CHAPTER 11

Dynamics of Subgroup Actions on Homogeneous Spaces of Lie Groups and Applications to Number Theory

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Introduction

This survey presents an exposition of homogeneous dynamics, that is, dynamical and ergodic properties of actions on homogeneous spaces of Lie groups. Interest in this area rose significantly during late eighties and early nineties, after the seminal work of Margulis (the proof of the Oppenheim conjecture) and Ratner (conjectures of Raghunathan, Dani and Margulis) involving unipotent flows on homogeneous spaces. Later developments were considerably stimulated by newly found applications to number theory. By the end of the 1990s this rather special class of smooth dynamical systems has attracted widespread attention of both dynamicists and number theorists, and became a battleground for applying ideas and techniques from diverse areas of mathematics.

The classical set-up is as follows. Given a Lie group G and a closed subgroup $\Gamma \subset G$, one considers the left action of any subgroup $F \subset G$ on G/Γ:

$$x \mapsto fx, \quad x \in G/\Gamma, \ f \in F.$$

Usually F is a one-parameter subgroup; the action obtained is then called a homogeneous (one-parameter) flow.

One may recall that many concepts of the modern theory of dynamical systems appeared in connection with the study of the geodesic flow on a compact surface of constant negative curvature. While establishing ergodicity and mixing properties of the geodesic flow, Hopf, Hedlund et al., initiated the modern theory of smooth dynamical systems. Essentially, they used the existence of what is now called strong stable and unstable foliations; later this transformed into the theory of Anosov and Axiom A flows. On the other hand, while studying orbits of the geodesic flow, Artin and Morse developed methods that later gave rise to symbolic dynamics.

The methods used in the 1920–1940s were of geometric or arithmetic nature. The fact that the geodesic flow comes from transitive G-action on G/Γ, Γ being a lattice in $G = \mathrm{SL}(2, \mathbb{R})$, was noticed and explored by Gelfand and Fomin in the early 1950s. This distinguished the class of homogeneous flows into a subject of independent interest; first results of general nature appeared in the early 1960s in the book [9] by Auslander, Green, and Hahn.

Since then, ergodic properties of homogeneous one-parameter flows with respect to Haar measure have been very well studied using the theory of unitary representations, more specifically, the so called Mautner phenomenon developed in papers of Auslander, Dani, and Moore. Applying the structure of finite volume homogeneous spaces, one can now give effective criteria for ergodicity and mixing properties, reduce the study to the ergodic case, calculate the spectrum, etc.; see Sections 1 and 2 for the exposition.

The algebraic nature of the phase space and the action itself allows one to obtain much more advanced results as compared to the general theory of smooth dynamical systems. This can be seen on the example of smooth flows with polynomial divergence of trajectories. Not much is known about these in the general case apart from the fact that they have zero entropy. The counterpart in the class of homogeneous flows is the class of the so called unipotent flows. Motivated by certain number theoretic applications (namely, by the Oppenheim conjecture on indefinite quadratic forms), in the late 1970s Raghunathan conjectured that any orbit closure of a unipotent flow on G/Γ must be itself homogeneous, i.e., must admit a transitive action of a subgroup of G. This and a related measure-theoretic conjecture by Dani, giving a classification of ergodic measures, was proved in the early 1990s by Ratner thus really providing a breakthrough in the theory. This was preceded by highly nontrivial results of Margulis–Dani on nondivergence of unipotent trajectories. Dynamics of unipotent action is exposed in Sections 3 of our survey.

On the other hand, a lot can be also said about partially hyperbolic homogeneous flows. What makes the study easier, is that the Lyapunov exponents are constant. This (and, of course, the algebraic origin of the flow) makes it possible to prove some useful results: to calculate the Hausdorff dimension of nondense orbits, to classify minimal sets, to prove the exponential and multiple mixing properties, etc. Needless to say that such results are impossible or very difficult to prove for smooth partially hyperbolic flows of general nature.

The substantial progress achieved due to Ratner's results for unipotent actions made it possible to study individual orbits and ergodic measures for certain multi-dimensional subgroups $F \subset G$. These and aforementioned results for non-unipotent actions are exposed in Section 4.

As was mentioned above, the interest in dynamical systems of algebraic origin in the last 20 years rose considerably due to remarkable applications to number theory. It was Margulis who proved the long standing Oppenheim conjecture in the mid 1980s settling a special case of Raghunathan's conjecture. Recently Eskin, Margulis and Mozes (preceded by earlier results of Dani and Margulis) obtained a quantitative version of the Oppenheim conjecture. Kleinbock and Margulis proved Sprindžuk's conjectures for Diophantine approximations on manifolds. Eskin, Mozes, and Shah obtained new asymptotics for counting lattice points on homogeneous manifolds. Skriganov established a typical asymptotics for counting lattice points in polyhedra. These and related results are discussed in Section 5.

The theory of homogeneous flows has so many diverse applications that it is impossible to describe all the related results, even in such an extensive survey. Thus for various reasons we do not touch upon certain subjects. In particular, all Lie groups throughout the survey are real; hence we are not concerned with dynamical systems on homogeneous spaces over other local fields; we only note that the Raghunathan–Dani conjectures are settled in this setup as well (see [147,148,190,191]). We also do not touch upon the analogy between actions on homogeneous spaces of Lie groups and the action of $\mathrm{SL}(2,\mathbb{R})$ on the moduli space of quadratic differentials, with newly found applications to interval exchange transformations and polygonal billiards. See [152] for an extensive account of the subject. The reader interested in a more detailed exposition of dynamical systems on homogeneous spaces may consult recent books [237] by Starkov and [13] by Bekka and Mayer, or surveys [57,59] by Dani. For a list of open problems we address the reader to a paper of Margulis [144].

1. Lie groups and homogeneous spaces

In this section we give an introduction into the theory of homogeneous actions. We start with basic results from Lie groups and algebraic groups (Sections 1.1 and 1.2). Then in

Section 1.3 we expose briefly the structure of homogeneous spaces of finite volume. For more details one can refer to books by Raghunathan [174], Margulis [141], Zimmer [264], and survey by Vinberg, Gorbatsevich and Shvartzman [249].

Homogeneous actions come into stage in Section 1.4. Basic examples include: rectilinear flow on a torus; solvable flows on a three-dimensional locally Euclidean manifold; suspensions of toral automorphisms; nilflows on homogeneous spaces of the three-dimensional Heisenberg group; the geodesic and horocycle flows on (the unit tangent bundle of) a constant negative curvature surface; geodesic flows on locally symmetric Riemannian spaces. Incidentally, in §1.3d we present the main link between homogeneous actions with number theory (Mahler's criterion and its consequences).

1.1. *Basics on Lie groups*

1a. *Notations and conventions.* Unless otherwise stated, all our Lie groups are assumed to be real (i.e., over $\mathbb{R}$). We use the following

NOTATION 1.1.1.
$\widetilde{G}$ – the universal cover of Lie group G;
$\mathfrak{g}$ or $\mathrm{Lie}(G)$ – the Lie algebra of Lie group G;
$\exp : \mathfrak{g} \mapsto G$ – the exponential map;
$e \in G$ – the identity element of G;
$\mathrm{Ad} : G \mapsto \mathrm{Aut}(\mathfrak{g})$ – the adjoint representation defined as

$$\mathrm{Ad}_g(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big(g \exp(tx) g^{-1}\big)|_{t=0}, \quad \forall g \in G,\ \forall x \in \mathfrak{g};$$

$\mathrm{ad} : \mathfrak{g} \mapsto \mathrm{Der}(\mathfrak{g})$ – the adjoint representation of $\mathfrak{g}$, i.e., the differential of Ad, defined as $\mathrm{ad}_x(y) = [x, y]$, $x, y \in \mathfrak{g}$;
$H^0 \subset G$ – the connected component of the identity of a closed subgroup $H \subset G$;
$Z_G(H) \subset G$ – the centralizer of a subgroup $H \subset G$;
$N_G(H) \subset G$ – the normalizer of a subgroup $H \subset G$;
$[H_1, H_2]$ – the commutant of subgroups $H_1, H_2 \subset G$, i.e., the subgroup generated by all commutators $\{h_1, h_2\} = h_1 h_2 h_1^{-1} h_2^{-1}$, $h_1 \in H_1$, $h_2 \in H_2$.
$g^{\mathbb{Z}} \subset G$ – the cyclic group generated by an element $g \in G$;
$g_{\mathbb{R}} \subset G$ – a one-parameter subgroup of G;
$\langle S \rangle$ – the subgroup generated by a subset S of G.
The notation $G = A \ltimes B$ stands for *semidirect product* of subgroups $A, B \subset G$. This means that B is normal in G, A and B generate G, and $A \cap B = \{e\}$.

It is well known that every Lie group admits a right-invariant measure m called *Haar measure*, defined uniquely up to a multiplicative constant. It arises as the volume measure for a right-invariant Riemannian metric d_G. A Lie group G is said to be *unimodular* if its Haar measure is bi-invariant. This is equivalent to saying that every operator Ad_g, $g \in G$, is unimodular, i.e., $|\det(\mathrm{Ad}_g)| = 1$.

DEFINITION 1.1.2. An element $g \in G$ is said to be:

unipotent if $(\mathrm{Ad}_g - \mathrm{Id})^k = 0$ for some $k \in \mathbb{N}$ (this is equivalent to saying that all eigenvalues of Ad_g are equal to 1);

quasi-unipotent if all eigenvalues of Ad_g are of absolute value 1;

partially hyperbolic if it is not quasi-unipotent;

semisimple if the operator Ad_g is diagonalizable over $\mathbb{C}$;

$\mathbb{R}$-*diagonalizable* if the operator Ad_g is diagonalizable over $\mathbb{R}$;

A subgroup $H \subset G$ is said to be *unipotent* (resp. *quasi-unipotent*) if all its elements are such.

Let g be an element of G. Then the subgroups

$$G^+ = \{h \in G \mid g^{-n} h g^n \to e,\ n \to +\infty\},$$
$$G^- = \{h \in G \mid g^{n} h g^{-n} \to e,\ n \to +\infty\}$$

are called the *expanding* and *contracting horospherical subgroups* of G relative to g, respectively. Any horospherical subgroup is connected and unipotent.

1b. *Nilpotent and solvable Lie groups.* Define $G_1 = G$ and $G_{k+1} = [G, G_k]$. G is said to be *nilpotent* if there exists $n \in \mathbb{N}$ such that $G_n = \{e\}$. For Lie groups this is equivalent to all elements of G being unipotent.

If a subspace V of a nilpotent Lie algebra $\mathfrak{g}$ is such that $\mathfrak{g} = V + [\mathfrak{g}, \mathfrak{g}]$, then V generates $\mathfrak{g}$. In particular, if subgroup $H \subset G$ is such that $G = H[G, G]$, then $H = G$.

A natural example of a nilpotent Lie group is the group $N(n) \subset \mathrm{SL}(n, \mathbb{R})$ of strictly upper-triangular matrices. Every connected simply connected nilpotent Lie group G is isomorphic to a subgroup of $N(n)$ for some $n \in \mathbb{N}$.

Define $G^1 = G$ and $G^{k+1} = [G^k, G^k]$. A Lie group G is said to be *solvable* if there exists $n \in \mathbb{N}$ such that $G^n = \{e\}$. Every nilpotent group is solvable.

Let G be a Lie group. The maximal connected nilpotent (resp. solvable) normal subgroup $N \subset G$ (resp. $R \subset G$) is said to be the *nilradical* (resp. the *radical*) of G.

If G is connected and solvable, then $[G, G] \subset N$, and hence the group G/N is Abelian.

Let G be a connected simply connected solvable Lie group. Then G is diffeomorphic to a vector space, and every connected subgroup of G is simply connected and closed. The map $\exp : \mathfrak{g} \mapsto G$ need not be a diffeomorphism, but if it is so then G is said to be an *exponential* Lie group. G is exponential iff for every eigenvalue λ of any operator Ad_g, $g \in G$, either $\lambda = 1$ or $|\lambda| \neq 1$. In particular, any connected simply connected nilpotent Lie group is exponential.

If all eigenvalues of any operator Ad_g, $g \in G$, are of absolute value 1, then G is said to be *of type* (I) (from 'imaginary'). Hence G is of type (I) iff all its elements are quasi-unipotent. If all eigenvalues of any operator Ad_g, $g \in G$, are purely real, then G is said to be a *triangular* group. Any connected simply connected triangular group is isomorphic to a subgroup of the group $T(n) \subset \mathrm{SL}(n, \mathbb{R})$ of upper-triangular matrices (for some $n \in \mathbb{N}$). Any triangular solvable group is exponential. A solvable group is nilpotent iff it is both exponential and of type (I).

Within the class of solvable Lie groups of type (I) one distinguishes the subclass of Euclidean Lie groups. A Lie group G is said to be *Euclidean* if G splits into a semidirect product $G = A \ltimes \mathbb{R}^n$, where $\mathbb{R}^n$ is the nilradical of G, A is Abelian, and the representation $\mathrm{Ad}: G \mapsto \mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}(n,\mathbb{R})$ sends A to a compact subgroup of $\mathrm{GL}(n,\mathbb{R})$.

A natural example of simply connected Euclidean group is the following. Let $G = g_{\mathbb{R}} \ltimes \mathbb{R}^2$, where the action of g_t on the plane $\mathbb{R}^2$ by conjugation is given by the matrix $\left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$. Then G is easily seen to be Euclidean; in fact, it is the universal cover of the group $G/Z(G) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ of Euclidean motions of the plane.

1c. *Semisimple Lie groups*. A connected Lie group G is said to be *semisimple* if its radical is trivial. G is said to be *simple* if it has no nontrivial proper normal connected subgroups. Every connected semisimple Lie group G can be uniquely decomposed into an *almost direct product* $G = G_1 G_2 \cdots G_n$ of its normal simple subgroups, called the *simple factors* of G. That is, G_i and G_j commute and the intersection $G_i \cap G_j$ is discrete if $i \neq j$. If in addition G is simply connected or center-free, then $G = G_1 \times G_2 \times \cdots \times G_n$.

If G is semisimple then its center $Z(G)$ is discrete (and coincides with the kernel of the adjoint representation if G is connected).

The universal cover of a compact semisimple Lie group is compact (Weyl). Compact semisimple Lie groups (e.g., $\mathrm{SO}(n)$, $n \geqslant 3$) have no unipotent one-parameter subgroups. On the contrary, noncompact simple connected Lie groups (e.g., $\mathrm{SL}(n,\mathbb{R})$, $n \geqslant 2$) are generated by unipotent one-parameter subgroups.

Let G be a connected semisimple Lie group. Then G is almost direct product $G = KS$ of its *compact* and *totally noncompact* parts, where $K \subset G$ is the product of all compact simple components of G, and S the product of all noncompact simple components. G is called *totally noncompact* if K is trivial.

For any unipotent subgroup $u_{\mathbb{R}} \subset G$ there exists a connected subgroup $H \subset G$ such that $u_{\mathbb{R}} \subset H$ and H is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ (*Jacobson–Morozov Lemma*).

Let G be a connected semisimple Lie group. A subgroup $H \subset G$ is said to be *Cartan* if H is a maximal connected Abelian subgroup consisting of semisimple elements. Any Cartan subgroup has a unique decomposition $H = T \times A$ into a direct product of a compact torus T and an $\mathbb{R}$-diagonalizable subgroup A.

All maximal connected $\mathbb{R}$-diagonalizable subgroups in G are conjugate and their common dimension is called the $\mathbb{R}$-*rank* of G. For instance, $\mathrm{rank}_{\mathbb{R}} \mathrm{SL}(n,\mathbb{R}) = n-1$; $\mathrm{rank}_{\mathbb{R}} G = 0$ iff G is compact. If maximal $\mathbb{R}$-diagonalizable subgroups are Cartan subgroups then G is said to be an $\mathbb{R}$-*split* group.

Of special importance for us will be the group $\mathrm{SO}(1,n)$ of all elements $g \in \mathrm{SL}(n+1,\mathbb{R})$ keeping the quadratic form $x_1^2 - x_2^2 - \cdots - x_{n+1}^2$ invariant. One knows that $\mathrm{SO}(1,n)$ is a simple Lie group of $\mathbb{R}$-rank one for any $n \geqslant 2$. Also $\mathrm{SO}(1,2)$ is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$.

Now assume that G has finite center. Let $C \subset G$ be a maximal connected compact subgroup, and $B \subset G$ a maximal connected triangular subgroup. Then $C \cap B = \{e\}$, and any element $g \in G$ can be uniquely decomposed as $g = cb$, $c \in C$, $b \in B$. Besides, $B = A \ltimes U$, where U is the nilradical of B called a *maximal horospherical subgroup* of G, and A is a maximal $\mathbb{R}$-diagonalizable subgroup. Decomposition $G = CB = CAU$ is called the *Iwasawa decomposition* of G. All maximal compact connected subgroups in G are conjugate. The same holds for maximal connected triangular subgroups; hence all maximal horospherical subgroups are conjugate. The normalizer of U is called *minimal parabolic subgroup* of G (clearly it contains B).

Any horospherical subgroup (see §1.1a for the definition) is contained in a maximal horospherical subgroup. The normalizer of a horospherical subgroup is called a *parabolic subgroup* of G. Any parabolic subgroup contains a minimal parabolic subgroup.

Any element $g \in G$ can be (non-uniquely) decomposed in the form $g = cac'$, $c, c' \in C$, $a \in A$. The formula $G = CAC$ is referred to as the *Cartan decomposition*.

1d. *Levi decomposition*. Let G be a connected Lie group, R the radical of G, and $L \subset G$ a maximal connected semisimple subgroup (called a *Levi subgroup* of G). Then the intersection $L \cap R$ is discrete and G is generated by L and R. The decomposition $G = LR$ is called the *Levi decomposition* of G. If G is simply connected then $G = L \ltimes R$.

Let $N \subset R$ be the nilradical of G. All Levi subgroups in G are conjugate by elements of N (Malcev).

1e. *Group actions and unitary representations*. If X is a topological space, we denote by $\mathcal{P}(X)$ the space of all Borel probability measures on X, with the weak-$*$ topology, that is, the topology induced by the pairing with the space $C_c(X)$ of continuous compactly supported functions on X. Any continuous action of a group G on X induces a continuous action of G on $\mathcal{P}(X)$ given by

$$g\mu(A) = \mu\big(g^{-1}A\big), \quad \text{where } g \in G \text{ and } A \text{ is a Borel subset of } X. \tag{1.1}$$

For a measure μ on X, we will denote by $\operatorname{Stab}(\mu)$ the set of elements $g \in G$ such that $g\mu = \mu$. The action of G on the measure space (X, μ) is called *measure-preserving* if $\operatorname{Stab}(\mu)$ coincides with G. In general, due to the continuity of the G-action on $\mathcal{P}(X)$, one has

LEMMA 1.1.3. $\operatorname{Stab}(\mu)$ *is a closed subgroup of G for any $\mu \in \mathcal{P}(X)$.*

A very important tool for studying homogeneous actions is the theory of unitary representations. By a *unitary representation* of a locally compact group G on a separable Hilbert space $\mathcal{H}$ we mean a continuous homomorphism of G into the group $U(\mathcal{H})$ of unitary transformations of $\mathcal{H}$ (supplied with the weak operator topology).

Clearly any measure-preserving G-action on a measure space (X, μ) leads to a unitary representation ρ of G on the space $L^2(X, \mu)$ of square-integrable functions on X, defined by $(\rho(g)f)(x) \overset{\text{def}}{=} f(g^{-1}x)$, and called the *regular representation* of G associated with the action. As is well known, many ergodic properties of the G-action on (X, μ) are expressed in terms of ρ: e.g., the action is ergodic iff any $\rho(G)$-invariant element of $L^2(X, \mu)$ is a constant function; the action of a one-parameter group $g_{\mathbb{R}}$ is weakly mixing iff any $f \in L^2(X, \mu)$ such that $\rho(g_t)f = \mathrm{e}^{i\lambda t} f$ for some $\lambda \in \mathbb{R}$ and all $t \in \mathbb{R}$ is a constant function.

Similarly to Lemma 1.1.3, one derives the following useful result from the continuity of the regular representation:

LEMMA 1.1.4. *Given any* $f \in L^2(X,\mu)$, *the stabilizer* $\mathrm{Stab}(f) \stackrel{\text{def}}{=} \{g \in G \mid \rho(g)f = f\}$ *is a closed subgroup of* G.

Let π be a unitary representation of G on $\mathcal{H}$. If a closed subspace $\mathcal{H}'$ of $\mathcal{H}$ is invariant under $\pi(G)$, then the restriction of π to $\mathcal{H}'$ is called a *sub-representation* of π. A representation is called *irreducible* if it has no nontrivial sub-representations. Two representations $\pi : G \mapsto U(\mathcal{H})$ and $\pi' : G \mapsto U(\mathcal{H}')$ are called *equivalent* if there exists an invertible operator $A : \mathcal{H} \mapsto \mathcal{H}'$ such that $\pi'(g) = A\pi(g)A^{-1}$ for each $g \in G$. The *Fell topology* on the set of all equivalence classes $[(\pi,\mathcal{H})]$ of unitary representations π of G on $\mathcal{H}$ is defined as follows: the sets

$$\left\{ [(\pi',\mathcal{H}')] \;\middle|\; \begin{array}{l} \text{there exist } \eta_1,\dots,\eta_n \in \mathcal{H}' \text{ such that} \\ \left|\langle \pi(g)\xi_i,\xi_j\rangle - \langle \pi'(g)\eta_i,\eta_j\rangle\right| < \varepsilon,\ \forall g \in K,\ 1 \leqslant i,j \leqslant n \end{array} \right\},$$

where K is a compact subset of G, $\varepsilon > 0$ and $\xi_1,\dots,\xi_n \in \mathcal{H}$, form a basis of neighborhoods of the class $[(\pi,\mathcal{H})]$. This topology is not Hausdorff; of special interest is the question of whether a certain representation[1] (or a family of representations) is isolated from the trivial (one-dimensional) representation I_G of G.

G is said to have *property*-(T) if the set of all its unitary representations without invariant vectors (that is, not having a trivial sub-representation) is isolated from I_G. Any compact group has property-(T), and an amenable group (see [3]) has property-(T) iff it is compact. If G is a connected semisimple Lie group with finite center, then G has property-(T) iff it has no factors locally isomorphic to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, $n \geqslant 2$. See [1] for more details and the proof of the aforementioned result.

1.2. *Algebraic groups*

We will give the definitions in the generality needed for our purposes. A subset $X \subset \mathrm{SL}(n,\mathbb{R})$ will be called *real algebraic* if it is the zero set for a family of real polynomials of matrix entries. This defines the *Zariski topology* on $\mathrm{SL}(n,\mathbb{R})$. In contrast, the topology on $\mathrm{SL}(n,\mathbb{R})$ as a Lie group will be called the *Hausdorff* topology.

To save words, real algebraic subgroups of $\mathrm{SL}(n,\mathbb{R})$ will be referred to as $\mathbb{R}$-*algebraic groups*. Any $\mathbb{R}$-algebraic group has a finite number of connected components relative to the Hausdorff topology. The intersection of $\mathbb{R}$-algebraic groups is an $\mathbb{R}$-algebraic group. If $G \subset \mathrm{SL}(n,\mathbb{R})$ is an $\mathbb{R}$-algebraic group and $H \subset G$ its subgroup, then the centralizer $Z_G(H)$ and the normalizer $N_G(H)$ are $\mathbb{R}$-algebraic. The product $AB \subset \mathrm{SL}(n,\mathbb{R})$ of two $\mathbb{R}$-algebraic groups is locally closed (in both topologies).

The smallest $\mathbb{R}$-algebraic group $\mathrm{Zcl}(H)$ containing a subgroup $H \subset \mathrm{SL}(n,\mathbb{R})$ is called the *Zariski closure* of H (and one says that H is *Zariski dense* in $\mathrm{Zcl}(H)$). If H is Abelian (resp. nilpotent, solvable) then $\mathrm{Zcl}(H)$ is Abelian (resp. nilpotent, solvable). If H normalizes a connected Lie subgroup $F \subset \mathrm{SL}(n,\mathbb{R})$ then so does $\mathrm{Zcl}(H)$.

[1] For brevity we will from now on write ‘representation’ meaning ‘equivalence class of representations’.

2a. *Classification of elements and subgroups.* An element $g \in \mathrm{SL}(n,\mathbb{R})$ is called *unipotent* if $(g - I_n)^k = 0$ for some $k \in \mathbb{N}$ (here and hereafter I_n stands for the $n \times n$ identity matrix). An element $g \in \mathrm{SL}(n,\mathbb{R})$ is called *semisimple* (resp. $\mathbb{R}$-*diagonalizable*) if it is diagonalizable over $\mathbb{C}$ (resp. over $\mathbb{R}$).[2] If G is $\mathbb{R}$-algebraic then any element $g \in G$ admits the decomposition $g = g_s \times g_u$, where $g_s \in G$ is semisimple and $g_u \in G$ is unipotent (*Jordan decomposition*).

Clearly, the group $D(n) \subset \mathrm{SL}(n,\mathbb{R})$ of all diagonal matrices consists of $\mathbb{R}$-diagonalizable elements. It contains (at most) countable number of $\mathbb{R}$-algebraic subgroups. An $\mathbb{R}$-algebraic subgroup $H \subset \mathrm{SL}(n,\mathbb{R})$ is called $\mathbb{R}$-*diagonalizable* if it is conjugate to a subgroup of $D(n)$.

An $\mathbb{R}$-algebraic subgroup $H \subset \mathrm{SL}(n,\mathbb{R})$ is called *unipotent* if all its elements are such (and then H is conjugate to a subgroup of $N(n)$). $H \subset N(n)$ is algebraic iff it is connected in the Hausdorff topology.

The maximal normal unipotent subgroup of an $\mathbb{R}$-algebraic group G is called the *unipotent radical* of G. If the unipotent radical is trivial then G is called *reductive*. Any Abelian reductive group is the direct product of a compact torus and an $\mathbb{R}$-diagonalizable group (and hence consists of semisimple elements). Any reductive group is an almost direct product of its Levi subgroup and an Abelian reductive group. Any compact subgroup of $\mathrm{SL}(n,\mathbb{R})$ is a reductive $\mathbb{R}$-algebraic group. Any semisimple subgroup of $\mathrm{SL}(n,\mathbb{R})$ is "almost $\mathbb{R}$-algebraic" (it is of finite index in its Zariski closure).

Any $\mathbb{R}$-algebraic group is a semidirect product of a maximal reductive subgroup and the unipotent radical. In particular, any solvable $\mathbb{R}$-algebraic group splits into a semidirect product of its maximal Abelian reductive subgroup and the unipotent radical.

2b. *Algebraic groups over* $\mathbb{Q}$. If a subgroup G of $\mathrm{SL}(n,\mathbb{R})$ is the zero set of a family of polynomials of matrix entries with coefficients in $\mathbb{Q}$, then it is called a $\mathbb{Q}$-*algebraic* group, or a $\mathbb{Q}$-*group*. If G is a $\mathbb{Q}$-algebraic group then the set $G(\mathbb{Q}) \stackrel{\mathrm{def}}{=} G \cap \mathrm{SL}(n,\mathbb{Q})$ of its rational points is Zariski dense in H. If $H \subset \mathrm{SL}(n,\mathbb{Q})$, then $\mathrm{Zcl}(H)$ is $\mathbb{Q}$-algebraic. The centralizer and the normalizer of a $\mathbb{Q}$-algebraic group are $\mathbb{Q}$-algebraic. The radical, the unipotent radical, and the center of $\mathbb{Q}$-algebraic group are $\mathbb{Q}$-algebraic. Any $\mathbb{Q}$-algebraic group admits a maximal reductive subgroup which is $\mathbb{Q}$-algebraic.

If $g \in \mathrm{SL}(n,\mathbb{Q})$ then its semisimple and unipotent parts in the Jordan decomposition are elements of $\mathrm{SL}(n,\mathbb{Q})$.

A $\mathbb{Q}$-*character* of a $\mathbb{Q}$-group $G \subset \mathrm{SL}(n,\mathbb{R})$ is a homomorphism χ from G to the multiplicative group $\mathbb{R}^*$ which can be written as a polynomial of matrix entries with coefficients in $\mathbb{Q}$. Unipotent $\mathbb{Q}$-groups admit no nontrivial $\mathbb{Q}$-characters. An Abelian reductive $\mathbb{Q}$-group is said to be $\mathbb{Q}$-*anisotropic* if it admits no nontrivial $\mathbb{Q}$-characters, and $\mathbb{Q}$-*split* if it has no nontrivial $\mathbb{Q}$-anisotropic subgroups. Clearly, any Abelian reductive $\mathbb{Q}$-group is the direct product of its $\mathbb{Q}$-anisotropic and $\mathbb{Q}$-split parts. Any $\mathbb{Q}$-split group is $\mathbb{R}$-diagonalizable.

Maximal $\mathbb{Q}$-split subgroups of a semisimple $\mathbb{Q}$-group G are conjugate and their common dimension is called the $\mathbb{Q}$-*rank* of G (clearly, $0 \leqslant \mathrm{rank}_{\mathbb{Q}} G \leqslant \mathrm{rank}_{\mathbb{R}} G$). Moreover, $\mathrm{rank}_{\mathbb{Q}} G = 0$ iff $G(\mathbb{Q})$ consists of semisimple elements.

[2]These properties are analogous to those for elements of Lie groups (cf. Definition 1.1.2). As a rule, it follows from the context which ones are used.

Let $G \subset \mathrm{SL}(n,\mathbb{R})$ be an $\mathbb{R}$-algebraic group. A homomorphism $\rho : G \mapsto \mathrm{SL}(N,\mathbb{R})$ is called an *algebraic linear representation* if its coordinate functions are polynomials of matrix entries. If all the polynomials have coefficients in $\mathbb{Q}$, then ρ is said to be a $\mathbb{Q}$-*algebraic linear representation*.

By Chevalley's theorem, for any $\mathbb{R}$-algebraic group $G \subset \mathrm{SL}(n,\mathbb{R})$ there exist an algebraic linear representation $\rho : \mathrm{SL}(n,\mathbb{R}) \mapsto \mathrm{SL}(N,\mathbb{R})$ and a vector $\mathbf{v} \in \mathbb{R}^N$ such that

$$G = \{g \in \mathrm{SL}(n,\mathbb{R}) \mid \rho(g)(\mathbb{R}\mathbf{v}) = \mathbb{R}\mathbf{v}\}.$$

Moreover, by [26, Proposition 7.7], if G is either unipotent or reductive then there exist ρ and $\mathbf{v}$ such that

$$G = \{g \in \mathrm{SL}(n,\mathbb{R}) \mid \rho(g)(\mathbf{v}) = \mathbf{v}\}.$$

If in addition G is a $\mathbb{Q}$-group then ρ can be chosen to be $\mathbb{Q}$-algebraic and $\mathbf{v}$ to be an element of $\mathbb{Q}^N$. In this case $\rho(\mathrm{SL}(n,\mathbb{Z}))$ has a finite index subgroup contained in $\mathrm{SL}(N,\mathbb{Z})$. Therefore $\rho(\mathrm{SL}(n,\mathbb{Z}))(\mathbf{v})$ is a discrete subset of $\mathbb{R}^N$. We derive the following (cf. [174, Proposition 10.15]):

PROPOSITION 1.2.1. *Let $G \subset \mathrm{SL}(n,\mathbb{R})$ be a connected real algebraic $\mathbb{Q}$-subgroup. Assume that either* (i) *G has no nontrivial $\mathbb{Q}$-characters, or* (ii) *G is a reductive subgroup. Then the G-orbit of the identity coset in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ is closed.*

1.3. *Homogeneous spaces*

Let G be a connected Lie group and $D \subset G$ its closed subgroup. In the survey we work mainly with *right* homogeneous spaces G/D, but occasionally (see §1.3c, §1.3e and §1.4e–§1.4f) we turn to *left* homogeneous spaces $D\backslash G$.

Any right-invariant Haar measure on G induces a smooth volume measure ν on G/D which will be also called a *Haar* measure. The space G/D is said to be of *finite volume* (we denote it by $\mathrm{vol}(G/D) < \infty$) if Haar measure ν on G/D is G-invariant and finite.

If $D \subset F \subset G$ then $\mathrm{vol}(G/D) < \infty \Leftrightarrow \mathrm{vol}(G/F) < \infty$ and $\mathrm{vol}(F/D) < \infty$. In particular, if $H \subset G$ is a normal subgroup, $\mathrm{vol}(G/D) < \infty$ and the product HD is closed then $\mathrm{vol}(H/D \cap H) < \infty$.

A closed subgroup $D \subset G$ is said to be *uniform* if the space G/D is compact.

A discrete subgroup $\Gamma \subset G$ is said to be a *lattice* in G if $\mathrm{vol}(G/\Gamma) < \infty$. Any uniform discrete subgroup is a lattice.

If G is a $\mathbb{Q}$-algebraic group, one has the following criterion of Borel and Harish-Chandra for its group of integer points $G_{\mathbb{Z}} \stackrel{\mathrm{def}}{=} G \cap \mathrm{SL}(n,\mathbb{Z})$ being a lattice in G:

THEOREM 1.3.1. *Let $G \subset \mathrm{SL}(n,\mathbb{R})$ be a $\mathbb{Q}$-algebraic group. Then $G(\mathbb{Z})$ is a lattice in G iff G admits no nontrivial $\mathbb{Q}$-characters.*

A closed subgroup $D \subset G$ is said to be a *quasi-lattice* in G if $\mathrm{vol}(G/D) < \infty$ and D contains no nontrivial normal connected subgroups of G.

Subgroups $\Gamma, \Gamma' \subset G$ are called *commensurable* if the intersection $\Gamma \cap \Gamma'$ is of finite index both in Γ and Γ'. For $\Gamma \subset G$ one defines the *commensurator* of Γ in G to be the subgroup

$$\mathrm{Comm}_G(\Gamma) \stackrel{\text{def}}{=} \{g \in G \mid \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}.$$

For example, the commensurator of $\mathrm{SL}(n,\mathbb{Z})$ in $\mathrm{SL}(n,\mathbb{R})$ is equal to $\mathrm{SL}(n,\mathbb{Q})$. A closed subset $X \subset G/D$ is said to be *homogeneous* if there exist a point $x \in X$ and a subgroup $H \subset G$ such that $X = Hx$.

3a. *Homogeneous spaces of nilpotent and solvable Lie groups*. Let $\rho: G \mapsto \mathrm{SL}(n,\mathbb{R})$ be a finite-dimensional representation. Subgroup $H \subset G$ is said to be *Zariski dense in* ρ if $\rho(H)$ and $\rho(G)$ have the same Zariski closures. ρ is said to be a *unipotent representation* if $\rho(G) \subset \mathrm{SL}(n,\mathbb{R})$ is a unipotent subgroup.

Homogeneous spaces of nilpotent (resp. solvable) Lie group are called *nilmanifolds* (resp. *solvmanifolds*).

In this subsection G is assumed to be a connected simply connected Lie group.

Let G be a nilpotent Lie group, and $D \subset G$ its closed subgroup. Then $\mathrm{vol}(G/D) < \infty \Leftrightarrow G/D$ is compact $\Leftrightarrow D$ is Zariski dense in any unipotent representation of G (Malcev [129]). In particular, any quasi-lattice in G is a lattice (the Lie subalgebra of D^0 is invariant under $\mathrm{Ad}(D)$ and hence under $\mathrm{Ad}(G)$). It is known that G contains a lattice iff it admits a structure of a $\mathbb{Q}$-algebraic group (Malcev), and not every nilpotent group with $\dim G > 6$ contains a lattice. If Γ is a lattice in G, then $\Gamma \cap [G,G]$ is a lattice in $[G,G]$.

Now let G be a solvable Lie group and $N \subset G$ its nilradical. Then $\mathrm{vol}(G/D) < \infty \Leftrightarrow G/D$ is compact; if D is a quasi-lattice in G, then $\mathrm{vol}(N/N \cap D) < \infty$ and $D^0 \subset N$ (Mostow [157]). In particular, any compact solvmanifold G/D bundles over a nontrivial torus (over G/ND if G is not nilpotent or over $G/[G,G]D$ if it is). The criterion for G to have at least one quasi-lattice is quite intricate (see [5] for details).

Homogeneous spaces of Euclidean Lie group are called *Euclidean* (see §1.4b for examples). A compact solvmanifold is Euclidean iff it is finitely covered by a torus (Brezin and Moore [30]). Every compact solvmanifold admits a unique maximal Euclidean quotient space.

3b. *Homogeneous spaces of semisimple Lie groups*. Let G be a semisimple Lie group. Then it has both uniform and non-uniform lattices (Borel). Let $G = K \times S$ be the decomposition into compact and totally noncompact parts. Assume that $\mathrm{vol}(G/D) < \infty$ and $G = \overline{SD}$. Then D is Zariski dense in any finite-dimensional representation of G (*Borel Density Theorem*). In particular, any quasi-lattice $D \subset G$ is a lattice, and the product $DZ(G)$ is closed (both statements do not hold if $G \neq \overline{SD}$).

Let Γ be a lattice in G and $H \subset G$ a Cartan subgroup. Then the set $\{g \in G \mid g\Gamma g^{-1} \cap H$ is a lattice in $H\}$ is dense in G (Mostow [158], Prasad and Raghunathan [173]).

It follows from Theorem 1.3.1 that $G(\mathbb{Z})$ is a lattice in G whenever G is a semisimple $\mathbb{Q}$-group. Moreover, it can be shown that $G(\mathbb{Z})$ is a uniform lattice in G iff $G(\mathbb{Q})$ consists of semisimple elements (Borel–Harish-Chandra). Note that the latter, as was mentioned in §1.2b, is equivalent to vanishing of $\mathrm{rank}_{\mathbb{Q}}\, G$.

The lattice $G(\mathbb{Z})$ presents an example of so called arithmetic lattices. More generally: a lattice Γ in a semisimple Lie group G is said to be *arithmetic* if there exist a $\mathbb{Q}$-algebraic group $H \subset \mathrm{SL}(n,\mathbb{R})$ and an epimorphism $\rho : H \mapsto \mathrm{Ad}(G)$ with compact kernel such that subgroups $\rho(H(\mathbb{Z}))$ and $\mathrm{Ad}(\Gamma)$ are commensurable.

Let G be a semisimple Lie group, and $\Gamma \subset G$ a Zariski dense lattice. Then Γ is said to be *irreducible* if given any noncompact connected normal subgroup $G' \subset G$ one has $G = \overline{G'\Gamma}$. For every lattice $\Gamma \subset G$ there exist a finite index subgroup $\Gamma' \subset \Gamma$ and an almost direct decomposition $G = G_1 G_2 \cdots G_n$ such that $\Gamma' = (\Gamma' \cap G_1)(\Gamma' \cap G_2) \cdots (\Gamma' \cap G_n)$ and $\Gamma' \cap G_i$ is an irreducible lattice in G_i for every $i = 1, \dots, n$ (*decomposition into irreducible components*).

The central result in the theory of discrete subgroups of semisimple Lie groups is the following *Margulis Arithmeticity Theorem*:

THEOREM 1.3.2. *Let G be a totally noncompact semisimple Lie group with* $\mathrm{rank}_{\mathbb{R}}\, G \geqslant 2$. *Then any irreducible lattice in G is arithmetic.*

3c. *Hyperbolic surfaces as homogeneous spaces.* Let $\mathbb{H}^2 = \{z = x + \mathrm{i}y \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ stand for the hyperbolic plane, which, as is well-known, is of constant curvature -1 relative to the metric $\mathrm{d}l^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$. The group $\mathrm{SL}(2,\mathbb{R})$ acts isometrically and transitively on $\mathbb{H}^2$ by linear-fractional transformations

$$f_g(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{H}^2,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

with the kernel $\mathbb{Z}_2 = \{\pm\,\mathrm{Id}\} \subset \mathrm{SL}(2,\mathbb{R})$. The quotient group $G = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \simeq \mathrm{SO}(1,2)^0$ acts effectively on $\mathbb{H}^2$, and the isotropy subgroup of the point $z_0 = \mathrm{i} \in \mathbb{H}^2$ is $C = \mathrm{PSO}(2) = \mathrm{SO}(2)/\mathbb{Z}_2$; hence $\mathbb{H}^2$ can be identified with G/C.

Let $S\mathbb{H}^2$ be the unit tangent bundle over $\mathbb{H}^2$. The differential of the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}^2$ defines a transitive and free action on $S\mathbb{H}^2$, so the latter can be naturally identified with G. Furthermore, any smooth surface M of constant curvature -1 is of the form $M = \Gamma\backslash\mathbb{H}^2 = \Gamma\backslash G/C$ for some discrete subgroup Γ of G (such groups are called *Fuchsian*) having no torsion. The unit tangent bundle SM is therefore the *left* homogeneous space $\Gamma\backslash S\mathbb{H}^2 = \Gamma\backslash G$. The volume form for the Riemannian metric on M induces a G-invariant measure on SM, which is called the *Liouville measure*, and clearly coincides with the appropriately normalized Haar measure on $\Gamma\backslash G$. In particular, a surface $M = \Gamma\backslash\mathbb{H}^2$ is of finite area iff Γ is a lattice in G (i.e., $SM = \Gamma\backslash G$ is of finite volume). The most important example is the *modular surface* $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, which is responsible for many number-theoretic applications of dynamics (see, e.g., the beginning of §5.2c).

3d. *The space of lattices.* Let us now consider a generalization of the aforementioned example, which also plays a very important role in applications to number theory. Namely define Ω_k to be the right homogeneous space G/Γ, where $G = \mathrm{SL}(k,\mathbb{R})$ and $\Gamma = \mathrm{SL}(k,\mathbb{Z})$. The standard action of Γ on $\mathbb{R}^k$ keeps the lattice $\mathbb{Z}^k \subset \mathbb{R}^k$ invariant, and, given any $g \in G$, the lattice $g\mathbb{Z}^k$ is *unimodular*, i.e., its fundamental set is of unit volume. On the other hand, any unimodular lattice Λ in $\mathbb{R}^k$ is of the form $g\mathbb{Z}^k$ for some $g \in \Gamma$. Hence the space Ω_k is identified with the space of all unimodular lattices in $\mathbb{R}^k$, and clearly this identification sends the left action of G on G/Γ to the linear action on lattices in $\mathbb{R}^k$: for a lattice $\Lambda \subset \mathbb{R}^k$, one has $g\Lambda = \{g\mathbf{x} \mid \mathbf{x} \in \Lambda\}$.

