu|T_N$ is N-invariant by the Lemma above, and every N-orbit in T_N contains a rectangular lattice (R_1, ρ_1). By Theorem 9.4, there is a positive measure set of $n \in N$ such that $\Phi(n \cdot (R_1, \rho_1))$ admits a splitting with only one cylinder. Thus if T_N has positive mass, so does T_G.

Next we show μ assigns full measure to T_G. By the Lemma, $\mu|T_G$ is G-invariant. By Theorem 9.5, for almost every $g \in G$, $\Phi_s(g \cdot (E_1, \omega_1))$ has *no*

2-cylinder splittings. Thus α assigns positive mass to the set of forms $M \subset \Omega_1\mathcal{M}_2(2)$ which have only 1-cylinder splittings. By ergodicity, α assigns full measure to M. Since $f^{-1}(M)$ is contained in $G \times T_G$, μ gives full measure to the space T_G.

In particular, μ is G-invariant. Thus each conditional measure μ_s is proportional to the unique G-invariant measure on $\Omega_{s^2}\mathcal{M}_1$.

We now analyze the measure ξ on $(0,1)$. Let $\xi = \xi^a + \xi^s$ denote the Lebesgue decomposition of ξ into measures absolutely continuous and singular with respect to ds respectively. We will show that $\xi^a \neq 0$.

Pick $0 < x < 1$ in the support of ξ, and pick a rectangular lattice $(R_1, \rho_1) \in T_x$. By G-invariance of μ, (R_1, ρ_1) is in the support of μ. Let $n_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$. By Theorems 9.2 and 9.4, we have an explicit second splitting

$$\Phi(R_1', \rho_1') = (F_1', \eta_1') \underset{I'}{\#} (F_2', \eta_2')$$

defined for all (R_1', ρ') in a neighborhood V of (R_1, ω_1) in T, such that $A(R_1', \rho_1') = \text{area}(F_1', \eta_1')$ satisfies

$$\frac{d}{du} A(n_u \cdot (R_1, \rho_1))\bigg|_{u=0} \neq 0.$$

Let $S(p) = A(p)^{1/2}$, so $S(R_1, \rho_1) = s$. By taking V small enough, we can insure that $dS(n_u \cdot p)/du \neq 0$ for all $p \in V$. Since $\mu(V) > 0$ and μ is N-invariant, this implies $S_*(\mu|V)$ is equivalent to Lebesgue measure ds on the open interval $J = S(V)$. On the other hand, the area of the first summand of a splitting chosen at random with respect to μ is controlled by the measure class of ξ, so we have

$$\xi \gg S_*(\mu|V) \asymp ds|J,$$

and thus $\xi^a \neq 0$.

Now let $\mu = \mu^a + \mu^s$ and $\alpha = \alpha^a + \alpha^s$ be the Lebesgue decompositions of μ and α. Since μ_s is absolutely continuous on $\Omega_{s^2}\mathcal{M}_1$, we have

$$\mu^a = \mu_s * \xi^a(s) \neq 0.$$

Since f is a locally covering map, we have $\mu^a \times dg = f^*(\alpha^a)$, and therefore $\alpha^a \neq 0$ as well. By ergodicity, this implies $\alpha = \alpha^a$. But the standard probability measure m on $\Omega_1\mathcal{M}_2(2)$ is also absolutely continuous, invariant and ergodic; therefore $\alpha = m$. ■

11 Dynamics on $\Omega\mathcal{M}_2(1,1)$

In this section we begin the analysis of forms with simple zeros. We will establish the following two results.

Theorem 11.1 *Let Z be the closure of the $\mathrm{SL}_2(\mathbb{R})$-orbit of $(X, \omega) \in \Omega_1\mathcal{M}_2(1,1)$. Then either:*

- *Z is contained in $\Omega_1 E_D$ for some $D > 0$, or*
- *$Z = \Omega_1 \mathcal{M}_2$.*

Theorem 11.2 *Any ergodic, $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on $\Omega_1\mathcal{M}_2(1,1)$ is either the standard measure on the full space, or a measure supported inside the eigenform locus $\Omega_1 E_D$ for some $D > 0$.*

The proofs parallel the case of double zeros.