The geometry of the space Ω_k is described by the so called *reduction theory* for lattices in $\mathbb{R}^k$; that is, following Minkowski and Siegel, one specifies an (almost) fundamental domain Σ for the right Γ-action on G. This way one can show that Γ is a non-uniform lattice in G; that is, Ω_k is not compact for every $k \geqslant 2$. See [174] or [26] for more details.

Bounded subsets of Ω_k can be described using the following *Mahler's Compactness Criterion*, which incidentally serves as one of the main links between number theory and the theory of homogeneous actions:

THEOREM 1.3.3. *A sequence of lattices $g_i \mathrm{SL}(k, \mathbb{Z})$ goes to infinity in Ω_k $\Leftrightarrow$ there exists a sequence $\{\mathbf{x}_i \in \mathbb{Z}^k \setminus \{0\}\}$ such that $g_i(\mathbf{x}_i) \to 0$, $i \to \infty$. Equivalently, fix a norm on $\mathbb{R}^k$ and define the function δ on Ω_k by*

$$\delta(\Lambda) \stackrel{\text{def}}{=} \max_{\mathbf{x}\in\Lambda\setminus\{0\}} \|\mathbf{x}\|. \tag{1.2}$$

Then a subset K of Ω_k is bounded iff the restriction of δ on K is bounded away from zero.

Denote by ν the probability Haar measure on Ω_k. The interpretation of points of Ω_k as lattices in $\mathbb{R}^k$ gives rise to the following important connection between ν and the Lebesgue measure on $\mathbb{R}^k$. Let us introduce the following notation: if φ is a function on $\mathbb{R}^k$, denote by $\tilde{\varphi}$ the function on Ω_k given by $\tilde{\varphi}(\Lambda) \stackrel{\text{def}}{=} \sum_{\mathbf{x}\in\Lambda\setminus\{0\}} \varphi(\mathbf{x})$. It follows from the reduction theory that $\tilde{\varphi}$ is integrable whenever φ is integrable, and moreover, the *Siegel summation formula* [208] holds: for any $\varphi \in L^1(\mathbb{R}^k)$ one has $\tilde{\varphi}(\Lambda) < \infty$ for ν-almost all $\Lambda \in \Omega_k$, and

$$\int_{\mathbb{R}^k} \varphi(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{\Omega_k} \tilde{\varphi}(\Lambda)\, \mathrm{d}\nu(\Lambda). \tag{1.3}$$

See [118, §7] for a modification and a generalization involving sums over d-tuples of vectors. This formula is very useful for counting lattice points inside regions in $\mathbb{R}^k$: indeed, if φ is a characteristic function of a subset B of $\mathbb{R}^k$, the integrand in the right-hand side of the above equality gives the cardinality of the intersection of $\Lambda \setminus \{0\}$ with B.

3e. *Locally symmetric spaces as infra-homogeneous spaces.* The examples considered in §1.3c are representatives of a large class of homogeneous spaces arising from considerations of locally symmetric spaces of noncompact type. Here we again switch to *left* homogeneous spaces. Let G be a semisimple noncompact Lie group with finite center, and $C \subset G$ a maximal compact connected subgroup. The Lie algebra $\mathfrak{g}$ is equipped with the nondegenerate bilinear $\mathrm{Ad}(C)$-invariant form $B(x, y) = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$, $x, y \in \mathfrak{g}$, called the *Killing form*. Let $\mathfrak{c} \subset \mathfrak{g}$ be the Lie subalgebra of C, and $\mathfrak{p} \subset \mathfrak{g}$ the orthogonal complement to $\mathfrak{c}$ in $\mathfrak{g}$, i.e., $B(\mathfrak{c}, \mathfrak{p}) \equiv 0$. Then $\mathfrak{g} = \mathfrak{c} + \mathfrak{p}$ and $\mathfrak{c} \cap \mathfrak{p} = 0$. The decomposition $\mathfrak{g} = \mathfrak{c} + \mathfrak{p}$ is called the *Cartan* or *polar* decomposition of $\mathfrak{g}$. For $G = \mathrm{SL}(n,\mathbb{R})$ and $C = \mathrm{SO}(n)$ this is simply the decomposition into subspaces of skew-symmetric and symmetric matrices.

One knows that $\mathfrak{p}$ is an $\mathrm{Ad}(C)$-invariant subspace of $\mathfrak{g}$ consisting of $\mathbb{R}$-diagonalizable elements. The restriction of B onto $\mathfrak{p}$ is a positively defined bilinear form which induces a C-invariant Riemannian metric on $X \stackrel{\text{def}}{=} G/C$ with nonpositive sectional curvature. The space X is a symmetric Riemannian space (i.e., Riemannian space with geodesic involution); moreover, it is known that any symmetric Riemannian space X of nonpositive sectional curvature is isometric to G/C, where G is a totally noncompact semisimple Lie group and the connected component C^0 of C is a maximal compact connected subgroup of G. The space X is of (strictly) negative curvature iff $\mathrm{rank}_{\mathbb{R}} G = 1$, and in this case the metric can be normalized so that the curvature takes values in the interval $[-4,-1]$. Moreover, the curvature is constant iff $X = \mathbb{H}^n = \mathrm{SO}(1,n)^0/\mathrm{SO}(n)$ (see §1.3c for the case $n = 2$).

Similarly, any locally symmetric Riemannian space M of nonpositive curvature can be realized as $M \cong \Gamma\backslash G/C$, where Γ is a discrete torsion-free subgroup of G. Again, the volume form on M comes from the appropriately normalized Haar measure on $\Gamma\backslash G$. Further, due to the compactness of C, the geometry of M, up to a bounded distortion, corresponds to that of $\Gamma\backslash G$. For example, if M is noncompact, of finite volume and irreducible,[3] one can apply reduction theory for arithmetic[4] groups [26] to study the asymptotic geometry of $\Gamma\backslash G$ and M. In particular, this way one can describe the behavior of the volume measure at infinity; more precisely, similarly to Theorem 1.3.5, the following can be proved (see [118] for details):

THEOREM 1.3.4. *For M as above, there exist $k, C_1, C_2 > 0$ such that for any $y_0 \in M$ and any $R > 0$,*

$$C_1\,\mathrm{e}^{-kR} \leqslant \mathrm{vol}\big(\{y \in M \mid \mathrm{dist}(y_0, y) \geqslant R\}\big) \leqslant C_2\,\mathrm{e}^{-kR}.$$

A more precise result for the case $G = \mathrm{SL}(k,\mathbb{R})$ and $\Gamma = \mathrm{SL}(k,\mathbb{Z})$ is obtained in [118] using (1.3): one describes the asymptotic behavior of the normalized Haar measure ν on Ω_k at infinity similarly to what was done in Theorem 1.3.4, but with the function

$$\Delta(\Lambda) \stackrel{\text{def}}{=} \log\big(1/\delta(\Lambda)\big) \tag{1.4}$$

in place of the distance function:

THEOREM 1.3.5. *There exist positive C_k, C_k' such that*

$$C_k\,\mathrm{e}^{-kR} \geqslant \nu\big(\{\Lambda \in \Omega_k \mid \Delta(\Lambda) \geqslant R\}\big) \geqslant C_k\,\mathrm{e}^{-kR} - C_k'\,\mathrm{e}^{-2kR} \quad \textit{for all } R \geqslant 0.$$

[3] That is, it cannot be compactly covered by a direct product of locally symmetric spaces; in this case Γ is an irreducible lattice in G.

[4] One can use Theorem 1.3.2 whenever $\mathrm{rank}_{\mathbb{R}} G > 1$; otherwise one uses the structure theory of lattices in rank-one Lie groups developed in [86].

3f. *Homogeneous spaces of general Lie groups.* Now let G be a connected simply connected Lie group. We will use a Levi decomposition $G = L \ltimes R$ together with the decomposition $L = K \times S$ of Levi subgroup L into compact and totally noncompact parts.

Assume that $\mathrm{vol}(G/D) < \infty$ and D^0 is solvable. Then there exists a torus $T \subset L$ such that $TR = (\overline{RD})^0$ and the space $TR/D \cap TR$ is compact (Auslander). Together with the Borel density theorem this easily implies that $\Gamma \cap KR$ is a lattice in KR for any lattice $\Gamma \subset G$.

In particular, if the Levi subgroup $L \subset G$ is totally noncompact, then any lattice in G intersects the radical R in a lattice (Wang [251]). More generally, let $D \subset G$ be a quasi-lattice and $G = \overline{SRD}$. Then $D \cap R$ is a quasi-lattice in R (Witte [255] and Starkov [223]).

Mostow's theorem for lattices in solvable Lie groups (see §1.3a) can be generalized as follows. Let N be the nilradical in G. Then any lattice in G intersects the normal subgroup $LN \subset G$ in a lattice (see [229]).

The Malcev and Borel density theorems were generalized by Dani [46] in the following way:

THEOREM 1.3.6. *Let $\rho : G \mapsto \mathrm{SL}(n,\mathbb{R})$ be a finite-dimensional representation of a connected Lie group G and $\mathrm{vol}(G/D) < \infty$. Then the Zariski closure $\mathrm{Zcl}(\rho(D))$ contains all unipotent and $\mathbb{R}$-diagonalizable one-parameter subgroups of the Zariski closure $\mathrm{Zcl}(\rho(G))$. In particular, $\mathrm{Zcl}(\rho(G))/\mathrm{Zcl}(\rho(D))$ is compact.*

This result can be deduced from the following more powerful result due to Dani [48, Corollary 2.6] (see also [257, Proof of Corollary 4.3]):

THEOREM 1.3.7. *Let $H \subset G \subset \mathrm{GL}(n,\mathbb{R})$ be real algebraic groups. Let μ be a finite measure on G/H. Then $\mathrm{Stab}(\mu) \subset G$ is a real algebraic subgroup of G containing*

$$J_\mu \stackrel{\mathrm{def}}{=} \{ g \in G \mid gx = x \ \forall x \in \operatorname{supp}\mu \}$$

as a normal subgroup, and $\mathrm{Stab}(\mu)/J_\mu$ is compact.

1.4. *Homogeneous actions*

Let G be a Lie group and $D \subset G$ its closed subgroup. Then G acts transitively by left translations $x \mapsto gx$ on the right homogeneous space G/D with D being the *isotropy subgroup*. Given a subgroup $F \subset G$ one studies the left action of F on G/D which is called *homogeneous action* and is denoted by $(G/D, F)$. The main assumptions we make in our exposition are as follows: G is a connected Lie group and $\mathrm{vol}(G/D) < \infty$. The classical case is that F is a one-parameter or a cyclic group. If F is a connected subgroup of G one calls the action a *homogeneous flow*.

Let $(G/D, F)$ be a homogeneous action on a finite volume space G/D, where G is a connected Lie group. Before giving examples, let us make some immediate observations.

First, with no harm G can be assumed to be simply connected. Otherwise one takes the universal covering $\alpha : \widetilde{G} \mapsto G$ and replaces the action $(G/D, F)$ by the action $(\widetilde{G}/\widetilde{D}, \ \widetilde{F})$, where $D = \alpha^{-1}(D)$, $\widetilde{F} = \alpha^{-1}(F)$. If the acting group has to be connected, one takes $\widetilde{F} = (\alpha^{-1}(F))^0$.

Second, one can assume that D is a quasi-lattice in G. Otherwise take the maximal connected normal in G subgroup $H \subset D$ and replace the action $(G/D, F)$ by the action $(G'/D', F')$, where $G' = G/H$, $D' = D/H$, $F' = FH/H$. If G is simply connected then so is G'. Thus it is natural in many cases to assume that D is a quasi-lattice in a connected simply connected Lie group G.

Third, sometimes (as we have seen in §1.3c and §1.3e) it is more natural to consider *right* actions on *left* homogeneous spaces. One easily checks that the map $gD \mapsto Dg^{-1}$ establishes isomorphism (both topological and measure-theoretic) between the left F-action on G/D and the right F-action on $D\backslash G$.

Finally, conjugate subgroups F and gFg^{-1} act isomorphically on G/D, and the isomorphism is given by the map $\beta_g(hD) = ghD$ (since $\beta_g(fhD) = gfhD = gfg^{-1}\beta_g(hD)$ for all $h \in G$, $f \in F$).

Let us now turn to various examples of homogeneous actions.

4a. *Flows on tori.* The simplest example of a homogeneous action is, of course, the rectilinear flow $(\mathbb{T}^n, \mathbb{R}\mathbf{v})$ on the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, where $\mathbf{v} \in \mathbb{R}^n$. Here $G = \mathbb{R}^n$, $D = \mathbb{Z}^n$ and $F = \mathbb{R}\mathbf{v}$ is a one-parameter subgroup of G. Kronecker's criterion for the flow to be ergodic is that the coordinates of $\mathbf{v}$ are linearly independent over $\mathbb{Q}$. The proof easily follows from considering the Fourier series on $\mathbb{T}^n$. We prefer somewhat more general way of proving ergodicity.

Let $f \in L^2(\mathbb{T}^n)$ be an F-invariant function. Then $\underline{\mathrm{Stab}}(f)$ contains both F and $\mathbb{Z}^n$, and by Lemma 1.1.4 it should contain the closure $A = \overline{F\mathbb{Z}^n}$. But one knows that $A = \mathbb{R}^n$ provided the coordinates of $\mathbf{v}$ are independent over $\mathbb{Q}$.

Moreover, let $\mu \in \mathcal{P}(\mathbb{T}^n)$ be any F-invariant probability measure. Then by Lemma 1.1.3, $\mathrm{Stab}(\mu) \supset A = \overline{F\mathbb{Z}^n}$. It follows that the rectilinear flow is uniquely ergodic whenever it is ergodic.

If it is not ergodic then one easily checks that the smooth partition of $\mathbb{T}^n$ into closed A-orbits defines the ergodic decomposition. All the ergodic components are homogeneous subspaces (subtori of $\mathbb{T}^n$) of the same dimension.

4b. *Flows on Euclidean manifolds.* Now let $G = g_{\mathbb{R}} \ltimes \mathbb{R}^2$ be the Euclidean 3-dimensional group from §1.1b with the center $Z(G) = g_{2\pi\mathbb{Z}}$. Then G is diffeomorphic to the 3-space $G^* = \mathbb{R} \times \mathbb{R}^2$ via the bijection

$$\varphi(g_t x) = t \times x, \quad t \in \mathbb{R},\ x = (x_1, x_2) \in \mathbb{R}^2.$$

One easily checks that $D = g_{2\pi\mathbb{Z}} \times \mathbb{Z}^2$ is a lattice in G. The bijection φ establishes diffeomorphism between G/D and the 3-torus G^*/D^*, where $D^* = \varphi(D) = (2\pi\mathbb{Z}) \times \mathbb{Z}^2$. Moreover, it induces an isomorphism (both smooth and measure-theoretic) between $(G/D, g_{\mathbb{R}})$ and $(G^*/D^*, \mathbb{R})$. All the orbits of the latter are periodic and hence the flows are not ergodic.

Any one-parameter subgroup of G either lies in the nilradical $\mathbb{R}^2 \subset G$ or is conjugate to $g_{\mathbb{R}}$. Hence G/D admits no ergodic homogeneous flows.

Clearly, the same holds if one replaces D with $D_1 = g_{\pi\mathbb{Z}} \ltimes \mathbb{Z}^2$. Then D is of index 2 in D_1 and G/D_1 is two-fold covered by the 3-torus G^*/D^*. Again, all orbits of the flow $(G/D_1, g_\mathbb{R})$ are periodic and the partition of G/D_1 into orbits forms the ergodic decomposition. One can check that (unlike the previous case) the partition is not a smooth bundle of G/D_1 (see [222] or [237]).

Now take $D_2 = a^\mathbb{Z} \times \mathbb{Z}^2$, where $a = g_{2\pi} \times (\sqrt{2}, 0)$. Then φ establishes an isomorphism between $(G/D_2, g_\mathbb{R})$ and the rectilinear flow $(G^*/\varphi(D_2), \mathbb{R})$. The latter is not ergodic and its ergodic components are 2-dimensional subtori. Hence ergodic components for $(G/D_2, g_\mathbb{R})$ are smooth 2-dimensional submanifolds in G/D_2. The manifolds are not homogeneous subspaces of G/D_2 (the reason is that φ is not a group isomorphism).

One can get an impression that no homogeneous space of G admits an ergodic homogeneous flow. However, given $D_3 = b^\mathbb{Z} \times \mathbb{Z}^2$, where $b = g_{2\pi} \times (\sqrt{2}, \sqrt{3})$, one easily checks that the flow $(G/D_3, g_\mathbb{R})$ is ergodic.

4c. *Suspensions of toral automorphisms.* Our next few examples come from the following construction. Let G/D be a homogeneous space of a connected simply connected Lie group G. Denote by $\mathrm{Aut}(G, D)$ the group of all automorphisms of G keeping the isotropy subgroup $D \subset G$ invariant. Then any $\sigma \in \mathrm{Aut}(G, D)$ defines an automorphism $\tilde{\sigma}$ of G/D via $\tilde{\sigma}(gD) = \sigma(g)D$.

Since $\mathrm{Aut}(G)$ is an algebraic group, some power of σ (say, σ^n) embeds into a one-parameter subgroup $g_\mathbb{R} \subset \mathrm{Aut}(G)$. Now take $G' = g_\mathbb{R} \ltimes G$ and $D' = \sigma^{n\mathbb{Z}} \ltimes D$. Then the flow $(G'/D', g_\mathbb{R})$ is the suspension of the automorphism $\tilde{\sigma}^n$.

Hence the dynamics of $\tilde{\sigma}$ can be studied via the flow $(G'/D', g_\mathbb{R})$ and vice versa. One may note that $g_\mathbb{R}$ and G' are not defined uniquely (there may exist countably many one-parameter subgroups $g_\mathbb{R}$ containing σ^n and one can get a countable number of pairwise nonisomorphic groups G').

EXAMPLE 1.4.1. Take $\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathbb{R}^2, \mathbb{Z}^2)$. Then σ is diagonalizable and hence there exists a one-parameter subgroup $g_\mathbb{R} \subset \mathrm{SL}(2, \mathbb{R})$ such that $\sigma = g_1$. Now $G' = g_\mathbb{R} \ltimes \mathbb{R}^2$ is a solvable triangular Lie group with a lattice $D' = \sigma^\mathbb{Z} \ltimes \mathbb{Z}^2$. The flow $(G'/D', g_\mathbb{R})$ is the suspension of the Anosov automorphism $\tilde{\sigma}$ of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ (hence $(G'/D', g_\mathbb{R})$ is an Anosov nonmixing flow). In particular, apart from everywhere dense and periodic orbits, the flow has orbits whose closures are not locally connected (and may have fractional Hausdorff dimension). Apart from the Haar measure and ergodic measures supported on periodic orbits, the flow has a lot of other ergodic measures (singular to the Haar measure).

More generally, take $\sigma \in \mathrm{SL}(n, \mathbb{Z})$. Then $\tilde{\sigma}$ is an automorphism of the n-torus $\mathbb{T}^n$. Using Fourier series one can prove the that $\tilde{\sigma}$ is ergodic iff it is weakly mixing, and the ergodicity criterion is that $\sigma \in \mathrm{SL}(n, \mathbb{Z})$ has no roots of 1 as eigenvalues (but may have eigenvalues of absolute value 1). Rokhlin proved that any ergodic $\tilde{\sigma}$ is a K-automorphism; the strongest result is that then $\tilde{\sigma}$ is isomorphic to a Bernoulli shift (Katznelson [110]). The conclusion is as follows:

THEOREM 1.4.2. *For $\sigma \in \mathrm{SL}(n, \mathbb{Z})$, the following conditions are equivalent*:

(1) *the automorphism $\tilde{\sigma}$ is ergodic*;

(2) $\tilde{\sigma}$ *is Bernoullian*;
(3) σ *has no roots of* 1 *as eigenvalues*.

4d. *Examples of nilflows*. Let $H = N(3)$ be the 3-dimensional Heisenberg group of strictly upper-triangular matrices with the lattice $\Gamma = H(\mathbb{Z})$. Let $F \subset H$ be the normal subgroup of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Any one-parameter subgroup of H is of the form

$$g_t = \begin{pmatrix} 1 & \alpha_1 t & \alpha_2 t + \frac{\alpha_1\alpha_3}{2} t^2 \\ 0 & 1 & \alpha_3 t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

Assume that $g_{\mathbb{R}} \not\subset F$, i.e., $\alpha_3 \neq 0$. Then the flow $(H/\Gamma, g_{\mathbb{R}})$ is the suspension of the skew shift[5] A on the 2-torus $\mathbb{T}^2 = F/\Gamma \cap F$, where

$$A(a, b) = (a + \beta_1, b - a + \beta_2 - \beta_1/2), \quad \beta_1 = \alpha_1/\alpha_3,\ \beta_2 = \alpha_2/\alpha_3.$$

One can use the following number-theoretic result to prove that all orbits of A are uniformly distributed in $\mathbb{T}^2$ and hence A is uniquely ergodic provided β_1 is irrational.

THEOREM 1.4.3 (Weyl). *Let $p(t) = a_0 t^n + \cdots + a_{n-1} t + a_n$ be a polynomial, where at least one of the coefficients $a_0, \ldots, a_{n-1}$ is irrational. Then the sequence $\{p(n) \bmod 1 \mid n \in \mathbb{Z}\}$ is uniformly distributed in* $[0, 1]$.

One derives from here that the nilflow $(H/\Gamma, g_{\mathbb{R}})$ is ergodic and uniquely ergodic provided $g_{\mathbb{R}}$ is in general position (i.e., α_1 and α_3 are independent over $\mathbb{Q}$); clearly, this is equivalent to the ergodicity on the maximal toral quotient $H/\Gamma[H, H]$.

4e. *Geodesic and horocycle flows*. Consider now flows on the *left* homogeneous space $\Gamma\backslash G$, where $G = \mathrm{PSL}(2, \mathbb{R})$ and Γ is a discrete subgroup of G (see §1.3c). First let us look at the geodesic flow $\{\gamma_t\}$ on $S\mathbb{H}^2$, which arises if one moves with unit speed along the geodesic line in $\mathbb{H}^2$ defined by a pair $(z, v) \in S\mathbb{H}^2$ (i.e., through $z \in \mathbb{H}^2$ in the direction of $v \in S_z\mathbb{H}^2$). Fix the unit vertical vector $w = (0, 1)$ at the point $i \in \mathbb{H}^2$. It is easy to check that the curve $\{\gamma_t(\mathrm{i}) = \mathrm{i}\,\mathrm{e}^t \mid t \in \mathbb{R}\}$ is the geodesic line in $\mathbb{H}^2$ issued from i in the direction of w. Since any isometry sends geodesic line to a geodesic line and $\mathrm{PSL}(2, \mathbb{R})$ acts transitively on $S\mathbb{H}^2$, all the geodesics are of the form $\{f_g(\mathrm{i}\,\mathrm{e}^t)\}$.

Let

$$A_t = \begin{pmatrix} \mathrm{e}^{t/2} & 0 \\ 0 & \mathrm{e}^{-t/2} \end{pmatrix}, \qquad H_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad U_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

[5] An affine automorphism of $\mathbb{T}^2$, i.e., a composition of an automorphism and a translation.

Then $\mathrm{i}\,\mathrm{e}^t = f_{A_t}(\mathrm{i})$ and hence any geodesic line is of the form $f_{gA_t}(\mathrm{i})$. It follows that $\gamma_t(z, v) = \mathrm{d} f_{gA_t}(\mathrm{i}, w)$, where $\mathrm{d} f_g(\mathrm{i}, w) = (z, v)$, and the geodesic flow corresponds to the right action of $A_{\mathbb{R}}$ on G.

Note that subgroups $A_{\mathbb{R}}, U_{\mathbb{R}}, H_{\mathbb{R}} \subset G$ are related as follows:

$$A_{-t} U_s A_t = U_{\mathrm{e}^{-t} s}, \qquad A_{-t} H_s A_t = H_{\mathrm{e}^t s}$$

and hence right $U_{\mathbb{R}}$-orbits on G form the contracting foliation for the right $A_{\mathbb{R}}$-action relative to the left-invariant metric on G (while $H_{\mathbb{R}}$-orbits form the expanding foliation).

The curves $\{\mathrm{d} f_{gU_t}(\mathrm{i}, w)\} \subset S\mathbb{H}^2$ are called the contracting horocycles. The flow $\{u_t\}$ on $S\mathbb{H}^2$ along the horocycles corresponds to the right $U_{\mathbb{R}}$-action on G. Similarly, the curves $\{\mathrm{d} f_{gH_t}(\mathrm{i}, w)\} \subset S\mathbb{H}^2$ are orbits of the expanding horocycle flow $\{h_t\}$ on $S\mathbb{H}^2$.

Now recall that $\Gamma \backslash S\mathbb{H}^2 = \Gamma \backslash G$, where Γ is a discrete torsion-free subgroup of G, can be realized as the unit tangent bundle SM to the surface $M = \Gamma \backslash \mathbb{H}^2 = \Gamma \backslash G/\mathrm{PSO}(2)$. The flows $\{\gamma_t\}$, $\{u_t\}$, $\{h_t\}$ on $S\mathbb{H}^2$ generate flows on SM called the geodesic, contracting horocycle and expanding horocycle flows respectively. They are nothing else but the homogeneous flows $(\Gamma \backslash G, A_{\mathbb{R}})$, $(\Gamma \backslash G, U_{\mathbb{R}})$ and $(\Gamma \backslash G, H_{\mathbb{R}})$.

It is a fact that two surfaces $M_1 = \Gamma_1 \backslash \mathbb{H}^2$ and $M_2 = \Gamma_2 \backslash \mathbb{H}^2$ of constant curvature -1 are isometric iff Γ_1 and Γ_2 are conjugate in G. On the other hand, M_1 and M_2 are diffeomorphic iff Γ_1 and Γ_2 are isomorphic.

Suppose that $M = \Gamma \backslash \mathbb{H}^2$ is of finite area. Then the geodesic flow on SM is well known to be ergodic (Hopf [99]) and mixing (Hedlund [96]). Moreover, it has Lebesgue spectrum of infinite multiplicity (Gelfand and Fomin [87]) and is a K-flow (Sinai [210]). All these results follow from a general theory of Anosov flows [4]. The strongest result from the ergodic point of view is that it is a Bernoullian flow (Ornstein and Weiss [169]).

Like any Anosov action (see Example 1.4.1)), the geodesic flow possesses many types of orbits and ergodic measures. Apart from everywhere dense and periodic orbits, it has orbit closures which are not locally connected (this concerns, for instance, Morse minimal sets obtained by means of symbolic dynamics [89]) and have fractional Hausdorff dimension. According to Sinai [212] and Bowen [32], it has uncountably many ergodic probability measures; both positive on open sets but singular to the Haar measure and those supported on nonsmooth invariant subsets.

The horocycle flow (both u_t and h_t) is also ergodic (Hedlund [96]) and has Lebesgue spectrum of infinite multiplicity (Parasyuk [170]). According to Marcus [130], it is mixing of all degrees. Unlike the geodesic flow, it has zero entropy (Gurevich [92] for compact case and Dani [42] for general case). If M is compact then the horocycle flow is minimal (Hedlund [95]) and uniquely ergodic (Furstenberg [84]). In the noncompact case every orbit is either dense or periodic (Hedlund [95]); apart from the Haar measure, all ergodic invariant measures are the length measures on periodic orbits (Dani and Smillie [68]).

4f. *Geodesic flows on locally symmetric spaces.* The geodesic flow on a surface of constant negative curvature is a special case of those on locally symmetric Riemannian spaces (see §1.3e). To present the general construction we need to generalize somehow the concept of a homogeneous flow. Let $g_{\mathbb{R}}$ be a subgroup of G and $C \subset G$ a compact subgroup which commutes with $g_{\mathbb{R}}$. Then the left action of $g_{\mathbb{R}}$ on a double coset space

$C\backslash G/D$ is well defined and called an *infra-homogeneous flow*. Similarly one can talk about right infra-homogeneous actions, such as the right action of $g_{\mathbb{R}}$ on a double coset space $D\backslash G/C$.

Now let G, Γ and C be as in §1.3e. The left action of G on $X = G/C$ induces an action on the unit tangent bundle SX over X. Observe that the tangent space to the identity coset eC of $X = G/C$ is isomorphic to the orthogonal complement $\mathfrak{p} \subset \mathfrak{g}$ to $\mathfrak{c} = \mathrm{Lie}(C)$. Fix a *Cartan subalgebra* $\mathfrak{a}$ of $\mathfrak{p}$, and let $\mathcal{C}$ be a positive Weyl chamber relative to a fixed ordering of the root system of the pair $(\mathfrak{g}, \mathfrak{a})$. Denote by $\mathcal{C}_1$ the set of vectors in $\mathcal{C}$ with norm one; it can be identified with a subset of $T_{eC}(G/C)$, and one can prove its closure $\overline{\mathcal{C}}$ to be a *fundamental set* for the G-action on SX; that is, every G-orbit intersects the set $\{(eC, a) \mid a \in \overline{\mathcal{C}}\}$ exactly once. In particular, the action of G on SX is transitive iff $\mathrm{rank}_{\mathbb{R}}\, G = 1$ (then $\overline{\mathcal{C}}$ consists of just one vector). In the latter case, $SX = G/C_1$, where $C_1 \subset C$ is the isotropy subgroup of a unit tangent vector at the point $C \in G/C$.

Now let $M = \Gamma\backslash G/C$ be a locally symmetric Riemannian manifold of nonpositive curvature and of finite volume, and $\{\gamma_t\}$ the geodesic flow on the unit tangent bundle SM over M. Then SM breaks into closed invariant manifolds and the restriction of the geodesic flow onto each of them is isomorphic to an infra-homogeneous flow of the form $(\Gamma\backslash G/C_a, \exp(\mathbb{R}a))$, where $a \in \overline{\mathcal{C}}$ and $C_a \subset C$ is the centralizer of a in C. If $\mathrm{rank}_{\mathbb{R}}\, G = 1$ then the geodesic flow is Anosov and hence ergodic. If $\mathrm{rank}_{\mathbb{R}}\, G > 1$ then the partition is nontrivial and hence the flow is not ergodic; however, one can show that the flow is ergodic being restricted to any of the components provided G is totally noncompact and Γ irreducible (in essence this was shown by Mautner [153]).

4g. *Actions on the space of lattices*. Finally let us mention the (important for number-theoretic applications) action of $\mathrm{SL}(k, \mathbb{R})$ and its subgroups on the space Ω_k of lattices in $\mathbb{R}^k$ considered in §1.3d. As a sneak preview of Section 5, consider a model number-theoretic situation when these actions arise. Let $Q(\mathbf{x})$ be a homogeneous polynomial in k variables (for example a quadratic form), and suppose that one wants to study nonzero integer vectors $\mathbf{x}$ such that the value $Q(\mathbf{x})$ is small. Let H_Q be the stabilizer of Q in $\mathrm{SL}(k, \mathbb{R})$, that is,

$$H_Q = \{g \in \mathrm{SL}(k, \mathbb{R}) \mid Q(g\mathbf{x}) = Q(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^k\}.$$

Then one can state the following elementary

LEMMA 1.4.4. *There exists a sequence of nonzero integer vectors $\mathbf{x}_n$ such that $Q(\mathbf{x}_n) \to 0$ iff there exists a sequence $h_n \in H_Q$ such that $\delta(h_n\mathbb{Z}^k) \to 0$.*

The latter condition, in view of Mahler's criterion, amounts to the orbit $H_Q\mathbb{Z}^k$ being unbounded in Ω_k. This gives a special number-theoretic importance to studying long-term behavior of various trajectories on Ω_k. See §§5.2c, 5.1a and Section 5.3 for details and specific examples.

2. Ergodic properties of flows on homogeneous spaces

Here we expose ergodic properties of a homogeneous flow $(G/D, g_{\mathbb{R}})$ with respect to a finite Haar measure on G/D. The subject is very well understood by now, and most problems have been solved already by the end of 1980s. This includes, for instance, criteria for ergodicity, mixing and K-property, calculating of spectrum, construction of ergodic decomposition, etc. We refer the reader to [3] for basic notions and constructions of ergodic theory.

To formulate the results, we need to introduce certain subgroups of G associated to $g_{\mathbb{R}}$. First in Section 2.1, in addition to horospherical subgroups G^+ and G^-, we define the neutral subgroup Q of G. Orbits of these groups on G/D form, respectively, the expanding, contracting, and neutral foliation for the flow $(G/D, g_{\mathbb{R}})$. Next we define normal subgroups $A \subset J \subset M$ associated with $g_{\mathbb{R}}$: those of Auslander, Dani, and Moore, respectively. They are related to important features of the flow such as partial hyperbolicity, distality and uniform continuity.

In Section 2.2 we discuss ergodicity and mixing criteria studying nilpotent, solvable, and semisimple cases separately. The general situation reduces to solvable and semisimple ones.

Spectrum calculation, multiple mixing, and K-property are discussed in Section 2.3. Such phenomenon as exponential mixing proved to be highly helpful in many applications; here the results are also almost definitive.

In Section 2.4 we construct explicitly the ergodic decomposition and reduce the study of homogeneous flows to ergodic case.

Related subjects such as topological rigidity, first cohomology group, and time changes are discussed in Section 2.5. Unlike the ergodic properties, cohomological questions are much less studied and here many interesting problems remain open.

To conclude, in Section 2.6 we show that in the case $\mathrm{vol}(G/D) = \infty$ the situation gets much more complicated. Whereas the case of solvable Lie group G presents no problems, ergodic properties in the semisimple case are related to delicate problems in the theory of Fuchsian and Kleinian groups.

2.1. *The Mautner phenomenon, entropy and K-property*

Let $(G/D, F)$ be a homogeneous action, where D is a quasi-lattice in a connected simply connected Lie group G. First we consider the simplest case: the acting subgroup F is normal in G. Then orbit closures of the action $(G/D, F)$ define a smooth G/D-bundle over the space $G/\overline{FD}$. It turns out that this bundle defines the ergodic decomposition for the action relative to the Haar probability measure on G/D [43,155]:

LEMMA 2.1.1. *Let F be a normal subgroup of G and* $\mathrm{vol}(G/D) < \infty$. *Then the measurable hull of partition of G/D into F-orbits coincides* (mod 0) *with the smooth partition of G/D into closed homogeneous subspaces $g\overline{FD}$, $g \in G$. The space*

$$\mathrm{Fix}(F) = \{f \in L^2(G/D) \mid f(hx) = f(x),\ h \in F,\ x \in G/D\}$$

is isomorphic to the space $L^2(G/\overline{FD})$.

1a. *Subgroups associated to an element* $g \in G$. Let us consider the linear operator Ad_g on the space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \bigotimes_{\mathbb{R}} \mathbb{C}$ and decompose the latter into *generalized root spaces*

$$\mathfrak{g}_{\mathbb{C}}^{\lambda} \stackrel{\text{def}}{=} \{x \in \mathfrak{g}_{\mathbb{C}} \mid \exists n \text{ such that } (\mathrm{Ad}_g - \lambda \,\mathrm{Id})^n x = 0\}.$$

If $\mathfrak{g}_{\mathbb{C}}^{\lambda} \neq 0$ then λ is said to be an *eigenvalue* of the operator Ad_g; the number $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^{\lambda}$ is called the *multiplicity* of λ.

Let Σ be the collection of eigenvalues. Note that $\lambda \in \Sigma \Leftrightarrow \overline{\lambda} \in \Sigma$. Since $\mathrm{Ad}_g[x, y] = [\mathrm{Ad}_g x, \mathrm{Ad}_g y]$, it follows that $[\mathfrak{g}_{\mathbb{C}}^{\lambda}, \mathfrak{g}_{\mathbb{C}}^{\theta}] \subset \mathfrak{g}_{\mathbb{C}}^{\lambda\theta}$. If $\lambda \in \mathbb{R}$ define $\mathfrak{g}^{\lambda} = \mathfrak{g}_{\mathbb{C}}^{\lambda} \cap \mathfrak{g}$, otherwise put $\mathfrak{g}^{\lambda} = \mathfrak{g}^{\overline{\lambda}} = (\mathfrak{g}_{\mathbb{C}}^{\lambda} + \mathfrak{g}_{\mathbb{C}}^{\overline{\lambda}}) \cap \mathfrak{g}$. One gets the decomposition $\mathfrak{g} = \sum_{\lambda \in \Sigma} \mathfrak{g}^{\lambda}$ into the sum of Ad_g-invariant subspaces. Now define three subalgebras

$$\mathfrak{g}^{+} = \sum_{|\lambda|>1} \mathfrak{g}^{\lambda}, \qquad \mathfrak{g}^{0} = \sum_{|\lambda|=1} \mathfrak{g}^{\lambda}, \qquad \mathfrak{g}^{-} = \sum_{|\lambda|<1} \mathfrak{g}^{\lambda}.$$

It is easily seen that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are nilpotent and $\mathfrak{g}^0$ normalizes both $\mathfrak{g}^+$ and $\mathfrak{g}^-$.

Let G^+, Q, G^- be the connected subgroups of G corresponding to the subalgebras $\mathfrak{g}^+$, $\mathfrak{g}^0$, $\mathfrak{g}^-$. One can check that G^+ and G^- are none other than the expanding and contracting horospherical subgroups of G defined in §1.1a. The subgroup Q is called *neutral*.

We also consider the subgroup $A \subset G$ generated by G^+ and G^-. It is easily seen to be normal in G. We call it the *Auslander subgroup*[6] associated with $g \in G$.

Clearly, A is nontrivial iff g is a partially hyperbolic element of G. In other words, A is the smallest normal subgroup of G such that the element $gA \in G/A$ is quasi-unipotent in G/A.

Now let $G = L \ltimes R$ be the Levi decomposition of G. The *Dani subgroup* associated with $g \in G$ is the smallest normal subgroup $J \subset G$ such that $gJ \in G/J$ is quasi-unipotent in G/J and gJR is semisimple in G/JR.

Finally, the *Moore subgroup*[7] associated with $g \in G$ is the smallest normal subgroup $M \subset G$ such that gM is both quasi-unipotent and semisimple in G/M.

Clearly, $A \subset J \subset M$. If G is nilpotent then $A = J = \{e\}$ and M is the smallest normal subgroup in G such that g projects to a central element in G/M. If G is solvable then $A = J \subset M \subset N$, the latter being the nilradical of G. If G is semisimple then $J = M$. If G is simple and has a finite center then M is trivial iff g belongs to a compact subgroup of G; otherwise $M = G$.

1b. *Auslander normal subgroup, K-property and entropy.* The Auslander, Dani and Moore subgroups for a one-parameter subgroup $g_{\mathbb{R}}$ are defined relative to any nontrivial

[6] It was introduced in [8] under the name of the unstable normal subgroup.

[7] Originally in [156] it was called the Ad-compact normal subgroup. Sometimes it is also called the Mautner subgroup.

element g_t, $t \neq 0$. They prove to be helpful in understanding the dynamics of the flow $(G/D, g_{\mathbb{R}})$.

Let d_G be a right-invariant metric on G which induces Riemannian metric on G/D. For any $u \in G^-$ one has

$$d_G(g_t, g_t u) = d_G(1, g_t u g_{-t}) \to 0, \quad t \to +\infty,$$

and hence the orbits $G^- hD \subset G/D$, $h \in G$, form the contracting foliation[8] of G/D for the flow $(G/D, g_{\mathbb{R}})$; the expanding one is given by G^+-orbits on G/D.

For example, given the diagonal subgroup $A_{\mathbb{R}}$ in the Lie group $G = \mathrm{SL}(2,\mathbb{R})$, one has $G^+ = U_{\mathbb{R}}$ and $G^- = H_{\mathbb{R}}$ (see §1.4e for the notations).[9]

If the subgroup $g \in G$ is partially hyperbolic then both foliations are nontrivial (since Haar measure on G/D is finite and G-invariant), and hence the entropy of the flow $(G/D, g_{\mathbb{R}})$ is positive. What is important to keep in mind is that homogeneous partially hyperbolic flows $(G/D, g_{\mathbb{R}})$ are *uniformly* partially hyperbolic (moreover, Lyapunov exponents are constant on all the space G/D).

If $g_{\mathbb{R}} \in G$ is quasi-unipotent then the rate of divergence of close orbits for the flow $(G/D, g_{\mathbb{R}})$ is clearly at most polynomial. A general result of Kushnirenko [123] implies that the entropy of the action is trivial.[10] It follows from Lemma 2.1.1 that the Pinsker partition (see [3]) for the flow $(G/D, g_{\mathbb{R}})$ is given by homogeneous subspaces $h\overline{AD}$, $h \in G$. Hence the following criterion holds ([41,42]):

THEOREM 2.1.2. *Let D be a quasi-lattice in G. Then the entropy of a flow $(G/D, g_{\mathbb{R}})$ is positive iff $g_{\mathbb{R}}$ is a partially hyperbolic subgroup. The flow has K-property iff $G = \overline{AD}$, where A is the Auslander normal subgroup for $g_{\mathbb{R}}$.*

Now we state the entropy formula for homogeneous flows.

THEOREM 2.1.3. *Let D be a quasi-lattice in G. Then the entropy $h(g_1)$ of the flow $(G/D, g_{\mathbb{R}})$ relative to the Haar probability measure is estimated as*

$$h(g_1) \leqslant \sum_{|\lambda_i|>1} m_i \log|\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the operator Ad_{g_1} on $\mathfrak{g}$ with corresponding multiplicities $m_1, \ldots, m_n$. Moreover, if D is a uniform lattice in G then the equality holds.

The entropy formula for D being a uniform lattice was proved by Bowen [31] and follows from a general Pesin formula relating entropy of a smooth dynamical system with

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[8]No doubt that the foliation of G/D into G^--orbits is smooth provided D is a discrete subgroup. A priori in the general case the dimension of the orbits may vary. But it is constant if $\mathrm{vol}(G/D) < \infty$; see [228].

[9]Notice that the left $U_{\mathbb{R}}$-orbits on G/D form the *expanding* foliation relative to the left $A_{\mathbb{R}}$-action, unlike the situation in §1.4e where the subgroups involved act on $D\backslash G$ on the right.