Orbit closures in $\boldsymbol{\mathcal{M}_2(1,1)}$. Let

$$(X,\omega) = (E_1,\omega_1)\underset{I}{\#}(E_2,\omega_2) \tag{11.1}$$

be a splitting of a form in $\Omega_1\mathcal{M}_2(1,1)$, and let $N_I \subset \mathrm{SL}_2(\mathbb{R})$ be the conjugate of N that stabilizes I. We then have

$$(n\cdot(E_1,\omega_1))\underset{I}{\#}(n\cdot(E_2,\omega_2)) = n\cdot(X,\omega)$$

for all $n \in N_I$.

We have seen in §2 that any pair of unimodular lattices (Λ_1,Λ_2) satisfies

$$\overline{N_\Delta\cdot(\Lambda_1,\Lambda_2)} = H\cdot(\Lambda_1,\Lambda_2) \subset (G\times G)/(\Gamma\times\Gamma) \tag{11.2}$$

for a unique subgroup $H \in \mathcal{H}$, where

$$\mathcal{H} = \{N_\Delta, N\times N, N\times G, G\times N, G\times G, G^u \ :\ u\in\mathbb{R}\}. \tag{11.3}$$

Applying this analysis to $N_I \cong N_\Delta$, we can now deduce:

Lemma 11.3 *Let Z be the $\mathrm{SL}_2(\mathbb{R})$-orbit closure of $(X,\omega) = (E_1,\omega_1)\underset{I}{\#}(E_2,\omega_2)$ in $\Omega_1\mathcal{M}_2(1,1)$. Then at least one of the following alternatives holds.*

1. *$\mathrm{SL}(X,\omega)\cap N_I \cong \mathbb{Z}$.*
2. *The tori (E_1,ω_1) and (E_2,ω_2) are isogenous.*
3. *There are rectangular tori such that*

$$(n\cdot(R_1,\rho_1))\underset{[0,1]}{\#}(R_2,\rho_2) \in Z$$

   for all $n \in N$.

(In (3) above, we have used the $\mathrm{SL}_2(\mathbb{R})$-invariance of Z to normalize so that the connected sum is over $I = [0,1]$.)

Proof. It suffices to treat the case $I = [0,1]$; then $N_I = N$. Let Λ_i be $\mathrm{Per}(\omega_i)$ rescaled to have determinant one, and let $H \in \mathcal{H}$ be the unique subgroup satisfying (11.2).

By Theorem 2.6, if $H = N_\Delta$ we are in case (1), if $H = G_\Delta = G^0$ we are in case (2), and if $H \supset N \times N$ we are in case (3). In the remaining case $H = G^u$ where $u \neq 0$. Then Z contains the forms

$$(X_g, \omega_g) = (g \cdot (e_1, \omega_2)) \underset{[0,1]}{\#} (n_u g n_u^{-1} \cdot (E_2, \omega_2))$$

for almost every $g \in G$. By Theorem 9.6, almost all of these forms have generic splittings (for which $H = G \times G$), and thus we are again in case (3). ■

Proof of Theorem 11.1. Any $(X, \omega) \in \Omega_1 \mathcal{M}_2(1,1)$ has infinitely many splittings. If alternative (1) of the Lemma holds for two of these splittings, then $\mathrm{SL}(X, \omega)$ contains a hyperbolic element and hence (X, ω) is an eigenform by Theorem 5.8. Similarly, if alternative (2) holds for two splittings, then (X, ω) is an eigenform by Theorem 8.2. In either case, we have $Z \subset \Omega_1 E_D$ for some discriminant D, completing the proof.

Now assume alternative (3) holds, and let $n_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$. By Theorem 9.4, for all u sufficiently small, we have a second splitting:

$$(n_u \cdot (R_1, \rho_1)) \underset{[0,1]}{\#} (R_2, \rho_2) = (F_1(u), \eta_1(u)) \underset{K(u)}{\#} (F_2, \eta_2(u)).,\tag{11.4}$$

where $A(u) = \mathrm{area}(F_1(u), \eta_1(u))$ is a nonconstant linear function of u. By Theorem 9.4, for almost every u, there is no cylinder on $F_1(u)$ and the summands of (11.4) are not isogenous. Since $\omega(u)$ has simple zeros, we can reverse the roles of R_1 and R_2 and conclude there is also, almost surely, no cylinder on $F_2(u)$.

Thus (11.4) provides a family of splittings, parameterized by a positive measure set of $u \in \mathbb{R}$, which fall into case (4) of the Lemma above. For each such u, Z contains all connected sums whose first summand has area $A(u)$. Since $A(u)$ itself varies in a set of positive measure, Z has positive measure; and thus Z contains the full stratum $\Omega_1 \mathcal{M}_2(1,1)$, by ergodicity. The stratum of forms with simple zeros is dense, so $Z = \Omega_1 \mathcal{M}_2$. ■

Splittings in $\mathcal{M}_2(1,1)$. To classify invariant measures, we again pull back to the space of splittings.

Let $I = [0,1] \subset \mathbb{C}$. For any $0 < s, t < 1$ with $s^2 + t^2 = 1$, let

$$T_s \subset \Omega_{s^2} \mathcal{M}_1 \times \Omega_{t^2} \mathcal{M}_1$$

denote the open set of pairs such that the connected sum

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)\tag{11.5}$$

is defined and has simple zeros. Let $T = \bigcup_{0<s<1} T_s$, equipped with the projection $\pi : T \to (0,1)$ sending T_s to $\{s\}$. Let

$$f : G \times T \to \Omega_1 \mathcal{M}_2(1,1)$$

be the local covering map defined by

$$f(g,((E_1,\omega_1),(E_2,\omega_2))) = g\cdot((E_1,\omega_1)\underset{I}{\#}(E_2,\omega_2)).$$

Then f presents $G\times T$ as the *splitting space* for $\Omega_1\mathcal{M}_2(1,1)$; its elements are forms (X,ω) with chosen splittings.

Types of splittings. Because N stabilizes $I=[0,1]$, T_s is invariant under the action of the diagonal subgroup N_Δ on

$$\Omega_{s^2}\mathcal{M}_1\times\Omega_{t^2}\mathcal{M}_1\cong(G\times G)/(\Gamma\times\Gamma),$$

where $\Gamma=\mathrm{SL}_2(\mathbb{Z})$. Let $T_H=\{x\in T\ :\ \overline{N_\Delta x}=Hx\}$. By the classification of orbit closures for N_Δ given in Theorem 2.6, we have:

$$T=\bigsqcup_{H\in\mathcal{H}}T_H. \tag{11.6}$$

Moreover, the splitting (11.5):

- is rational, for points in T_{N_Δ};
- has summands with (E_1,ω_1) isogenous to $n_u\cdot(E_,\omega_2)$, but no cylinders, for points in T_{G^u};
- is irrational, with two cylinders, for points in $T_{N\times N}$;
- has exactly one cylinder, for points in $T_{N\times G}$ and $T_{G\times N}$; and
- is generic, for points in $T_{G\times G}$.