[10]The entropy estimate of [123] was proved for diffeomorphisms of *compact* manifolds. In the case of G/D being noncompact but of finite volume the necessary details are given in [42].

Lyapunov exponents. Most probably the same formula holds for any lattice $D \subset G$; for instance, it was proved in [147] for $\mathbb{R}$-diagonalizable subgroups $g_{\mathbb{R}} \subset G$. The estimate from above for D being a quasi-lattice may be obtained using Theorem 2.4.1 below.

The connection between homogeneous flows and automorphisms of homogeneous spaces described in §1.4c provides an entropy formula for the latter. Namely, if $\sigma \in \mathrm{Aut}(G, D)$ where D is a uniform lattice in G, then the entropy $h(\tilde{\sigma})$ of the automorphism $\tilde{\sigma}$ of G/D is given by the same formula: $h(\tilde{\sigma}) = \sum_{|\lambda_i|>1} m_i \log|\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the differential $d\sigma$ on $\mathfrak{g}$ with corresponding multiplicities $m_1, \ldots, m_n$. For toral automorphisms it was first proved by Sinai [209] and for those of nilmanifolds by Parry [171].

1c. *Dani normal subgroup and the distal property.* We have seen that taking the quotient flow $(G/\overline{AD}, g_{\mathbb{R}})$ kills both (partial) hyperbolicity and entropy of the flow $(G/D, g_{\mathbb{R}})$. Now we turn to the Dani normal subgroup $J \subset G$.

It is well known (see [9]) that any homogeneous flow on G/D is distal[11] whenever G is nilpotent. On the other hand, if G is semisimple and the unipotent part of a subgroup $g_{\mathbb{R}}$ of G is nontrivial then the flow $(G/D, g_{\mathbb{R}})$ is not distal. From here one can deduce that the flow $(G/D, g_{\mathbb{R}})$ is distal iff the Dani normal subgroup for $g_{\mathbb{R}}$ is trivial. In other words, the quotient flow $(G/\overline{JD}, g_{\mathbb{R}})$ is the maximal distal (homogeneous) quotient of the original flow.

Later in §2.1f we will see that the Dani normal subgroup plays an important role in constructing the ergodic decomposition for homogeneous actions.

1d. *Moore normal subgroup and the Mautner phenomenon.* Let ρ be the regular unitary representation of G on the space $L^2(G/D)$. If the flow $(G/D, g_{\mathbb{R}})$ is ergodic then every $g_{\mathbb{R}}$-fixed function $f \in L^2(G/D)$ should be constant, i.e., G-fixed. Clearly, it is important to know which subgroup of G keeps fixed every $g_{\mathbb{R}}$-fixed element of $L^2(G/D)$. The answer is given by the following result called the *Mautner phenomenon* for unitary representations due to Moore [156]:

THEOREM 2.1.4. *Let π be a continuous unitary representation of Lie group G on a Hilbert space $\mathcal{H}$. Let $M \subset G$ be the Moore normal subgroup for a subgroup $g_{\mathbb{R}} \subset G$. Then $\pi(M)v = v$ whenever $\pi(g_{\mathbb{R}})v = v$, $v \in \mathcal{H}$.*

The same result clearly holds if one replaces $g_{\mathbb{R}}$ with any subgroup $F \subset G$ and takes the Moore normal subgroup $M = M(F)$ generated by Moore normal subgroups for elements $g \in F$.

An elementary proof (not involving representation theory) of the theorem was sketched by Margulis in [142]; for details see [13,237].

It is very easy to demonstrate that $\pi(A)v = v$ whenever $\pi(g_{\mathbb{R}})v = v$ (this was observed by Mautner [153]). In fact, take $h \in G^-$. Then

$$\bigl(\pi(h)v, v\bigr) = \bigl(\pi(g_t h g_{-t})\pi(g_t)v, \pi(g_t)v\bigr)$$

[11] Recall that a continuous flow (X, φ_t) on a metric space is called *distal* if for each $x, y \in X$, $x \neq y$, there exists $\varepsilon > 0$ such that $d(\varphi_t x, \varphi_t y) > \varepsilon$ for all $t \in \mathbb{R}$.

$$= \bigl(\pi(g_t h g_{-t})v, v\bigr) \to (v, v), \quad t \to +\infty,$$

and hence $(\pi(h)v, v) = (v, v)$. Since π is unitary it follows that $\pi(h)v = v$. Then $\pi(G^-)v = v$ and similarly $\pi(G^+)v = v$; hence $\pi(A)v = v$.

1e. *Ergodicity and weak mixing.* Now we can give a criterion for the flow $(G/D, g_\mathbb{R})$ to be ergodic. Suppose that $\rho(g_\mathbb{R})f \equiv f$ for some $f \in L^2(G/D)$. Then $\rho(M)f \equiv f$ and hence f is constant on almost all leaves $g\overline{MD} \subset G/D$. Lemma 2.1.1 implies that f defines an element $\bar{f} \in L^2(G/\overline{MD})$. We derived the following ([30]):

THEOREM 2.1.5. *The flow $(G/D, g_\mathbb{R})$ is ergodic if and only if the quotient flow $(G/\overline{MD}, g_\mathbb{R})$ is ergodic.*

Suppose that the Moore normal subgroup for $g_\mathbb{R}$ is trivial. This means that the subgroup $\mathrm{Ad}_{g_\mathbb{R}} \subset \mathrm{Aut}(\mathfrak{g})$ is relatively compact. Hence there exists a G-right-invariant metric on G which is $g_\mathbb{R}$-left-invariant. Relative to the induced metric on G/D the flow $(G/D, g_\mathbb{R})$ is isometric (and uniformly continuous relative to the metric induced by any other G-right-invariant metric on G). But an isometric flow is ergodic iff it is minimal.

It follows that given an arbitrary subgroup $g_\mathbb{R} \subset G$, the quotient flow $(G/\overline{MD}, g_\mathbb{R})$ is ergodic iff it is minimal.

COROLLARY 2.1.6. *The flow $(G/D, g_\mathbb{R})$ is ergodic iff it is topologically transitive.*

On the contrary, if the flow $(G/D, g_\mathbb{R})$ is uniformly continuous and $D \subset G$ is a quasi-lattice then the Moore subgroup of $g_\mathbb{R}$ is trivial. It follows that in the general case the quotient flow $(G/\overline{MD}, g_\mathbb{R})$ is the maximal uniformly continuous (homogeneous) quotient of the original flow.

Analogously to Theorem 2.1.5 one proves the following criterion ([30]):

THEOREM 2.1.7. *The flow $(G/D, g_\mathbb{R})$ is weakly mixing iff $G = \overline{MD}$.*

1f. *Ergodic decomposition into invariant submanifolds.* Since the quotient flow $(G/\overline{MD}, g_\mathbb{R})$ is uniformly continuous, it follows that its orbit closures are compact minimal sets (in fact, they are diffeomorphic to tori). The partition of $G/\overline{MD}$ into these minimal sets forms the ergodic decomposition for the quotient action. Now using Theorem 2.1.4 one easily derives that the partition $\widetilde{E}$ of G/D into closed submanifolds $\widetilde{E}_h(g_\mathbb{R}) = \overline{g_\mathbb{R} MhD} \subset G/D$ forms the ergodic decomposition $\widetilde{E}$ of the original flow $(G/D, g_\mathbb{R})$ with respect to the Haar probability measure ν. One can prove (see [226]) that every submanifold $\widetilde{E}_h(g_\mathbb{R})$ is equipped with a smooth $\overline{g_\mathbb{R} M}$-invariant measure $\tilde{\nu}_h$ such that $\mathrm{d}\nu = \int_{\widetilde{E}} \mathrm{d}\tilde{\nu}_h$.

On the other hand, the quotient flow $(G/\overline{JD}, g_\mathbb{R})$ is distal. By Ellis' theorem (cf. [9]) any distal topologically transitive flow on a compact manifold is minimal. Hence orbit closures of the flow $(G/\overline{JD}, g_\mathbb{R})$ are also minimal sets (in fact, they are diffeomorphic to nilmanifolds) which form the ergodic decomposition for the quotient flow. It follows that the partition E of G/D into closed submanifolds $E_h(g_\mathbb{R}) = \overline{g_\mathbb{R} JhD} \subset G/D$ also forms the ergodic decomposition E of the original flow $(G/D, g_\mathbb{R})$.

Moreover, unlike $\widetilde{E}_h(g_{\mathbb{R}})$, the flow on $E_h(g_{\mathbb{R}})$ is always ergodic with respect to a smooth $\overline{g_{\mathbb{R}} J}$-invariant measure ν_h (this is so because the Dani normal subgroup of $g_{\mathbb{R}}$ in the Lie group $\overline{g_{\mathbb{R}} J}$ is again J) such that $\mathrm{d}\nu = \int_E \mathrm{d}\nu_h$.

Clearly, partition E is a refinement of $\widetilde{E}$ and since the ergodic decomposition is defined uniquely (mod 0), it follows that the components of E and $\widetilde{E}$ coincide almost everywhere. In §2.4a we will see that the components are covered (together with flows) by homogeneous spaces of finite volume.

2.2. *Ergodicity and mixing criteria*

2a. *Nilpotent case.* Let Γ be a lattice in a *nilpotent* Lie group G. Clearly, given any subgroup $g_{\mathbb{R}} \subset G$ its Moore normal subgroup M lies in $[G, G]$ and $g_{\mathbb{R}} M$ is normal in G. If $G = \overline{g_{\mathbb{R}}[G, G]\Gamma}$, then G contains no proper subgroup $H \subset G$ such that $g_{\mathbb{R}} \subset H$ and the product $H\Gamma$ is closed (otherwise the product $H[G, G]\Gamma$ is closed and hence $G = H[G, G] = H$). It follows that $G = \overline{g_{\mathbb{R}} M\Gamma}$ and we have proved Green's ergodicity criterion [9] for nilflows:

THEOREM 2.2.1. *A nilflow $(G/\Gamma, g_{\mathbb{R}})$ is ergodic iff the quotient flow on the maximal toral quotient $(G/[G, G]\Gamma, g_{\mathbb{R}})$ is ergodic.*

Note that the image of $g_{\mathbb{R}}$ in $G' = G/(\overline{M\Gamma})^0$ is a central subgroup of G' and hence it can act ergodically on $G/\overline{M\Gamma}$ only if G' is Abelian; hence $(\overline{M\Gamma})^0 = [G, G]$.

Since all nilflows are distal, it follows from Ellis' theorem (see [9]) that any ergodic nilflow is minimal (Auslander [9]). A stronger result due to Furstenberg [82] is as follows.

THEOREM 2.2.2. *An ergodic nilflow $(G/\Gamma, g_{\mathbb{R}})$ is uniquely ergodic (hence all its orbits are uniformly distributed).*

Applying this result to a special nilpotent group one can derive Weyl's Theorem 1.4.3 on uniform distribution (see [9,82,237]).

By an induction argument and Theorems 2.2.1, 2.2.2 one easily derives (cf. [222]) the following:[12]

THEOREM 2.2.3. *$H_1 = (\overline{g_{\mathbb{R}}\Gamma})^0$ is a subgroup of G and the action $(H_1/\Gamma \cap H_1, g_{\mathbb{R}})$ is minimal and uniquely ergodic.*

Hence the ergodic decomposition for the nilflow $(G/\Gamma, g_{\mathbb{R}})$ is the partition into closed orbits $E_s(g_{\mathbb{R}}) = sH_s\Gamma$, where $H_s = (\overline{s^{-1} g_{\mathbb{R}} s\Gamma})^0$. One can show that the components may be of different dimension and hence the ergodic decomposition may not form a smooth bundle (even for the Heisenberg group).

[12] This incidentally proves topological and measure conjectures of Raghunathan and Dani (to be discussed in Section 3) for the class of nilflows.

On the other hand, the group $H = (\overline{g_{\mathbb{R}} M \Gamma})^0$ is normal in N and hence $\widetilde{E}_s(g_{\mathbb{R}}) = \overline{g_{\mathbb{R}} M s \Gamma} = sH\Gamma$ for all $s \in N$. It follows that $\widetilde{E}$ is the homogeneous bundle of G/Γ over $G/H\Gamma$.

2b. *Solvable case*. Let $(G/D, g_{\mathbb{R}})$ be a flow on a *solvmanifold*. Recall that any compact solvmanifold admits a unique maximal Euclidean quotient G/P, where $P \subset G$ is a closed subgroup containing D (actually $P = \overline{HD}$, where $H \subset G$ is the smallest normal subgroup such that G/H is a Euclidean Lie group). The following result of Brezin and Moore [30] reduces the ergodicity criterion for homogeneous flows on solvmanifolds to those on Euclidean manifolds.

THEOREM 2.2.4. *A homogeneous flow $(G/D, g_{\mathbb{R}})$ on a compact solvmanifold is ergodic iff its maximal Euclidean quotient flow $(G/P, g_{\mathbb{R}})$ is such.*

Clearly, if G is nilpotent then the maximal toral quotient $G/D[G, G]$ is the maximal Euclidean one. Hence this theorem generalizes Green's ergodicity criterion (Theorem 2.2.1).

Now let D be a quasi-lattice in a simply connected *Euclidean* Lie group $G = A \ltimes \mathbb{R}^n$, where the Abelian group A acts with discrete kernel on the nilradical $\mathbb{R}^n$ in such a way that $\mathrm{Ad}(A) = T \subset \mathrm{GL}(n, \mathbb{R}) = \mathrm{Aut}(\mathbb{R}^n)$ is a torus. Actions of T and A on $\mathbb{R}^n$ commute and hence one can define the group $G^* = (T \times A) \ltimes \mathbb{R}^n$. Note that $G^* = T \ltimes W$, where $W = \Delta \times \mathbb{R}^n$ is the nilradical of G^* and $\Delta = \{(\mathrm{Ad}(a), a^{-1}) \in T \times A,\ a \in A\}$ is a central subgroup of G^*.

If $g_{\mathbb{R}}$ is a one-parameter subgroup of G then one can assume (replacing $g_{\mathbb{R}}$ with its conjugate if necessary) that its Jordan decomposition in G^* is such that $g_t = c_t \times u_t$, where $c_{\mathbb{R}} \subset T$ and $u_{\mathbb{R}} \subset W$. Let

$$p : G^* = T \ltimes W \mapsto T, \qquad \pi : G^* = T \ltimes W \mapsto W$$

be the corresponding projections. Note that the restriction $p : G \mapsto T$ is a group epimorphism while $\pi : G \mapsto W$ is only a diffeomorphism.

Using Mostow's theorem (see §1.3a) one can prove that the image $p(D)$ is a finite subgroup of the torus T. If the flow $(G/D, g_{\mathbb{R}})$ is ergodic then clearly $T = \overline{c_{\mathbb{R}}}$. Hence T commutes with $u_{\mathbb{R}}$ and one can easily prove that π induces smooth isomorphism of flows

$$\Pi : (G/D \cap W, g_{\mathbb{R}}) \mapsto (W/D \cap W, u_{\mathbb{R}}).$$

Besides, one can prove that if the second flow is ergodic, then $D \subset W$ and $T = \overline{c_{\mathbb{R}}}$. Hence the following holds ([30]):

THEOREM 2.2.5. *A flow $(G/D, g_{\mathbb{R}})$ on a compact Euclidean manifold is ergodic iff the rectilinear flow $(W/D \cap W, u_{\mathbb{R}})$ is ergodic. Moreover, in the case of ergodicity the flows are smoothly isomorphic. In particular, the flow $(G/D, g_{\mathbb{R}})$ is uniquely ergodic whenever it is ergodic.*

Note that this result is a particular case of Auslander's theorem [7] for type-(I) solvmanifolds. Combining Theorems 2.2.4 and 2.2.5 one gets a criterion for homogeneous flows on solvmanifolds to be ergodic. Clearly, such a flow is never mixing (since any solvmanifold admits a torus as a quotient space).

2c. *Semisimple case.* Now one can use the Mautner phenomenon (Theorem 2.1.4) to find an ergodicity criterion for a (simply connected) *semisimple* Lie group G. Let $G = K \times S$ be the decomposition of G into compact and totally noncompact parts, and $p : G \mapsto K$, $q : G \mapsto S$ the corresponding projections. Define $H = \overline{p(D)}$, $T = \overline{p(g_{\mathbb{R}})} \subset K$. Clearly if the flow $(G/D, g_{\mathbb{R}})$ is ergodic, then the action of the torus T on K/H is ergodic. But K/H has finite fundamental group and hence cannot be an orbit of a T-action. Hence $G = \overline{SD}$ and by the Borel density theorem we derive that D is a Zariski-dense lattice in G, provided D is a quasi-lattice in G and G/D admits an ergodic homogeneous flow.

Suppose that the flow $(G/D, g_{\mathbb{R}})$ is ergodic, and $M \subset G$ is the Moore normal subgroup for $g_{\mathbb{R}}$. The group $P = (\overline{MD})^0$ is normal in G, and one can consider the projection $\alpha : G \mapsto G' = G/P$ and the quotient flow $(G'/D', g'_{\mathbb{R}})$, where $D' = \alpha(\overline{MD})$, $g'_{\mathbb{R}} = \alpha(g_{\mathbb{R}})$. Note that $g'_{\mathbb{R}}$ has a trivial Moore subgroup and hence is relatively compact modulo the center $Z(G')$. But the product $Z(G')D'$ is closed and hence all orbit closures of the flow $(G'/D', g'_{\mathbb{R}})$ are tori. It follows that the flow cannot be ergodic unless $G' = 1$. We have proved the following result due to Moore [155]:

THEOREM 2.2.6. *Let $(G/D, g_{\mathbb{R}})$ be a flow on a finite volume semisimple homogeneous space, and $M \subset G$ the Moore normal subgroup for $g_{\mathbb{R}}$. Then the following conditions are equivalent*:
(1) *the flow is ergodic*;
(2) *the flow is weakly mixing*;
(3) $G = \overline{MD}$.

The ergodicity criterion in the semisimple case may be made more effective in the following way. Suppose that the flow $(G/D, g_{\mathbb{R}})$ is ergodic. Then D is a Zariski-dense lattice in G and one can decompose it into irreducible components: $G = \prod_i G_i$, where $G_i \cap D$ is irreducible G_i for each i and $\prod_i (D \cap G_i)$ is of finite index in D. Let $q_i : G \mapsto G_i$ be the corresponding projection. Clearly, every quotient flow $(G_i/q_i(D), q_i(g_{\mathbb{R}}))$ is ergodic. Since $q_i(D)$ is irreducible in G_i, this is so iff the Moore subgroup $q_i(M)$ of $q_i(g_{\mathbb{R}})$ is nontrivial. On the contrary, if every quotient flow is ergodic then M projects nontrivially to any irreducible component G_i. Thus one gets the following ([155]):

COROLLARY 2.2.7. *Let D be a quasi-lattice in a semisimple Lie group G. Then the flow $(G/D, g_{\mathbb{R}})$ is ergodic iff D is a Zariski-dense lattice in G and* $\mathrm{Ad}(q_i(g_{\mathbb{R}}))$ *is an unbounded subgroup of* $\mathrm{Ad}(G_i)$ *for every i, where $q_i : G \mapsto G_i$ is the projection onto the i-th irreducible component relative to D.*

2d. *General case.* Now let $G = L \ltimes R$ be an *arbitrary* (connected simply connected) Lie group. Let $q : G \mapsto L$ be the projection of G onto its Levi subgroup $L = K \times S$. The following makes finding an ergodicity criterion for homogeneous flows on G/D much easier (see [223,255]):

LEMMA 2.2.8. *If G/D admits an ergodic one-parameter homogeneous flow, then $G = \overline{SRD}$, and hence (see §1.3f) the product RD is closed.*

Clearly, $G/\overline{RD}$ is the maximal semisimple quotient space of G/D. The maximal solvable quotient space has the form $G/\overline{G_\infty D}$, where G_∞ is defined to be the minimal normal subgroup of G containing the Levi subgroup L.

With the help of Lemma 2.2.8 one gets the following criteria first proved by Dani [43] for D being a lattice in G (see [223,232] in the general case):

THEOREM 2.2.9.

(1) *The flow $(G/D, g_{\mathbb{R}})$ is ergodic iff so are the flows $(G/\overline{RD}, g_{\mathbb{R}})$ and $(G/\overline{G_\infty D}, g_{\mathbb{R}})$ on the maximal semisimple and solvable quotient spaces.*

(2) *The flow $(G/D, g_{\mathbb{R}})$ is weakly mixing iff $G = \overline{JD} = \overline{MD}$, where J and M are the Dani and Moore normal subgroups for $g_{\mathbb{R}}$ respectively.*

(3) *If $G = \overline{G_\infty D}$, then any ergodic homogeneous flow on G/D is weakly mixing; otherwise G/D admits no weakly mixing homogeneous flows.*

Similarly to Lemma 2.2.8, one can prove the following [255]:

THEOREM 2.2.10. *If G/D admits a weakly mixing one-parameter homogeneous flow, then the radical R of G is nilpotent and D is Zariski-dense in the* Ad*-representation. In particular, D is a lattice in G if it is a quasi-lattice.*

The following criterion of Brezin and Moore [30] is a consequence of Theorem 2.2.4 and Theorem 2.2.9:

THEOREM 2.2.11. *The flow $(G/D, g_{\mathbb{R}})$ is ergodic iff so are the flows $(G/\overline{RD}, g_{\mathbb{R}})$ and $(G/P, g_{\mathbb{R}})$ on the maximal semisimple and Euclidean quotient spaces.*

Originally, this theorem was proved for so called *admissible* spaces of finite volume. Since then it was independently proved by many authors (cf. [259]) that any finite volume homogeneous space is admissible.

2.3. *Spectrum, Bernoullicity, multiple and exponential mixing*

3a. *Spectrum.* The following stronger formulation of the Mautner phenomenon (Theorem 2.1.4) plays a central role in calculation of spectra of homogeneous flows [156]:

THEOREM 2.3.1. *Let π be a unitary representation of a Lie group G on a Hilbert space $\mathcal{H}$, and let $M \subset G$ be the Moore normal subgroup for $g_{\mathbb{R}} \subset G$. Let* $\mathrm{Fix}(M) \subset \mathcal{H}$ *be the closed subspace of M-fixed vectors. Then the spectrum of $\pi(g_{\mathbb{R}})$ is absolutely continuous on the orthogonal complement* $\mathrm{Fix}(M)^\perp$ *of* $\mathrm{Fix}(M)$.

Using Theorem 2.3.1 one can derive the following general result due to Brezin and Moore [30]:

THEOREM 2.3.2. *The spectrum of a homogeneous flow* $(G/D, g_{\mathbb{R}})$ *is the sum of a discrete component and a Lebesgue component of infinite multiplicity.*

Since any subgroup $g_{\mathbb{R}} \subset G$ with trivial Moore subgroup induces a uniformly continuous flow on G/D, the latter can be ergodic only if G/D is diffeomorphic to a torus, i.e., G/D is a Euclidean manifold. Now mixing criterion (Theorem 2.2.9) implies the following ([30]):

THEOREM 2.3.3. *The spectrum of an ergodic homogeneous flow* $(G/D, g_{\mathbb{R}})$ *is discrete iff* G/D *is a Euclidean manifold. It is Lebesgue of infinite multiplicity iff the flow is weakly mixing* (*in which case* G/D *has no nontrivial Euclidean quotients*).

Let us consider several special cases of Theorem 2.3.3. If G is a non-Abelian nilpotent Lie group and D a lattice in G, then the spectrum of any ergodic flow on G/D is discrete on the space $L^2(G/D[G,G])$ viewed as a subspace of $L^2(G/D)$ and is of infinite multiplicity on its orthogonal complement (cf. §1.2a). The spectrum in this case was found by Green in [9] using the representation theory for nilpotent Lie groups (see [237] for corrections and [171] for an alternative proof); the multiplicity of the Lebesgue component was calculated by Stepin [238].

In the solvable case the spectrum was found by Safonov [194]. Basically, this case reduces to the nilpotent one via the following general result due to Sinai [211] and Parry [171]:

THEOREM 2.3.4. *Let* π *be the Pinsker partition for a measure-preserving flow on a Lebesgue space* (X, μ) *and* $\mathrm{Fix}(\pi) \subset L^2(X, \mu)$ *the subspace of functions constant along* π. *Then the spectrum is Lebesgue of infinite multiplicity on* $\mathrm{Fix}(\pi)^{\perp}$.

From here one deduces that the spectrum of a homogeneous flow $(G/D, g_{\mathbb{R}})$ is Lebesgue of infinite multiplicity on the orthogonal complement to the subspace $\mathrm{Fix}(A) \simeq L^2(G/\overline{AD})$.

Since any compact solvmanifold bundles over a torus, the discrete component of a homogeneous flow thereon is always nontrivial.

For a semisimple G the spectrum of ergodic flows on G/D was calculated by Moore [155] and Stepin [239]. As follows from Theorem 2.2.6 and Theorem 2.3.2, it is also Lebesgue of infinite multiplicity.

3b. *Bernoulli property.* No criterion for a homogeneous flow $(G/D, g_{\mathbb{R}})$ to be Bernoullian is known in the general case. Surely, such a flow should have the K-property and the criterion for the latter is known (see Theorem 2.1.2). The following theorem due to Dani [40] gives a sufficient condition.

THEOREM 2.3.5. *Let $g_\mathbb{R} \subset G$ be a subgroup such that the operator* Ad_{g_t} *is semisimple when restricted to the neutral subalgebra* $\mathfrak{g}^0$ *(see* §2.1a*). Then the flow* $(G/D, g_\mathbb{R})$ *is Bernoullian whenever it is a K-flow.*

It is not known whether the K-property for homogeneous flows implies the Bernoullicity. Note that for toral automorphisms the answer is affirmative (see Theorem 1.4.2).

3c. *Decay of matrix coefficients.* Let $L^2_0(G/D)$ stand for the space of square-integrable functions on G/D with zero average, and ρ stand for the regular representation of G on $L^2_0(G/D)$. Let F be a closed subgroup of G. Recall that the action $(G/D, F)$ is mixing iff given any $u, v \in L^2_0(G/D)$, the *matrix coefficient* $(\rho(g)u, v)$ vanishes as $g \to \infty$ in F. If F is one-parameter then F-action on G/D is mixing whenever it is weakly mixing (since it has Lebesgue spectrum of infinite multiplicity).

Now assume that G and Γ satisfy the following conditions:

$$G \text{ is a connected semisimple totally noncompact Lie group with finite center,} \tag{2.1}$$

and

$$\Gamma \text{ is an irreducible lattice in } G. \tag{2.2}$$

One knows (see Theorem 2.2.6) that any unbounded subgroup $g_\mathbb{R} \subset G$ induces a weakly mixing flow on G/Γ. In fact, there is a stronger result due to Howe and Moore [101] (see [264]):

THEOREM 2.3.6. *Let π be an irreducible unitary representation of a semisimple Lie group G with finite center on a Hilbert space $\mathcal{H}$. Then given any $u, v \in \mathcal{H}$, one has $(\pi(g)u, v) \to 0$ as $g \to \infty$ in G.*

In particular, one can deduce that the F-action on G/Γ is mixing if G and Γ are as in (2.1), (2.2) and F is not relatively compact.

3d. *Exponential mixing.* It turns out that in many cases one can have a good control of the rate of decay of the matrix coefficients $(\pi(g)u, v)$. Namely, fix a maximal compact subgroup C of G, and denote by $\mathfrak{c}$ its Lie algebra. Take an orthonormal basis $\{Y_j\}$ of $\mathfrak{c}$, and set $\Upsilon = 1 - \sum Y_j^2$. Then Υ belongs to the center of the universal enveloping algebra of $\mathfrak{c}$ and acts on smooth vectors of any representation space of G. We also fix a right-invariant and C-bi-invariant Riemannian metric d_G on G.

The following theorem was deduced by Katok and Spatzier [107] from earlier results of Howe and Cowling (see [100] and [39]):

THEOREM 2.3.7. *Let G be as in* (2.1)*, and let Π be a family of unitary representations of G such that the restriction of Π to any simple factor of G is isolated* (*in the Fell topology*) *from the trivial representation I_G. Then there exist constants $E > 0$, $l \in \mathbb{N}$* (*dependent only* *on* G) *and* $\alpha > 0$ (*dependent only on* G *and* Π) *such that for any* $\pi \in \Pi$, *any* C^∞-*vectors* v, w *in a representation space of* π, *and any* $g \in G$ *one has*

$$\left|\left(\pi(g)v, w\right)\right| \leqslant E\,\mathrm{e}^{-\alpha d_G(e,g)}\left\|\Upsilon^l(v)\right\|\left\|\Upsilon^l(w)\right\|.$$

One can apply the above result to the action of G on G/Γ as follows:

COROLLARY 2.3.8. *Let* G *be as in* (2.1), Γ *as in* (2.2), ρ *as in* §2.3c. *Assume that*

$$\text{The restriction of } \rho \text{ to any simple factor of } G \text{ is isolated from } I_G. \tag{2.3}$$

Then there exist $E, \beta > 0$ *and* $l \in \mathbb{N}$ *such that for any two compactly supported functions* $\varphi, \psi \in C^\infty(G/\Gamma)$ *and any* $g \in G$ *one has*

$$\left|(g\varphi, \psi) - \int\varphi\int\psi\right| \leqslant E\,\mathrm{e}^{-\beta\,\mathrm{dist}(e,g)}\left\|\Upsilon^l(\varphi)\right\|\left\|\Upsilon^l(\psi)\right\|.$$

Condition (2.3) can be checked in the following (overlapping) cases: all simple factors of G has property-(T) (by definition), or G is simple (see [12]), or Γ is non-uniform (the proof in [118] is based on the results of Burger and Sarnak [35] and Vigneras [248]), and widely believed to be true for any G and Γ satisfying (2.1) and (2.2).

3e. *Multiple mixing*. Let F be a locally compact group which acts on a Lebesgue probability space (X, μ) by measure-preserving transformations. Then the action is said to be k-*mixing* if given any $k+1$ measurable subsets $A_1, \ldots, A_{k+1} \subset X$ and any $k+1$ sequences $\{g_i(1)\}, \ldots, \{g_i(k+1)\}$ of elements in F such that $g_i(l)g_i(m)^{-1} \to \infty$, $i \to \infty$, for all $l \neq m$, one has

$$\lim_{i\to\infty}\mu\left(\bigcap_{l=1}^{k+1} g_i(l)A_l\right) = \prod_{l=1}^{k+1}\mu(A_l).$$

One can check that 1-mixing for a one-parameter F is equivalent to the usual definition of mixing.

Clearly, k-mixing implies l-mixing for all $l \leqslant k$. Also, if $H \subset F$ is a closed subgroup then the H-action is k-mixing whenever so is the F-action.

An old problem of Rokhlin is whether 1-mixing implies k-mixing for all k. In the classical case ($F \simeq \mathbb{R}$ or $\mathbb{Z}$) this problem is only solved under rather restrictive conditions on the action. For instance, the answer to the Rokhlin problem is affirmative for one-parameter measurable actions with *finite rank of approximation* (Ryzhikov [193]). Also, the K-property implies mixing of all degrees. On the other hand, Ledrappier [125] constructed a $\mathbb{Z}^2$-action which is 1-mixing but not 2-mixing.

For a non-Abelian F one has the following result of Mozes [159,161]. A Lie group F is said to be Ad-*proper* if the center $Z(F)$ is finite and $\mathrm{Ad}(F) = F/Z(F)$ is closed in $\mathrm{Aut}(\mathrm{Lie}(F))$.

THEOREM 2.3.9. *If F is an* Ad-*proper Lie group then any mixing measure-preserving F-action on a Lebesgue probability space is mixing of all degrees.*

One immediately gets the following:

COROLLARY 2.3.10. *If G and Γ are as in* (2.1), (2.2) *then the G-action on G/Γ is mixing of all degrees.*

One can derive from here the following result of Marcus [130]: if G is a semisimple Lie group then any ergodic flow $(G/D, g_{\mathbb{R}})$ is mixing of all degrees.

Marcus conjectured in [130] that Rokhlin problem has the affirmative answer for any one-parameter homogeneous flow $(G/D, g_{\mathbb{R}})$. This was proved by Starkov [229] reducing the problem to the case of $g_{\mathbb{R}}$ being unipotent and then combining Ratner's measure theorem (Theorem 3.3.2) with joinings technique.

THEOREM 2.3.11. *Any mixing homogeneous flow $(G/D, g_{\mathbb{R}})$ is mixing of all degrees.*

The same result was conjectured in [229] for any action $(G/D, F)$, where F is a closed connected subgroup of G.

2.4. *Ergodic decomposition*

4a. *Reduction to the ergodic case.* As was said in §2.1f, there are two versions of an ergodic decomposition for a homogeneous flow $(G/D, g_{\mathbb{R}})$ relative to the Haar probability measure. The first one, $\widetilde{E}$, is the partition into closed submanifolds $\widetilde{E}_h(g_{\mathbb{R}}) = \overline{g_{\mathbb{R}} MhD} \subset G/D$; the second one, E, is a refinement of $\widetilde{E}$ and consists of closed submanifolds $E_h(g_{\mathbb{R}}) = \overline{g_{\mathbb{R}} JhD}$. Here as always J is the Dani normal subgroup for $g_{\mathbb{R}}$, and M the Moore subgroup. Note that components of $\widetilde{E}$ are defined everywhere on G/D (not a.e. as in the abstract ergodic decomposition for measure-preserving flows, cf. [3, Theorem 4.2.4]), and they either coincide or do not intersect. The same is true for E.

If G is nilpotent then $\widetilde{E}$ forms a homogeneous bundle of G/D (see §2.2a). The components of E are also homogeneous subsets of G/D but their dimension is not constant everywhere (it may drop on a dense subset of zero measure). This can be generalized as follows: if $g_{\mathbb{R}}$ is a subgroup of G such that all eigenvalues of Ad_{g_t}, $t > 0$, are real, then there exists a subgroup $F \subset G$ such that $\widetilde{E}_h(g_{\mathbb{R}}) = hFD$ for all $h \in G$ (this is so because $g_{\mathbb{R}}$ projects to a central subgroup in G/M). If the homogeneous flow $(hFD, g_{\mathbb{R}})$ is not ergodic, one can repeat the procedure and finally find an invariant homogeneous subspace of G/D containing the point $hD \in G/D$ and such that the flow thereon is ergodic relative to the Haar measure; one can show that this subspace is nothing else but $E_h(g_{\mathbb{R}})$. Hence all ergodic components are homogeneous subspaces of G/D provided $g_{\mathbb{R}}$ has only real eigenvalues with respect to the Ad-representation (in particular, if $g_{\mathbb{R}}$ is $\mathbb{R}$-diagonalizable or unipotent).

The existence of nonreal eigenvalues of Ad_{g_t} complicates the matters in the sense that ergodic components need not be homogeneous. For instance, if G is Euclidean then E and

$\widetilde{E}$ coincide and their components may not be homogeneous subsets of G/D. However, the restriction of the flow onto each component is smoothly isomorphic to an ergodic rectilinear flow on a torus (see §1.4b for examples).

This is a particular case of the following general phenomenon: the restriction of the flow onto each ergodic component is compactly covered by a homogeneous flow (see [226]).

THEOREM 2.4.1. *Let G be a connected simply connected Lie group and $(G/D, g_{\mathbb{R}})$ a homogeneous flow on a finite volume space. Then G can be embedded as a normal subgroup into a connected Lie group G^* such that G^*/G is a torus and G^* contains a connected subgroup F and a one-parameter subgroup $f_{\mathbb{R}}$ such that*

(1) *for every $h \in G$ one has a smooth equivariant covering with compact leaves*

$$\big(F/D \cap F, h^{-1} f_{\mathbb{R}} h\big) \mapsto \big(\widetilde{E}_h(g_{\mathbb{R}}), g_{\mathbb{R}}\big)$$

where the covering is finitely sheeted for almost all $h \in G$;

(2) *for every $h \in G$ there exists a finitely sheeted equivariant covering*

$$\big(F_h/D \cap F_h,\, f_{\mathbb{R}}(h)\big) \mapsto \big(E_h(g_{\mathbb{R}}), g_{\mathbb{R}}\big),$$

where F_h is a connected subgroup of F, and one has $F_h = F$ and $f_{\mathbb{R}}(h) = h^{-1} f_{\mathbb{R}} h$ for almost all $h \in G$;

(3) *the subgroup D^0 is normal in F.*

This theorem implicitly states that $f_{\mathbb{R}}$ and $g_{\mathbb{R}}$ have the same Moore normal subgroup M; hence $F = (\overline{h f_{\mathbb{R}} h^{-1} M D})^0 \subset G^*$ for all $h \in G$.

The above theorem reduces the study of non-ergodic homogeneous flow to that of a family of ergodic homogeneous flows on homogeneous spaces with discrete isotropy subgroups. In particular, one has the following ([226]):

COROLLARY 2.4.2. *An ergodic homogeneous flow $(G/D, g_{\mathbb{R}})$ is finitely covered by a homogeneous flow $(H/\Gamma, f_{\mathbb{R}})$, where Γ is a lattice in H.*

For the proof one takes $H = F/D^0$ and $\Gamma = D \cap F/D^0$, where F is as in Theorem 2.4.1. Corollary 2.4.2 generalizes a similar result of Auslander [6] for homogeneous flows on solvmanifolds.

4b. *Typical orbit closures.* We have seen that flows on $\widetilde{E}_h(g_{\mathbb{R}})$ are covered by homogeneous flows on the same space $F/D \cap F$, and for almost all $h \in G$ the covering map is finitely sheeted. One can show that the finite fibers are the same almost everywhere and hence almost all manifolds $\widetilde{E}_h(g_{\mathbb{R}})$ are diffeomorphic to each other. Moreover, a stronger result holds ([226]).

THEOREM 2.4.3. *There exists a closed invariant subset $Q \subset G/D$ of zero measure such that outside Q the partition $\widetilde{E}$ forms a smooth bundle.*

As an example in §1.4b shows, one cannot claim that $\widetilde{E}$ is a smooth bundle of G/D. Another example comes from homogeneous flows of the form $(K/T, g_{\mathbb{R}})$, where K is a compact semisimple Lie group and T a maximal torus in K.

The above theorem sharpens an earlier result of Brezin and Moore [30] that a nonergodic homogeneous flow admits a nonconstant C^∞-smooth invariant function.

One knows that the smooth flow $(\widetilde{E}_h(g_{\mathbb{R}}), g_{\mathbb{R}})$ is ergodic for almost all $h \in G$. On the other hand, almost all orbits of an ergodic flow on manifold with a smooth invariant measure are dense (Hedlund's lemma). This allows one to deduce the following [226]:

THEOREM 2.4.4. *Almost all orbit closures of a homogeneous flow* $(G/D, g_{\mathbb{R}})$ *on a finite volume space are smooth manifolds. Moreover, almost all of them are diffeomorphic to the same manifold* (*the typical ergodic submanifold given by Theorem* 2.4.3).

2.5. *Topological equivalence and time change*

5a. *Topological equivalence.* Let Γ (resp. Γ') be a lattice in a Lie group G (resp. G'). Let us say that a homeomorphism $\sigma : G/\Gamma \mapsto G'/\Gamma'$ is an *affine isomorphism* if there exist an epimorphism $\alpha : G \mapsto G'$ and an element $h \in G'$ such that $\alpha(\Gamma) = \Gamma'$ and $\sigma(g\Gamma) = h\alpha(g)\Gamma'$ for all $g \in G$.

Recall that given subgroups $g_{\mathbb{R}} \subset G$ and $g'_{\mathbb{R}} \subset G'$, a homeomorphism $\sigma : G/\Gamma \mapsto G'/\Gamma'$ is a *topological equivalence* of corresponding flows if σ sends $g_{\mathbb{R}}$-orbits to $g'_{\mathbb{R}}$-orbits. If in addition σ is an affine isomorphism then σ is said to be an *affine equivalence* of the corresponding flows (one can check that then there exists $c \neq 0$ such that $\sigma(g_t x) = g'_{ct}\sigma(x)$ for all $x \in G/\Gamma$, $t \in \mathbb{R}$).

Clearly, topological equivalence not always implies the existence of an affine equivalence (even if the flows are ergodic). For example, any ergodic flow on a Euclidean manifold G/Γ is smoothly isomorphic (and hence topologically equivalent) to a rectilinear flow on a torus and the spaces involved are not affinely isomorphic unless G is Abelian. On the other hand, geodesic flows on homeomorphic compact surfaces of constant negative curvature are always topologically equivalent, though the surfaces may not be isometric [4].

However, in a broad class of Lie groups topological equivalence does imply the existence of an affine one. Following Benardete [14], let us demonstrate the idea on the classical case $G/\Gamma = G'/\Gamma' = \mathbb{R}^n/\mathbb{Z}^n$. Taking a composition of homeomorphism σ with a translation if necessary, one may assume that σ induces a homeomorphism $\tilde{\sigma}$ of $\mathbb{R}^n/\mathbb{Z}^n$ such that $\tilde{\sigma}(x + \mathbb{Z}^n) = \tilde{\sigma}(x) + \mathbb{Z}^n$, $x \in \mathbb{R}^n$. Since $\tilde{\sigma}|_{\mathbb{Z}^n} = s \in \mathrm{SL}(n, \mathbb{Z})$, it follows that $s^{-1}\tilde{\sigma}$ is a homeomorphism of $\mathbb{R}^n$ which is identical on $\mathbb{Z}^n$ and sends $g_{\mathbb{R}}$-orbits into $s^{-1}(g'_{\mathbb{R}})$-orbits. Since $\mathbb{R}^n/\mathbb{Z}^n$ is compact, there exists a $c > 0$ such that $d(x, s^{-1}\tilde{\sigma}(x)) \leqslant c$, $x \in \mathbb{R}^n$. On the other hand, any two distinct one-parameter subgroups of $\mathbb{R}^n$ diverge from each other. Hence $g_{\mathbb{R}} = s^{-1}(g'_{\mathbb{R}})$ and the flows are affinely equivalent.

This way one can prove the same result provided: (a) Γ and Γ' are uniform lattices in G and G' respectively, (b) any isomorphism between Γ and Γ' can be extended to the Lie groups involved (rigidity of lattices), (c) any two distinct one-parameter subgroups in G diverge from each other. In fact, the uniformness of lattices is not necessary. The condition (b) stands for rigidity of lattices and it is satisfied if both G and G' are triangular Lie groups (Saito), or both G and G' are semisimple and satisfy the conditions of Mostow–Margulis–Prasad rigidity theorem (see [141]). As for the last condition, one can replace it with ergodicity. One obtains the following result due to Benardete [14]:

THEOREM 2.5.1. *Let* $\sigma:(G/\Gamma, g_{\mathbb{R}}) \mapsto (G'/\Gamma', g'_{\mathbb{R}})$ *be a topological equivalence of ergodic flows. Assume that any one of the following conditions holds*:

(1) *G and G' are simply connected triangular Lie groups.*
(2) *G and G' are totally noncompact semisimple center-free Lie groups, and G contains no normal subgroup S locally isomorphic to* $\mathrm{SL}(2,\mathbb{R})$ *such that the product $S\Gamma$ is closed.*

Then σ is a composition of an affine equivalence and a homeomorphism of G'/Γ' preserving $g'_{\mathbb{R}}$-orbits.

Witte [256] extended this result to homogeneous actions of multi-parameter subgroups:

THEOREM 2.5.2. *Let* $\sigma:(G/\Gamma, F) \mapsto (G'/\Gamma', F')$ *be a topological equivalence of homogeneous flows, where F and F' are connected unimodular subgroups. Assume that G and G' are the same as in Theorem* 2.5.1. *Then σ is a composition of an affine equivalence and a homeomorphism of G'/Γ' preserving F'-orbits.*

Clearly, Theorem 2.5.2 is not true for $G = G' = \mathrm{SL}(2,\mathbb{R})$ if, for instance, $F = F' = D(2)$: if Γ and Γ' are uniform and isomorphic then the geodesic flows are topologically equivalent. As it was shown in [256], the same holds for $F = F' = \mathrm{SO}(2)$ and $F = F' = T(2)$.

On the other hand, in [131] Marcus proved that horocycle flows are topologically rigid:

THEOREM 2.5.3. *Let* $\sigma:(G/\Gamma, U_{\mathbb{R}}) \mapsto (G/\Gamma', U_{\mathbb{R}})$ *be a topological equivalence of horocycle flows, where both Γ and Γ' are uniform. Then σ is a composition of an affine equivalence and a homeomorphism of G/Γ' preserving $U_{\mathbb{R}}$-orbits.*

Having in mind Theorems 2.5.1 and 2.5.3 one can conjecture that the topological equivalence $(G/\Gamma, u_{\mathbb{R}}) \mapsto (G'/\Gamma', u'_{\mathbb{R}})$ of two ergodic unipotent flows is always a composition of affine equivalence and homeomorphism of G'/Γ' preserving $u'_{\mathbb{R}}$-orbits.

5b. *The first cohomology group*. We have seen that under favorable conditions the topological equivalence of flows is a composition of an affine equivalence and time change of the second flow.

Let us consider time changes for homogeneous actions $(G/\Gamma, F)$, where F is isomorphic to $\mathbb{R}^k$ for some $k \geqslant 1$. The simplest case is the *linear* time change arising from an automorphism of F. Clearly, any time change for $(G/\Gamma, F)$ gives one a cocycle $v: F \times (G/\Gamma) \mapsto F \simeq \mathbb{R}^k$ and the description of time changes reduces to the study of the first cohomology group over the flow $(G/\Gamma, F)$ with values in $\mathbb{R}$. Here the smoothness class of time change defines the smoothness of the cocycle; hence one can study the first cohomology group in the category of continuous, Hölder, C^∞-smooth functions etc. For instance, if the first cohomology group *trivializes*, i.e., every cocycle is cohomological to a homomorphism $\alpha : F \mapsto \mathbb{R}$, then any time change is equivalent (in the corresponding category) to a linear one.

Not much is known about the first cohomology group over homogeneous flows. Let us first consider the question for one-parameter flows. It is well known that the first continuous (or smooth) cohomology group over a smooth flow does not trivialize if the flow has at least two distinct periodic orbits. In other words, the family of periodic orbits presents an obstruction to the trivialization of the first cohomology group. It is important that for Anosov flows on compact Riemannian manifolds there are no other obstructions in the categories of Hölder and C^∞-functions (theorem of Livsiĉ, see [107]).

Surprisingly, the situation is quite different for multi-parameter Anosov flows. All known nontrivial examples of such actions (except those arising from operations with one-parameter Anosov flows) come from the following construction. Let $C \subset G$ be a compact subgroup which commutes with $F \simeq \mathbb{R}^k$. Then an infra-homogeneous flow $(C\backslash G/\Gamma, F)$ is said to be *Anosov* if F contains a partially hyperbolic element $a \in F$ for which F-orbits on $C\backslash G/\Gamma$ form the neutral foliation (i.e., $C \times F$ is the neutral subgroup for $a \in G$). Katok and Spatzier studied different classes of Anosov actions (called the *standard* actions) and discovered that their first (Hölder and C^∞) cohomology groups do trivialize. The common in all the standard actions is the absence of quotient actions which degenerate to one-parameter Anosov actions (see [107] for more details), and this is probably the criterion for the first cohomology groups to trivialize (see [233,235] for a progress in proving this conjecture). The most important class (of so called *Weyl chamber flows*) is obtained by taking G to be a totally noncompact semisimple Lie group with $\mathrm{rank}_{\mathbb{R}} G \geqslant 2$, $\Gamma \subset G$ an irreducible lattice, $F \subset G$ a maximal $\mathbb{R}$-diagonalizable subgroup, and $C \subset G$ the compact part of the centralizer of F in G. Note that the key role in establishing the trivialization of the first cohomology group is played by the exponential mixing property (see §2.3d).

5c. *Time changes*. Now let us again consider one-parameter homogeneous flows. Assume that the flow has no periodic orbits and, moreover, is minimal. Then the study of time changes becomes very complicated. Partial results are known only for rectilinear flows on $\mathbb{T}^2$ and for the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/\Gamma$.

First we describe shortly some results for the rectilinear flow $(\mathbb{T}^2, \mathbb{R}v_\alpha)$ given by a vector $v_\alpha = (1,\alpha) \in \mathbb{R}^2$ (see [38,213,237] for more information). It turns out that the possibility to linearize time changes depends on Diophantine properties of the rotation number α. We have the following result due to Kolmogorov [120]:

THEOREM 2.5.4. *For almost all irrational numbers α, any positive function $\tau \in C^\infty(\mathbb{T}^2)$ determines a time change for the rectilinear flow $(\mathbb{T}^2, \mathbb{R}v_\alpha)$ which is C^∞-equivalent to a linear scaling of time. On the other hand, there exist uncountably many irrational numbers α for which not every C^∞-smooth time change can be linearized.*

In fact, the numbers α with nontrivial time changes are those with very fast rate of approximation by rational numbers (see §5.2a). Moreover, as was indicated in [120], there exist irrational number α and an analytical time change for the flow $(\mathbb{T}^2, \mathbb{R}v_\alpha)$ which determines a flow on $\mathbb{T}^2$ with continuous spectrum (hence the flows are not conjugate even via a measure-preserving transformation of $\mathbb{T}^2$); earlier a similar example with a continuous time change was constructed by von Neumann.

Finally, we mention some results related to the horocycle flow $(G/\Gamma, U_{\mathbb{R}})$, where Γ is a uniform lattice in $G = \mathrm{SL}(2,\mathbb{R})$. Such a flow is Kakutani equivalent to a rectilinear ergodic flow on $\mathbb{T}^2$ (Ratner [175]).[13] On the other hand, any two ergodic rectilinear flows on $\mathbb{T}^2$ are Kakutani equivalent (Katok [106]). Hence given any two uniform lattices Γ_1 and Γ_2, there exists a measurable time change of the horocycle flow on G/Γ_1 which leads to a flow isomorphic to the horocycle flow on G/Γ_2.

On the other hand, even a mild smoothness condition preserves rigidity properties of the horocycle flow (cf. Section 3.8). Namely, let $K(G/\Gamma) \subset L^2(G/\Gamma)$ be the class of positive functions τ such that τ and $1/\tau$ are bounded and τ is Hölder along the SO(2)-orbits. The following was proved by Ratner [181]:

THEOREM 2.5.5. *Let the flows* $(G/\Gamma_1, U_t^{\tau_1})$ *and* $(G/\Gamma_2, U_t^{\tau_2})$ *be obtained from horocycle flows via time changes* $\tau_i \in K(G/\Gamma_i)$, $i = 1, 2$, *where* $\int \tau_1 = \int \tau_2$. *Then any measure-theoretic isomorphism of the flows implies conjugacy of* Γ_1 *and* Γ_2 *in* G (*hence the horocycle flows are also isomorphic*).

It is not known whether any C^∞-smooth time change of the horocycle flow is equivalent to a linear one. On the other hand, there are no results similar to Theorem 2.5.5 for other unipotent ergodic flows. For instance (see [189]), is it true that isomorphism of two flows obtained via C^∞-smooth time changes of two ergodic unipotent flows implies isomorphism of the latter?

2.6. *Flows on arbitrary homogeneous spaces*

6a. *Reduction to the case of discrete isotropy subgroup*. Homogeneous actions on spaces of general nature (possibly not of finite volume) are not well studied. Still, one can mention a few results in this setting.

First, while studying one-parameter flows $(G/D, g_{\mathbb{R}})$ one can assume with no loss of generality that the isotropy subgroup D is discrete. This is better for several reasons: (a) considering local structure of G/D in this case presents no difficulties; (b) the flow itself has no fixed points and orbits of contracting/expanding subgroups form smooth foliations; (c) a right Haar measure on G induces a G-semi-invariant smooth measure ν on G/D (i.e., there exists a multiplicative character $\chi : G \to \mathbb{R}^*$ such that $\nu(gX) = \chi(g)\nu(X)$, $g \in G$, $X \subset G/D$) which may be helpful (note that in the general case ν is only G-quasi-invariant, i.e., left translations on G/D preserve the subalgebra of sets of zero measure). One has the following ([225]):

THEOREM 2.6.1. *Given a one-parameter flow* $(G/D, g_{\mathbb{R}})$, *the space* G/D *is a disjoint union of two invariant subsets* O *and* P *such that*

(1) *O is open and all orbits inside O are locally closed,*

[13] Surprisingly enough, this is not the case for the Cartesian square of the horocycle flow [176].

(2) *the closed subset P is a disjoint union of closed invariant submanifolds $P_h(g_{\mathbb{R}})$ such that every flow $(P_h(g_{\mathbb{R}}), g_{\mathbb{R}})$ is finitely covered by a homogeneous flow $(F_h/\Gamma_h, f_{\mathbb{R}}(h))$, where Γ_h is a discrete subgroup of F_h.*

Clearly, the above theorem reduces the study of orbit closures and finite ergodic measures on G/D to the case of discrete isotropy subgroup. In particular, one has the following ([225]:

COROLLARY 2.6.2. *Any topologically transitive homogeneous flow on a space of dimension greater than* 1 *is finitely covered by a homogeneous flow on a space with discrete isotropy subgroup.*

Here the dimension condition is put to avoid the degenerate case when the flow is both topologically transitive and dissipative. (Example: $G = \mathrm{SL}(2, \mathbb{R})$, $D = T(2)$, $g_{\mathbb{R}} = D(2)$.)

6b. *Flows on arbitrary solvmanifolds.* One may note an analogy between Theorem 2.6.1 and the ergodic decomposition Theorem 2.4.1. In fact, in the case $\mathrm{vol}(G/D) < \infty$ the set O is empty and ergodic submanifolds $E_h(g_{\mathbb{R}})$ play the role of $P_h(g_{\mathbb{R}})$. Clearly, Corollary 2.6.2 generalizes Corollary 2.4.2.

Another situation when all $P_h(g_{\mathbb{R}})$ are ergodic manifolds is given by the solvable case. In this case Theorem 2.6.1 may be significantly refined and the study of homogeneous flows on G/D may be completely reduced to the case when D is a *lattice* (not just a discrete subgroup) in G. One has the following ([224]):

THEOREM 2.6.3. *Assume that G is a solvable Lie group. Then, in the notation of Theorem* 2.6.1, *all submanifolds $P_h(g_{\mathbb{R}})$ are compact and the covering homogeneous flows $(F_h/\Gamma_h, f_{\mathbb{R}}(h))$ are ergodic flows on compact solvmanifolds with discrete isotropy subgroups.*

One can also prove that any orbit inside O goes to infinity at least in one direction. Besides, solvable flows are either dissipative (when P is of zero Haar measure) or conservative (when O is empty).

6c. *Geodesic and horocycle flows.* As we have seen in Theorem 2.6.1, it suffices to consider homogeneous flows of the form $(G/\Gamma, g_{\mathbb{R}})$, where Γ is a discrete subgroup of G. Unlike the solvable case, for a semisimple Lie group G the study of one-parameter homogeneous flows gets much more complicated and there is no hope to reduce it to the finite volume case. This can be seen on the simplest case $G = \mathrm{SL}(2, \mathbb{R})$.

We describe briefly some known results for the geodesic and horocycle flows on G/Γ (where Γ is an arbitrary discrete subgroup of G) with respect to the Haar measure on G/Γ. The results below come from the theory of Fuchsian groups and for more details one can consult survey [230].

First, the geodesic flow is either dissipative or ergodic. Unlike the solvable case (where dissipative flow possesses an open conull subset consisting entirely of locally closed orbits), the geodesic flow may be both dissipative (in particular, almost all its orbits go to infinity in both directions) and topologically transitive (hence the set of dense orbits is of the second category).

The behavior of the horocycle flow is much more sensitive to the structure of the Fuchsian group Γ. We assume that Γ is non-elementary (i.e., Γ is not almost cyclic). Let $\Omega \subset G/\Gamma$ be the nonwandering closed set for the horocycle flow. Note that Ω is invariant under the geodesic flow. One knows that either $\Omega = G/\Gamma$ or Ω is a nowhere locally connected closed noncompact set whose Hausdorff dimension is equal to $2 + \delta$ (where δ is the Hausdorff dimension of the limit set of Γ).

Almost all orbits inside Ω are dense therein. In particular, the horocycle flow may have orbit closures which are not submanifolds. Moreover, the set Ω can be minimal (and noncompact). Apart from dense orbits the set Ω can contain horocycle orbits of other types: periodic, going to infinity in both directions, locally closed but nonclosed, recurrent but nondense in Ω, etc.

Both Ω and its complement can be of infinite measure; hence the horocycle flow can be neither dissipative nor conservative (clearly, in this case the geodesic flow is dissipative). On the other hand, the horocycle flow can be conservative and non-ergodic. Recently Kaimanovich [105] proved that the horocycle flow on G/Γ is ergodic if and only if the Borel subgroup $T(2) \subset G$ acts ergodically.

Note that all the pathologies do not happen if the Fuchsian group $\Gamma \subset \mathrm{SL}(2, R)$ is finitely generated. However, if one proceeds to $G = \mathrm{SO}(1, n)$, $n > 2$, the situation is not well understood even for finitely generated discrete subgroups $\Gamma \subset G$ (cf. [166]).

This brief exposition shows how complicated can be the situation for arbitrary homogeneous flows. Apparently, in the general case one cannot hope to find, for instance, the ergodicity criterion or to describe even typical orbit closures. Still, some results of general nature can be proved for any discrete subgroup Γ of Lie group G (cf. §§3.3b, 3.3d, 4.1a, 4.3b, 4.3c).

3. Unipotent flows and applications

From the developments in last 15 years it is apparent that many questions in number theory, more specifically those related to Diophantine analysis, can be reformulated in terms of ergodic properties of *individual* orbits of certain flows on $\Omega_k = \mathrm{SL}(k, \mathbb{R})/\mathrm{SL}(k, \mathbb{Z})$, the space of unimodular lattices in $\mathbb{R}^k$. In most of the interesting cases one encounters a situation where the collection of trajectories, whose dynamical behavior yields useful information for the problem at hand, has total measure zero. Therefore one is unable to effectively use the results which are true for almost all trajectories. This is the main reason one is interested in the study of individual trajectories. The ergodic properties to be considered for this purpose are closures and limiting distributions of trajectories. One also needs to understand how much 'percentage of time' a given trajectory spends in a sequence of chosen neighborhoods of either the point at infinity, or a fixed lower dimensional submanifold in the homogeneous space. To illustrate this view point we consider the following

EXAMPLE 3.0.1. Based on an idea of Mostow the following result was proved in [67], where the condition of ergodicity and the additional algebraic structure lead to a statement about density of individual orbits: *Let G be a Lie group and Γ be a lattice in G. Let $g \in G$ be such that* $\operatorname{Ad} g$ *is semisimple and the action of g on G/Γ is ergodic. Let G^+ be the expanding horospherical subgroup associated to g. Let H be the subgroup generated by G^+ and g. Then every H-orbit on G/Γ is dense.*

Using this observation the following number theoretic statement was obtained by Dani and Raghavan [67]: *Let $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_p)$ be a p-frame over $\mathbb{R}^k$ with $1 \leqslant p \leqslant k-1$. Then its orbit under* $\mathrm{SL}(k, \mathbb{Z})$ *is dense in the space of p-frames over $\mathbb{R}^k$ if and only if $\mathbf{v}$ is 'irrational'* (*that is, the real span of* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *contains no nonzero rational vector*).

It is well known that individual trajectories of actions of a partially hyperbolic one-parameter subgroup on a homogeneous space can have strange behavior, see §1.4e and §4.1a. On the other hand, by the contributions of Dani, Margulis, and Ratner, to name a few, the behavior of individual orbits of actions of unipotent subgroups is now very well understood. Here the two most important results are: (1) the nondivergence of a unipotent trajectory on a finite volume homogeneous space (exposed in Section 3.1 and Section 3.2), and (2) the homogeneity of finite ergodic invariant measures and orbit closures of a unipotent flow (see Section 3.3, and also Section 3.5 for actions of subgroups generated by unipotent elements). Putting together these basic results provides a powerful tool to investigate ergodic properties of individual trajectories. For example, it can be shown that every unipotent trajectory on a finite volume homogeneous space is uniformly distributed with respect to a unique invariant (Haar) probability measure on a closed orbit of a (possibly larger) subgroup (see Section 3.3 for the main result, and Section 3.6 for refinements and generalizations). One of the aims of this section is to describe a variety of ergodic theoretic results which were motivated by number theoretic questions and which can be addressed using the two basic results. Those include limiting distributions of sequences of measures (Section 3.7) and equivariant maps, joinings and factors of unipotent flows (Section 3.8). We will also give some idea of the techniques involved in applying the basic results, see Section 3.4. Other applications and generalizations can be found in Section 4, see Sections 4.2, 4.3 and 4.4.

3.1. *Recurrence property*

The most crucial property of a one-parameter group of unipotent linear transformations, say $\rho : \mathbb{R} \mapsto \mathrm{SL}(k, \mathbb{R})$, is that its matrix coefficients, that is, the (i, j)-th entries of ρ, are polynomials of degree at most $k-1$.

One of the first striking applications of this property was made by Margulis in 1970, see [132], to prove the following nondivergence result:

THEOREM 3.1.1. *Let $G = \mathrm{SL}(k, \mathbb{R})$, $\Gamma = \mathrm{SL}(k, \mathbb{Z})$, and let u be a unipotent element of G. Then for any $\Lambda \in G/\Gamma$, there exists a compact set $K \subset G/\Gamma$ such that the set $\{n \in \mathbb{N} \mid u^n \Lambda \in K\}$ is infinite.*

This result was used in Margulis' proof [133] of the arithmeticity of non-uniform irreducible lattices in semisimple groups of $\mathbb{R}$-rank $\geqslant 2$. Its proof is based on a combinatorial inductive procedure, utilizing some observations about covolumes of sublattices of $u^n\mathbb{Z}^k$, and also growth properties of positive valued real polynomials. Later a quantitative version of it was obtained by Dani [45,51,55]; the statement given below appears in [62, Proposition 1.8].

THEOREM 3.1.2. *Let G be a Lie group and Γ a lattice in G. Then given a compact set $C \subset G/\Gamma$ and an $\varepsilon > 0$, there exists a compact set $K \subset G/\Gamma$ such that the following holds*: *For any one-parameter unipotent subgroup $u_{\mathbb{R}}$ of G, any $x \in C$ and any $T > 0$,*

$$\frac{1}{T}\left|\left\{t \in [0,T] \mid u_t x \in K\right\}\right| > 1 - \varepsilon,$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

The result was proved first for $G = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. It was extended for semisimple groups G of $\mathbb{R}$-rank $\geqslant 2$ and irreducible lattices Γ using the Margulis Arithmeticity Theorem (Theorem 1.3.2). It was proved for semisimple groups G of $\mathbb{R}$-rank one in [51], using the cusp structure of fundamental domains for lattices in these groups [86]. The two results were combined to obtain the theorem for all semisimple Lie groups. The general case follows from the fact that any finite volume homogeneous space has a semisimple homogeneous space as an equivariant factor with compact fibers [51,55] (see also §1.3f).

1a. *Property-(D) and the Mautner phenomenon.*

DEFINITION 3.1.3. A subgroup H of a Lie group G is said to have *property*-(D) on a homogeneous space X of G if for any locally finite H-invariant measure σ on X there exists a countable partition of X into H-invariant measurable subsets $\{X_i\}_{i\in\mathbb{N}}$ such that $\sigma(X_i) < \infty$ for all $i \in \mathbb{N}$.

In particular, if H has property-(D) on X, then every locally finite H-invariant H-ergodic measure on X is finite. The validity of the converse can be deduced by using the technique of ergodic decomposition, see, e.g., [3, Theorem 4.2.4].

Combining Theorem 3.1.2 and Birkhoff's ergodicity theorem, Dani [45] showed the following:

COROLLARY 3.1.4. *Any unipotent subgroup of a connected Lie group G has property-(D) on any finite volume homogeneous space of G.*

In view of the Mautner phenomenon [156] (see Theorem 2.1.4), one can show the following [136,201]:

PROPOSITION 3.1.5. *Let G be a Lie group and W be a subgroup generated by one-parameter unipotent subgroups. Then there exists a connected unipotent subgroup $U \subset W$ such that the triple (G, W, U) has the following property*: *for any* (*continuous*) *unitary* *representation G on a Hilbert space $\mathcal{H}$, if a vector $v \in \mathcal{H}$ is fixed by U then v is fixed by W.*

In particular, if μ is a finite G-invariant Borel measure on a G-space X, and if μ is W-ergodic then μ is U-ergodic.

It was observed by Margulis [136] that from Corollary 3.1.4 and Proposition 3.1.5 one can deduce the following:

THEOREM 3.1.6. *Let G be a Lie group, and H a subgroup of G such that H/H_1 is compact, where H_1 is the closed normal subgroup of H generated by all one-parameter unipotent subgroups contained in H. Then H has property-(D) on any finite volume homogeneous space of G.*

In particular, if X is a finite volume homogeneous space of G, and $x \in X$ is such that Hx is closed, then $Hx \cong H/H_x$ admits a finite H-invariant measure, where H_x denotes the stabilizer of x in H (cf. [174, Theorem 1.3]).

From the above result, Margulis [136] derived an alternative proof of Theorem 1.3.1. By similar techniques one obtains the following [201, §2]:

THEOREM 3.1.7. *Let G be a Lie group, Γ a lattice in G, and W a subgroup of G which is generated by one-parameter unipotent subgroups. Given any $x \in G/\Gamma$, let H be the smallest closed subgroup of G containing W such that the orbit Hx is closed (such a subgroup exists). Then:*

(1) *H/H_x has finite volume;*
(2) *W acts ergodically on $Hx \cong H/H_x$ with respect to the H-invariant measure;*
(3) $\mathrm{Ad}_G(H) \subset \mathrm{Zcl}(\mathrm{Ad}_G(H_x))$.

The Zariski density statement in the above corollary is a consequence of Theorem 1.3.6.

1b. *Compactness of minimal closed sets.* The following result due to Margulis [140] can be seen as a topological analogue of the property-(D) of unipotent flows:

THEOREM 3.1.8. *Let G be a Lie group, Γ a lattice in G, and U a unipotent subgroup of G. Then any minimal closed U-invariant subset of G/Γ is compact.*

This result was proved by Dani and Margulis [61] for the case of one-parameter unipotent subgroups. The proof of the general case is based on Theorem 3.1.2 and the following compactness criterion, which was proved using some new interesting observations about nilpotent Lie groups.

THEOREM 3.1.9. *Let G be a nilpotent Lie group of finite type; that is, G/G^0 is finitely generated. Let a continuous action of G on a locally compact space X be given. Suppose that there exists a relatively compact open subset $V \subset X$ with the following properties:* (a) *$\forall x \in X$ and $\forall g \in G$ the trajectory $g^{\mathbb{N}}x$ does not tend to infinity, and* (b) *$GV = X$. Then X is compact.*

3.2. *Sharper nondivergence results*

Now we will state some results which sharpen and generalize the nondivergence results of Dani and Margulis in different directions.

2a. *Quantitative sharpening.* In [117] Kleinbock and Margulis affirmatively resolved the conjectures of Baker and Sprindžuk (see §5.2e and §5.3b) by applying the following sharpened version of Theorem 3.1.2; in order to state the results in greater generality, we need some definitions. The results stated in this subsection appeared in [117].

DEFINITION 3.2.1. Let $V \subset \mathbb{R}^d$ be open, and $f \in \mathrm{C}(V)$. For $C > 0$ and $\alpha > 0$, say that f is (C, α)-*good* on V if for any open ball $B \subset V$ and any $\varepsilon > 0$, one has

$$\frac{1}{|B|}\left|\left\{\mathbf{x} \in B \;\Big|\; \big|f(\mathbf{x})\big| \leqslant \varepsilon \cdot \sup_{\mathbf{x} \in B}\big|f(\mathbf{x})\big|\right\}\right| \leqslant C\varepsilon^{\alpha},$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^d$.

One can show that

PROPOSITION 3.2.2. *For any $k \in \mathbb{N}$, any real polynomial f of degree not greater than k is (C, α)-good for $C = 2k(k+1)^{1/k}$ and $\alpha = 1/k$.*

A version of this result was also noted earlier in [66]. The following can be thought of as a generalization. Let U be an open subset of $\mathbb{R}^d$. Say that an n-tuple $\mathbf{f} = (f_1, \ldots, f_n)$ of C^l functions, where $f_i : U \to \mathbb{R}$, is *nondegenerate at $x \in U$* if the space $\mathbb{R}^n$ is spanned by partial derivatives of $\mathbf{f}$ at x of order up to l.

PROPOSITION 3.2.3. *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a C^l map from an open subset U of $\mathbb{R}^d$ to $\mathbb{R}^n$, and let $x_0 \in U$ be such that $\mathbf{f}$ is nondegenerate at x_0. Then there exists a neighborhood $V \subset U$ of x_0 and $C > 0$ such that any linear combination of $1, f_1, \ldots, f_n$ is $(C, 1/dl)$-good on V.*

Let us say that map $h : V \mapsto \mathrm{GL}(k, \mathbb{R})$ is (C, α)-good if for any linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\} \subset \mathbb{R}^k$ and $\mathbf{w} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j$, the map $x \mapsto \|h(x)\mathbf{w}\|$[14] is (C, α)-good.

THEOREM 3.2.4. *Let $d, k \in \mathbb{N}$, $C, \alpha > 0$, $0 < \rho \leqslant 1$, and let a ball $B = B(x_0, r_0) \subset \mathbb{R}^d$, and a (C, α)-good map $h : B(x_0, 3^k r_0) \mapsto \mathrm{GL}(k, \mathbb{R})$ be given. Then one of the following conditions is satisfied*:

(1) *There exists $x \in B$ and $\mathbf{w} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j$, where $\{\mathbf{v}_i\}_{i=1}^{j} \subset \mathbb{Z}^k$ are linearly independent and $1 \leqslant j \leqslant k$, such that*

$$\big\|h(x)\mathbf{w}\big\| < \rho.$$

[14] Here one fixes a norm on $\mathbb{R}^k$ and extends it to $\wedge^j \mathbb{R}^k$, $1 \leqslant j \leqslant k$.

(2) *For any* $\varepsilon < \rho$,

$$\left|\left\{x \in B\colon\ \delta\big(h(x)\mathbb{Z}^k\big) < \varepsilon\right\}\right| \leqslant kCN_d^k(\varepsilon/\rho)^{\alpha}|B|,$$

where δ *is as in* (1.2), *and* N_d *is a constant depending only on* d.

In view of Mahler's Compactness Criterion (Theorem 1.3.3) the following special case of the above result sharpens Theorem 3.1.2 for $G = \mathrm{SL}(k,\mathbb{R})$ and $\Gamma = \mathrm{SL}(k,\mathbb{Z})$:

THEOREM 3.2.5. *For any lattice* Λ *in* $\mathbb{R}^k$ *there exists* $0 < \rho = \delta(\Lambda) \leqslant 1$ *such that for any one-parameter unipotent subgroup* $u_{\mathbb{R}}$ *of* $\mathrm{SL}(k,\mathbb{R})$, *for any* $\varepsilon \leqslant \rho$ *and any* $T > 0$, *one has*

$$\frac{1}{T}\left|\left\{0 < t < T \mid \delta(u_t\Lambda) < \varepsilon\right\}\right| \leqslant C_k(\varepsilon/\rho)^{1/k^2},$$

where $C_k = 2k^3 2^k (k^2+1)^{1/k^2}$.

2b. *Qualitative strengthening*. In this section we will consider polynomial trajectories on finite volume homogeneous spaces. We will choose a large compact set in the homogeneous space depending only on the degree and the number of variables of polynomial trajectories and a positive $\varepsilon > 0$, so that if the projection of a 'long piece' of a polynomial trajectory on the homogeneous space does not visit the chosen compact set with relative probability at least $1-\varepsilon$, then it satisfies an algebraic condition, which is analogous to possibility (1) of Theorem 3.2.4.

Let G be a real algebraic semisimple group defined over $\mathbb{Q}$, and let Γ be a lattice in G commensurable with $G(\mathbb{Z})$. Let r denote the $\mathbb{Q}$-rank of G (as was mentioned in §1.2b, G/Γ is noncompact if and only if $r > 0$). Fix a minimal parabolic subgroup P of G defined over $\mathbb{Q}$. There are exactly r distinct proper maximal parabolic $\mathbb{Q}$-subgroups of G containing P, say $P_1, \ldots, P_r$.

Take any $1 \leqslant i \leqslant r$. Let U_i denote the unipotent radical of P_i. Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{U}_i = \mathrm{Lie}(U_i)$, and $d_i = \dim U_i$. Now $\mathfrak{g}$ has a $\mathbb{Q}$-structure, so that $\mathfrak{g}(\mathbb{Q})$ is $\mathrm{Ad}(G(\mathbb{Q}))$-invariant, and the Lie subalgebra associated to any $\mathbb{Q}$-subgroup of G is a $\mathbb{Q}$-subspace. Since U_i is defined over $\mathbb{Q}$, one can choose a nonzero $\mathbf{q}_i \in (\wedge^{d_i}\mathfrak{U}_i) \cap (\wedge^{d_i}\mathfrak{g}(\mathbb{Q}))$. Consider the action of G on $\wedge^{d_i}\mathfrak{g}$ given by the $\wedge^{d_i}\mathrm{Ad}_G$-representation. Then for any $a \in P_i$, one has $a \cdot \mathbf{q}_i = \det(a|_{\mathfrak{U}_i})\mathbf{q}_i$.

Let K be a maximal compact subgroup of G such that $G = KP$. Fix a K-invariant norm $\|\cdot\|$ on $\wedge^{d_i}\mathfrak{g}$. Now for any $g \in G$, write $g = ka$, where $k \in K$ and $a \in P$. Then $\|g \cdot \mathbf{q}_i\| = |\det(a|_{\mathfrak{U}_i})| \cdot \|\mathbf{q}_i\|$.

By [26, Theorem 15.6], there exists a finite set $F \subset G(\mathbb{Q})$ such that

$$G(\mathbb{Q}) = \Gamma \cdot F \cdot P(\mathbb{Q}).$$

DEFINITION 3.2.6. For $d, m \in \mathbb{N}$, let $\mathcal{P}_{d,m}(G)$ denote the set of continuous maps $\Theta : \mathbb{R}^m \mapsto G$ with the following property: for all $c, a \in \mathbb{R}^m$ and $X \in \mathfrak{g}$, the map

$$\mathbb{R} \ni t \mapsto \mathrm{Ad}\big(\Theta(tc+a)\big)(X) \in \mathfrak{g}$$

is a polynomial of degree at most d in each coordinate, with respect to any basis of $\mathfrak{g}$.

THEOREM 3.2.7. *For any $d, m \in \mathbb{N}$, $\nu > 0$ and $\varepsilon > 0$ there exists a compact set $\mathcal{C} \subset G/\Gamma$ such that for any bounded open convex set $B \subset \mathbb{R}^m$ and any $\Theta \in \mathcal{P}_{d,m}(G)$, one of the following conditions is satisfied*:

(1) $\frac{1}{|B|}\left|\left\{\mathbf{x} \in B\colon \pi\left(\Theta(\mathbf{x})\right) \in \mathcal{C}\right\}\right| < 1 - \varepsilon.$

(2) *There exists $i \in \{1, \ldots, r\}$ and $g \in \Gamma F$ such that*

$$\left\|\Theta(\mathbf{x})g \cdot \mathbf{q}_i\right\| < \nu, \quad \forall \mathbf{x} \in B.$$

Note that a similar conclusion can be obtained for any Lie group G and a lattice Γ in G, see [203, Theorem 2.2]. Note also that in Theorem 3.2.7, we have not related ε and the compact set $\mathcal{C}$ as in Theorem 3.2.4. The methods in [117] are general enough to provide sharp relations between ε and $\mathcal{C}$ and for (C, α)-good maps in place of polynomial maps as above. For this purpose one can use the algebraic description of compact subsets of G/Γ as given by [64, Proposition 1.8].

Finally, one can observe that the Γ-orbit of any element of $\wedge^{d_i}\mathfrak{g}(\mathbb{Q})$ is contained in $\frac{1}{k}\wedge^{d_i}\mathfrak{g}(\mathbb{Z})$ for some $k \in \mathbb{N}$. Therefore the set $\{g \cdot \mathbf{q}_i \mid g \in \Gamma F\}$ is discrete. As a consequence of this remark one deduces the following:

COROLLARY 3.2.8. *Given any $d, m \in \mathbb{N}$ and $\varepsilon > 0$ there exists a compact set $\mathcal{C} \subset G/\Gamma$ such that the following holds*: *for any $\Theta \in \mathcal{P}(d, m)$ there exists a $T_0 > 0$ such that one of the following conditions is satisfied*:

(1) *For any open convex set B containing $B(0, T_0)$ in $\mathbb{R}^m$*,

$$\frac{1}{|B|}\left|\left\{\mathbf{x} \in B\colon \pi\left(\Theta(\mathbf{x})\right) \in \mathcal{C}\right\}\right| < 1 - \varepsilon;$$

(2) *There exists $i \in \{1, \ldots, r\}$ and $g \in \Gamma F$ such that*

$$\Theta(0)^{-1}\Theta\left(\mathbb{R}^m\right) \subset g^{-1}\left({}^0P_i\right)g,$$

where ${}^0P_i = \{a \in P_i \mid \det(\mathrm{Ad}\, a|_{\mathfrak{U}_i}) = 1\}$.

3.3. *Orbit closures, invariant measures and equidistribution*

In this section we will describe the fundamental results on unipotent flows on homogeneous spaces of Lie groups.

3a. *Horospherical flows and conjectures of Raghunathan, Dani and Margulis.* The most naturally occurring unipotent subgroup associated to geometric constructions is the horospherical subgroup. For a (contracting or expanding) horospherical subgroup U of a Lie group G, the action of U on a homogeneous space of G is called a *horospherical flow*. In fact, orbits of a horospherical flow on a homogeneous space of a Lie group are precisely the leaves of the strongly stable foliation of a partially hyperbolic action of an element of this group.

We recall Hedlund's theorem [95] that any nonperiodic orbit of a horocycle flow on a finite volume homogeneous space of $\mathrm{SL}(2,\mathbb{R})$ is dense. In [84], Furstenberg showed that the horocycle flow on a compact homogeneous space of $\mathrm{SL}(2,\mathbb{R})$ is uniquely ergodic, and hence every orbit is equidistributed with respect to the $\mathrm{SL}(2,\mathbb{R})$-invariant measure. The result was generalized by Veech [247] and Bowen [33] for ergodic actions of horospherical subgroups of semisimple groups G acting on G/Γ, where Γ is a uniform lattice in G.

The first results in this direction for the non-uniform lattice case were obtained by Dani, who classified the ergodic invariant measures for actions of horospherical subgroups on noncompact finite volume homogeneous spaces of semisimple Lie groups [44,47].

As a very general approach to resolve the Oppenheim conjecture on values of quadratic forms at integral points (see §5.1a), in the late 1970's Raghunathan formulated the following conjecture (in oral communication), which was also motivated by the earlier works of Margulis and Dani:

CONJECTURE 3.3.1. *The closure of any orbit of a unipotent flow on a finite volume quotient space of a Lie group is* homogeneous*, that is, it is itself an orbit of a* (*possibly larger*) *Lie subgroup*.

In [56], Dani showed the validity of Raghunathan's conjecture for horospherical flows in the semisimple case. Using methods of [56] Starkov [228] gave an explicit formula for orbit closures of a horospherical flow on compact homogeneous space of arbitrary Lie group. It may be noted that Raghunathan's conjecture was generalized by Margulis [136], with 'unipotent flow' replaced by an action of a subgroup W generated by one-parameter unipotent subgroups. Namely, it was conjectured that for any $x \in G/\Gamma$ there exists a closed subgroup F of G (containing W) such that $\overline{Wx} = Fx$, and the closed orbit Fx supports a finite F-invariant measure.

Using the results of Auslander and Green (see [9] or §2.2a) about nilflows, in [222] the latter statement was proved for solvable Lie groups. But as far as homogeneous spaces of semisimple Lie groups are concerned, until 1986 Conjecture 3.3.1 was proved only for actions of horospherical subgroups. Since these are geometrically defined subgroups, various techniques from the theory of unitary representations, geometry, and Markov partitions could be used to study their properties.

A major breakthrough came from Margulis' work on Oppenheim conjecture. In [138] he proved the orbit closure conjecture in the case when $G = \mathrm{SL}(3,\mathbb{R})$, $W = \mathrm{SO}(2,1)$, and the closure of the given orbit of W is compact; the case of a noncompact orbit closure for this action, as well as for the action of a unipotent subgroup of $\mathrm{SO}(2,1)$, was resolved by Dani and Margulis in [61,62]. Note that $\mathrm{SO}(2,1)$ does not contain any horospherical subgroup of $\mathrm{SL}(3,\mathbb{R})$.

The methods of Dani and Margulis were exploited to prove the generalized Raghunathan conjecture for the case of $G = \mathrm{SO}(n,1)$, see [201, Remark 7.4].

3b. *Ergodic invariant measures for unipotent flows*. The aesthetically most pleasing, deep and fundamental result about dynamics of subgroup actions on homogeneous spaces of Lie groups is the algebraic classification of finite ergodic invariant measures of unipotent flows due to Ratner [182]; this result also provides the crucial ingredient for the proof of the aforementioned conjectures on orbit closures.

Unless changed later, we fix a Lie group G, a discrete subgroup Γ of G, and the natural quotient map $\pi : G \mapsto G/\Gamma$.

Now we state the description of finite ergodic invariant measures for unipotent flows on homogeneous spaces due to Ratner [186]:

THEOREM 3.3.2 (Ratner). *Let U be a one-parameter unipotent subgroup of G. Then any finite U-invariant U-ergodic Borel measure, say μ, on G/Γ is a* homogeneous measure; *that is, if one defines*

$$F = \{g \in G \mid \textit{the left action of } g \textit{ on } G/\Gamma \textit{ preserves } \mu\}, \tag{3.1}$$

then F acts transitively on $\operatorname{supp}(\mu)$.

We note that the subgroup F of G defined by (3.1) is closed, and hence it is a Lie subgroup of G.

This theorem was proved through a series of papers: [184] contains the proof in the case when G is solvable, [183] contains the proof when G is semisimple and G/Γ is compact, and [186] completes the proof in the general case. It may be noted that the measure classification for unipotent flows on solvmanifolds can also be deduced from the results of Starkov [224].

Ratner's proof of this theorem does not use any major results apart from the Birkhoff's ergodic theorem, but it is a very long and involved proof. One of the basic observations about polynomial divergence of unipotent orbits, known as *R-property*, plays an important role in Ratner's proof.

R-property. Consider a right G-invariant Riemannian metric d_G on G. Given $\varepsilon > 0$, there exists $\eta > 0$, such that if $\mathcal{S}$ is a large rectangular box (cf. Corollary 3.6.8) in a closed connected simply connected unipotent subgroup U of G with $e \in \mathcal{S}$ and $\sup_{u \in \mathcal{S}} d_G(u, gU) = \theta$ for some small $\theta > 0$ and some $g \notin U$, $d_G(e, g) \leqslant \theta$, then there exists another rectangular box $\mathcal{A} \subset \mathcal{S}$ such that $(1-\varepsilon)\theta \leqslant d_G(u, gU) \leqslant \theta$, $\forall u \in \mathcal{A}$, and $\lambda(\mathcal{A}) \geqslant \eta\lambda(\mathcal{S})$, where λ denotes the Haar measure on U. Moreover, if $u \in \mathcal{S}$ and $d_G(u, gU) = d_G(u, ur(u))$ for some $r(u) \in G$ with $ur(u) \in gU$ and $d_G(e, r(u)) \leqslant \theta$ then $r(u)$ is close to the normalizer of U in G, and this closeness tends to zero as the sides of the rectangular box $\mathcal{S}$ tend to infinity. (If $U = u_{\mathbb{R}}$, then $\mathcal{S} = u_{[0,T]}$ for a large $T > 0$.)

Some versions of R-property for the horocycle flows, like the H-property, were observed and used in the earlier works of Ratner on rigidity of equivariant maps of horocycle flows [178] and ergodic joinings [179,180] for the diagonal embedding of a horocycle subgroup of $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathrm{SL}(2,\mathbb{R})/\Gamma_1 \times \cdots \times \mathrm{SL}(2,\mathbb{R})/\Gamma_m$, where $\Gamma_1, \ldots, \Gamma_m$ are lattices in $\mathrm{SL}(2,\mathbb{R})$. The H-property was generalized and used by Witte [254] for extending Ratner's result on equivariant maps to actions of unipotent elements on homogeneous spaces of Lie groups.