Pullback of measures. Now let α be an ergodic, G-invariant probability measure on $\Omega_1\mathcal{M}_2(1,1)$. Then we have

$$f^*(\alpha)=dg\times\mu,$$

where dg is Haar measure on G and μ is a locally finite measure on T. Since N stabilizes $I=[0,1]$, μ is N-invariant. Disintegrating over the projection $\pi:T\to(0,1)$, we can write μ as the convolution

$$\mu=\mu_s*\xi(s)$$

of a family of locally finite, N-invariant conditional measures μ_s on T_s, over a locally finite measure ξ on $(0,1)$. By Theorem 2.8, we then have:

Lemma 11.4 *The restriction of μ to $T_H\subset T$ is H-invariant.*

Proof of Theorem 11.2. First suppose the measure μ on T assigns full measure to $T_{N_\Delta}\cup T_{G_\Delta}$ (recall $G_\Delta=G^0$). Then α-almost every (X,ω) either has two rational splittings or two splittings with isogenous summands. In either case,

(X,ω) is an eigenform, and thus $\alpha(\bigcup \Omega_1 E_D) = 1$. By ergodicity, $\alpha(\Omega_1 E_D) = 1$ for some discriminant D, completing the proof in this case.

Now suppose μ assigns positive mass to the union of the remaining blocks in the partition (11.6) of T. We claim μ assigns positive mass to $T_{G\times G}$.

Indeed, if $\mu(T_{N\times N}) > 0$, then $\mu|T_{N\times N}$ is $(N\times N)$-invariant by the Lemma above. Every $(N\times N)$-orbit in $T_{N\times N}$ contains a pair of rectangular tori. By Theorem 9.4, for every such pair of tori, there is a positive measure set of $(n_1,n_2) \in (N\times N)$ such that

$$(n_1 \cdot (R_1,\rho_1)) \underset{I}{\#} (n_2 \cdot (R_2,\rho_2))$$

has a second splitting with no cylinders and with non-isogenous summands. Such splittings belong to $T_{G\times G}$; since μ is $N\times N$-invariant, we have $\mu(T_{G\times G}) > 0$. The same conclusion holds if μ assigns positive mass to $T_{N\times G}$ or $T_{G\times N}$, by Theorem 9.5. Finally suppose μ assigns positive mass to $E = \bigcup_{u\neq 0} T_{G^u}$. Then $\mu|E$ is invariant under the action of $G \cong G^u$, by the Lemma above. By Theorem 9.6, μ-almost every point in E also admits a generic splitting, and thus $\mu(T_{G\times G}) > 0$ in this case as well.

Thus we may assume $\mu(T_{G\times G}) > 0$. By the Lemma, the measure $\mu|T_{G\times G}$ is $G\times G$-invariant. By Theorem 9.8, for almost every $(g_1,g_2) \in G\times G$, the form

$$(g_1 \cdot (E_1,\omega_1)) \underset{I}{\#} (g_2 \cdot (E_2,\omega_2))$$

is generic: all of its splittings are generic. Thus α assigns positive mass to the set $A \subset \Omega_1\mathcal{M}_2(1,1)$ of generic forms. But α is ergodic, so it assigns full measure to A; consequently, μ assigns full measure to $T_{G\times G}$. It follows that each conditional measure μ_s is an absolutely continuous, $G\times G$-invariant measure on T_s.

By considering pairs of splittings in a neighborhood of a pair of rectangular tori (as in the proof of Theorem 10.2), we deduce from the absolute continuity of μ_s that the absolutely continuous component ξ^a of ξ is nontrivial. Thus μ^a and α^a are also nontrivial, and thus $\alpha = \alpha^a$ by ergodicity. But the standard invariant probability measure m on $\Omega_1\mathcal{M}_2(1,1)$ is also absolutely continuous and ergodic, so $\alpha = m$, completing the proof. ∎

12 Dynamics on ΩE_D

In this section we complete the proof of our main results by studying the space of eigenforms. We will show:

Theorem 12.1 *Let Z be the closure of the* $\mathrm{SL}_2(\mathbb{R})$*-orbit of $(X,\omega) \in \Omega_1 E_D$. Then either:*

- *$Z = \Omega_1 V$ for some Teichmüller curve $V \to \mathcal{M}_2$, or*

- $Z = \Omega_1 E_D$.