REMARK 3.3.3. Let N be a connected nilpotent group, and μ a finite N-invariant and N-ergodic measure on an N-space. Then there exists a one-parameter subgroup V of N such that V acts ergodically with respect to μ.

REMARK 3.3.4. Note that if Γ is a lattice in G, and U is a connected unipotent subgroup of G, then by Corollary 3.1.4 any locally finite U-invariant U-ergodic measure on G/Γ is finite. Hence by Theorem 3.3.2 it is a homogeneous measure.

From Proposition 3.1.5 and Remark 3.3.3 one concludes the following:

COROLLARY 3.3.5. *Let W be a subgroup of G generated by one-parameter unipotent subgroups. Then any finite W-invariant W-ergodic measure on G/Γ is ergodic with respect to a one-parameter unipotent subgroup contained in W, and hence it is a homogeneous measure. In particular, if Γ is a lattice in G, then any locally finite W-ergodic measure on G/Γ is finite, and hence is a homogeneous measure.*

We would like to note that motivated by a question of A. Borel, in [147] Margulis and Tomanov gave another (shorter and simpler) proof of the analog of Theorem 3.3.2 for the case when G is a finite product of linear Lie groups; see also [148] for the general case. The p-adic and S-arithmetic analogues of Theorem 3.3.2 have also been obtained; see [190,191] and [146–148,244].

3c. *Closures of unipotent trajectories.* For orbit closures of unipotent flows on finite volume homogeneous spaces, Ratner [187] proved the following, which in particular settled the conjectures of Raghunathan and Margulis (see §3.3a) in affirmation:

THEOREM 3.3.6. *Let G be a Lie group and Γ a lattice in G. Let W be a subgroup of G generated by one-parameter unipotent subgroups. Then for any $x \in G/\Gamma$ there exists a closed subgroup F of G (containing W) such that $\overline{Wx} = Fx$, and the closed orbit Fx supports a finite F-invariant measure.*

The approach to the proof of this result using Ratner's measure classification is discussed in the next section.

In the arithmetic situation one has the following more concrete description of orbit closures (see [201, Proposition 3.2]):

PROPOSITION 3.3.7. *Let G be a real algebraic group defined over $\mathbb{Q}$ and with no nontrivial $\mathbb{Q}$-characters, and let x_0 denote the coset of identity in $G/G(\mathbb{Z})$. Let W be a subgroup of G generated by one-parameter unipotent subgroups. For any $g \in G$, let H be the smallest real algebraic $\mathbb{Q}$-subgroup of G containing $g^{-1}Wg$. Then $\overline{Wg\Gamma} = gH\Gamma$. Moreover the radical of H is unipotent.*

In view of Theorem 1.3.2, the closures of orbits of subgroups generated by unipotent one-parameter subgroups acting on G/Γ, where G is a semisimple group and Γ is an irreducible lattice in G, can be described more precisely using the above proposition and the following observation (see [204, Lemma 7.3]).

LEMMA 3.3.8. *Let G be a Lie group, and Γ_i be a closed subgroup such that G/Γ_i has a finite G-invariant measure, where $i = 1, 2$. Suppose that $\Gamma_1 \cap \Gamma_2$ contains a subgroup of finite index which is cocompact and normal in Γ_i for $i = 1, 2$. Let x_i be the coset of the identity in G/Γ_i for $i = 1, 2$. Then for any closed subgroup H of G and any $g \in G$:*

$$Hgx_1 \text{ is closed } \Leftrightarrow Hgx_2 \text{ is closed.}$$

In particular, the above applies to any two commensurable lattices Γ_1 and Γ_2 in G.

Closures of totally geodesic immersions of symmetric spaces. A weaker form of Ratner's orbit closure theorem has an interesting formulation in the Riemannian geometric set-up (see Payne [172]):

THEOREM 3.3.9. *Let $\varphi: M_1 \mapsto M_2$ be a totally geodesic immersion, where M_i is a connected locally symmetric space of noncompact type for $i = 1, 2$. Suppose that M_2 has finite Riemannian volume. Then $\overline{\varphi(M_1)}$ is an immersed submanifold of M_2. Further, if* $\mathrm{rank}(M_1) = \mathrm{rank}(M_2)$, *then $\overline{\varphi(M_1)}$ is a totally geodesic immersed submanifold of M_2.*

Such a geometric reformulation of Ratner's theorem in the case of $\mathrm{rank}\, M_1 = \mathrm{rank}\, M_2 = 1$ was suggested by Ghys in oral communication (see [200] or [88]).

3d. *Limiting distributions of unipotent trajectories.* The orbit closure theorem was proved by Ratner [187] by obtaining the following stronger result, which in turn uses her classification of ergodic invariant measures:

THEOREM 3.3.10. *Let G be a Lie group and Γ a lattice in G. Let $U = u_{\mathbb{R}}$ be a one-parameter unipotent subgroup of G. Then for any $x \in G/\Gamma$ there exists a closed subgroup of F of G (containing U) such that the orbit Fx is closed and the trajectory $u_{\mathbb{R}_+}x$ is uniformly distributed with respect to a (unique) F-invariant probability measure, say μ, supported on Fx. In other words, for any bounded continuous function f on G/Γ,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t x)\,\mathrm{d}t = \int_{Fx} f\,\mathrm{d}\mu.$$

The result for $G = \mathrm{SL}(2,\mathbb{R})$ and Γ a uniform lattice is an immediate consequence of the unique ergodicity of the horocycle flow due to Furstenberg [84]. For $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ the result was proved by Dani [49], and for any non-uniform lattice it was proved by Dani and Smillie [68]. The statement of Theorem 3.3.10 was roughly conjectured in [68,136].

Using Ratner's classification of ergodic invariant measures, in [201] the above equidistribution result was obtained in the case when G is a semisimple group of real rank one; and also in the case when G is a reductive group, U is a generic one-parameter unipotent subgroup of G, and Γ is a uniform lattice in G.

Later Theorem 3.3.10 was extended to the infinite volume case by Dani and Margulis [66] as follows:

THEOREM 3.3.11. *Let G be a Lie group and Γ a discrete subgroup of G. Let $U = u_{\mathbb{R}}$ be a one-parameter unipotent subgroup of G and $x \in G/\Gamma$. Suppose that the following condition is satisfied*: *given any $\varepsilon > 0$ there exists a compact set $K \subset G/\Gamma$ such that*

$$\frac{1}{T}\left|\left\{t \in [0,T] \mid u_t x \in K\right\}\right| \geqslant 1-\varepsilon \quad \forall T > 0. \tag{3.2}$$

Then there exists a closed subgroup F of G such that the orbit Fx is closed and the trajectory $u_{\mathbb{R}_+}x$ is uniformly distributed with respect to a (*unique*) *F-invariant probability measure on Fx.*

Note that condition (3.2) is satisfied if one knows that the orbit Ux is relatively compact.

3e. *An approach to proofs of the equidistribution results.* Let μ_T denote the push-forward of the normalized Lebesgue measure of $[0,T]$ onto the trajectory $u_{[0,T]}x$. In the case of Theorem 3.3.10, the condition (3.2) is satisfied due to Theorem 3.1.2. Therefore, in the setting of Theorems 3.3.10 and 3.3.11, there exists a subsequence $T_i \to \infty$ such that μ_{T_i} converges to a probability measure, say λ, with respect to the weak-$*$ topology on the space of probability measures on G/Γ. It is straightforward to verify that λ is $u_{\mathbb{R}}$-invariant.

By Theorem 3.3.2, one can have a description of the ergodic components of λ. Further analysis of this information, which is carried out in the next section, will show that λ is either G-invariant, or it is strictly positive on the image of a 'strictly lower dimensional algebraic or analytic subvariety of G' in G/Γ. If λ is G-invariant the proof is completed. Otherwise, from the polynomial behavior of the unipotent trajectories, one concludes that the entire trajectory must lie on one of those lower dimensional subvarieties. From this fact and Theorem 3.1.7, one can reduce the problem to a strictly lower dimensional homogeneous space, and use the induction argument to complete the proof (see §3.4e).

3.4. *Techniques for using Ratner's measure theorem*

In this section we shall describe some concepts and techniques that can be applied to study a large class of problems, like Theorem 3.3.10, where one needs to analyze the limiting distributions that are invariant under unipotent subgroups.

4a. *Finite invariant measures for unipotent flows.* The main consequence of Ratner's classification of ergodic invariant measures is that the non-G-invariant ergodic components of a $u_{\mathbb{R}}$-invariant measure on G/Γ are concentrated on the image of a countable union of lower dimensional 'algebraic subvarieties' of G; this is explained below.

Let G be a Lie group, Γ a discrete subgroup of G, and let $\pi : G \mapsto G/\Gamma$ be the natural quotient map. Let $\mathcal{H}_\Gamma$ denote the collection of all closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H, and the subgroup generated by the one-parameter unipotent subgroups of G contained in H acts ergodically on $H/H \cap \Gamma$ with respect to the H-invariant probability measure. In view of Theorem 3.1.7, $\mathrm{Ad}_G(H)$ is contained in the Zariski closure of $\mathrm{Ad}_G(H \cap \Gamma)$ for each $H \in \mathcal{H}_\Gamma$.

The following fact was observed by Ratner [186, Theorem 1.1] (cf. [201, Lemma 5.2]):

THEOREM 3.4.1. *The collection $\mathcal{H}_\Gamma$ is countable.*

For a simpler proof of this result see [66, Proposition 2.1]. In the arithmetic situation, the proof follows easily:

REMARK 3.4.2. Suppose that G be a real algebraic group defined over $\mathbb{Q}$ and $\Gamma = G(\mathbb{Z})$. Then any $H \in \mathcal{H}_\Gamma$ is a real algebraic group defined over $\mathbb{Q}$ (cf. [201, Proposition 3.2]). Therefore $\mathcal{H}_\Gamma$ is countable in this case.

Let W be a subgroup of G which is generated by one-parameter unipotent subgroups. For any $H \in \mathcal{H}$, one defines:

$$N(W,H) \stackrel{\text{def}}{=} \{g \in G \mid W \subset gHg^{-1}\};$$

$$S(W,H) \stackrel{\text{def}}{=} \bigcup_{F \in \mathcal{H}_\Gamma,\ F \subset H,\ \dim F < \dim H} N(W,F);$$

$$N^*(W,H) \stackrel{\text{def}}{=} N(W,H) \setminus S(W,H).$$

REMARK 3.4.3. Let $\mathfrak{g}$, $\mathfrak{H}$ and $\mathfrak{W}$ denote the Lie algebras of G, H and $\overline{W}$, respectively. Let $d = \dim \mathfrak{H}$, and consider the $\wedge^d$ Ad-action of G on $\wedge^d \mathfrak{g}$. Let $\mathbf{p}_H \in \wedge^d \mathfrak{H} \setminus \{0\}$. Then

$$N(W,H) = \{g \in G \mid X \wedge (g \cdot \mathbf{p}_H) = 0 \in \wedge^{d+1}\mathfrak{g}\ \forall X \in \mathfrak{W}\}.$$

In particular, if G is a real algebraic group, then $N(W,H)$ is an algebraic subvariety of G.

LEMMA 3.4.4 [162, Lemma 2.4]. *For any $g \in N^*(W,H)$, if F is any closed subgroup of G such that $W \subset F$ and $F\pi(g)$ is closed, then $gHg^{-1} \subset F$. In particular,*

$$\pi\big(N^*(W,H)\big) = \pi\big(N(W,H)\big) \setminus \pi\big(S(W,H)\big). \tag{3.3}$$

Using Ratner's description (Corollary 3.3.5) of finite W-invariant W-ergodic Borel measures on G/Γ, Theorem 3.4.1, and the ergodic decomposition of invariant measures ([136, Section 1.2] or [3, Theorem 4.2.4]) one can reformulate Ratner's theorem to describe finite W-invariant measures as follows:

THEOREM 3.4.5. *Let μ be a finite W-invariant measure on G/Γ. Then there exists $H \in \mathcal{H}$ such that*

$$\mu\big(\pi\big(N(W,H)\big)\big) > 0, \quad \textit{and} \quad \mu\big(\pi\big(S(W,H)\big)\big) = 0. \tag{3.4}$$

Moreover, almost every W-ergodic component of the restriction of the measure μ to $\pi(N(W,H))$ is concentrated on $g\pi(H)$ for some $g \in N^*(W,H)$, and it is invariant under gHg^{-1}.

See [162, Theorem 2.2] or [57, Corollary 5.6] for its deduction.

COROLLARY 3.4.6. *If μ is a finite W-invariant measure on G/Γ, $H \in \mathcal{H}$ is normal in G, and $\mu(\pi(S(W, H))) = 0$, then μ is H-invariant. In particular, if $\mu(\pi(S(W, G))) = 0$ then μ is G-invariant.*

4b. *Self-intersections of $N(W, H)$ under $\pi : G \mapsto G/\Gamma$.* The theme of this subsection is that the self-intersection of $N(W, H)$ under π occurs only along $S(W, H)$. Since μ vanishes on this set in view of (3.4), one can, in some sense, lift the restriction of μ on $\pi(N(W, H))$ back to $N(W, H)$ and analyze it there.

Since in the applications one obtains W-invariant measures as limiting distributions of sequences of algebraically defined measures, one would like to understand the geometry of thin neighborhoods of compact subsets of $\pi(N^*(W, H))$ and the behavior of these limiting sequence of measures in those neighborhoods. As we remarked in the above paragraph, the lifting procedure helps in this regard. This is done in the next subsection.

The geometry of thin neighborhoods of $\pi(N^*(S(W, H))$ is understood via the the following facts:

PROPOSITION 3.4.7. *For $H_1, H_2 \in \mathcal{H}$ and $\gamma \in \mathrm{Comm}_G(\Gamma)$, suppose that*

$$N^*(W, H_1)\gamma \cap N(W, H_2) \neq \emptyset.$$

Then $H_1 \subset \gamma H_2 \gamma^{-1}$. In particular, if $H_1 = H_2$ then $\gamma \in N_G(H_1)$.

We note that:

$$N(W, H) = N_G(W) N(W, H) N_G(H), \tag{3.5}$$

$$N(W, H)\gamma = N(\gamma^{-1} H \gamma, W) \quad \forall \gamma \in \mathrm{Comm}_G(\Gamma), \tag{3.6}$$

$$N^*(W, H) = N_G(W) N^*(W, H) \big(N_G(H) \cap \Gamma\big). \tag{3.7}$$

The injectivity property in the following fact is a useful technical observation.

COROLLARY 3.4.8. *The natural map*

$$N^*(W, H)/N_G(H) \cap \Gamma \mapsto G/\Gamma$$

is injective.

4c. *A method of analyzing W-invariant measures.* The combination of Theorem 3.4.5, Corollary 3.4.8 and Theorem 1.3.7 provides a very useful method for studying a finite measure on a homogeneous space of a Lie group which is invariant and ergodic for the action of a connected subgroup containing a 'nontrivial' unipotent one-parameter subgroup. The technique can be illustrated as follows:

We are given: (a) A real algebraic subgroup G of $\mathrm{GL}(n, \mathbb{R})$ and a discrete subgroup Γ of G, (b) a closed connected subgroup F of G, and (c) an F-invariant F-ergodic Borel probability measure μ on G/Γ.

Let W be the subgroup generated by all algebraic unipotent one-parameter subgroups contained in F. Let $H \supset W$ be as in Theorem 3.4.5. Let μ^* denote the restriction of μ to $\pi(N(W, H))$. Since μ is F-invariant, $\lambda_F(\{g \in F \mid gx \in \pi(N(H, W))\}) > 0$ for μ^*-a.e. x, where λ_F denotes a Haar measure on F. Since F is an analytic subgroup of G, Γ is discrete, and $N(H, W)$ is an analytic subvariety of G, we have that $Fx \subset \pi(N(H, W))$ for μ^*-a.e. x (see [204, Lemma 5.3]). Thus μ^* is F-invariant, and hence by the ergodicity of μ, $\mu = \mu^*$.

Replacing Γ by a suitable conjugate subgroup, we may assume that

$$e \in N^*(W, H) \quad \text{and} \quad \pi(e) \in \mathrm{supp}(\mu).$$

By Corollary 3.4.8, we can lift μ to $N^*(W, H)/N_G(H) \cap \Gamma$, and call it $\tilde{\mu}$. Let $L = \mathrm{Zcl}(N_G(H) \cap \Gamma)$. We project the measure $\tilde{\mu}$ onto $N^*(H, W)/L \subset G/L$, and call it ν. In the notation of Theorem 1.3.7, G_ν and J_ν are algebraic subgroups of G, J_ν is normal in G_ν, and G_ν/J_ν is compact. Since $\pi(e) \in \mathrm{supp}(\mu)$, $J_\nu \subset L$. Clearly $F \subset G_\nu$, and since F acts ergodically with respect to ν, we have that $G_\nu = \overline{F J_\nu}$. Also $W \subset J_\nu$.

Suppose we are also given: (d) $\mathrm{Zcl}(F)$ is generated by $\mathbb{R}$-diagonalizable and algebraic unipotent subgroups.

Then $F \subset J_\nu \subset L$. Therefore $\mu(\pi(L)) = 1$. By our choice of H, we have that $\mu(\pi(S(W, H))) = 0$. Therefore by Corollary 3.4.6 applied to L in place of G, we have that μ is H-invariant.

We conclude: (1) μ is concentrated on $L/L \cap \Gamma$, (2) $F \subset L$, (3) there exists a closed connected normal subgroup H of L containing all algebraic unipotent subgroups of F, such that μ is H-invariant, and (4) $H \cap \Gamma$ is a lattice in H. Note that all W-ergodic components of μ are H-invariant and their supports are closed H-orbits.

The above method provides group theoretic restrictions on μ if we know that $W \neq \{e\}$. In other words, by imposing various algebraic conditions on the subgroup F one can obtain more information about μ. See [206, Proof of Theorem 7.2] for a refined version of the above argument. The results in [148] providing algebraic information about invariant measures for actions of connected subgroups also employ this method. Again the same method is applied in [243]. This question is further discussed in §4.4b.

4d. *Linear presentations*. As mentioned in previous subsections, the dynamics of unipotent or polynomial trajectories in thin neighborhood of $\pi(N^*(W, H))$ are studied via lifting them to G and then studying them via suitable linear representations of G on vector spaces.

Let $V = \bigoplus_{k=1}^{\dim \mathfrak{g}} \wedge^k \mathfrak{G}$ be the direct sum of exterior powers of $\mathfrak{g}$, and consider the linear action of G on V via the direct sum of the exterior powers of the adjoint representation. Fix any norm on V.

For any nontrivial connected Lie subgroup H of G, and its Lie algebra $\mathfrak{H}$, let us choose a nonzero vector $\mathbf{p}_H$ in the one-dimensional subspace $\wedge^{\dim \mathfrak{H}} \mathfrak{H} \subset V$.

For $H \in \mathcal{H}_\Gamma$, let $V(W,H)$ denote the linear span of the set $N(W,H)\cdot \mathbf{p}_H$ in V. We note that (cf. Remark 3.4.3, see [162])

$$N(W,H) = \{g \in G \mid g\cdot \mathbf{p}_H \in V(W,H)\}, \quad \text{and} \tag{3.8}$$

$$N^1_G(H) \stackrel{\text{def}}{=} \{g \in N_G(H) \mid \det(\operatorname{Ad} g|_{\mathfrak{H}}) = 1\}$$
$$= \{g \in G \mid g\cdot \mathbf{p}_H = \mathbf{p}_H\}. \tag{3.9}$$

REMARK 3.4.9. Due to Corollary 3.4.8, for any $g \in N^*(W,H)$ and $\gamma_i \in \Gamma$ $(i = 1,2)$, if $g\gamma_i \cdot \mathbf{p}_H \in V(W,H)$, then $g\gamma_1\cdot \mathbf{p}_H = \pm g\gamma_2 \cdot \mathbf{p}_H$.

One of very important facts, due to which the methods work, is the following generalization of Theorem 3.4.1 due to Dani and Margulis [66, Theorem 3.4]:

THEOREM 3.4.10. *For $H \in \mathcal{H}_\Gamma$, the orbit $\Gamma\cdot \mathbf{p}_H$ is discrete. In particular, $N^1_G(H)\Gamma$ is closed in G/Γ.*

Note that, as in Remark 3.4.2 for the case of an arithmetic lattice, this result is a consequence of the fact that for an algebraic linear representation of the ambient group defined over $\mathbb{Q}$, the orbit any rational point under the lattice consists of rational points with bounded denominators, and hence it is a discrete subset.

Combining Remark 3.4.9 and Theorem 3.4.10 one obtains the following useful fact:

COROLLARY 3.4.11. *Given a compact subset K of G/Γ and a compact set $D \subset V(W,H)$, there exists a neighborhood Φ of D in V such that for any $g\Gamma \in K$, and $\gamma_1,\gamma_2 \in \Gamma$, if $g\gamma_i \cdot \mathbf{p}_H \in \Phi$ $(i = 1,2)$ then $\gamma_1 \cdot \mathbf{p}_H = \pm\gamma_2 \cdot \mathbf{p}_H$.*

Combining Corollary 3.4.11 and the growth properties of polynomial trajectories (cf. Proposition 3.2.2), one obtains the next result. It is one of the basic technical tools used for applying Ratner's measure classification to the study of limiting distributions of algebraically defined sequences of measures (see [57,66,162,201,202]).

PROPOSITION 3.4.12. *Let $H \in \mathcal{H}$, $d,m \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then for any compact set $C \subset \pi(N^*(W,H))$ there exists a compact set $D \subset V(W,H)$ with the following property: for any neighborhood Φ of D in V, there exists a neighborhood Ψ of C in G/Γ, such that for any $\Theta \in \mathcal{P}_{d,m}(G)$ (see Definition 3.2.6), and any bounded open convex set $B \subset \mathbb{R}^m$, one of the following holds:*

(1) *$\Theta(B)\gamma\cdot \mathbf{p}_H \subset \Phi$ for some $\gamma \in \Gamma$;*
(2) *$\frac{1}{|B|}|\{\mathbf{t} \in B \mid \pi(\Theta(\mathbf{t})) \in \Psi\}| < \varepsilon$.*

This proposition is an analogue of Theorem 3.2.7, where the point at infinity plays the role of the singular set $\pi(S(W,H))$.

Using Proposition 3.4.12 one obtains the following:

THEOREM 3.4.13. *Let G be a connected Lie group and let Γ be a discrete subgroup of G. Let W be any closed connected subgroup of G which is generated by unipotent elements contained in it. Let F be a compact subset of $G/\Gamma \setminus \pi(S(W,G))$. Then for any $\varepsilon > 0$ and natural numbers m and d, there exists a neighborhood Ω of $\pi(S(W,G))$ such that for any $\Theta \in P_{m,d}(G)$, any $x \in F$, and a ball B in $\mathbb{R}^m$ centered at* 0, *one has*

$$\left|\left\{\mathbf{t} \in B \mid \Theta(\mathbf{t})x \in \Omega\right\}\right| \leqslant \varepsilon \cdot |B|.$$

It may be noted that proofs of both the above results do not involve Ratner's measure classification.

4e. *Proof of Theorems 3.3.10 and 3.3.11.* If $Ux \subset Fx$ for a closed subgroup F of G with Fx having a finite F-invariant measure and $\dim F < \dim G$, then one can appeal to the induction argument. Therefore one can assume that $x \notin \pi(S(U,G))$.

Consider the measure λ as in §3.3e. By Theorem 3.4.13, $\lambda(\pi(S(U,G))) = 0$. Therefore, since λ is U-invariant, by Corollary 3.4.6, λ is G-invariant. This completes the proof. □

The above method can be applied in the proofs of many of the results stated below.[15] In most applications, the main part is to show that the first alternative in Proposition 3.4.12 corresponds to a certain algebraic group theoretic or rationality condition on the group actions under consideration.

3.5. *Actions of subgroups generated by unipotent elements*

The descriptions of invariant measures and orbit closures for actions of (possibly discrete) subgroups generated by unipotent elements on homogeneous spaces of Lie groups are very similar to those for the actions of subgroups generated by unipotent one-parameter subgroups [204].

THEOREM 3.5.1. *Let G be a Lie group and Γ a closed subgroup of G. Let W be a subgroup of G with the following property: there exists a subset $S \subset W$ such that S consists of unipotent elements and $\mathrm{Ad}_G(W) \subset \mathrm{Zcl}(\mathrm{Ad}_G(\langle S\rangle))$. Then any finite W-invariant W-ergodic measure on G/Γ is a homogeneous measure.*

Note that if W is a subgroup of G such that W is generated by unipotent elements contained in it, then W satisfies the condition of Theorem 3.5.1.

THEOREM 3.5.2. *Let the notation be as in Theorem* 3.5.1. *Suppose that G/Γ admits a finite G-invariant measure. Then for any $x \in G/\Gamma$ there exists a closed subgroup F of G containing W such that $\overline{Wx} = Fx$, and F^0x admits a finite F^0-invariant measure* (*see also Corollary* 3.5.4 *below*).

[15] See also [244] for applications in the S-arithmetic case.

It may be noted that Theorem 3.5.1 and Theorem 3.5.2 were proved by Ratner [186,187] in the following special case: G is connected, W is of the form $W = \bigcup_{i=1}^{\infty} w_i W^0$, where w_i is unipotent, $i = 1, 2, \ldots,$ W/W^0 is finitely generated, and W^0 is generated by one-parameter unipotent subgroups. In the case when G is not connected and W is a nilpotent unipotent subgroup of G, Theorem 3.5.1 was proved by Witte [257, Theorem 1.2].

The proofs of the above theorems use Ratner's classification of finite ergodic measures and orbit closures for actions of unipotent one-parameter subgroups, and some additional results like Proposition 3.4.7 and Theorem 1.3.6. The proof of Theorem 3.5.2 also involves a version of Theorem 3.7.1 given in Remark 3.7.3.

The following result generalizes Corollary 3.1.6 and leads to an improved conclusion of Theorem 3.5.2.

THEOREM 3.5.3. *Let G, Γ, and W be as in Theorem* 3.5.1. *Suppose that G/Γ admits a finite G-invariant measure. Then any locally finite W-invariant W-ergodic measure on G/Γ is a finite homogeneous measure. In particular, W has property-(D) on G/Γ.*

For the proof one starts with a locally finite W-ergodic invariant measure μ on G/Γ. By the method of the proof of Theorem 3.5.1, one can show that there exists a closed subgroup H of G and $x \in G/\Gamma$ such that μ is H-invariant, $\operatorname{supp}(\mu) = Hx$, and the restriction of μ to H^0x is finite. Now it remains to prove that Hx has finitely many connected components. By certain arguments using the Margulis Arithmeticity Theorem and handling the case of real rank one semisimple groups separately, the question can be reduced to the case of $G = G(\mathbb{R})^0$ and $\Gamma = G(\mathbb{Z})$, where G is an algebraic semisimple group defined over $\mathbb{Q}$, W is Zariski dense discrete subgroup of G, and $W \subset G(\mathbb{Q})$. In this situation one can apply an interesting recent result due to Eskin and Margulis [74] on nondivergence property of random walks on finite volume homogeneous spaces to show that $W/W \cap \Gamma$ is finite.

Combining Theorems 3.5.2 and 3.5.3 one obtains the following:

COROLLARY 3.5.4. *Let the notation be as in Theorem* 3.5.2. *Then Fx admits a finite F-invariant measure, and hence it has finitely many connected components.*

Using the suspension argument, one obtains the following interesting extension of all the above results:

THEOREM 3.5.5. *Let the notations and conditions be as in any one of the three Theorems* 3.5.1–3.5.3 *or Corollary* 3.5.4, *and let Λ be a lattice in $\overline{W}$. Then the corresponding conclusions of the aforementioned theorems or the corollary hold for Λ in place of W.*

In particular, the above result applies to the subgroup action of an irreducible lattice in a noncompact semisimple group, which is acting on a homogeneous space of a larger Lie group containing it.

3.6. *Variations of Ratner's equidistribution theorem*

In applications one needs to have more robust versions of the equidistribution results, where one has the flexibility of suitably perturbing the base point and the acting one-parameter subgroup involved in the limiting process.

6a. *Uniformity in speed of convergence with respect to the starting point.* The following result was proved by Dani and Margulis [66] (see also Ratner [189, Theorem 8]):

THEOREM 3.6.1. *Let G be a Lie group, Γ a lattice in G, and μ the G-invariant probability measure on G/Γ. Let $u_{\mathbb{R}}^{(i)}$ be a sequence of one-parameter unipotent subgroups converging to a unipotent subgroup $u_{\mathbb{R}}$; that is, $u_t^{(i)} \to u_t$ for all t. Then for any sequence $x_i \to x$ in G/Γ with $x \notin \pi(S(u_{\mathbb{R}}, G))$, any sequence $T_i \to \infty$, and any $\varphi \in C_c(G/\Gamma)$, one has*

$$\frac{1}{T_i}\int_0^{T_i} \varphi\big(u_t^{(i)} x_i\big)\,\mathrm{d}t \to \int_{G/\Gamma} \varphi\,\mathrm{d}\mu.$$

In the next result, also due to Dani and Margulis [66], one has a uniform rate of convergence with greater flexibility in choosing the base point.

THEOREM 3.6.2. *Let G be a connected Lie group, Γ a lattice in G, and μ the G-invariant probability measure on G/Γ. Let $u_{\mathbb{R}}$ be a one-parameter unipotent subgroup of G and let $\varphi \in C_c(G/\Gamma)$. Then given a relatively compact subset $K \subset G/\Gamma$ and an $\varepsilon > 0$, there exist finitely many proper subgroups $H_i \in \mathcal{H}_\Gamma$ $(i = 1, \ldots, k)$ and a compact set $C \subset \bigcup_{i=1}^k N(u_{\mathbb{R}}, H_i)$ such that the following holds: for any compact set $F \subset K \setminus \pi(C)$ there exists a $T_0 \geqslant 0$ such that*

$$\left|\frac{1}{T}\int_0^{T} \varphi(u_t x)\,\mathrm{d}t - \int_{G/\Gamma} \varphi\,\mathrm{d}\mu\right| < \varepsilon \quad \forall x \in F,\ \forall T > T_0.$$

6b. *Limit distributions of polynomial trajectories.* For polynomial trajectories on finite volume homogeneous spaces of linear Lie groups, we have the following results about their closures and limit distributions [202]:

THEOREM 3.6.3. *Let G and $\Gamma \subset G$ be closed subgroups of $\mathrm{SL}(n, \mathbb{R})$ such that G/Γ has a finite G-invariant measure. Let $\pi : G \mapsto G/\Gamma$ denote the natural quotient map. Let $\Theta : \mathbb{R}^k \mapsto G$ be a polynomial map; that is, each coordinate function is a polynomial in k variables. Suppose that $\Theta(0) = e$. Let F be the minimal closed subgroup of G containing $\Theta(\mathbb{R}^k)$ such that $\pi(F)$ is closed and supports a finite F-invariant measure. Then $\overline{\pi(\Theta(\mathbb{R}^k))} = \pi(F)$, and the following holds: for any $f \in C_c(G/\Gamma)$ and any sequence $T_n \to \infty$,*

$$\lim_{n\to\infty} \frac{1}{|B_n|}\int_{\mathbf{t}\in B_n} f\big(\pi\big(\Theta(\mathbf{t})\big)\big)\,\mathrm{d}\mathbf{t} = \int_{\pi(F)} f\,\mathrm{d}\mu_F, \tag{3.10}$$

where $\mathrm{d}\mathbf{t}$ *denotes the integral corresponding to the Lebesgue measure* $|\cdot|$ *on* $\mathbb{R}^k$ *and* B_n *denotes the ball of radius* $T_n > 0$ *in* $\mathbb{R}^k$ *around* 0.

One can also choose to integrate over large boxes in $\mathbb{R}^k$:

THEOREM 3.6.4. *Let the notation be as in Theorem* 3.6.3. *Suppose further there exist polynomial maps* $\theta_1, \ldots, \theta_k$ *from* $\mathbb{R}$ *to* G *such that*

$$\Theta(t_1, \ldots, t_k) = \theta(t_1) \cdots \theta(t_k), \quad \forall t_i \in \mathbb{R}. \tag{3.11}$$

Then for any sequences $T_n^{(1)} \to \infty, \ldots, T_n^{(k)} \to \infty$, *the limit* (3.10) *holds for* $B_n = [0, T_n^{(1)}] \times \cdots \times [0, T_n^{(k)}]$, $\forall n \in \mathbb{N}$.

This result is a generalization of Ratner's equidistribution theorem for actions of algebraic unipotent one-parameter subgroups on finite volume homogeneous spaces of real algebraic groups, because one-parameter unipotent subgroups are polynomial maps. Conversely, if a polynomial map $\theta : \mathbb{R} \mapsto \mathrm{SL}(n, \mathbb{R})$ is a group homomorphism, then its image consists of unipotent elements.

The proof of Theorem 3.6.4 begins with the following observation motivated by [62, Proposition 2.4]:

LEMMA 3.6.5. *Let* $\theta : \mathbb{R} \mapsto \mathrm{SL}(n, \mathbb{R})$ *be a nonconstant polynomial map. Then there exists a* $q \geqslant 0$ *and a nontrivial unipotent one-parameter subgroup* $u_{\mathbb{R}} \subset \mathrm{SL}(n, \mathbb{R})$ *such that*

$$\lim_{t \to \infty} \theta\big(t + st^q\big)\theta(t)^{-1} = u_s, \quad \forall s \in \mathbb{R}.$$

Next consider a sequence of probability measures μ_T as in §3.3e. Then Theorem 3.2.7 is applicable, and there exists a sequence $T_i \to \infty$ such that $\mu_{T_i} \to \lambda$ in the space of probability measures on G/Γ. Then, by the above lemma, one can easily show that λ is $u_{\mathbb{R}}$-invariant. Now the arguments as in §3.4e are also applicable. This will provide a proof in the case $k = 1$. The proof of Theorem 3.6.3 can be obtained by radially fibering B_n and applying the one-dimensional case for each fiber. Certain Baire's category type arguments yield the result. Although the fibering argument is not applicable for Theorem 3.6.4, the method used for the one-dimensional case can be generalized.

REMARK 3.6.6. In the above theorems, if G is a real algebraic group defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ is an arithmetic lattice with respect to the $\mathbb{Q}$-structure on G, then F is the smallest real algebraic subgroup of G containing $\Theta(\mathbb{R}^k)$ and defined over $\mathbb{Q}$.

REMARK 3.6.7. The uniform convergence result for Θ in place of U as in Theorem 3.6.2 is valid, if one replaces $N(U, H)$ by

$$N(\Theta, H) \stackrel{\text{def}}{=} \big\{g \in G \mid \Theta(\mathbb{R}^k) \subset gHg^{-1}\big\},$$

and the integration can be carried out over the balls B_T with $T > T_0$ for arbitrary Θ, or over the boxes B_n with each $T_n^{(i)} > T_0$ in the case when Θ is as in (3.11).

Using Theorem 3.6.4 one obtains the equidistribution theorem for higher dimensional unipotent flows on homogeneous spaces of Lie group [202]. In order to describe the sets on which we intend to average, we need the following definition:

Let N be a simply connected nilpotent group with Lie algebra $\mathfrak{N}$. Let $B = \{b_1, \ldots, b_k\}$ be a basis in $\mathfrak{N}$. Say that the basis B is *triangular* if $[b_i, b_j] \in \operatorname{Span}\{b_l \mid l > \max\{i, j\}\}$ for all $i, j = 1, \ldots, k$. Any permutation of a triangular basis is called a *regular* basis.

COROLLARY 3.6.8. *Let G be a Lie group, Γ a closed subgroup of G such that G/Γ admits a finite G-invariant measure, and let N be a simply connected unipotent subgroup of G. Let $\{b_1, \ldots, b_k\}$ be a regular basis in $\mathfrak{N}$. For $s_1, \ldots, s_k > 0$ define*

$$\mathcal{B}(s_1, \ldots, s_k) = \bigl\{(\exp t_k b_k) \cdots (\exp t_1 b_1) \in N \mid 0 \leqslant t_j \leqslant s_j,\ j = 1, \ldots, k\bigr\}.$$

Given $x \in G/\Gamma$, let F be the minimal closed subgroup of G containing N such that the orbit Fx is closed and admits a unique F-invariant probability measure, say μ_F. Then for any $f \in C_c(G/\Gamma)$,

$$\lim_{s_1 \to \infty, \ldots, s_k \to \infty} \frac{1}{\lambda(\mathcal{B}(s_1, \ldots, s_k))} \int_{h \in \mathcal{B}(s_1, \ldots, s_k)} f(hx)\, \mathrm{d}\lambda(h) = \int_{Fx} f \, \mathrm{d}\mu_F,$$

where λ denotes a Haar measure on N.

3.7. *Limiting distributions of sequences of measures*

Now we will note some results on limiting distributions of sequences of homogeneous measures. The common theme of the results is that, under somewhat more general conditions, the limits of sequences of homogeneous measures are also homogeneous measures, and under appropriate algebraic conditions such sequences become equidistributed in the ambient homogeneous space.

7a. *Closure of the set of ergodic invariant measures.* Let G be a Lie group and Γ be a discrete subgroup of G. Let $\mathcal{Q} \stackrel{\text{def}}{=} \{\mu \in \mathcal{P}(G/\Gamma) \mid \mu$ is U-invariant and U-ergodic for some unipotent one-parameter subgroup U of $G\}$.

The following was proved by Mozes and Shah [162]:

THEOREM 3.7.1. *$\mathcal{Q}$ is a closed subset of $\mathcal{P}(G/\Gamma)$. Further, if $\{\mu_i\} \subset \mathcal{Q}$ is a sequence converging to μ, then it converges algebraically in the following sense: there exists a sequence $g_i \to e$ in G such that for some $i_0 \in \mathbb{N}$,*

$$g_i \operatorname{supp}(\mu_i) \subset \operatorname{supp}(\mu), \quad \forall i \geqslant i_0. \tag{3.12}$$

COROLLARY 3.7.2. *Suppose that Γ is lattice in G. For any $x_0 \in G/\Gamma$, define*

$$\mathcal{Q}_{x_0} \stackrel{\text{def}}{=} \bigl\{\mu \in \mathcal{Q} \mid x_0 \in \operatorname{supp}(\mu)\bigr\}.$$

Then $\mathcal{Q}_{x_0}$ is compact.

This follows from Theorem 3.7.1, because using Theorem 3.1.2 it is straightforward to verify that $\mathcal{Q}_{x_0}$ is relatively compact in $\mathcal{P}(G/\Gamma)$.

REMARK 3.7.3. Suppose that G/Γ is connected. Take any unipotent element $u \in G$. If μ_i is a u-invariant and u-ergodic probability measure on G/Γ for all $i \in \mathbb{N}$ and $\mu_i \to \mu$ in $\mathcal{P}(G/\Gamma)$ as $i \to \infty$, then μ is a homogeneous measure, and there exists a sequence $g_i \to e$ such that $g_i \operatorname{supp}(\mu_i) \subset \operatorname{supp}(\mu)$. See [204, Theorem 7.16] for more details.

7b. *Limits of translates of algebraic measures.* In the case of arithmetic lattices, one has the following analogue of Theorem 3.7.1 due to Eskin, Mozes and Shah [78].

THEOREM 3.7.4. *Let G be a connected real algebraic group defined over $\mathbb{Q}$, let Γ be a subgroup of $G(\mathbb{Z})$, and let $\pi : G \mapsto G/\Gamma$ be the natural quotient map. Let $H \subset G$ be a connected real algebraic $\mathbb{Q}$-subgroup such that $\pi(H)$ admits an H-invariant probability measure, say μ_H. Let $\{g_i\}$ be a sequence such that the $g_i\mu_H \to \mu$ in $\mathcal{P}(G/\Gamma)$ as $i \to \infty$. Then there exists a connected real algebraic $\mathbb{Q}$-subgroup L of G such that $L \supset H$ and the following holds*:

(1) *There exists $c \in G$ such that μ is cLc^{-1}-invariant and* $\operatorname{supp}(\mu) = c\pi(L)$. *In particular, μ is a homogeneous measure.*
(2) *There exists a sequence $c_i \to c$ in G such that $g_i\pi(H) \subset c_i\pi(L)$ and $c_i^{-1}g_i \in \Gamma H$ for all but finitely many $i \in \mathbb{N}$.*

In the above situation, it can be shown that if all but finitely many elements of the sequence $\{g_i\}$ are not contained in any set of the form $CZ_G(H)$, where C is a compact set, then there exists a sequence $X_i \to 0$ in $\operatorname{Lie}(H)$ such that $\operatorname{Ad} g_i(X_i) \to X$ for some $X \neq 0$. Now it is straightforward to deduce that $U = \exp(\mathbb{R}X)$ is a unipotent subgroup of G and μ is U-invariant. Then one applies Ratner's measure classification and Proposition 3.4.12 for further analysis.

Theorem 3.7.4 is better applicable if one has a criterion to decide when a sequence $\{g_i\mu_H\}$ has a convergent subsequence in the space of probability measures on G/Γ. Suppose that $\pi(Z_G(H))$ is not compact. Since $Z_G(H)$ is a reductive $\mathbb{Q}$-subgroup, $\pi(Z_G(H))$ is closed, and hence there exists a sequence $z_i \in Z_G(H)$ such that $\pi(z_i)$ has no convergent subsequence. It is easily verified that the sequence $z_i\mu_H$ is divergent; that is, for any compact set $K \subset G/\Gamma$, $z_i\mu_H(K) \to 0$ as $i \to \infty$. The next result due to Eskin, Mozes and Shah [79] says that if $\pi(Z_G(H))$ is compact, then the orbit $G\mu_H$ of μ_H is relatively compact in the space of probability measures on G/Γ.

THEOREM 3.7.5. *Let G be a connected reductive algebraic group defined over $\mathbb{Q}$, and H a connected real reductive $\mathbb{Q}$-subgroup of G, both admitting no nontrivial $\mathbb{Q}$-characters. Suppose that H is not contained in any proper parabolic $\mathbb{Q}$-subgroup of G defined over $\mathbb{Q}$. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic lattice in G and $\pi : G \mapsto G/\Gamma$ the natural quotient map. Let μ_H denote the H-invariant probability measure on $\pi(H)$. Then given any $\varepsilon > 0$ there exists a compact set $\mathcal{K} \subset G/\Gamma$ such that $g\mu_H(\mathcal{K}) > 1 - \varepsilon$ for all $g \in G$.*

We note that the condition of H not being contained in a proper parabolic $\mathbb{Q}$-subgroup is equivalent to the condition that $\pi(Z_G(H))$ is compact (see [79, Lemma 5.1]).

The proof of Theorem 3.7.5 is based on a version of Theorem 3.2.7.

7c. *Limits of expanding translations of horospheres*. The result in Example 3.0.1 can be extended using Ratner's measure theorem as follows [203]:

THEOREM 3.7.6. *Let G be a connected semisimple totally noncompact Lie group. Let $g \in G$ be such that*

$$\begin{aligned} &\operatorname{Ad} g \text{ is semisimple and the projection of} \\ &g^{\mathbb{Z}} \text{ on any simple factor of } G \text{ is unbounded.} \end{aligned} \tag{3.13}$$

Let G^+ be the expanding horospherical subgroup of G corresponding to g, and let H be the subgroup of G generated by g and G^+. Now suppose that G is realized as a closed subgroup of a Lie group L and let Λ be a lattice in L. Then any closed H-invariant subset of L/Λ, as well as any (locally finite) H-invariant Borel measure on L/Λ, is G-invariant.

In particular, (locally finite) H-invariant H-ergodic measures on L/Λ are finite and homogeneous, and the closures of orbits of H on L/Λ are homogeneous. Thus H has property-(D) on L/Λ.

One of the motivations for the above result was the following very general result due to Furstenberg (oral communication, 1990):

THEOREM 3.7.7. *Let notation be as in Theorem 3.7.6. Let X be a locally compact Hausdorff space with a continuous G-action. Suppose that there exists a finite G-invariant Borel measure on X which is strictly positive on every non-empty open subset of X. Then for any $x \in X$, if $\overline{Gx} = X$ then $\overline{Px} = X$, where $P = N_G(G^+)$.*

In particular, if the action of G on X is minimal, then P acts minimally on X.

Note that, in view of Ratner's orbit closure theorem, if L is a Lie group containing G, Λ is a lattice in L, then closure of every G-orbit on L/Λ has a finite G-invariant measure which is positive on every open subset. Hence by Theorem 3.7.7, the closure of every P-orbit on L/Λ is G-invariant (cf. Theorem 4.4.5).

Theorem 3.7.6 was proved by Ratner [188] in the case when each simple factor of G is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ (see Theorem 4.4.2).

Theorem 3.7.6 is a direct consequence of the following limiting distribution result [203]:

THEOREM 3.7.8. *Let the notation be as in Theorem 3.7.6. Take any $x \in L/\Lambda$, and let F be a closed subgroup of L containing G such that $\overline{Gx} = Fx$ (see Theorem 3.3.6). Let λ be the F-invariant probability measure supported on Fx, and let ν be any probability measure on G^+ which is absolutely continuous with respect to a Haar measure on G^+. Then for $x \in L/\Lambda$ and any $f \in C_c(L/\Lambda)$,*

$$\lim_{n\to\infty} \int_{G^+} f\left(g^n u x\right) \mathrm{d}\nu(u) = \int_{Fx} f \,\mathrm{d}\lambda.$$

The proof of this result involves Ratner's measure theorem, Proposition 3.4.12 and the following observation ([203, Lemma 5.2]), which allows one to contradict possibility (1) of the proposition:

LEMMA 3.7.9. *For any continuous linear representation of G on a finite dimensional vector space V and any G^+-fixed vector $\mathbf{v} \in V$, either $\mathbf{v}$ is G-fixed or $g^n\mathbf{v} \to \infty$ as $n \to \infty$.*

REMARK 3.7.10. Note that a result similar to Theorem 3.7.8 (with $L = G$ and $\Lambda = \Gamma$ a lattice in G) was proved in [116] without using the classification of ergodic invariant measure or orbit closures for unipotent flows, and with assumption (3.13) replaced by the partial hyperbolicity of g. It was derived from mixing properties of the g-action on G/Γ, see §2.3c, and the conjugation $u \mapsto g^n u g^{-n}$ being an expanding automorphism of G^+. Further, the result is valid for Hölder square-integrable functions f, and the convergence becomes exponentially fast if one knows that the g-action is exponentially mixing, in the sense of (4.3) below. For example, (2.3) would be a sufficient condition.

Using a slightly stronger version of Theorem 3.7.8 the following can be deduced [203]:

COROLLARY 3.7.11. *Let G be a totally noncompact semisimple Lie group, and let K be a maximal compact subgroup of G. Let L be a Lie group which contains G as a Lie subgroup, and let Λ be a lattice in L. Let ν be any probability measure on K which is absolutely continuous with respect to a Haar measure on K. Let $\{g_n\}$ be a sequence of elements of G without accumulation points. Then for any $x \in L/\Lambda$ and any $f \in C_c(L/\Lambda)$,*

$$\lim_{n\to\infty} \int_K f(g_n k x)\nu(k) = \int_{Fx} f \, d\lambda,$$

where F is a closed subgroup of L such that $\overline{Gx} = Fx$, and λ is a (unique) F-invariant probability measure on Fx.

In the case of $L = G$, the result was first proved by Duke, Rudnick and Sarnak [73] using methods of unitary representation theory. Soon afterwards a much simpler proof was obtained by Eskin and McMullen [75] using the mixing property of geodesic flows (cf. Remark 3.7.10). As in [73,75] an appropriate analogue of the above theorem is also valid for any affine symmetric subgroup of G in place of the maximal compact subgroup K of G.

3.8. *Equivariant maps, ergodic joinings and factors*

It turns out that equivariant maps and ergodic joinings of unipotent flows are algebraically rigid. The first two results are direct consequences of Ratner's description of ergodic invariant measures for unipotent flows (see [186,189]).

THEOREM 3.8.1. *Let G_i be a connected Lie group and Γ_i be a lattice in G_i containing no nontrivial normal subgroups of G_i, where $i = 1, 2$. Let u_i be a unipotent element of G_i, $i =$*

1, 2. *Assume that the action of u_1 on G_1/Γ_1 is ergodic with respect to the G_1-invariant probability measure. Suppose that there is a measure-preserving map $\psi : G_1/\Gamma_1 \mapsto G_2/\Gamma_2$ such that $\psi(u_1 x) = u_2\psi(x)$ for a.e. $x \in G_1/\Gamma_1$. Then there is $c \in G_2$ and a surjective homomorphism $\alpha : G_1 \mapsto G_2$ such that $\alpha(\Gamma_1) \subset c\Gamma_2 c^{-1}$ and $\psi(h\Gamma_1) = \alpha(h)c\Gamma_2$ for a.e. $h\Gamma_1 \in G/\Gamma_1$.*

Also α is a local isomorphism whenever ψ is finite-to-one or G_1 is simple. Further it is an isomorphism whenever ψ is one-to-one or G_1 is simple with trivial center.

THEOREM 3.8.2. *Let G_i be a connected Lie group, Γ_i be a lattice in G, ν_i be the G_i-invariant probability measure on G_i/Γ_i, and u_i be a unipotent element of G_i, where $i = 1, 2$. Let $x_i = e\Gamma_i \in G_i/\Gamma_i$ for $i = 1, 2$. Suppose that μ is a joining of $\{(G_i/\Gamma_i, \nu_i, u_i) \mid i = 1, 2\}$ which is ergodic with respect to $u_1 \times u_2$ (see [3, §3.3.h]), and that $(x_1, x_2) \in \operatorname{supp}(\mu)$. Then there exist a closed normal subgroup N_i of G_i, where $i = 1, 2$, and a group isomorphism $\alpha : \overline{G}_1 \mapsto \overline{G}_2$, where $\overline{G}_i = G_i/N_i$ for $i = 1, 2$, such that the following holds:*

1. *$\overline{\Gamma_i} = \Gamma_i N_i/N_i$ is a lattice in $\overline{G}_i$ for $i = 1, 2$;*
2. *If one writes $G = \{(g, \alpha(g)) \mid g \in \overline{G}_1\}$ then $G \cap \overline{\Gamma_1} \times \overline{\Gamma_2}$ is a lattice in G;*
3. *If $\varphi : G_1/\Gamma_1 \times G_2/\Gamma_2 \mapsto \overline{G}_1/\overline{\Gamma_1} \times \overline{G}_2/\overline{\Gamma_2}$ is the standard quotient map, and if $\bar{\mu}$ denotes the G-invariant probability measure on $G\varphi(x_1, x_2)$, then μ is the canonical lift of $\bar{\mu}$ with respect to φ;*
4. *$\operatorname{supp}(\bar{\mu})$ is a finite covering of $\overline{G}_2/\overline{\Gamma_2}$ of index equal to $[\overline{\Gamma_2} : \alpha(\overline{\Gamma_1}) \cap \overline{\Gamma_2}]$.*

In particular, if G_i is a semisimple totally noncompact Lie group, Γ_i is an irreducible lattice in G_i, where $i = 1, 2$, and the joining μ as above is nontrivial, then G_1 and G_2 are locally isomorphic, and $\operatorname{supp}(\mu)$ is a finite covering of G_i/Γ_i for $i = 1, 2$.

For $G_1 = G_2 = \mathrm{SL}(2, \mathbb{R})$ both the above results were proved earlier by Ratner [178,179]. Using Ratner's method, Theorem 3.8.1 was proved in the general case by Witte [254].

It turns out that the measure theoretic factors for actions of subgroups generated by unipotent elements on homogeneous spaces can be described algebraically.

DEFINITION 3.8.3. (1) Suppose G is a connected Lie group and Γ is a discrete subgroup of G. A homeomorphism $\tau : G/\Gamma \mapsto G/\Gamma$ is called an *affine automorphism* of G/Γ if there exists $\sigma \in \mathrm{Aut}(G)$ such that $\tau(gx) = \sigma(g)\tau(x)$ for all $x \in G/\Gamma$.

Put $\mathrm{Aut}(G)_\Gamma = \{\sigma \in \mathrm{Aut}(G) \mid \sigma(\Gamma) = \Gamma\}$. Define a map $\pi : G \rtimes \mathrm{Aut}(G)_\Gamma \mapsto \mathrm{Aff}(G/\Gamma)$ by $\pi(h, \sigma)(g\Gamma) = h\sigma(g)\Gamma$ for all $g \in G$. Observe that π is a surjective map. Hence $\mathrm{Aff}(G/\Gamma)$ has the structure of a Lie group acting differentiably on G/Γ.

(2) Suppose G is a connected Lie group and Γ is a closed subgroup of G such that Γ^0 is normal in G. Put $\overline{G} = G/\Gamma^0$ and $\overline{\Gamma}$ the image of Γ in $\overline{G}$. One defines $\mathrm{Aff}(G/\Gamma) \overset{\mathrm{def}}{=} \mathrm{Aff}(\overline{G}/\overline{\Gamma})$.

(3) Suppose G is a Lie group and Γ is a closed subgroup of G such that $G = G^0\Gamma$ and Γ^0 is normal in G^0. One defines $\mathrm{Aff}(G/\Gamma) \overset{\mathrm{def}}{=} \mathrm{Aff}(G^0/G^0 \cap \Gamma)$.

The next result is obtained by combining Theorem 3.5.1 and [257].

THEOREM 3.8.4. *Let G be a Lie group and Γ a closed subgroup of G such that G/Γ is connected and has a finite G-invariant measure. Let W be a subgroup of G and a subset*

$S \subset W$ be such that S consists of unipotent elements and $\mathrm{Ad}_G(W) \subset \mathrm{Zcl}(\mathrm{Ad}_G(\langle S \rangle))$. Suppose W acts ergodically on G/Γ. Then any W-equivariant measurable quotient of G/Γ is of the form $K\backslash G/\Lambda$, where Λ is a closed subgroup of G containing Γ, and K is a compact subgroup of the centralizer of the image of W in $\mathrm{Aff}(G/\Lambda)$.

REMARK 3.8.5. Let G be a connected semisimple totally noncompact Lie group and Γ be an irreducible lattice in G. Then $[\Lambda : \Gamma] < \infty$. Further suppose that $Z_G(W)$ is finite. Then K is a finite group.

It may be noted that the above result was proved by Ratner [177] in the case of $G = \mathrm{SL}(2, \mathbb{R})$.

The description of topological factors for unipotent flow is very similar to the measurable factors, but it is somewhat more involved (see Shah [203]).

THEOREM 3.8.6. *Let G and Γ be as in Theorem 3.8.4. Let W be a subgroup of G generated by unipotent one-parameter subgroups contained in it. Suppose that W acts ergodically on G/Γ. Let X be a Hausdorff locally compact space with a continuous W-action and $\varphi : G/\Gamma \mapsto X$ a continuous surjective W-equivariant map. Then there exists a closed subgroup Λ containing Γ, a compact group K contained in the centralizer of the image of W in $\mathrm{Aff}(G/\Lambda)$, and a W-equivariant continuous surjective map $\psi : K\backslash G/\Lambda \mapsto X$ such that the following statements hold*:

(1) *Define the W-equivariant map $\rho : G/\Gamma \mapsto K\backslash G/\Lambda$ by $\rho(g\Gamma) \stackrel{\text{def}}{=} Kg\Lambda$ for all $g \in G$. Then $\varphi = \psi \circ \rho$.*
(2) *Given a neighborhood Ω of e in $Z_G(W)$, there exists an open dense W-invariant subset X_0 of G/Λ such that for any $x \in X_0$ and $y \in G/\Lambda$ one has removing brackets $y \in K\Omega x$ whenever $\psi(Kx) = \psi(Ky)$. In this situation, if $K\overline{Wx} = K\backslash G/\Lambda$, then $Ky = Kx$.*

REMARK 3.8.7. Let the notation be as in Theorem 3.8.6. Further suppose that G and Γ are as in Remark 3.8.5, and the subgroup W is a proper maximal connected subgroup of G with discrete center. Then $[\Lambda : \Gamma] < \infty$, K is finite, $\Omega = \{e\}$, and $C \setminus X_0$ is contained in a union of finitely many closed orbits of W for any compact subset C of G/Λ.

Apart from Theorem 3.3.6, a new ingredient in the proof of Theorem 3.8.6 is the use of the conclusion (3.12) of Theorem 3.7.1.

8a. *Disjointness of actions on boundaries and homogeneous spaces.* Consider a connected Lie group L and its Lie subgroup G which is a simple Lie group of $\mathbb{R}$-rank > 1. Let P be a proper parabolic subgroup of G and Λ a lattice in L. Then the actions of G on G/P and on L/Λ are 'topologically disjoint' in the following sense (see [203]):

THEOREM 3.8.8. *Let G be a real algebraic semisimple Lie group whose all simple factors have $\mathbb{R}$-rank > 1. Let L be a Lie group and Λ be a lattice in L. Suppose that G is realized as a Lie subgroup of L and it acts ergodically on L/Λ. Let P be a parabolic subgroup of G and consider the diagonal action of G on $L/\Lambda \times G/P$. Let Y be locally compact second* *countable space with a continuous G-action and $\varphi : L/\Lambda \times G/P \mapsto Y$ be a continuous surjective G-equivariant map. Then there exist a parabolic subgroup $Q \supset P$ of G, a locally compact Hausdorff space X with a continuous G-action, a continuous surjective G-equivariant map $\varphi_1 : L/\Lambda \mapsto X$ (see Theorem 3.8.6), and a continuous G-equivariant map $\psi : X \times G/Q \mapsto Y$ such that the following holds*:

(1) *If one defines $\rho : L/\Lambda \times G/P \mapsto X \times G/Q$ as $\rho(x, gP) = (\varphi_1(x), gQ)$ for all $x \in L/\Lambda$ and $g \in G$, then $\varphi = \psi \circ \rho$.*

(2) *There exists an open dense G-invariant set $X_0 \subset L/\Lambda$ such that if one puts $Z_0 = \varphi_1(X_0) \times G/Q$ and $Y_0 = \psi(Z_0)$, then*
   (a) *Y_0 is open and dense in Y,*
   (b) *$Z_0 = \psi^{-1}(Y_0)$, and*
   (c) *$\psi|_{Z_0}$ is a homeomorphism onto Y_0.*

In particular, if the action of G on L/Λ is minimal then ψ is a homeomorphism.

In the case when $L = G$, the result was proved by Dani [50] in order to obtain a topological analogue of a result of Margulis [134] on Γ-equivariant measurable factors of G/P. The above result was formulated to answer a question of Stuck [241] for actions on homogeneous spaces.

The proof of the result is based on the ideas from [50] combined with Ratner's orbit closure theorem, and Theorem 3.7.6.

4. Dynamics of non-unipotent actions

As we saw in Section 3, unipotent orbits on homogeneous spaces exhibit somewhat 'regular' behavior. On the other hand, in the complementary partially hyperbolic case there always exist orbits with complicated behavior. After describing several examples of partially hyperbolic one-parameter flows, we prove that any such flow has an orbit with nonsmooth closure. Further, one can find orbits avoiding any pre-selected point, or even a small enough subset, of the space. Results of this type are reviewed in §4.1a and §4.1b. Then in §4.1c we specialize to noncompact homogeneous spaces and look at excursions of trajectories to infinity, obtaining a generalization and strengthening of Sullivan's logarithm law for geodesics. Finally, in §4.1d we consider trajectories exiting to infinity.

On the other hand, the results of Section 3 simplify considerably the study of arbitrary (not necessarily unipotent) actions. For example, one can extend a lot of results to the class of quasi-unipotent actions. It turns out that the algebraicity of orbit closures and ergodic measures is to be replaced by 'quasi-algebraicity'. This implies, in particular, that all orbit closures of a one-parameter flow are smooth iff the flow is quasi-unipotent. We discuss this dichotomy and related results in Section 4.2. Although measure rigidity does not take place for ergodic quasi-unipotent flows, it does so for mixing quasi-unipotent flows. As for the ergodic case, one can establish 'fiber-wise' rigidity.

After proving a stronger form of Ratner's topological theorem in Section 4.3 we give a 'classification' of minimal sets of homogeneous one-parameter flows. Then we comment on the structure of rectifiable invariant sets.

In Section 4.4 we consider the structure of ergodic measures and orbit closures for actions of multi-dimensional connected subgroups $F \subset G$. Here the results are far from being definitive. Nevertheless, some classes of connected subgroups $F \subset G$ not generated by unipotent elements, whose ergodic measures and orbit closures are of algebraic origin, have been found. Incidentally, we show how the study of F-ergodic invariant measures for a connected subgroup F can be reduced to the case when F is Abelian and consists of partially hyperbolic elements. For such a subgroup F with $\dim F > 1$, many important problems are open; we only list some of them. Finally, we discuss the relation between minimality and unique ergodicity of homogeneous actions.

4.1. *Partially hyperbolic one-parameter flows*

Throughout Section 4.1, Γ is a lattice in a Lie group G (perhaps sometimes just a discrete subgroup), and $g_{\mathbb{R}}$ is a partially hyperbolic one-parameter subgroup of G. The basic examples are those listed in §1.4e and §1.4f: that is, geodesic flows on locally symmetric spaces of noncompact type. In this setting, G will be as in (2.1); as we saw in Sections 1 and 2, many problems involving homogeneous flows can be reduced to this case. Moreover, the situation when the space essentially splits into a product of smaller spaces can be reduced to studying the factors; therefore it will be often natural to assume (2.2).

While studying ergodic properties of partially hyperbolic one-parameter actions, one easily notices similarities to those of Anosov flows. Informally speaking, both classes of dynamical systems grew out of the examples considered in §1.4e, that is, geodesic flows on surfaces of constant negative curvature, and then two different ways of generalization are chosen. Indeed, the local decomposition of G as the product of the neutral subgroup Q and the expanding/contracting horospherical subgroups G^+ and G^- is a direct analogue of the local product structure induced by an Anosov flow. However, the methods of studying these classes are strikingly different. Most important results in Anosov flows are achieved by means of symbolic dynamics of Markov partitions of the phase space (see [2] for details). The absence of symbolic representation in the higher rank partially hyperbolic case calls for other methods, making heavy use of the underlying rich algebraic structure and the uniformity of the geometry of G/Γ.

Let us briefly describe a particular important example of a partially hyperbolic homogeneous flow.

EXAMPLE 4.1.1. Take $G = \mathrm{SL}(k, \mathbb{R})$ and a subgroup $g_{\mathbb{R}}$ of G, where $g_t = \mathrm{diag}(e^{w_1 t}, \dots, e^{w_k t})$ and $w_1, \dots, w_k$ are real numbers with $\sum_{i=1}^k w_i = 0$. The flow is partially hyperbolic iff at least one (and hence at least two) of the numbers w_i are nonzero. Arrange them so that $w_i \geqslant w_j$ if $i \leqslant j$; then the expanding horospherical subgroup G^+ corresponding to g_1 is a subgroup of the group of unipotent upper-triangular matrices, and is exactly equal to the latter iff all the values of w_i are different. In the latter case, G^- is the group of lower-triangular matrices, and the neutral subgroup Q (the centralizer of $g_{\mathbb{R}}$) consists of diagonal matrices. Another extreme case is when g_1 has only two (multiple) real eigenvalues, i.e., $k = m + n$ and

$$g_t = \mathrm{diag}(\underbrace{e^{t/m}, \dots, e^{t/m}}_{m \text{ times}}, \underbrace{e^{-t/n}, \dots, e^{-t/n}}_{n \text{ times}}). \tag{4.1}$$

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Then all elements of G^+ are of the form

$$L_A \stackrel{\text{def}}{=} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}, \quad A \in M_{m,n}, \tag{4.2}$$

and similarly

$$G^- = \left\{ \begin{pmatrix} I_m & 0 \\ B & I_n \end{pmatrix} \,\middle|\, B \in M_{n,m} \right\},$$

while Q is the group of block-diagonal matrices

$$Q = \left\{ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \,\middle|\, C \in M_{m,m},\ D \in M_{n,n},\ \det(C) \cdot \det(D) = 1 \right\}.$$

As was mentioned in Section 2, partially hyperbolic homogeneous flows are precisely those of positive entropy. Also, a partially hyperbolic one-parameter subgroup $g_{\mathbb{R}}$ is clearly unbounded. Assume that the conditions (2.1) and (2.2) hold. Then the flow is mixing (hence ergodic) by Moore's Theorem 2.3.6. Moreover, one can show that it follows from the partial hyperbolicity of $g_{\mathbb{R}}$ that $\operatorname{dist}(e, g_t)$ is bounded from below by const $\cdot\, t$. Therefore from Corollary 2.3.8 one can deduce the following

THEOREM 4.1.2. *Assume that conditions* (2.1), (2.2) *and* (2.3) *hold. Then there exist constants* $\gamma > 0$, $E > 0$, $l \in \mathbb{N}$ *such that for any two functions* $\varphi, \psi \in C^\infty(G/\Gamma)$ *with zero mean and compact support and for any* $t \geqslant 0$ *one has*

$$\left|(g_t\varphi, \psi)\right| \leqslant E\, \mathrm{e}^{-\gamma t} \|\varphi\|_l \|\psi\|_l, \tag{4.3}$$

where $\|\cdot\|_l$ *means the norm in the Sobolev space* $W_l^2(G/\Gamma)$.

It is worthwhile to compare the above result with the recent work of Dolgopyat on exponential decay of correlations for special classes of Anosov flows [72].

1a. *Nondense orbits*. Our goal here is to look at orbit closures of a partially hyperbolic homogeneous flow $(G/\Gamma, g_{\mathbb{R}})$. We will see in later that the results of this section differ drastically from the case when the subgroup $g_{\mathbb{R}}$ is quasiunipotent. Roughly speaking, the dynamics in the partially hyperbolic case has many chaotic features (whatever this means), while quasi-unipotent flows have somewhat regular behavior.

As was known for a long time (see §1.4e), orbits of the geodesic flow can be 'very bad'. Namely, an orbit can be nonrecurrent and not exiting to infinity or, on the contrary, it can be recurrent and have nowhere locally connected closure (as the Morse minimal set). The following observation of Margulis generalizes this fact (see [227]):

LEMMA 4.1.3. *Any uniformly partially hyperbolic flow on a manifold with a finite smooth invariant measure always has a nonrecurrent orbit which does not exit to infinity* (*and hence its closure is not a submanifold*).

This is certainly in contrast with homogeneity of *all* orbit closures of unipotent flows, as explained in Section 3. Further, if a unipotent flow is ergodic, one can use Theorem 3.4.1 to conclude that the exceptional set of points with nondense orbits of a unipotent flow belongs to a countable union of proper algebraic subvarieties.

We now look more closely at that exceptional set in the partially hyperbolic case. Motivated by similar results in nonhomogeneous hyperbolic dynamics [246], one should expect it to be quite big. Moreover, in many particular cases the set of points with orbits escaping a fixed set Z of the phase space is rather big. More precisely, if F is a set of self-maps of a metric space X, and Z a subset of X, let us denote by $E(F,Z)$ the set $\{x \in X \mid \overline{F(x)} \cap Z = \emptyset\}$ of points with F-orbits escaping Z, and say that Z is *escapable relative to* F (or, briefly, F*-escapable*) if $E(F,Z)$ is *thick* in X, that is, has full Hausdorff dimension at any point of X.

Here are some results to compare with: X is a Riemannian manifold, $F = f_{\mathbb{R}_+}$ where $f_t : X \mapsto X$ is a C^2 Anosov flow. Urbanski [246] (essentially) proved that any one-element set is F-escapable, while Dolgopyat [71], under some additional assumptions (e.g., if X has a smooth F-invariant measure), showed that any countable subset of X is F-escapable.

The following theorem is proved in [114]:

THEOREM 4.1.4. *Let G be a unimodular Lie group, Γ a discrete subgroup of G, $F = g_{\mathbb{R}}$ a partially hyperbolic one-parameter subgroup of G. Then any compact C^1 submanifold Z of G/Γ of dimension less than* $\min(\dim(G^+), \dim(G^-))$*, which is transversal to the F-orbit foliation, is F-escapable.*

In particular, this shows that if the flow is partially hyperbolic, points with nondense orbits form a thick set. Note that analogous results in the setting of Anosov flows and diffeomorphisms, see [71,246], are proved via symbolic dynamics of Markov partitions of the manifold. This tool is not available in the higher rank case, when the dimension of the neutral leaf is bigger than one. As a replacement, one considers natural 'rectangular' partitions of the expanding horospherical subgroup G^+ of G (called *tessellations* in [116] and [114]) and studies their behavior under the automorphism $h \mapsto g_t h g_{-t}$ of G^+. Then the Hausdorff dimension of the set of points escaping Z is estimated using the fact that the F-translates of small neighborhoods of Z are 'topologically small', that is, they can be covered by relatively small number of rectangles from the aforementioned partitions. Those rectangles can be used to create a Cantor set consisting of points with orbits avoiding a neighborhood of Z. In fact this argument yields a more precise description of the set $E(F^+, Z)$ of points with positive semi-orbits escaping Z (here $F^+ = g_{\mathbb{R}_+}$) as follows:

THEOREM 4.1.5. *Let G, Γ, $F = g_{\mathbb{R}}$ and Z be as in the above theorem. Then for any $x \in G/\Gamma$, the set $\{h \in G^+ \mid hx \in E(F^+, Z)\}$ is thick in G^+.*

In other words, the intersections of $E(F^+, Z)$ with open subsets of any unstable leaf have full Hausdorff dimension. The previous theorem follows from this one by a standard slicing argument. See [114] for details and other results, including similar statement for actions of cyclic partially hyperbolic subgroups $g^{\mathbb{Z}}$ of G.

1b. *Bounded orbits*. A special class of nondense orbits of homogeneous flows happened to be very important in view of a connection with Diophantine approximation, and, in fact, served as a motivation for this circle of problems. Namely, consider $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, and let $g_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$ (this is clearly a special case of Example 4.1.1). Using the fact that *badly approximable* real numbers form a thick subset of $\mathbb{R}$ [102], one can show (see §5.2a for more details on the connection with number theory) that the set of points of the noncompact space G/Γ with *bounded* (i.e., relatively compact) $g_{\mathbb{R}}$-orbits is thick. Let us modify the definition from the previous section: if F is a set of self-maps of a metric space X, and Z a subset of the one-point compactification $X^* \stackrel{\text{def}}{=} X \cup \{\infty\}$ of X, denote by $E(F, Z)$ the set $\{x \in X \mid \overline{F(x)} \cap Z = \emptyset\}$ (with the closure taken in the topology of X^*) and say that Z is F-*escapable* if $E(F, Z)$ is thick in X. With this terminology, the abundance of bounded orbits reads as '$\{\infty\}$ is $g_{\mathbb{R}}$-escapable'.

One might ask whether a similar statement holds for more general G and Γ. In the 1980s Dani used the results and methods of Schmidt in simultaneous Diophantine approximation (see §5.2c) to prove the escapability of $\{\infty\} \subset (G/\Gamma)^*$ in the following two cases:

[52] $G = \mathrm{SL}(k, \mathbb{R})$, $\Gamma = \mathrm{SL}(k, \mathbb{Z})$, and $g_{\mathbb{R}}$ is of the form (4.1);

[54] G is a connected semisimple Lie group of $\mathbb{R}$-rank 1, Γ a lattice in G, and $g_{\mathbb{R}}$ is partially hyperbolic.

These results were substantially refined in [116], where essentially the following result was established:

THEOREM 4.1.6. *Let G be a Lie group, Γ a lattice in G, $F = g_{\mathbb{R}}$ a partially hyperbolic one-parameter subgroup of G. Assume that either*

(1) *F consists of semisimple elements and the F-action on G/Γ is mixing, or*

(2) *the F-action on G/Γ is exponentially mixing, in the sense of* (4.3).

Denote by F^+ the semigroup $g_{\mathbb{R}_+}$, and let Z be a closed null subset of G/Γ. Then:

(a) *if Z is F^+-invariant, then for any $x \in G/\Gamma$ the set $\{h \in G^+ \mid hx \in E(F^+, Z)\}$ is thick in G^+*;

(b) *if Z is F-invariant, then $Z \cup \{\infty\}$ is F-escapable. That is, points $x \in E(F, Z)$ for which Fx is bounded form a thick set*.

As was mentioned in §2.3d, the above condition (2) is satisfied for any partially hyperbolic one-parameter subgroup $F = g_{\mathbb{R}}$ whenever G satisfies (2.1) and $\Gamma \subset G$ is an irreducible non-uniform lattice. This proves the abundance of bounded F-orbits under the assumptions (2.1), (2.2). More generally, by reducing to the case (2.1), (2.2) it is essentially shown in [116] that the set of bounded orbits for an ergodic homogeneous flow is thick iff the maximal quotient space $G/\overline{A\Gamma}$ of zero entropy (here A is the Auslander subgroup associated to g_1, see §2.1a) is compact.

Since all orbit closures of a unipotent flow are homogeneous, it follows by dimensional considerations that minimal sets for such a flow exist, and by Theorem 3.1.8, they are compact. Now, if $h_{\mathbb{R}}$ is a one-parameter subgroup with purely real eigenvalues of the operators Ad_{h_t}, it follows that the quotient flow $(G/\overline{A\Gamma}, h_{\mathbb{R}})$ is unipotent. Given a compact minimal subset $X \subset G/\overline{A\Gamma}$, let $Y \subset G/\Gamma$ be its full inverse. Then Y is a homogeneous space of finite volume and by above statement, the set of bounded orbits inside Y is thick therein. In particular, the flow $(G/\Gamma, h_{\mathbb{R}})$ has a bounded orbit.

By Lemma 4.3.2 below, one can reduce the general case to the case of purely real eigenvalues and derive the following

COROLLARY 4.1.7. *If* $\mathrm{vol}(G/\Gamma) < \infty$, *then every flow* $(G/\Gamma, g_{\mathbb{R}})$ *has a bounded orbit.*

The methods of proof of Theorem 4.1.6 are similar to those of Theorem 4.1.4; the only difference is that the F-translates of small neighborhoods of $Z \cup \{\infty\}$ are not 'topologically small' anymore (here small neighborhoods of ∞ are complements to large compact subsets of G/Γ). However, they are 'measure-theoretically small', and one can, as before, use a tessellation of G^+ to create a Cantor set which will have big enough Hausdorff dimension by virtue of mixing properties (1) and (2). See [116] or [114] for details.

Other results worth mentioning for comparison: the paper [71] of Dolgopyat on Anosov flows (by symbolic dynamics of Markov partitions) and work of Bishop and Jones [25], Stratmann [240] and Fernández and Melián [81] on bounded geodesics on rank-1 locally symmetric spaces (hyperbolic geometry being the main ingredient). Note that in all these cases the Hausdorff dimension of the set of points with bounded orbits is calculated in the infinite volume case as well. In particular, for geodesics on a rank-1 locally symmetric space $C\backslash G/\Gamma$, this dimension is equal to the *critical exponent* of Γ. The analogous question in the higher rank case remains untouched.

1c. *Excursions to infinity.* This subsection is, in some sense, complementary to §4.1b, since we are going to consider unbounded orbits; more precisely, unbounded with a certain 'rate of unboundedness'. This class of problems is motivated by the paper of Sullivan [242] about geodesic excursions to infinity on hyperbolic manifolds. We state the next theorem in the setting of the geodesic flow on the unit tangent bundle SM to a locally symmetric space $M \cong C\backslash G/\Gamma$. For $y \in M$, we denote by S_yM the set of unit vectors tangent to M at y, and, for $\xi \in S_yM$, we let $\gamma_t(y,\xi)$ be the geodesic on M through y in the direction of ξ. We have the following result [116]:

THEOREM 4.1.8. *Let M be a noncompact locally symmetric space of noncompact type and finite volume. Fix $y_0 \in M$ and let $\{r_t \mid t \in \mathbb{N}\}$ be an arbitrary sequence of real numbers. Then for any $y \in M$ and almost every* (*resp. almost no*) *$\xi \in S_yM$ there are infinitely many $t \in \mathbb{N}$ such that*

$$\mathrm{dist}\big(y_0, \gamma_t(y,\xi)\big) \geqslant r_t, \tag{4.4}$$

provided the series $\sum_{t=1}^{\infty} \mathrm{e}^{-kr_t}$, with $k = k(M)$ as in Theorem 1.3.4, *diverges* (*resp. converges*).

It follows from (4.4) that the above series is, up to a constant, the sum of volumes of the sets $A(r_t) \stackrel{\mathrm{def}}{=} \{y \in M \mid \mathrm{dist}(y_0, y) \geqslant r_t\}$; the latter sets can be viewed as a 'target shrinking to ∞' (cf. [98]), and Theorem 4.1.8 says that if the shrinking is slow enough (read: the sum of the volumes is infinite), then almost all geodesics approach infinity faster than the sets $A(r_t)$. Here it is convenient to introduce the following

DEFINITION 4.1.9. Let (X, μ) be a probability space. Say that a family $\mathcal{B}$ of measurable subsets of X is *Borel–Cantelli* for a sequence $g_{\mathbb{N}}$ of self-maps of X if for every sequence $\{A_t \mid t \in \mathbb{N}\}$ of sets from $\mathcal{B}$ one has

$$\mu\big(\big\{x \in X \mid g_t(x) \in A_t \text{ for infinitely many } t \in \mathbb{N}\big\}\big) = \begin{cases} 0 & \text{if } \sum_{t=1}^{\infty} \mu(A_t) < \infty, \\ 1 & \text{if } \sum_{t=1}^{\infty} \mu(A_t) = \infty. \end{cases}$$

(Note that the statement on top is always true in view of the classical Borel–Cantelli Lemma.)

One can see that Theorem 4.1.8 essentially amounts to the family of sets $\{(y, \xi) \in SM \mid \operatorname{dist}(y_0, y)) \geqslant r\}$, $r > 0$, being Borel–Cantelli for $\gamma_{\mathbb{N}}$. A choice $r_t = \frac{1}{\varkappa} \log t$, where $\varkappa$ is arbitrarily close to k, yields the following special case, which has been referred to (by Sullivan [242] in the setting of hyperbolic manifolds) as the *logarithm law for geodesics*:

COROLLARY 4.1.10. *For M as above, any $y \in M$ and almost all $\xi \in S_y M$,*

$$\limsup_{t \to \infty} \frac{\operatorname{dist}(y, \gamma_t(y, \xi))}{\log t} = 1/k.$$

In other words, almost all geodesics have a logarithmic rate or growth.

As was mentioned in §1.3e, the geodesic flow on SM can be seen as a special case of a partially hyperbolic infra-homogeneous flow (more precisely, SM is foliated by infra-homogeneous manifolds on which the geodesic flow is realized via action of one-parameter diagonalizable subgroups of G). It is therefore natural to expect a generalization of Theorem 4.1.8 written in the language of homogeneous flows on G/Γ. Throughout the end of the section we let G and Γ be as in (2.1), (2.2) (although some of the results below hold for reducible lattices as well), and denote G/Γ by X and the normalized Haar measure on X by ν. To describe sequences of sets 'shrinking to infinity' in X, we replace the distance function $\operatorname{dist}(y_0, \cdot)$ by a function $\Delta : X \mapsto \mathbb{R}$ satisfying certain properties. Namely, say that Δ is *DL* (an abbreviation for 'distance-like') if it is uniformly continuous, and the measure of sets $\{x \mid \Delta(x) \geqslant z\}$ does not decrease very fast as $z \to +\infty$, more precisely, if

$$\exists c, \delta > 0 \text{ such that } \nu\big(\{x \mid \Delta(x) \geqslant z + \delta\}\big) \geqslant c \cdot \nu\big(\{x \mid \Delta(x) \geqslant z\}\big)\ \forall z \geqslant 0. \tag{4.5}$$

For $k > 0$, we will also say that Δ is *k-DL* if it is uniformly continuous and in addition

$$\exists C_1, C_2 > 0 \text{ such that } C_1 \mathrm{e}^{-kz} \leqslant \nu\big(\big\{x \mid \Delta(x) \geqslant z\big\}\big) \leqslant C_2 \mathrm{e}^{-kz} \quad \forall z \in \mathbb{R}. \tag{4.6}$$

It is easy to show that (4.6) implies (4.5). The most important example is the distance function on X; Theorem 1.3.4 is exactly the k-DL property. Another example, important for number-theoretical applications, is the function Δ on $\Omega_k = \mathrm{SL}(k, \mathbb{R})/\mathrm{SL}(k, \mathbb{Z})$ introduced in §1.3d; the fact that it is k-DL immediately follows from Theorem 1.3.5.

The next theorem [118], essentially a generalization of Theorem 4.1.8, gives a way, using such a function Δ, to measure growth rate of almost all orbits of partially hyperbolic flows.

THEOREM 4.1.11. *Let Δ be a DL function on X and $g_{\mathbb{R}}$ a one-parameter partially hyperbolic subgroup of G. Then the family $\{\{x \in X \mid \Delta(x) \geqslant r\} \mid r > 0\}$ is Borel–Cantelli for $g_{\mathbb{N}}$.*

As before, it follows from the above statement that the function Δ evaluated on $g_{\mathbb{R}}$-orbit points grows logarithmically for almost all trajectories. Besides the proof of Theorem 4.1.8, it provides a new proof of the Khintchine–Groshev theorem in metric Diophantine approximation (see §5.2b).

A few words about the proof of Theorem 4.1.11: the main tool is a 'quasi-independent' Borel–Cantelli lemma: if a sequence $\{A_t\}$ of subsets of X satisfies a certain quasi-independence assumption (see [117, Lemma 2.6]) and the sum of their measures is infinite, then the 'upper limit' $\bigcap_n \bigcup_{t \geqslant n} A_t$ has full measure. One needs to show that the sequence $A_t = g_{-t}(\{x \mid \Delta(x) \geqslant r_t\})$ satisfies this assumption. This is done in two steps: first, using the DL property of Δ, one approximates the sets $\{x \mid \Delta(x) \geqslant r_t\}$ by smooth functions, and then one applies the exponential decay estimates of Theorem 4.1.2 to those functions and verifies the aforementioned assumption.

1d. *Divergent trajectories.* In the previous section we considered unbounded trajectories and studied their excursions to neighborhoods of infinity. Now we turn to *divergent* trajectories, i.e., eventually leaving every bounded subset of the space.

We start with the geodesic flow $(\Gamma\backslash G, g_{\mathbb{R}})$, where $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and $g_t = \mathrm{diag}(\mathrm{e}^t, \mathrm{e}^{-t})$. Let $U = U_{\mathbb{R}}$ be the subgroup of all strictly upper-triangular matrices (cf. §1.4e). Since the orbit ΓU is closed in $\Gamma\backslash G$, it follows that the positive trajectory $\{\Gamma g_t \mid t \geqslant 0\}$ is divergent (otherwise the contracting horocycle flow given by the U-action would have a fixed point). For the same reason, the negative trajectory $\{\Gamma g_t \mid t \leqslant 0\}$ is divergent as well. Hence the orbit $\Gamma g_{\mathbb{R}}$ is closed and noncompact.

It is not hard to show that, more generally, given a matrix $H_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$, the trajectory $\Gamma H_\alpha g_{\mathbb{R}_+}$ is divergent iff the number α is rational. This implies that divergent positive trajectories of the geodesic flow correspond precisely to periodic orbits of the contracting horocycle flow; hence nondivergent geodesic trajectories correspond to dense horocycle orbits.

This observation was generalized by Dani [56] as follows (here as before we switch to left actions on right homogeneous spaces):

THEOREM 4.1.12. *Let $g_{\mathbb{R}}$ be a reductive subgroup of a semisimple Lie group G, and let H be the contracting horospherical subgroup for $g_{\mathbb{R}}$. Assume that the horospherical flow $(G/\Gamma, H)$ is ergodic. Then given a point $x \in G/\Gamma$, the orbit Hx is dense in G/Γ whenever the positive trajectory $g_{\mathbb{R}_+}x$ is not divergent.*

(It follows from the theorem that horospherical flow on a compact homogeneous space is minimal whenever it is ergodic; see [247] and [33] for a stronger result.)

Now one can deduce the following result of Dani [52].