Theorem 12.2 *Any ergodic* $\mathrm{SL}_2(\mathbb{R})$*-invariant probability measure* α *on* $\Omega_1 E_D$ *is either the standard measure on* $\Omega_1 E_D(1,1)$*, or the standard measure on* $\Omega_1 V$ *for some Teichmüller curve* V*.*

These theorems, in concert with the results of §10 and §11, complete the classification of orbits closures and invariant measures for $\Omega_1\mathcal{M}_2$.

Orbit closures in $\boldsymbol{\Omega_1 E_D}$. Let $\Omega_1 E_D(1,1)$ and $\Omega_1 E_D(2)$ denote the eigenforms with simple and double zeros, respectively. We begin by studying what happens when one $\mathrm{SL}_2(\mathbb{R})$-orbit is a limit of others.

Lemma 12.3 *Let* $Y = \mathrm{SL}_2(\mathbb{R}) \cdot (X,\omega)$ *be a single orbit in a closed,* $\mathrm{SL}_2(\mathbb{R})$*-invariant set* $Z \subset \Omega_1 E_D$*. If* Y *is not open in* Z*, then* $\mathrm{int}(Z) \neq \emptyset$ *and* $Y \subset \mathrm{int}(Z) \cup \Omega_1 E_D(2)$*.*

Proof. By assumption, (X,ξ) is the limit of a sequence $(X_n,\xi_n) \in Z - Y$. Since $\Omega_1 E_D(2)$ is a finite union of closed orbits (by Corollary 5.11), we can assume $(X_n,\xi_n) \in \Omega_1 E_D(1,1)$.

Choose a splitting
$$(X,\xi) = (E,\omega) \underset{I}{\#} (F,\eta)$$
with $I = [0,v]$. If ξ happens to have a double zero, order the summands so I embeds in E.

For all $n \gg 0$ there exist corresponding splittings
$$(X_n,\xi_n) = (E_n,\omega_n) \underset{I_n}{\#} (F_n,\eta_n),$$
with $I_n = [0,v_n]$ and $v_n \to v$. By Corollary 8.4, up to the action of $\mathrm{SL}_2(\mathbb{R})$, only a finite number of pairs of tori are required to express all connected sums in $\Omega_1 E_D$. Thus after passing to a subsequence, we can find a fixed pair of tori and a sequence $g_n \in \mathrm{SL}_2(\mathbb{R})$ such that
$$(X_n,\xi_n) = (g_n \cdot (E',\omega')) \underset{I_n}{\#} (g_n \cdot (F',\eta')). \tag{12.1}$$

Since $H_1(E,\mathbb{Z}) \subset H_1(X,\mathbb{Z})$ has a pair of generating cycles disjoint from I, there are corresponding nearby cycles generating $H_1(E_n,\mathbb{Z}) \subset H_1(X_n,\mathbb{Z})$. By continuity of the periods of ω_n along these cycles, we have
$$(E_n,\omega_n) = g_n \cdot (E',\omega') \to (E,\omega)$$
in $\Omega\mathcal{M}_1$. Thus we may choose $(E',\omega') = (E,\omega)$ and $(F',\eta') = (F,\eta)$ as our fixed pair of tori, and arrange that $g_n \to \mathrm{id}$. But $g_n^{-1}(X_n,\xi_n) \in Z$ also converges to (X,ξ), so we can further modify the sequence so that $g_n = \mathrm{id}$ for all n. Thus (X,ξ) is the limit of a sequence
$$(X_n,\xi_n) = (E,\omega) \underset{I_n}{\#} (F,\eta) \in Z, \tag{12.2}$$

consisting of connected sums of a fixed pair of tori.