THEOREM 4.1.13. *Let $g_{\mathbb{R}}$ be a reductive subgroup in a Lie group G, and let Γ be a non-uniform lattice in G. Then the flow $(G/\Gamma, g_{\mathbb{R}})$ has divergent trajectories whenever it is ergodic.*

In fact, the proof can be easily reduced to the case when G and Γ are as in (2.1) and (2.2). Since $g_{\mathbb{R}}$ is reductive and acts ergodically on G/Γ, it follows that the horospherical flow $(G/\Gamma, H)$ is also ergodic. Note that according to Theorem 3.1.8 horospherical flow on a noncompact space cannot be minimal. Hence by Theorem 4.1.12, the flow $(G/\Gamma, g_{\mathbb{R}})$ has a divergent trajectory.

The existence of divergent trajectories for a nonreductive subgroup $g_{\mathbb{R}} \subset G$ has not been proved. However, the case $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ is well understood, see [52]:

THEOREM 4.1.14. *A one-parameter subgroup $g_{\mathbb{R}} \subset \mathrm{SL}(n, \mathbb{R})$ has a divergent trajectory on $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ if and only if it is partially hyperbolic.*

The proof of the 'if' part easily follows from Mahler's compactness criterion; the 'only if' direction is discussed in Section 3.1. Note that given $g_{\mathbb{R}}$ as in (4.1) and L_A as in (4.2), it was proved by Dani that the trajectory $g_{\mathbb{R}_+} L_A \mathrm{SL}(m+n, \mathbb{Z})$ is divergent iff the matrix A is *singular*. A real number (= one-by-one matrix) is singular iff it is rational. But a one-by-two matrix (a_1, a_2) can be singular even if the numbers $1, a_1$ and a_2 are linearly independent over $\mathbb{Q}$. This produces examples of so called *nondegenerate* divergent trajectories (see [52] or [237] for more details).

4.2. *Quasi-unipotent one-parameter flows*

2a. *Smoothness of orbit closures.* In what follows, Γ is a discrete subgroup of a Lie group G. We recall that the dynamics of a homogeneous flow $(G/\Gamma, g_{\mathbb{R}})$ depends drastically on whether or not the subgroup $g_{\mathbb{R}}$ is quasi-unipotent. Relative to the metric induced by a right-invariant metric on G, the rate of divergence of trajectories of a quasi-unipotent flow $(G/\Gamma, g_{\mathbb{R}})$ is polynomial whereas a partially hyperbolic flow has the exponential divergence of trajectories and is uniformly partially hyperbolic .

As was already said in Lemma 4.1.3, any partially hyperbolic one-parameter flow has an orbit closure that is not a manifold. On the contrary, all orbit closures of a quasi-unipotent flow are manifolds.

In fact, one can always treat orbits of a quasi-unipotent flow as 'twisted' orbits of a unipotent flow. For example, if G is a semisimple group with finite center, then for any quasi-unipotent subgroup $g_{\mathbb{R}} \subset G$ there exist a torus $T \subset G$ and a unipotent subgroup $u_{\mathbb{R}} \subset G$ such that $g_t = c_t \times u_t$, $t \in \mathbb{R}$, where $c_{\mathbb{R}}$ is a dense subgroup of the torus T (this is a Jordan decomposition of the subgroup $g_{\mathbb{R}}$ into reductive and unipotent parts). Thus, the subgroup $g_{\mathbb{R}}$ can be viewed as diagonal in the 'cylinder' $T \times u_{\mathbb{R}}$ with vertical axis $u_{\mathbb{R}}$.

In general, G can fail to admit such a decomposition. However, one can extend 'slightly' the group G to a group G^* which does admit the decomposition in question (see [227]).

LEMMA 4.2.1. *Let $g_{\mathbb{R}}$ be a quasi-unipotent subgroup in a connected simply connected Lie group G. Then there exist*

(1) *a torus $T \subset \mathrm{Aut}(G)$ commuting with $g_{\mathbb{R}}$, and*

(2) *a unipotent subgroup $h_{\mathbb{R}} \subset G^* = T \ltimes G$*

such that the projection $\pi : G^* \to G$ along T induces a compact cover of the quasi-unipotent flow $(G/\Gamma, g_{\mathbb{R}})$ by the unipotent flow $(G^*/\Gamma, h_{\mathbb{R}})$, $\Gamma \subset G$ being an arbitrary discrete subgroup. If $M' = \overline{h_{\mathbb{R}}x} \subset G^*/\Gamma$ is a homogeneous space of finite volume, then $M = \overline{g_{\mathbb{R}}x} \subset G/\Gamma$ is an (infra-homogeneous) submanifold with finite smooth $g_{\mathbb{R}}$-ergodic measure.

It is obvious that $\mathrm{vol}(G/\Gamma) < \infty \Leftrightarrow \mathrm{vol}(G^*/\Gamma) < \infty$. Hence given a lattice $\Gamma \subset G$, all orbits of the unipotent flow $(G^*/\Gamma, h_{\mathbb{R}})$ have homogeneous closures and are uniformly distributed. This implies that any orbit $g_{\mathbb{R}}x$ of quasi-unipotent flow on a finite volume space is also uniformly distributed (relative to the smooth measure on the manifold $M = \overline{g_{\mathbb{R}}x} \subset G/\Gamma$ obtained from H-invariant measure on the space $Hx = \overline{h_{\mathbb{R}}x} \subset G^*/\Gamma$). We recall that the manifold M can fail to be a homogeneous subspace (see §1.4b).

Now we can formulate a general criterion [227].

THEOREM 4.2.2. *Let Γ be a lattice in G. Then all orbit closures of a flow $(G/\Gamma, g_{\mathbb{R}})$ are smooth manifolds $\Leftrightarrow$ the subgroup $g_{\mathbb{R}}$ is quasi-unipotent. All orbits of a quasi-unipotent flow are uniformly distributed in their closures.*

2b. *Smoothness of ergodic measures.* Now let μ be a finite ergodic measure for a quasi-unipotent flow $(G/\Gamma, g_{\mathbb{R}})$ and let $M \subset G/\Gamma$ be its support. Then the flow $(M, g_{\mathbb{R}})$ is ergodic with respect to a strictly positive measure and hence possesses a dense orbit. By Lemma 4.2.1, the flow $(M, g_{\mathbb{R}})$ is compactly covered by a topologically transitive flow $(M', h_{\mathbb{R}})$, where $M' \subset G^*/\Gamma$ and $h_{\mathbb{R}}$ is a unipotent subgroup of G^*. Weil's construction of the semidirect product of measures (cf. [174]) provides us with a finite $h_{\mathbb{R}}$-invariant measure on M' that projects to the measure μ. Let μ'' be the ergodic component of μ' that projects to μ. Then μ'' is an H-invariant measure on a closed orbit $Hx \subset M' \subset G^*/\Gamma$. Hence μ is a smooth measure on the manifold $M \subset G/\Gamma$. We have proved the following statement:

COROLLARY 4.2.3. *Let μ be a finite ergodic measure for a quasi-unipotent flow $(G/\Gamma, g_{\mathbb{R}})$. Then there exists a smooth manifold $M \subset G/\Gamma$ such that μ is supported on M and the restriction of μ onto M is a smooth measure.*

We recall once more that according to Sinai [212] and Bowen [32], any Anosov flow has uncountably many ergodic measures, those with their support not locally connected, and those that are strictly positive (but singular to smooth volume measure).

One can formulate the following

CONJECTURE 4.2.4. *Any partially hyperbolic flow on a finite volume homogeneous space has a finite ergodic measure whose support is not a smooth manifold. If the flow is ergodic with respect to Haar measure, then there exists an ergodic strictly positive singular measure.*

2c. *Measure rigidity.* Here we consider the measure rigidity property of quasi-unipotent flows. According to Theorem 3.8.1, ergodic unipotent flows are *measure rigid*: any measure-theoretical isomorphism is (almost everywhere) an affine map of homogeneous spaces. On the contrary, partially hyperbolic flows do not possess this property. In fact, given any two lattices $\Gamma_1, \Gamma_2 \subset \mathrm{SL}(2, \mathbb{R})$, the geodesic flows on the corresponding homogeneous spaces, being Bernoullian flows with equal entropies, are always measure-theoretically isomorphic [168]. But Γ_1 and Γ_2 need not be conjugate in $\mathrm{SL}(2, \mathbb{R})$.

On the other hand, by Theorem 2.2.5, an ergodic quasi-unipotent flow on a compact Euclidean manifold is smoothly and measure-theoretically isomorphic to a rectilinear flow on a torus. Hence measure rigidity does not take place in the quasi-unipotent case either.

However, as Witte [255] proved, measure rigidity holds if the quasi-unipotent flows involved are mixing:

THEOREM 4.2.5. *Any measure-theoretic isomorphism of mixing quasi-unipotent flows is affine almost everywhere.*

It turns out that although an isomorphism of quasi-unipotent flows need not be an affine map globally, it is 'fiber-wise' affine [232]. In what follows, G_∞ is the minimal normal subgroup of G such that the factor-group G/G_∞ is solvable.

THEOREM 4.2.6. *Let $f : (G/\Gamma, g_{\mathbb{R}}) \to (G'/\Gamma', g'_{\mathbb{R}})$ be a measure-theoretic isomorphism of ergodic quasi-unipotent flows. Then for almost all $g\Gamma \in G/\Gamma$ one has $f(\overline{G_\infty g\Gamma}) = \overline{G'_\infty f(g\Gamma)}$, and the restriction of f onto almost every homogeneous subspace $\overline{G_\infty g\Gamma}$ is affine. In particular, f induces an isomorphism of ergodic quotient flows*

$$\hat{f} : \left(G/\overline{G_\infty \Gamma}, g_{\mathbb{R}}\right) \to \left(G'/\overline{G'_\infty \Gamma'}, g'_{\mathbb{R}}\right)$$

on solvable homogeneous spaces.

Note that Witte's Theorem (Theorem 4.2.5) follows immediately because solvable homogeneous spaces admit no mixing homogeneous flows.

4.3. *Invariant sets of one-parameter flows*

3a. *Nondivergence inside the neutral leaf.* Sometimes one can ensure that an individual orbit of a partially hyperbolic flow $(G/\Gamma, g_{\mathbb{R}})$ has smooth (or even algebraic) closure. For instance, this applies to orbits that 'do not exit to infinity inside the neutral leaf' (this generalizes Ratner's topological theorem). The result proves to be helpful for classification of minimal sets of homogeneous flows.

More precisely, let Q be the neutral subgroup for $g_{\mathbb{R}}$ (see §2.1a). Note that $g_{\mathbb{R}} \subset Q$ and $g_{\mathbb{R}}$ is always quasi-unipotent in Q. We say that an orbit $g_{\mathbb{R}}x \subset G/\Gamma$ *does not exit to infinity inside the neutral leaf* $Qx \subset G/\Gamma$ if there exist a sequence $\{t_n\} \to \infty$ and a compact subset $K \subset Q$ such that $g_{t_n}x \in Kx$ for all n. One can prove the following ([231]):

THEOREM 4.3.1. *Let* $\mathrm{vol}(G/\Gamma) < \infty$ *and let an orbit* $g_{\mathbb{R}}x \subset G/\Gamma$ *do not exit to infinity inside the neutral leaf* $Qx \subset G/\Gamma$. *Then the closure* $\overline{g_{\mathbb{R}}x} \subset G/\Gamma$ *is a smooth manifold with finite smooth ergodic measure*.

The proof involves a statement similar to Lemma 4.2.1.

LEMMA 4.3.2. *Let* $g_{\mathbb{R}}$ *be a one-parameter subgroup in a simply connected Lie group* G. *Then there exist a torus* $T \subset \mathrm{Aut}(G)$ *and a subgroup* $h_{\mathbb{R}} \subset G^* = T \ltimes G$ *such that all eigenvalues of the operators* Ad_{h_t} *on* $\mathfrak{g}^*$ *are purely real, and each closure* $\overline{g_{\mathbb{R}}x} \subset G/\Gamma$ *is compactly covered by the closure* $\overline{h_{\mathbb{R}}x} \subset G^*/\Gamma$. *If the second closure is homogeneous and has finite volume, then the first closure is a manifold and carries a finite smooth ergodic measure*.

Now Theorem 4.3.1 is a corollary of the following [231]:

THEOREM 4.3.3. *Let* $\mathrm{vol}(G/\Gamma) < \infty$ *and let a subgroup* $g_{\mathbb{R}}$ *be such that all eigenvalues of the operators* Ad_{g_t} *on* $\mathfrak{g}$ *are purely real*. *Then if an orbit* $g_{\mathbb{R}}x \subset G/\Gamma$ *does not exit to infinity inside the neutral leaf* $Qx \subset G/\Gamma$, *then the closure* $\overline{g_{\mathbb{R}}x} \subset G/\Gamma$ *is a homogeneous space of finite volume*.

Note that this result generalizes Ratner's topological Theorem 3.3.6 for one-parameter unipotent flows. In fact, if $\mathrm{vol}(G/\Gamma) < \infty$, then by Theorem 3.1.1, no trajectory of a unipotent flow exits to infinity. Theorem 4.3.3 is not an immediate corollary of Ratner's theorem because the neutral leaf $Qx \subset G/\Gamma$ can be nonclosed and as a homogeneous Q-space it may have infinite Q-invariant measure.

Theorem 4.3.3 is proved via multiple application of Ratner's theorem and reducing the semisimple, solvable and general cases to the arithmetic case. Note that an orbit of the horocycle flow on a surface of constant curvature -1 and of infinite area may not exit to infinity and at the same time not come back to a compact set with positive density of times (this concerns, for example, recurrent nonperiodic orbits of the horocycle flow on a surface of the second kind, see [230]). Hence Theorem 4.3.3 does not follow from Theorem 3.3.11.

3b. *Minimal sets*. Now we are ready to give a classification of minimal sets of homogeneous flows. First we introduce a new class of invariant sets.

Let $(X, g_{\mathbb{R}})$ be a continuous flow on a locally compact space X. Then a closed invariant set $M \subset X$ is called *birecurrent* if each orbit inside M is recurrent (comes back arbitrarily close to the original point) in both directions.

It is clear that a compact minimal set is birecurrent. On the other hand, a compact birecurrent set need not be minimal even if it is topologically transitive. In fact, a compact homogeneous space of finite volume is birecurrent relative to any unipotent flow, but an ergodic unipotent flow can fail to be minimal. (An example: take a uniform lattice $\Gamma \subset \mathrm{SL}(4,\mathbb{R})$ that intersects $\mathrm{SL}(2,\mathbb{R})$ in a lattice, and let $g_{\mathbb{R}} \subset \mathrm{SL}(2,\mathbb{R})$ be a unipotent subgroup. Then the flow $(\mathrm{SL}(4,\mathbb{R})/\Gamma, g_{\mathbb{R}})$ is ergodic and not minimal.)

The proof of the result to follow is modelled over Lemma 4.1.3, see [231].

LEMMA 4.3.4. *Let $(X, g_{\mathbb{R}})$ be a uniformly partially hyperbolic flow on a manifold X with birecurrent set $M \subset X$. Let $O(x) \subset X$ be a small enough neighborhood of a point $x \in M$. Then the connected component of the point x in the intersection $M \cap O(x)$ belongs to the neutral leaf of $x \in X$.*

Theorem 4.3.3 and Lemma 4.3.4 immediately imply the following [231]:

THEOREM 4.3.5. *Let M be a topologically transitive birecurrent set for a flow $(G/\Gamma, g_{\mathbb{R}})$ on a homogeneous space of finite volume. Then either M is nowhere locally connected, or M is a smooth submanifold in G/Γ.*

Now assume that M is a minimal set of a continuous flow $(X, g_{\mathbb{R}})$. Then either all semiorbits inside M do not exit to infinity, or one of them does. Theorem 3.1.9 asserts that in the first case M is a compact set. Now we can give a classification of minimal sets of homogeneous flows [231].

THEOREM 4.3.6. *Let M be a minimal set of a homogeneous flow $(G/\Gamma, g_{\mathbb{R}})$. Then one of the following statements holds*:

(1) *$M = g_{\mathbb{R}}x$ is an orbit exiting to infinity in both directions*;
(2) *there exists a point $x \in M$ such that one of its semiorbits exits to infinity, and the other semiorbit is recurrent*;
(3) *M is a compact nowhere locally connected set*;
(4) *M is a compact smooth submanifold in G/Γ such that the flow $(M, g_{\mathbb{R}})$ is compactly covered by a uniquely ergodic unipotent flow.*

Note that the theorem holds true for arbitrary discrete subgroups $\Gamma \subset G$. If G/Γ fails to have finite volume, then instead of Ratner's theorem one can apply Theorem 3.3.11 of Dani and Margulis which ensures that compact closure of a unipotent orbit is always homogeneous.

Orbits of the geodesic flow exiting to infinity in both directions demonstrate the first type of minimal sets in our classification. Morse minimal sets illustrate the third type, and periodic orbits the last type. It has been recently proved in [69] that, assuming the surface has finite area and is noncompact, the geodesic flow has minimal sets of the second type as well.

Now let $(G/\Gamma, g_{\mathbb{R}})$ be an ergodic flow on a homogeneous space of finite volume. The existence of compact minimal sets for any homogeneous flow follows from Corollary 4.1.7. The question arises: does an ergodic partially hyperbolic flow have minimal sets of all possible types? Namely, does it have minimal sets of the third and forth types in the compact case and of all four types in the noncompact case (provided that the flow has the K-property)?

Note that the closure of a horocycle orbit on a surface of infinite area need not be homogeneous even if it is minimal. In fact, following Hedlund it is easy to prove (see [230]) that the nonwandering set Ω for the horocycle flow $(\mathrm{SL}(2, \mathbb{R})/\Gamma, u_{\mathbb{R}})$ is minimal if the Fuchsian group Γ is finitely generated and has no parabolic elements (the same follows from [34]). If Γ is not a uniform lattice then the minimal set Ω is of the second type in our classification: otherwise it would be compact and hence consist of one periodic orbit (which is impossible because Γ contains no unipotent elements). We note also that in this case Ω is nowhere locally connected.

Combining Corollary 4.1.7 and Theorem 4.3.6 one derives the following statement [231]:

THEOREM 4.3.7. *A one-parameter homogeneous flow on a space of finite volume is minimal if and only if it is uniquely ergodic*.

We emphasize a contrast between the class of homogeneous flows and the more general class of smooth flows. As was mentioned, a homogeneous flow on a compact space can be minimal only if it has zero entropy (i.e., is quasi-unipotent). On the other hand, Herman [97] has constructed a minimal diffeomorphism of a compact manifold with positive entropy. Further, according to Theorem 4.3.6, any compact minimal set of a homogeneous flow is locally connected either everywhere or nowhere. The first example of a homeomorphism of the plane with a minimal set that is locally connected on a proper non-empty subset was given in [192]. A smooth flow with a similar minimal set was constructed by Johnson [104] by making use of quasi-periodic systems of Millionshchikov [154] and Vinograd [250].

3c. *Rectifiable sets*. Now we describe results of Zeghib on 'rectifiable' invariant subsets for the action of an $\mathbb{R}$-diagonalizable subgroup $g_{\mathbb{R}} \subset G$. Ergodic components of Hausdorff measure (of the corresponding dimension) on such a subset turn out to be algebraic. Sometimes this enables one to establish the algebraicity of closed invariant sets under rather weak smoothness conditions.

First, several definitions from geometric measure theory [80] are in order. Let (X, d) be a measure space and $n \in \mathbb{N}$. One says that a subset $Y \subset X$ is *n-rectifiable* if $Y = f(A)$, where $A \subset \mathbb{R}^n$ is a bounded subset and $f : A \mapsto X$ is a Lipschitz map.[16] A countable union of n-rectifiable subsets of X is called *σ-n-rectifiable*.

As an illustration, we consider the case $n = 1$. It is easily seen that a 1-rectifiable curve in $\mathbb{R}^2$ has finite length. At the same time, one can construct a homeomorphism of the segment $[0, 1]$ to the image in $\mathbb{R}^2$ which is of infinite length (such a curve can be σ-1-rectifiable). The square $[0, 1] \times [0, 1]$ is not a σ-1-rectifiable set but it is a continuous image of the segment $[0, 1]$ (the Peano curve).

Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure on X. A subset $Y \subset X$ is called *$\mathcal{H}^n$-rectifiable*, if $\mathcal{H}^n(Y) < \infty$ and there exists a σ-n-rectifiable subset $Y' \subset X$ such that $\mathcal{H}^n(Y - Y') = 0$. If in addition $\mathcal{H}^n(Y) > 0$, then the Hausdorff dimension of Y is n (but not every subset $Y \subset X$ of Hausdorff dimension n is n-rectifiable).

A nontrivial example of an $\mathcal{H}^1$-rectifiable subset Y of the square $[0, 1]^2$ is given by a countable union $\bigcup_i \{a_i\} \times B_i$, where $a_i \in [0, 1]$ and $B_i \subset [0, 1]$ is a measurable subset with $\sum_i l(B_i) < \infty$.

[16] A map $f : (A, d_A) \mapsto (X, d_X)$ of two metric spaces is said to be *Lipschitz* if there exists a constant $C > 0$ such that $d_X(f(a), f(b)) \leqslant C d_A(a, b)$ for all $a, b \in A$.

Now let X be a Riemannian manifold. There holds [80] the following criterion: a subset $Y \subset X$ is $\mathcal{H}^n$-rectifiable $\Leftrightarrow$ it belongs modulo a set of $\mathcal{H}^n$-measure 0 to a countable union of n-dimensional C^1-submanifolds in X.

Now we are in a position to formulate results of [262] on the structure of invariant rectifiable subsets for a flow $(K\backslash G/\Gamma, a_{\mathbb{R}})$, where $a_{\mathbb{R}}$ is an $\mathbb{R}$-diagonalizable subgroup that commutes with a compact subgroup $K \subset G$. Here Γ is an arbitrary discrete subgroup of G.

THEOREM 4.3.8. *Let* $(K\backslash G/\Gamma, a_{\mathbb{R}})$ *be an* $\mathbb{R}$*-diagonalizable flow and let* $Y \subset K\backslash G/\Gamma$ *be an* $\mathcal{H}^n$*-rectifiable invariant subset for some* $n \leqslant \dim(K\backslash G)$. *Then the flow* $(Y, \mathcal{H}^n, a_{\mathbb{R}})$ *is measure-preserving and all its ergodic components are algebraic.*

Having in mind Ratner's measure theorem and Corollary 4.2.3, it is natural to formulate the following.

CONJECTURE 4.3.9. *Let* $(K\backslash G/\Gamma, g_{\mathbb{R}})$ *be an infra-homogeneous flow and let* $Y \subset K\backslash G/\Gamma$ *be an* $\mathcal{H}^n$*-rectifiable invariant subset. Then the flow* $(Y, \mathcal{H}^n, g_{\mathbb{R}})$ *is measure-preserving and all its ergodic components are smooth manifolds.*

Theorem 4.3.8 can be made more precise. More specifically, one can describe the subset Y modulo a zero measure set. Let $R(a_{\mathbb{R}}, \Gamma)$ be the collection of closed subgroups $H \subset G$ such that $\Gamma \cap H$ is a lattice in H, and $g^{-1} a_{\mathbb{R}} g \subset H$ for some $g \in G$. The following was proved by Zeghib [262,263]:

THEOREM 4.3.10. *Under the assumptions of Theorem* 4.3.8 *there exist countable collections of elements* $g_i \in G$, *subgroups* $H_i \in R(a_{\mathbb{R}}, \Gamma)$ *and subsets* $S_i \subset G$ *such that each* S_i *commutes with* $a_{\mathbb{R}}$ *and is* $\mathcal{H}^k$*-rectifiable for some* $k = k(i)$, *and* $Y = \bigcup_i K S_i g_i H_i \Gamma$ *modulo a set of zero* $\mathcal{H}^n$*-measure.*

It is clear that ergodic components in the theorem are (locally homogeneous) subspaces of finite volume $K y g_i H_i \Gamma$, $y \in S_i$.

Here is an example illustrating the theorem. Let $a_{\mathbb{R}} \subset H \subset G$, where $\Gamma \cap H$ is a lattice in H, and $\overline{a_{\mathbb{R}^+}\Gamma} = H\Gamma \subset G/\Gamma$. Suppose that $\dim H = n > 1$ and that outside H there exists an element $u \in G$ for which $a_t u a_{-t} \to 1$, $t \to +\infty$ (such example can be easily constructed). Then the limit set for the semiorbit $a_{\mathbb{R}^+} u \Gamma$ is the homogeneous subspace $H\Gamma$. The closure $Y = \overline{a_{\mathbb{R}^+} u \Gamma}$ is an $\mathcal{H}^n$-rectifiable set invariant under the semigroup $a_{\mathbb{R}^+}$. The restriction of $\mathcal{H}^n$ onto Y is an ergodic measure supported on the homogeneous subspace $H\Gamma$. But the set Y is not homogeneous. In a similar fashion one can construct a full orbit $a_{\mathbb{R}} x$ having two distinct limit sets which are homogeneous subspaces of finite volume of dimension $n > 1$. Then the closure $Y = \overline{a_{\mathbb{R}} x}$ is an $\mathcal{H}^n$-rectifiable set with two ergodic homogeneous components.

In the case when $(K\backslash G/\Gamma, a_{\mathbb{R}})$ is an Anosov flow, the results obtained have especially simple formulation. For instance, let G be a simple group of $\mathbb{R}$-rank 1 and let $K \subset G$ be a maximal compact subgroup that commutes with $a_{\mathbb{R}}$ (the flow is none other than the geodesic flow on the unit tangent bundle $T^1 M$ of locally symmetric manifold M of negative curvature). Since in this case the centralizer of $a_{\mathbb{R}}$ in G is $Ka_{\mathbb{R}}$, it follows that an $\mathcal{H}^n$-invariant subset Y is a countable union of closed subspaces of finite volume:

$$Y = \bigcup_i Y_i \ (\mathrm{mod}\ 0), \quad \text{where } Y_i = Kg_i H_i \Gamma.$$

Moreover, it is proved in [262] that the decomposition has finitely many members (this follows from the condition $\mathcal{H}^n(Y) < \infty$), and each component is the unit tangent bundle of some totally geodesic submanifold of finite volume $W_i \subset M$. Hence $Y = T^1 W$ (mod 0), where $W = \bigcup_i W_i$ is a closed (not necessarily connected) totally geodesic submanifold in M.

4.4. *On ergodic properties of actions of connected subgroups*

We know that given a connected subgroup $H \subset G$ generated by unipotent elements, all finite ergodic measures and orbit closures on G/Γ (if $\mathrm{vol}(G/\Gamma) < \infty$) have an algebraic origin. If H is generated by quasi-unipotent elements, then it is not difficult to construct a compact extension G^* of the group G such that our subgroup H belongs to a compact extension of a connected subgroup $U \subset G^*$ generated by unipotent elements. This easily implies the following:

THEOREM 4.4.1. *Let H be a connected subgroup of G generated by quasi-unipotent elements. Then any finite H-invariant ergodic measure on G/Γ is a smooth measure supported on a submanifold in G/Γ. If Γ is a lattice in G then any orbit closure of the H-action on G/Γ is a smooth manifold.*

Now we consider the 'mixed' case where H is generated by unipotent and $\mathbb{R}$-diagonalizable elements.

Let $U \subset G$ be a unipotent one-parameter subgroup, and $A \subset G$ an $\mathbb{R}$-diagonalizable one-parameter subgroup that normalizes U. One says that A is *diagonal* for U if there exists a connected subgroup $S = S(U, A) \subset G$ locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ that contains $F = AU$. In the course of the proof of Theorem 3.3.2 Ratner discovered that any finite F-invariant ergodic measure μ on G/Γ is S-invariant. Since S is generated by unipotent one-parameter subgroups, it follows that μ is an algebraic measure. This can be generalized as follows [188].

THEOREM 4.4.2. *Assume that a connected subgroup $U \subset G$ is generated by unipotent elements and that subgroups $A_1, \ldots, A_n \subset G$ are diagonal for unipotent subgroups $U_1, \ldots, U_n \subset U$ respectively. Let $F \subset G$ be generated by U and all $A_1, \ldots, A_n$. Then each finite F-ergodic invariant measure on G/Γ is algebraic.*

If $\mathrm{vol}(G/\Gamma) < \infty$, then all closures of the F-orbits on G/Γ are homogeneous subspaces of finite volume in G/Γ.

Clearly, here ergodic measures and orbit closures for F are those for the subgroup $H \subset G$ generated by U and all $S(U_i, A_i)$.

Note that the assumptions of the theorem are verified if $F \subset G$ is a parabolic subgroup of a totally noncompact semisimple group $H \subset G$; see Theorem 3.7.6 for an extension of this result.

4a. *Epimorphic subgroups*. Ratner's Theorem 4.4.2 can be generalized to a larger class of subgroups. We start with a definition.

DEFINITION 4.4.3. Let $H \subset \mathrm{SL}(n, \mathbb{R})$ be a real algebraic group. A subgroup F of H is called *epimorphic* in H if any F-fixed vector is also H-fixed for any finite-dimensional algebraic linear representation of H.

Epimorphic subgroups were introduced by Bergman [18], and their in-depth study was made by Bien and Borel [23,24]. We note some examples of epimorphic subgroups: (i) a parabolic subgroup of totally noncompact semisimple group; (ii) a Zariski dense subgroup of a real algebraic group; (iii) the subgroup $H = g^{\mathbb{Z}} G^{+}$ of G as in the statement of Theorem 3.7.6 (see Lemma 3.7.9). It may be noted that any $\mathbb{R}$-split simple real algebraic group contains an algebraic epimorphic subgroup $F = A \ltimes U$, where A is $\mathbb{R}$-diagonalizable and U is the unipotent radical of F with $\dim A = 1$ and $\dim U \leqslant 2$ (see [23, 5(b)]).

It was observed by Mozes [160] that the above definition of epimorphic subgroups reflects very well in the ergodic properties of the subgroup actions on homogeneous spaces:

THEOREM 4.4.4. *Let G be real algebraic group. Let H be a real algebraic subgroup of G which is generated by algebraic unipotent one-parameter subgroups. Let F be a connected epimorphic subgroup of H which is generated by $\mathbb{R}$-diagonalizable and algebraic unipotent subgroups. Then any finite F-invariant Borel measure on G/Γ is H-invariant, Γ being a discrete subgroup of G.*

The above theorem is proved using Ratner's measure theorem and the Poincaré recurrence theorem. Clearly, the invariant measure part of Theorem 4.4.2 is contained in it as a particular case.

The topological counterpart of Theorem 4.4.4 was proved by Shah and Weiss [206] (cf. Theorem 3.7.6).

THEOREM 4.4.5. *Let $F \subset H \subset G$ be an inclusion of real algebraic groups such that F is epimorphic in H. Let Γ be a lattice in G. Then any F-invariant closed subset of G/Γ is also H-invariant. Further, if H is generated by unipotent one-parameter subgroups, then F has property-(D) on G/Γ (see* §3.1a).

Earlier this was proved by Weiss [252] in the case when $G = H$ is defined over $\mathbb{Q}$ and $\Gamma = G_{\mathbb{Z}}$ (not necessarily a lattice). In this case the action of F on G/Γ was shown to be minimal. From this Weiss derived the following (cf. Proposition 1.2.1):

THEOREM 4.4.6. *Let $F \subset G$ be an inclusion of real algebraic $\mathbb{Q}$-groups. Then $\overline{F G_{\mathbb{Z}}} = H G_{\mathbb{Z}}$, where H is the largest algebraic subgroup of G such that F is epimorphic in H.*

Note that for non-algebraic epimorphic subgroups $F \subset G$, the action of F on G/Γ need not be minimal.

EXAMPLE 4.4.7 (*Raghunathan*). Let $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$, and $F = g_{\mathbb{R}} \ltimes (U_1 \times U_2)$, where U_i is the unipotent subgroup of strictly upper-triangular matrices in the corresponding copy of $\mathrm{SL}(2,\mathbb{R})$ and

$$g_t = \mathrm{diag}\big(\mathrm{e}^t, \mathrm{e}^{-t}, \mathrm{e}^{-\alpha t}, \mathrm{e}^{\alpha t}\big),$$

where $\alpha > 0$ is an irrational number. Note that the conjugation $x \mapsto g_1 x g_{-1}$ expands U_1 and contracts U_2, and the orbits $U_1\Gamma$ and $U_2\Gamma$ are compact. Hence the orbit $F\Gamma \subset G/\Gamma$ is closed (otherwise an orbit of some subgroup U_i would degenerate to a point). Therefore the action $(G/\Gamma, F)$ is not minimal. On the other hand, since the number α is irrational, the subgroup F is Zariski dense in the 4-dimensional parabolic subgroup $B \subset G$ and hence epimorphic in G. Hence by Theorem 4.4.4, the action is uniquely ergodic. Note that in this example the homogeneous subspace $F\Gamma$ is of infinite volume.

For a uniform lattice $\Gamma \subset G$, Weiss [253] obtained additional information:

THEOREM 4.4.8. *Let $F \subset H \subset G$ be as in Theorem* 4.4.5. *In addition, assume that $F = AU$, where A is an $\mathbb{R}$-diagonalizable Abelian group and U is the unipotent radical of F, and that H is normal in G. Let Γ be a uniform lattice in G. Then any U-invariant closed subset in G/Γ, as well as any finite U-invariant measure, is also H-invariant.*

In particular, if the H-action on G/Γ is topologically transitive, then the U-action is minimal and uniquely ergodic.

It follows that any $\mathbb{R}$-split simple real algebraic group contains a unipotent subgroup U of dimension at most 2 which acts uniquely ergodically on any space G/Γ, where Γ is a uniform lattice. Note that $G = \mathrm{SL}(3,\mathbb{R})$ has no one-dimensional unipotent subgroups with this property (see [253]).

4b. *Reduction to the Abelian case.* Let $H \subset G$ be a connected subgroup, and $\Gamma \subset G$ a discrete subgroup. We will expose a reduction of the study of H-invariant probability measures on G/Γ to the case when H is Abelian.

Let $H_u \subset H$ be the normal subgroup of H generated by all its unipotent elements. Due to Ratner's theorem (Corollary 3.3.5), all H_u-invariant ergodic probability measures are homogeneous.

Let us say that subgroup $H \subset G$ is of Ad-*triangular type* if H is generated by elements $h \in H$ such that all eigenvalues of Ad_h are purely real. It is easily seen that H is of Ad-triangular type $\Leftrightarrow$ the Zariski closure $\mathrm{Zcl}(\mathrm{Ad}(H)) \subset \mathrm{Aut}(\mathfrak{g})$ decomposes into a semidirect product $(S \times A) \ltimes U$, where U is the unipotent radical, S is a totally noncompact semisimple Lie group, and A is an $\mathbb{R}$-diagonalizable Abelian group. Clearly, for such an H one has $\mathrm{Ad}[H,H] \subset \mathrm{Ad}(H_u)$ and hence the quotient group H/H_u is Abelian.

One also has the following [148,160,234]:

THEOREM 4.4.9. *Let Γ be a discrete subgroup of G, $H \subset G$ a connected subgroup of* Ad*-triangular type, and μ an H-invariant ergodic probability measure on G/Γ. Then there exists a connected closed subgroup $L \subset G$ such that μ-a.e. H_u-ergodic component of μ is an L-invariant probability measure on a closed L-orbit in G/Γ. Now let*

$$P = \{p \in N_G(L) \mid \det(\operatorname{Ad} p|_{\mathrm{Lie}(L)}) = 1\}.$$

Then $H \subset P$ and there exists a point $x \in G/\Gamma$ such that the orbit $Px \subset G/\Gamma$ is closed and $\mu(Px) = 1$.

It follows that the homogeneous space Px is H-invariant and supports our measure μ. Let $\alpha : P \mapsto \widehat{P} = P/L$ and $\widehat{H} = \alpha(H)$. Note that all L-orbits inside Px are closed and form a bundle over a homogeneous $\widehat{P}$-space. Also μ is L-invariant and projects to an $\widehat{H}$-invariant measure $\hat{\mu}$ on the $\widehat{P}$-space. Therefore, the study of the H-invariant measure μ reduces to that of the $\widehat{H}$-invariant measure $\hat{\mu}$, where $\widehat{H}$ is Abelian because $H_u \subset L$.

Note that here $\widehat{H}$ is Ad-triangular in P. If it contains unipotent elements in P, one can repeat the procedure. As a result, one can always come to the situation when the subgroup in question contains no unipotent elements (i.e., all its elements are partially hyperbolic and have purely real eigenvalues).

Apparently, a similar reduction must exist for the study of the closures of H-orbits on finite volume spaces.

Now let $H \subset G$ be an arbitrary connected subgroup. Sometimes H splits into a semidirect product $H = C \ltimes H'$ of a compact subgroup C and a normal subgroup H' of Ad-triangular type. If so, this allows one to describe H-invariant measures via H'-invariant ones.

In the general case one can play the following game. Assume with no loss of generality that G is connected and simply connected. Suppose first that H is solvable. Consider the algebraic hull $\mathrm{Zcl}(\mathrm{Ad}(H)) \subset \mathrm{Aut}(\mathfrak{g})$. Then $\mathrm{Zcl}(\mathrm{Ad}(H)) = T \ltimes M$, where T is a compact torus and M is a normal triangular subgroup. Now take $\widetilde{G} = T \ltimes G$ and $\widetilde{H} = T \ltimes H$. It can be easily proved that $\widetilde{H} = T \ltimes H'$, where $H' \subset \widetilde{H}$ is a normal subgroup of Ad-triangular type. Note that H-invariant measures on G/Γ are in one-to-one correspondence with $\widetilde{H}$-invariant measures on $\widetilde{G}/\Gamma$. The same concerns orbits.

For a general $H = L_H R_H$ one does the same with its radical R_H and notes that the torus T can be chosen to commute with $L_H = K_H S_H$. Moreover, $T \ltimes R_H = T \ltimes R'_H$, where R'_H is a normal subgroup of $\widetilde{H} = T \ltimes H$ of Ad-triangular type. Then $\widetilde{H} = CH'$, where $C = K_H \times T$ is compact and $H' = S_H R'_H \subset \widetilde{H}$ is a normal subgroup of Ad-triangular type. Again, the study of invariant measures and orbit closures for the action $(G/\Gamma, H)$ is equivalent to that for the action $(\widetilde{G}/\Gamma, \widetilde{H})$.

4c. *Actions of multi-dimensional Abelian subgroups.* In the previous subsection we came (from the measure-theoretic point of view) to the situation when the acting subgroup (call it A) is Abelian and consists of partially hyperbolic elements.

If $\dim A \geqslant 2$, the A-action is called a *higher rank Abelian action* and very little is known about it in the general case. We concentrate on the most interesting case. Take $G = \mathrm{SL}(k, \mathbb{R})$, $\Gamma = \mathrm{SL}(k, \mathbb{Z})$ and let A be the group of all positive diagonal matrices in G.

If $k=2$ then $\dim A=1$ and the A-action on G/Γ is none other than the geodesic flow. In particular, the A-action in this case has a lot of 'bad' orbits (and ergodic measures); see §1.4e.

On the other hand, for $k \geqslant 3$, only few types of A-orbits (including closed and dense) are known, and all of them have homogeneous closures. It was conjectured by Margulis that any relatively compact A-orbit is compact (see Conjecture 5.3.3). As was (implicitly) observed in [37], this would prove an old number-theoretic conjecture of Littlewood, which we state in the next chapter as Conjecture 5.3.1; see Section 5.3 for more details. The following result due to Weiss and Lindenstrauss [127] provides interesting evidence in this direction:

THEOREM 4.4.10. *Let A denote the group of all positive diagonal matrices in* $\mathrm{SL}(k,\mathbb{R})$, $k \geqslant 3$. *Let $x \in \Omega_k$ be such that for some point $y \in \overline{Ax}$, the orbit Ay is compact. Then there are integers l and d with $k = ld$ and a permutation matrix σ such that $\overline{Ax} = Fy$, where*

$$F = \left\{\sigma \operatorname{diag}(B_1,\ldots,B_d)\sigma^{-1} \mid B_i \in \mathrm{GL}(l,\mathbb{R})\right\} \cap \mathrm{SL}(k,\mathbb{R}).$$

Moreover, if $\overline{Ax}$ is compact then Ax is compact.[17]

On the other hand, given an arbitrary lattice $\Gamma \subset G = \mathrm{SL}(k,\mathbb{R})$, sometimes one can find other types of A-orbits on G/Γ. For instance, as was observed by Rees (see [127] or [108]), one can construct a uniform lattice $\Gamma \subset \mathrm{SL}(3,\mathbb{R})$ that intersects two commuting subgroups $L, A' \subset G$ in lattices, where $L \simeq \mathrm{SL}(2,\mathbb{R})$ and A' is $\mathbb{R}$-diagonalizable and 1-dimensional. Now if $A'' \subset L$ is $\mathbb{R}$-diagonalizable then the A''-action on $L\Gamma$ is the geodesic flow. Clearly, if $A = A' \times A''$ then any 'bad' A''-orbit (or A''-ergodic measure) inside $L\Gamma \subset G/\Gamma$ gives rise to a 'bad' A-orbit (or A-ergodic measure) inside $(L \times A')\Gamma$.

This led Margulis to the following conjecture[18] (see [144]):

CONJECTURE 4.4.11. *Given a Lie group G with a lattice $\Gamma \subset G$ and an $\mathbb{R}$-diagonalizable connected subgroup $A \subset G$, either*

(1) *the orbit closure $\overline{Ax} \subset G/\Gamma$ is homogeneous, or*
(2) *$\overline{Ax}$ is embedded into a closed subspace $Fx \subset G/\Gamma$ admitting a quotient F-space such that the A-action on this quotient space degenerates to a one-parameter homogeneous flow.*

If μ is an A-invariant ergodic probability measure on G/Γ such that for every $x \in \operatorname{supp}(\mu)$ the statement (2) *does not hold, then μ is homogeneous.*

Earlier, a related measure-theoretic conjecture was formulated by Katok and Spatzier [108] who made some progress in this direction. In particular, they proved the following (see [109] for corrections):

[17] This theorem has been recently generalized in [245] to the case when Γ is any *inner type* lattice in $\mathrm{SL}(k,\mathbb{R})$ and the closure of Ax contains an orbit Ay whose closure is homogeneous with a finite invariant measure.