Now suppose the slopes of the vectors v_n defining I_n form an infinite set. Let Γ be the lattice $\mathrm{SL}(E,\omega)\cap\mathrm{SL}(F,\eta)$. Since Z is $\mathrm{SL}_2(\mathbb{R})$-invariant, we have

$$(E,\omega)\underset{gI_n}{\#}(F,\eta)\in Z$$

for all $g\in\Gamma$. By Theorem 2.10, $\bigcup\Gamma v_n$ is dense in $\mathbb{R}^2$. Thus $\overline{\bigcup\Gamma v_n}$ contains the open dense set D consisting of all $w\in\mathbb{R}^2$ such that

$$(E,\omega)\underset{[0,w]}{\#}(F,\eta)$$

is defined and has simple zeros. Since Z is closed, it also contains all these sums, and hence $\mathrm{int}(Z)\neq\emptyset$. Finally if (X,ω) has simple zeros, then $v\in D$ and thus $Y\subset\mathrm{int}(Z)$; otherwise $Y\subset\Omega_1E_D(2)$. This completes the proof in the case of infinitely many slopes.

Now suppose v_n assumes only finitely many slopes. We will show that after resplitting, we obtain infinitely many slopes.

Since $v_n\to v$, we can pass to a subsequence such that $v_n=t_nv$ where $t_n\to 1$ and $t_n\neq 1$ (because $(X_n,\xi_n)\in Z-Y$). Choose a second splitting

$$(X,\xi)=(A,\alpha)\underset{J}{\#}(B,\beta)$$

with $J=[0,w]\neq I$. Then for all $n\gg 0$, there are corresponding splittings

$$(X_n,\xi_n)=(A,\alpha)\underset{J_n}{\#}(B,\beta),$$

$J_n=[0,w_n]$. (The old summands in (12.2) were independent of n, so the new summands are as well.)

Since both v_n and w_n are relative periods of saddle connections joining the zeros of ξ_n, we can assume (after adjusting signs) that w_n-v_n is an absolute period of ξ_n. But the absolute periods of ξ_n are independent of n, so $w_n=v_n+p=t_nv+p$ for a fixed absolute period p. Since I and J have different slopes, so do p and v, and therefore the vectors w_n assume infinitely many slopes. The desired conclusions then follow from the argument above. ■

Proof of 12.1. If the $\mathrm{SL}_2(\mathbb{R})$-orbit of $(X,\xi)\in\Omega_1E_D$ is already closed, then (X,ξ) generates a Teichmüller curve, by Theorem 3.4. Otherwise, its closure Z contains a non-isolated orbit, and thus Z has nonempty interior. But any orbit $Y\subset\overline{\mathrm{int}(Z)}\cap\Omega_1E_D(1,1)$ fails to be open in Z, so by Lemma 12.3 it satisfies $Y\subset\mathrm{int}(Z)$. Therefore $\mathrm{int}(Z)\cap\Omega_1E_D(1,1)$ is both open and closed. Since $\Omega_1E_D(1,1)$ is connected, and dense in Ω_1E_D, this implies $Z=\Omega_1E_D$. ■

Splittings of eigenforms. To study invariant measures, we once more pull back to the space of splittings.

Let $((E_i, \omega_i), (F_i, \eta_i))$, $i = 1, \ldots, n_D$ be the finite list of pairs of tori provided by Corollary 8.4. Let $T_i \subset \mathbb{R}^2$ be the open dense set of vectors v such that the connected sum of the ith pair of tori over $I = [0, v]$ is well-defined and has simple zeros. Let $G = \mathrm{SL}_2(\mathbb{R})$, and let

$$f_i : G \times T_i \to \Omega_1 E_D(1,1)$$

be the local covering map defined by

$$f_i(g, v) = g \cdot ((E_i, \omega_i) \underset{[0,v]}{\#} (F_i, \eta_i)). \tag{12.3}$$

Note that

$$f_i(g, h \cdot v) = f_i(gh, v) \tag{12.4}$$

for all h in the lattice $\Gamma_i = \mathrm{SL}(E_i, \omega_i) \cap \mathrm{SL}(F_i, \eta_i)$. By Corollary 8.4, the open sets $f_i(G \times T_i)$ cover $\Omega_1 E_D(1,1)$.

Proof of Theorem 12.2. Let α be an ergodic, G-invariant probability measure on $\Omega_1 E_D$. Then we can write

$$f_i^*(\alpha) = dg \times \mu_i,$$

where dg is Haar measure on G and μ_i is a locally finite measure on $T_i \subset \mathbb{R}^2$. By (12.4), μ_i is Γ_i-invariant.

Recall $\Omega_1 E_D(2) = \bigcup_1^n \Omega_1 V_i$ by Corollary 5.11. If α assigns positive mass to $\Omega_1 E_D(2)$, then by ergodicity it assigns full measure to a single component $\Omega_1 V_i$, and we are done.

Otherwise, at least one measure μ_i is nonzero. Let $\alpha^a + \alpha^s$ and $\mu_i^a + \mu_i^s$ be the Lebesgue decompositions of α and μ_i. By Theorem 2.10, μ_i^a is a multiple of the standard area measure on $\mathbb{R}^2$, while μ_i^s is supported on the set

$$R_i = \{v \in \mathbb{R}^2 \ : \ \Gamma_i v \text{ is discrete}\}.$$

Concretely, $R_i = \mathbb{R} \cdot \mathrm{Per}(\omega_i)$ is a countable union of lines through the origin, consisting of those vectors v such that the splitting (12.3) has two cylinders.

Now suppose $\mu_j^a \neq 0$ for some j. We claim α coincides with the standard invariant probability measure m on $\Omega_1 E_D(1,1)$. Indeed, since $dg \times \mu_j^a$ is a constant multiple of $f_j^*(m)$, the measure $\alpha^a | f_j(G \times T_i)$ is a constant multiple of m. By ergodicity, $\alpha = \alpha^a$. Applying the same reasoning to the open sets $f_i(G \times T_i)$ in $\Omega_1 E_D(1,1)$, we conclude that α/m is locally constant on the connected space $\Omega_1 E_D(1,1)$. Since both α and m are probability measures, we have $\alpha = m$.

Finally suppose $\mu_i^a = 0$ for all i. This implies μ_i is supported on R_i for all i, so α is supported on the set of eigenforms for which every splitting has two cylinders. Consequently μ_i is supported on the set P_i of v such that every splitting of $(X, \xi) = f(\mathrm{id}, v)$ has 2 cylinders.

We claim that P_i is countable. Indeed, for any $v \in P_i$, we have a distinct second splitting of $(X,\xi) = f(\mathrm{id}, v)$ over $J = [0, w]$, where $w = p_1 + v$ for some absolute period

$$p_1 \in \mathrm{Per}(\xi) = \mathrm{Per}(\omega_i) + \mathrm{Per}(\eta_i) = \Lambda_i.$$

Since both splittings have cylinders, there are periods $p_2, p_3 \in \Lambda_i$ such that $v \in \mathbb{R}p_2$ and $w \in \mathbb{R}p_3$. But distinct splittings cannot be parallel, so we have

$$\{v\} = \mathbb{R}p_2 \cap (-p_1 + \mathbb{R}p_3).$$

There are only countably many possibilities for $(p_1, p_2, p_3) \in \Lambda_i^3$, so P_i itself is a countable set.

Consequently α is supported on a countable union of $\mathrm{SL}_2(\mathbb{R})$-orbits. Since α is ergodic, it must be supported on a single orbit $\mathrm{SL}_2(\mathbb{R}) \cdot (X,\xi)$, and since α has finite measure, $\mathrm{SL}(X,\xi)$ must be a lattice. Therefore (X,ξ) generates a Teichmüller curve $V \to \mathcal{M}_2$, and α coincides with the standard measure on $\Omega_1 V$. ■

Proof of Theorems 1.2 and 1.5. The classification of orbit closures in $\Omega_1\mathcal{M}_2$ now follows, by combining Theorems 10.1, 11.1 and 12.1 on orbits in $\Omega_1\mathcal{M}_2(2)$, $\Omega_1(1,1)$ and $\Omega_1 E_D$ respectively. Similarly, the classification of invariant measures follows from Theorems 10.2, 11.2 and 12.2. ■

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