[18] Actually, the conjecture was formulated by Margulis for any connected Ad-triangular subgroup H in place of A.

THEOREM 4.4.12. *Let G be as in* (2.1)*, Γ be as in* (2.2)*, and A be a maximal $\mathbb{R}$-diagonalizable subgroup of G, where* $\dim A = \operatorname{rank}_{\mathbb{R}} G \geqslant 2$. *Let μ be a weakly mixing measure for the A-action on G/Γ which is of positive entropy with respect to some element of A. Then μ is homogeneous.*

Apparently, Katok and Spatzier were the first to relate remarkable results of Furstenberg on higher rank $\mathbb{Z}_+^k$-actions by circle endomorphisms with homogeneous $\mathbb{R}^k$-actions. In his landmark paper [83] Furstenberg proved that given a semigroup $\mathbb{Z}_+^2$ of circle endomorphisms generated by multiplications by integers p and q, where $p^n \neq q^m$ unless $p = q = 0$, any infinite semigroup orbit is dense.[19] Later his results were extended by Berend [15] for $\mathbb{Z}_+^k$-actions by toral endomorphisms. Most probably, using his approach one can prove that all orbit closures of the action are smooth submanifolds of the torus if and only if there is no quotient action (of algebraic origin) degenerating to a $\mathbb{Z}_+$-action.

Recently new results on non-Abelian toral automorphism groups were obtained. In particular, Muchnik [165] and Starkov [236] proved that a Zariski dense subgroup of $\mathrm{SL}(k,\mathbb{Z})$ has only closed and dense orbits on the k-torus. Moreover, Muchnik [164] obtained a criterion for a subsemigroup of $\mathrm{SL}(k,\mathbb{Z})$ to possess this property. Similar result was proved by Guivarch and Starkov [91] using methods of random walk theory.

4d. *Minimality and unique ergodicity.* First nontrivial results showing that minimal homogeneous flows must be uniquely ergodic are due to Furstenberg. He established this for nilflows and for the horocycle flow (see §2.2a and §3.3a). Theorem 4.3.7 claims that for one-parameter homogeneous flows these two properties are in fact equivalent. This answers positively Furstenberg's question raised in [84].

As follows from Ratner's results, minimality and unique ergodicity of the H-action on G/Γ, H being any connected subgroup of G, are equivalent whenever H is generated by unipotent elements. On the other hand, the following holds [234]:

LEMMA 4.4.13. *If the group H is connected, then every H-invariant compact subset $M \subset G/\Gamma$ carries an H-invariant probability measure.*

In fact, let $H = (K_H \times S_H) \ltimes R_H$ be a Levi decomposition for H (here as usual $K_H \times S_H$ is the decomposition of a Levi subgroup into compact and totally noncompact parts, and R_H is the radical of H). Let $P \subset S_H$ be a parabolic subgroup of S_H. Then the group $A = (K_H \times P) \ltimes R_H$ is amenable and hence M carries an A-invariant probability measure μ. Note that the subgroup P is epimorphic in S_H, and S_H is generated by unipotent elements. Now by Theorem 4.4.4, any P-invariant probability measure is S_H-invariant. Hence μ is H-invariant.

As a corollary, we deduce that the action $(G/\Gamma, H)$ is minimal whenever it is uniquely ergodic and the space G/Γ is compact. As Example 4.4.7 shows, this breaks down in the noncompact case. However, apparently the converse statement holds. In this direction one has the following result [234,243]:

[19] Notice that still it is not known whether any non-atomic $\mathbb{Z}_+^2$-ergodic measure on the circle is Lebesgue.

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THEOREM 4.4.14. *Let G be a connected Lie group with a connected subgroup H, and let* $\mathrm{vol}(G/\Gamma) < \infty$. *Assume that* $\mathrm{Ad}(H) \subset \mathrm{Aut}(\mathfrak{g})$ *splits into the semidirect product of a reductive subgroup and the unipotent radical. Then the action* $(G/\Gamma, H)$ *is uniquely ergodic whenever it is minimal.*

Earlier, this result was proved by Mozes and Weiss [163] for real algebraic groups G and H.

5. Applications to number theory

In this chapter we discuss applications of various results from earlier chapters to Diophantine approximation. The discussion should not be thought of as a complete reference guide to number-theoretical applications of homogeneous dynamics. Instead, we are going to concentrate on several directions where significant progress has been achieved during recent years, thanks to interactions between number theory and the theory of homogeneous flows.

It is also important to keep in mind that these interactions go both ways – as was mentioned before, many problems involving homogeneous spaces (e.g., in the theory of unipotent flows) came to light due to their connections with Diophantine approximation. The most striking example is given by the Oppenheim conjecture on density of the set of values $Q(\mathbb{Z}^k)$ of an indefinite irrational quadratic form Q in $k \geqslant 3$ variables. In §5.1a we discuss the original conjecture and its reduction to a special case of Ratner's topological theorem (Theorem 3.3.6). Quantitative versions of the Oppenheim conjecture have also been studied using the methods of homogeneous dynamics; we review most of the results in §5.1b.

Then we switch from quadratic to linear forms, and define the circle of questions in metric theory of Diophantine approximations that happen to correspond to orbit properties of certain homogeneous flows. Results to be mentioned are: abundance of badly approximable objects, a dynamical proof of the Khintchine–Groshev Theorem (§5.2c), inhomogeneous approximation (§5.2d), approximation on manifolds (§5.2e). Here the main role is played by actions of partially hyperbolic one-parameter subgroups of $\mathrm{SL}(k, \mathbb{R})$ on the space Ω_k of lattices in $\mathbb{R}^k$.

A modification of the standard set-up in the theory of Diophantine approximations, the so called *multiplicative approximation*, deserves a special treatment. Several results in the standard theory have their multiplicative extensions, which are obtained by considering multi-parameter partially hyperbolic actions. However, the non-existence of 'badly multiplicatively approximable' vectors is an open problem (Littlewood's Conjecture), which happens to be a special case of conjectures on higher rank actions mentioned in §4.4c.

Finally we review some applications of homogeneous dynamics to counting problems, such as counting integer points on homogeneous affine varieties (§5.4a), and estimating the error term in the asymptotics of the number of lattice points inside polyhedra (§5.4b).

5.1. *Quadratic forms*

Here the main objects will be real quadratic forms in k variables, and their values at integer points. Naturally if such a form Q is positive or negative definite, the set $Q(\mathbb{Z}^k \setminus \{0\})$

has empty intersection with some neighborhood of zero. Now take an indefinite form, and call it *rational* if it is a multiple of a form with rational coefficients, and *irrational* otherwise. For $k = 2, 3, 4$ it is easy to construct rational forms which do not attain small values at nonzero integers;[20] therefore a natural assumption to make is that the form is irrational. Let us start from the case $k = 2$. Then one can easily find an irrational λ for which $\inf_{(x_1,x_2)\in\mathbb{Z}\setminus\{0\}} |x_1^2 - \lambda x_2^2| > 0$, i.e., the set of values of such a form also has a gap at zero. We will see an explanation of this phenomenon in Section 5.2: the stabilizer of the form $|x_1^2 - \lambda x_2^2|$ consists of semisimple elements, and forms with nondense integer values correspond to nondense orbits. The situation is however quite different in higher dimensions.

1a. *Oppenheim Conjecture.* In 1986, Margulis [135,138] proved the following result, which resolved then a 60 year old conjecture due to Oppenheim:

THEOREM 5.1.1. *Let Q be a real indefinite nondegenerate irrational quadratic form in $k \geqslant 3$ variables.*[21] *Then given any $\varepsilon > 0$ there exists an integer vector $\mathbf{x} \in \mathbb{Z}^k \setminus \{0\}$ such that $|Q(\mathbf{x})| < \varepsilon$.*

By analytic number theory methods this conjecture was proved in the 1950s for $k \geqslant 21$; the history of the problem is well described in [143]. In particular, it has been known that the validity of the conjecture for some k_0 implies its validity for all $k \geqslant k_0$; in other words, Theorem 5.1.1 reduces to the case $k = 3$.

The turning point in the history was the observation (implicitly made in [37] and later by Raghunathan) that Theorem 5.1.1 is equivalent to the following

THEOREM 5.1.2. *Consider $Q_0(\mathbf{x}) = 2x_1x_3 + x_2^2$, and let H be the stabilizer of Q_0 in* $\mathrm{SL}(3,\mathbb{R})$. *Then any relatively compact orbit $H\Lambda$, Λ a lattice in $\mathbb{R}^3$, is compact.*

The above result was proved by Margulis [135] in 1986, which later led to the first instance of establishing Conjecture 3.3.1 for actions of nonhorospherical subgroups (in the semisimple case). See §3.3a for related historical comments, and [13,237] for a good account of the original proof.

To derive Theorem 5.1.2 from Theorem 3.3.6, one needs to observe that H as above is generated by its unipotent one-parameter subgroups, and that there are no intermediate subgroups between H and $\mathrm{SL}(3,\mathbb{R})$. As for the equivalence of Theorems 5.1.1 and 5.1.2, first note that the latter theorem can be restated in the following way: if Q is as in Theorem 5.1.1 and H_Q is the stabilizer of Q in $\mathrm{SL}(3,\mathbb{R})$ defined as in §1.4g, then the orbit $H_Q\mathbb{Z}^3$ is compact whenever it is relatively compact. (Indeed, by a linear change of variables one can turn Q into a form proportional to Q_0, which makes H_Q conjugate to H via an element $g \in \mathrm{SL}(3,\mathbb{R})$ and establishes a homeomorphism between the orbits $H_Q\mathbb{Z}^3$ and $H\Lambda$ where $\Lambda = g\mathbb{Z}^3$.) The rest of the argument boils down to Lemma 1.4.4. Indeed,

[20] However, by Meyer's Theorem [36] if Q is nondegenerate indefinite rational quadratic form in $k \geqslant 5$ variables, then Q represents zero over $\mathbb{Z}$ nontrivially, i.e., there exists a nonzero integer vector $\mathbf{x}$ such that $Q(\mathbf{x}) = 0$.

[21] The original conjecture of Oppenheim assumed $k \geqslant 5$; later it was extended to $k \geqslant 3$ by Davenport.

suppose for some $\varepsilon > 0$ one has $\inf_{\mathbf{x}\in\mathbb{Z}^3\setminus\{0\}} |Q(\mathbf{x})| \geqslant \varepsilon$. Then by Lemma 1.4.4 the orbit $H_Q\mathbb{Z}^3$ is bounded in the space Ω_3 of unimodular lattices in $\mathbb{R}^3$, hence it is compact in view of Theorem 5.1.2. But since this orbit can be identified with $H_Q/H_Q \cap \mathrm{SL}(3,\mathbb{Z})$, this shows that $H_Q \cap \mathrm{SL}(3,\mathbb{Z})$ is a lattice in H_Q, hence is Zariski dense by the Borel Density Theorem (see §1.3b). The latter is not hard to show to be equivalent to H_Q being defined over $\mathbb{Q}$, which, in turn, is equivalent to Q being proportional to a form with rational coefficients. The reverse implication is proved in a similar way. See [13,27,237] for more detail.

In [137] Margulis has proved a stronger version of the conjecture: under the same assumptions, for any $\varepsilon > 0$ there exists $\mathbf{x} \in \mathbb{Z}^k$ such that $0 < |Q(\mathbf{x})| < \varepsilon$. It had been known (see [126, §5]) by the work of Oppenheim [167] that if 0 is the right limit point for the set $Q(\mathbb{Z}^k)$, then it is the left one as well. Moreover, $Q(\mathbb{Z}^k)$ is clearly invariant under multiplication by any square integer. Hence the following holds:

THEOREM 5.1.3. *Let Q be as in Theorem* 5.1.1. *Then the set $Q(\mathbb{Z}^k)$ is dense in $\mathbb{R}$.*

Another extension is due to Dani and Margulis [61]. One says that an integer vector $\mathbf{x} \in \mathbb{Z}^k$ is *primitive* if there is no vector $\mathbf{y} \in \mathbb{Z}^k$ such that $\mathbf{x} = m\mathbf{y}$ for some $m \in \mathbb{Z}$, $m \neq 1, -1$. We denote by $P(\mathbb{Z}^k)$ the set of all primitive vectors. Note that the set $P(\mathbb{Z}^k)$ is invariant under the action of $\mathrm{SL}(k,\mathbb{Z})$. The following statement holds:

THEOREM 5.1.4. *Let Q be as in Theorem* 5.1.1. *Then the set $Q(P(\mathbb{Z}^k))$ is dense in $\mathbb{R}$.*

The above two theorems can be deduced from the stronger version of Theorem 5.1.2, namely that any orbit $H_Q\Lambda$ is either closed or everywhere dense in Ω_3 (as before, a simple reduction to the case $k = 3$ is in order). For simplified proofs of Theorem 5.1.4, see [60,63, 139]

Using Ratner's general results, Theorem 5.1.4 was sharpened in an elegant way by Borel and Prasad [29]:

THEOREM 5.1.5. *Let Q be as above. Then for any $c_1, \ldots, c_{k-1} \in \mathbb{R}$ and $\varepsilon > 0$ there exist vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1} \in \mathbb{Z}^k$ extendable to a basis of $\mathbb{Z}^k$* (*and hence primitive*) *such that* $|Q(\mathbf{x}_i) - c_i| < \varepsilon$, $i = 1, \ldots, k-1$.

See [244] for generalization of the result of Borel–Prasad to the case of Hermitian forms over division algebras, [58] for simultaneous solution of linear and quadratic inequalities, and [258,260,261] for 'prehomogeneous' analogues of Ratner's theorems and their number-theoretic applications.

1b. *Quantitative versions of the Oppenheim Conjecture.* By obtaining appropriate uniform versions (see Theorem 3.6.2) of Ratner's equidistribution result, Dani and Margulis [65,66] proved the following quantitative analogue of Theorem 5.1.1.

For any real indefinite nondegenerate quadratic form Q on $\mathbb{R}^k$, an open interval $I \subset \mathbb{R}$ and $T > 0$, define

$$V_Q(I,T) = \{\mathbf{x} \in \mathbb{R}^k\colon\ \|\mathbf{x}\| < T,\ Q(\mathbf{x}) \in I\}$$

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and

$$N_Q(I,T) \stackrel{\text{def}}{=} \#\big(\mathbb{Z}^k \cap V_Q(I,T)\big).$$

Write $k = m + n$ with $m \geqslant n$, and let $F(m,n)$ denote the space of quadratic forms on $\mathbb{R}^k$ with discriminant ± 1 and signature (m,n).

THEOREM 5.1.6. *Given a relatively compact set $\mathcal{K}$ of $F(m,n)$, an open interval I in $\mathbb{R}$, and $\theta > 0$, there exists a finite subset $\mathcal{S}$ of $\mathcal{K}$ such that each element of $\mathcal{S}$ is rational, and for any compact subset $\mathcal{C}$ of $\mathcal{K} \setminus \mathcal{S}$ the following holds*:

$$\liminf_{T\to\infty}\left(\inf_{Q\in\mathcal{C}} \frac{N_Q(I,T)}{\operatorname{vol}(V_Q(I,T))}\right) \geqslant 1-\theta. \tag{5.1}$$

Obtaining the upper bound posed new kind of difficulties. The question was resolved by Eskin, Margulis and Mozes [76] to obtain the following:

THEOREM 5.1.7. *Let the notation be as in Theorem 5.1.6, and further suppose that $(m,n) \neq (2,1)$ or $(2,2)$. Then given $\mathcal{K}$, I, and θ as before, there exists a finite set $\mathcal{S} \subset \mathcal{K}$ such that $\mathcal{S}$ consists of rational forms, and that for any compact set $\mathcal{C} \subset \mathcal{K} \setminus \mathcal{S}$ the following holds*:

$$\limsup_{T\to\infty}\left(\sup_{Q\in\mathcal{C}} \frac{N_Q(I,T)}{\operatorname{vol}(V_Q(I,T))}\right) \leqslant 1+\theta. \tag{5.2}$$

The proofs of both of the above results are based on Siegel's formula (see §1.3d) and Theorem 3.6.2. It may be noted that Theorem 3.6.2 is valid only for bounded continuous functions φ; but to apply the theorem to count the number of integral points in $V_Q(I,T)$ using Siegel's formula one needs its analogue for unbounded continuous L^1-functions on Ω_k. It may be noted that the lower bound in (5.1) can be obtained by truncating the unbounded function, and applying Theorem 3.6.2 to the truncated one.

The upper bound in (5.2) was obtained as a consequence of the next theorem due to Eskin, Margulis and Mozes [76], whose proof involves significantly new ideas and techniques.

DEFINITION 5.1.8. For $\Lambda \in \Omega_k$, we define

$$\alpha(\Lambda) = \sup\big\{1/\operatorname{vol}(L/L\cap\Lambda) \mid L \text{ is a subspace of } \mathbb{R}^k \text{ and } L/L\cap\Lambda \text{ is compact}\big\}.$$

Now consider the quadratic form

$$Q_0(x_1,\ldots,x_k) = 2x_1x_k + \sum_{i=2}^{m} x_i^2 - \sum_{i=m+1}^{k-1} x_i^2$$

on $\mathbb{R}^k$. Let H be the stabilizer of Q_0, consider a one-parameter subgroup $a_{\mathbb{R}}$ of H, where $a_t = \mathrm{diag}(e^t, 1, \ldots, 1, e^{-t})$, and let $K = \mathrm{SO}(k) \cap H$. Denote by σ the normalized Haar measure on K. (We note that, as in §5.1a, every form $Q \in F(m,n)$ reduces to Q_0 by changing variables.)

THEOREM 5.1.9. *Let integers $m > 2$ and $k \geqslant 4$, and a real $0 < s < 2$ be given. Then for any $\Lambda \in \Omega_k$,*

$$\limsup_{t\to\infty} \int_K \alpha(a_t g \Lambda)^s \,\mathrm{d}\sigma(g) < \infty.$$

Using this theorem and Theorem 3.6.2 one obtains the following result, which leads to the simultaneous proof of Theorems 5.1.6 and 5.1.7:

THEOREM 5.1.10. *Let the notation be as in Theorem 5.1.9. Let f be any continuous function on Ω_k which is dominated by α^s for some $0 < s < 2$. Then given a relatively compact set $C \subset \Omega_k$ and an $\varepsilon > 0$, there exist finitely many lattices $\Lambda_1, \ldots, \Lambda_l \in C$ such that the following holds: $H\Lambda_i$ is closed for $i = 1, \ldots, l$, and for any compact set $C_1 \subset C \setminus \bigcup_{i=1}^{l} H\Lambda_i$, there exists $t_0 > 0$ such that*

$$\left| \int_K f(a_t g \Lambda)\,\mathrm{d}\sigma(g) - \int f \,\mathrm{d}\nu \right| < \varepsilon, \quad \forall \Lambda \in C_1,\ t > t_0,$$

where ν denotes the $\mathrm{SL}(k,\mathbb{R})$-invariant probability measure on Ω_k.

Combining Theorems 5.1.6 and 5.1.7 one obtains the following result due to [76]:

COROLLARY 5.1.11. *If $(m,n) \neq (2,1)$ or $(2,2)$ then as $T \to \infty$,*

$$\lim_{T\to\infty} \frac{N_Q(I,T)}{\mathrm{vol}(V_Q(I,T))} = 1, \tag{5.3}$$

for all irrational $Q \in F(m,n)$ and any bounded open interval $I \subset \mathbb{R}$.

1c. *Upper bounds for the case of signature* $(2,2)$. The analysis of upper bounds in the cases of quadratic forms with signatures $(2,1)$ and $(2,2)$ is very delicate. The case of signature $(2,2)$ is now well understood by recent works of Eskin, Margulis, and Mozes [77, 145]:

DEFINITION 5.1.12. Fix a norm $\|\cdot\|$ on the space $F(2,2)$. We say that a quadratic form $Q \in F(2,2)$ is *extremely well approximable by split rational forms*, to be abbreviated as *EWAS*, if for any $N > 0$ there exist an integral form $Q' \in F(2,2)$ which is split over $\mathbb{Q}$, and a real number λ such that $\|\lambda Q - Q'\| \leqslant \lambda^{-N}$. (A rational $(2,2)$-form is *split over* $\mathbb{Q}$ if and only if there is a 2-dimensional subspace of $\mathbb{R}^4$ defined over $\mathbb{Q}$ on which the form vanishes.)

It may be noted that if the ratio of two nonzero coefficients of Q is badly approximable (see §5.2a) then Q is not EWAS.

THEOREM 5.1.13. *The formula* (5.3) *holds if* $Q \in F(2,2)$ *is not EWAS and* $0 \notin I$.

We note that forms of signature $(2,2)$, i.e., differences of two positive definite quadratic forms in 2 variables, naturally arise in studying pair correlations of eigenvalues of the Laplacian on a flat torus. Thus the above theorem provides examples of flat metrics on tori for which asymptotics of pair correlations agrees with conjectures made by Berry and Tabor. Before [77] it was proved by Sarnak that (5.3) holds for almost all forms within the family

$$\left(x_1^2 + 2bx_1x_2 + cx_2^2\right) - \left(x_3^2 + 2bx_3x_4 + cx_4^2\right).$$

One can also prove that (5.3) is valid for almost all $Q \in F(2,1)$, and it is not valid for those Q which are EWAS (correspondingly defined for $Q \in F(2,1)$).

See also the papers by Marklof [149–151] for further applications of the study of flows on homogeneous spaces to results related to quantum chaos.

5.2. *Linear forms*

We start with a brief introduction to the metric theory of Diophantine approximations.

2a. *Basics of metric number theory.* The word ‘metric’ here does not refer to the metric distance, but rather to the measure. The terminology originated with the Russian school (Khintchine, Luzin) and now is quite customary. Roughly speaking, by ‘usual’ Diophantine approximation one could mean looking at numbers of a very special kind (e.g., e or algebraic numbers) and studying their approximation properties. On the other hand, metric Diophantine approximation starts when one fixes a certain approximation property and wants to characterize the set of numbers (vectors) which share this property. A big class of problems arises when one has to decide whether a certain property is satisfied for almost all or almost no numbers, with respect to Lebesgue measure. Hence the term “metric” in the heading. A quote from Khintchine’s ‘Continued fractions’ [113]: ‘metric theory ... inquires into the measure of the set of numbers which are characterized by a certain property’.

Here is a typical example of an approximation property one can begin with. Let $\psi(x)$ be a non-increasing function $\mathbb{R}_+ \mapsto \mathbb{R}_+$. Say that a real number α is *ψ-approximable* if there are infinitely many integers q such that the distance between αq and the closest integer is not greater than $\psi(|q|)$; in other words, such that

$$|\alpha q + p| \leqslant \psi\big(|q|\big) \quad \text{for some } p \in \mathbb{Z}.$$

The basic question is now as follows: given a function ψ as above, what can one say about the set of all ψ-approximable numbers? The whole theory starts from a positive result of Dirichlet (involving the pigeon-hole principle), where one considers the function $\psi_0(x) \stackrel{\text{def}}{=} 1/x$:

THEOREM 5.2.1. *Every* $\alpha \in \mathbb{R}$ *is* ψ_0*-approximable*.

It is natural to guess that the faster $\psi(x)$ decays as $x \to +\infty$, the fewer there are ψ-approximable numbers. In particular, one can replace ψ_0 in the above theorem by $(1/\sqrt{5})\psi_0$ (Hurwitz), but not by $c\psi_0$ with $c < 1/\sqrt{5}$. Numbers which are not $c\psi_0$-approximable for some $c > 0$ are called *badly approximable*, and numbers which are not badly approximable are called *well approximable*. It is a theorem of Jarnik [102] that the set of badly approximable numbers has full (i.e., equal to 1) Hausdorff dimension; moreover, this set is *thick* in $\mathbb{R}$ (that is, has full Hausdorff dimension at every point). However almost all numbers are well approximable. This is a special case of the following theorem due to Khintchine [112], which gives the precise condition on the function ψ under which the set of ψ-approximable numbers has full measure:

THEOREM 5.2.2. *Almost no* (*resp. almost every*) $\alpha \in \mathbb{R}$ *is* ψ*-approximable, provided the integral* $\int_1^\infty \psi(x)\,\mathrm{d}x$ *converges* (*resp. diverges*).

Note that the first statement, usually referred to as the convergence part of the theorem, immediately follows from the Borel–Cantelli Lemma, while the second one (the divergence part) is nontrivial.

EXAMPLE 5.2.3. Consider $\psi_\varepsilon(x) \stackrel{\text{def}}{=} x^{-(1+\varepsilon)}$. It immediately follows from the above theorem that almost no numbers are ψ_ε-approximable if $\varepsilon > 0$. One says that α is *very well approximable* (abbreviated as VWA) if it is ψ_ε-approximable for some $\varepsilon > 0$; the set of VWA numbers has measure zero. Note that the Hausdorff dimension of this set is equal to one – this follows from a theorem by Jarnik and Besicovitch [22,103].

It is important that most of the results cited above have been obtained by the method of *continued fractions* (thus implicitly – using dynamics of the Gauss map $x \mapsto 1/x \bmod 1$ of the unit interval). For more results from the theory of metric Diophantine approximation on the real line, see [94,199]. Our goal now is to consider some higher-dimensional phenomena which are generally much less understood.

2b. *Simultaneous approximation*. In order to build a multi-dimensional generalization for the notions discussed above, one views $\alpha \in \mathbb{R}$ as a linear operator from $\mathbb{R}$ to $\mathbb{R}$, and then changes it to a linear operator A from $\mathbb{R}^n$ to $\mathbb{R}^m$. That is, the main object is now a matrix $A \in M_{m,n}(\mathbb{R})$ (interpreted as a system of m linear forms A_i on $\mathbb{R}^n$).

Denote by $\|\cdot\|$ the norm on $\mathbb{R}^k$ given by $\|\mathbf{y}\| = \max_{1 \leqslant i \leqslant k} |y_i|$. It turns out that in order to mimic the one-dimensional theory in the best way one needs to raise norms to powers equal to the dimension of the ambient space. In other words, fix $m, n \in \mathbb{N}$ and, for ψ as above, say that $A \in M_{m,n}(\mathbb{R})$ is ψ*-approximable* if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p}\|^m \leqslant \psi\big(\|\mathbf{q}\|^n\big) \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m.$$

Then one has

THEOREM 5.2.4. *Every* $A \in M_{m,n}(\mathbb{R})$ *is* ψ_0*-approximable.*

(This was also proved by Dirichlet in 1842 and in fact served the occasion to introduce the pigeon-hole principle.)

The function ψ_0 in this theorem can be replaced by $c\psi_0$ for some $c < 1$, but the infimum of numbers c such that every $A \in M_{m,n}(\mathbb{R})$ is $c\psi_0$-approximable (the top of the *Markov spectrum* for simultaneous approximation) is not known when $(m, n) \neq (1, 1)$ (it is estimated to be not greater than $m^m n^n (m+n)!/(m+n)^{mn} m!n!$ by Minkowski, see [199]). However this infimum is known to be positive; in other words for every m and n there exist badly approximable systems of m linear forms A_i on $\mathbb{R}^n$ (where as before badly approximable means not $c\psi_0$-approximable for some $c > 0$). This was shown by Perron in 1921, and in 1969 Schmidt [198] proved that the set of badly approximable $A \in M_{m,n}(\mathbb{R})$ is thick. The fact that this set has measure zero had been known before: Khintchine's theorem (Theorem 5.2.2) has been generalized to the setting of systems of linear forms by Groshev [90] and is now usually referred to as the Khintchine–Groshev Theorem:

THEOREM 5.2.5. *Almost no* (*resp. almost every*) $A \in M_{m,n}(\mathbb{R})$ *is* ψ*-approximable, provided the integral* $\int_1^\infty \psi(x)\,\mathrm{d}x$ *converges* (*resp. diverges*).

Note that the above condition on ψ does not depend on m and n – this is an advantage of the normalization that we are using (that is, employing $\|\cdot\|^n$ instead of $\|\cdot\|$). The reader is referred to [70] or [220] for a good exposition of the proof, and to [195,196,220,242] for a quantitative strengthening and further generalizations.

EXAMPLE 5.2.6. Again, almost no $A \in M_{m,n}(\mathbb{R})$ are ψ_ε-approximable if $\varepsilon > 0$. As in the case $m = n = 1$, one says that A is VWA if it is ψ_ε-approximable for some $\varepsilon > 0$, and the set of VWA matrices has measure zero. It follows from a result of Dodson (see [70]) that this set also has full Hausdorff dimension (more precisely, Dodson, as well as Jarnik in the case $m = n = 1$, computed the Hausdorff dimension of the set of ψ_ε-approximable matrices to be equal to $mn(1 - \frac{\varepsilon}{m+n+n\varepsilon})$).

2c. *Dani's correspondence.* As far as one-dimensional theory of Diophantine approximation is concerned, it has been known for a long time (see [207] for a historical account) that Diophantine properties of real numbers can be coded by the behavior of geodesics on the quotient of the hyperbolic plane by $\mathrm{SL}(2,\mathbb{Z})$. In fact, the geodesic flow on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ can be viewed as the suspension flow of the Gauss map. There have been many attempts to construct a higher-dimensional analogue of the Gauss map so that it captures all the features of simultaneous approximation, see [121,124] and references therein. On the other hand, it seems much more natural and efficient to generalize the suspension flow itself, and this is where one needs higher rank homogeneous dynamics.

As we saw in the preceding section, in the theory of simultaneous Diophantine approximation one takes a system of m linear forms $A_1, \ldots, A_m$ on $\mathbb{R}^n$ and simultaneously looks at the values of $|A_i(\mathbf{q}) + p_i|$, $p_i \in \mathbb{Z}$, when $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$ is far from 0. The trick is to put together

$$A_1(\mathbf{q}) + p_1, \ldots, A_m(\mathbf{q}) + p_m \quad \text{and} \quad q_1, \ldots, q_n,$$

and consider the collection of vectors

$$\left\{ \begin{pmatrix} A\mathbf{q} + \mathbf{p} \\ \mathbf{q} \end{pmatrix} \,\middle|\, \mathbf{p} \in \mathbb{Z}^m,\ \mathbf{q} \in \mathbb{Z}^n \right\} = L_A \mathbb{Z}^k,$$

where $k = m + n$, A is the matrix with rows $A_1, \ldots, A_m$ and L_A is as in (4.2).

This collection is a unimodular lattice in $\mathbb{R}^k$, i.e., an element of Ω_k. Our goal is to keep track of vectors in such a lattice having very small projections onto the first m components of $\mathbb{R}^k$ and very big projections onto the last n components. This is where dynamics comes into the picture. Denote by g_t the one-parameter subgroup of $\mathrm{SL}(k, \mathbb{R})$ given by (4.1). One watches the vectors mentioned above as they are moved by the action of g_t, $t > 0$, and in particular looks at the moment t when the "small" and "big" projections equalize. The following observation of Dani [52] illustrates this idea, and can be thought of as a generalization of the aforementioned geodesic-flow approach to continued fractions.

THEOREM 5.2.7. *$A \in M_{m,n}(\mathbb{R})$ is badly approximable iff the trajectory*

$$\{g_t L_A \mathbb{Z}^k \mid t \in \mathbb{R}_+\}, \tag{5.4}$$

is bounded in the space Ω_k.

Let us sketch a short proof, which is basically a rephrasing of the original proof of Dani. From Mahler's Compactness Criterion it follows that the orbit (5.4) is unbounded iff there exist sequences $t_i \to +\infty$ and $(\mathbf{p}_i, \mathbf{q}_i) \in \mathbb{Z}^k \setminus \{0\}$ such that

$$\max\left(\mathrm{e}^{t_i/m} \|\mathbf{p}_i + A\mathbf{q}_i\|, \mathrm{e}^{-t_i/n} \|\mathbf{q}_i\|\right) \to 0 \quad \text{as } k \to \infty. \tag{5.5}$$

On the other hand, A is well approximable iff there exist sequences $\mathbf{p}_i \in \mathbb{Z}^m$ and $\mathbf{q}_i \in \mathbb{Z}^n$ such that $\|\mathbf{q}_i\| \to \infty$ and

$$\|\mathbf{p}_i + A\mathbf{q}_i\|^m \|\mathbf{q}_i\|^n \to 0 \quad \text{as } k \to \infty. \tag{5.6}$$

Therefore for well approximable A one can define $\mathrm{e}^{t_i} \overset{\text{def}}{=} \sqrt{\|\mathbf{q}_i\|^n / \|\mathbf{p}_i + A\mathbf{q}_i\|^m}$ and check that $t_i \to +\infty$ and (5.5) is satisfied. Also (5.6) obviously follows from (5.5). To finish the proof, first exclude the trivial case when $A\mathbf{q} + \mathbf{p} = 0$ for some nonzero $(\mathbf{p}, \mathbf{q})$ (such A is clearly well approximable and the trajectory (5.4) diverges). Then it is clear that (5.6) can only hold if $\|\mathbf{q}_i\| \to \infty$, i.e., unboundedness of the trajectory (5.4) forces A to be well approximable. □

From the results of [116] on abundance of bounded orbits one can then deduce the aforementioned result of Schmidt [198]: the set of badly approximable $A \in M_{m,n}(\mathbb{R})$ is thick in $M_{m,n}(\mathbb{R})$. Indeed, as was observed in Section 2, $\{L_A \mid A \in M_{m,n}(\mathbb{R})\}$ is the expanding horospherical subgroup of $\mathrm{SL}(k, \mathbb{R})$ relative to g_1, so an application of Theorem 4.1.6(a) proves the claim.

The above correspondence has been made more general in [118], where the goal was to treat ψ-approximable systems similarly to the way $c\psi_0$-approximable ones were treated

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in Theorem 5.2.7. Roughly speaking, the faster ψ decays, the 'more unbounded' is the trajectory (5.4) with A being ψ-approximable. We will need to transform ψ to another function which will measure the 'degree of unboundedness' of the orbit. The following was proved in [118]:

LEMMA 5.2.8. *Fix $m, n \in \mathbb{N}$ and $x_0 > 0$, and let $\psi : [x_0, \infty) \mapsto (0, \infty)$ be a non-increasing continuous function. Then there exists a unique continuous function $r : [t_0, \infty) \mapsto \mathbb{R}$, where $t_0 = \frac{m}{m+n} \log x_0 - \frac{n}{m+n} \log \psi(x_0)$, such that*

$$\text{the function } t - nr(t) \text{ is strictly increasing and tends to } \infty \text{ as } t \to +\infty, \tag{5.7}$$

$$\text{the function } t + mr(t) \text{ is nondecreasing}, \tag{5.8}$$

and

$$\psi\left(\mathrm{e}^{t-nr(t)}\right) = \mathrm{e}^{-(t+mr(t))}, \quad \forall t \geqslant t_0. \tag{5.9}$$

Conversely, given $t_0 \in \mathbb{R}$ and a continuous function $r : [t_0, \infty) \mapsto \mathbb{R}$ such that (5.7) *and* (5.8) *hold, there exists a unique continuous non-increasing function $\psi : [x_0, \infty) \mapsto (0, \infty)$, with $x_0 = \mathrm{e}^{t_0 - nr(t_0)}$, satisfying* (5.9). *Furthermore,*

$$\int_{x_0}^{\infty} \psi(x)\,\mathrm{d}x < \infty \quad \textit{iff} \quad \int_{t_0}^{\infty} \mathrm{e}^{-(m+n)r(t)}\,\mathrm{d}t < \infty. \tag{5.10}$$

A straightforward computation using (5.9) shows that the function $c\psi_0$ corresponds to $r(t) \equiv$ const. Now recall one of the forms of Mahler's Compactness Criterion: $K \subset \Omega_{m+n}$ is bounded iff $\Delta(\Lambda) \leqslant$ const for all $\Lambda \in K$, where Δ is the 'distance-like' function introduced in §1.3d. Thus the following statement, due to [118], is a generalization of Theorem 5.2.7:

THEOREM 5.2.9. *Let ψ, m and n be as in Lemma* 5.2.8, *Δ as in* (1.4), *$g_{\mathbb{R}}$ as in* (4.1). *Then $A \in M_{m,n}(\mathbb{R})$ is ψ-approximable iff there exist arbitrarily large positive t such that*

$$\Delta\left(g_t L_A \mathbb{Z}^{m+n}\right) \geqslant r(t).$$

EXAMPLE 5.2.10. Take $\psi(x) = \psi_\varepsilon(x) = 1/x^{1+\varepsilon}$, $\varepsilon > 0$; then $r(t)$ is a linear function, namely

$$r(t) = \gamma t, \quad \text{where } \gamma = \frac{\varepsilon}{(1+\varepsilon)m + n}.$$

Thus $A \in M_{m,n}(\mathbb{R})$ is VWA iff $\Delta(L_A \mathbb{Z}^{m+n})$ grows at least linearly; that is, for some $\gamma > 0$ there exist arbitrarily large positive t such that

$$\Delta\left(g_t L_A \mathbb{Z}^{m+n}\right) \geqslant \gamma t.$$

Recall that Δ is a $(m+n)$-DL function (see Theorem 1.3.5), and so by Theorem 4.1.11, for any sequence $\{r(t) \mid t \in \mathbb{N}\}$ of real numbers the following holds: for almost every (resp. almost no) $\Lambda \in \Omega_{m+n}$

$$\begin{aligned}&\text{there are infinitely many } t \in \mathbb{N} \text{ such that } \Delta(g_t\Lambda) \geqslant r(t),\\ &\text{provided the series } \sum_{t=1}^{\infty} \mathrm{e}^{-(m+n)r(t)} \text{ diverges (resp. converges).}\end{aligned} \tag{5.11}$$

Assuming in addition that $r(t)$ is close to being monotone increasing (for example that (5.7) holds), it is not hard to derive that (5.11) holds for almost every Λ in any expanding (with respect to g_1) leaf, e.g., for lattices of the form $L_A\mathbb{Z}^{m+n}$ for a.e. $A \in M_{m,n}(\mathbb{R})$. In view of the above correspondence (Theorem 5.2.9) and (5.10), this proves the Khintchine–Groshev Theorem 5.2.5.

2d. *Inhomogeneous approximation.* In this subsection, as well as in §5.2e, we mention several new results that can be obtained by means of the aforementioned correspondence or its modifications. First let us discuss *inhomogeneous* analogues of the above notions. By an *affine form* we will mean a linear form plus a real number. A system of m affine forms in n variables will be then given by a pair $\langle A, \mathbf{b}\rangle$, where $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$. We will denote by $\widetilde{M}_{m,n}(\mathbb{R})$ the direct product of $M_{m,n}(\mathbb{R})$ and $\mathbb{R}^m$. As before, let us say that $\langle A, \mathbf{b}\rangle \in \widetilde{M}_{m,n}(\mathbb{R})$ is *ψ-approximable* if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{b} + \mathbf{p}\|^m \leqslant \psi\big(\|\mathbf{q}\|^n\big) \quad \text{for some } \mathbf{p} \in \mathbb{Z}^m.$$

A 'doubly metric' inhomogeneous analogue of Theorem 5.2.5 can be found already in Cassels's monograph [36]. The singly metric strengthening is due to Schmidt [196]: for any $\mathbf{b} \in \mathbb{R}^m$, the set of $A \in M_{m,n}(\mathbb{R})$ such that $\langle A, \mathbf{b}\rangle$ is ψ-approximable,

$$\text{has} \quad \begin{cases}\text{full measure} & \text{if } \sum_{k=1}^{\infty} \psi(k) = \infty,\\ \text{zero measure} & \text{if } \sum_{k=1}^{\infty} \psi(k) < \infty.\end{cases}$$

Similarly one can say that a system of affine forms given by $\langle A, \mathbf{b}\rangle \in \widetilde{M}_{m,n}(\mathbb{R})$ is *badly approximable* if

$$\liminf_{\mathbf{p} \in \mathbb{Z}^m,\ \mathbf{q} \in \mathbb{Z}^n,\ \mathbf{q} \to \infty} \|A\mathbf{q} + \mathbf{b} + \mathbf{p}\|^m \|\mathbf{q}\|^n > 0,$$

and *well approximable* otherwise. In other words, badly approximable means not $c\psi_0$-approximable for some $c > 0$. It follows that the set of badly approximable $\langle A, \mathbf{b}\rangle \in \widetilde{M}_{m,n}(\mathbb{R})$ has Lebesgue measure zero. Similarly to what was discussed in §§5.2a and 5.2b, one can try to measure the magnitude of this set in terms of the Hausdorff dimension. All examples of badly approximable systems known before happen to belong to a countable union of proper submanifolds of $\widetilde{M}_{m,n}(\mathbb{R})$ and, consequently, form a set of positive Hausdorff codimension. Nevertheless one can prove

THEOREM 5.2.11. *The set of badly approximable $\langle A, \mathbf{b}\rangle \in \widetilde{M}_{m,n}(\mathbb{R})$ is thick in $\widetilde{M}_{m,n}(\mathbb{R})$.*

This is done in [115] by a modification of the correspondence discussed in §5.2c. Namely, one considers a collection of vectors

$$\left\{\begin{pmatrix} A\mathbf{q}+\mathbf{b}+\mathbf{p} \\ \mathbf{q}\end{pmatrix} \,\middle|\, \mathbf{p}\in\mathbb{Z}^m,\ \mathbf{q}\in\mathbb{Z}^n\right\} = L_A\mathbb{Z}^k + \begin{pmatrix}\mathbf{b}\\0\end{pmatrix},$$

which is an element of the space $\widehat{\Omega}_k = \widehat{G}/\widehat{\Gamma}$ of *affine lattices* in $\mathbb{R}^k$, where

$$\widehat{G} \stackrel{\text{def}}{=} \mathrm{Aff}(\mathbb{R}^k) = \mathrm{SL}(k,\mathbb{R}) \ltimes \mathbb{R}^k \quad \text{and} \quad \widehat{\Gamma} \stackrel{\text{def}}{=} \mathrm{SL}(k,\mathbb{Z}) \ltimes \mathbb{Z}^k$$

(as before, here we set $k = m+n$). In other words,

$$\widehat{\Omega}_k \cong \{\Lambda + \mathbf{w} \mid \Lambda \in \Omega_k,\ \mathbf{w}\in\mathbb{R}^k\}.$$

Note that the quotient topology on $\widehat{\Omega}_k$ coincides with the natural topology on the space of affine lattices: that is, $\Lambda_1 + \mathbf{w}_1$ and $\Lambda_2 + \mathbf{w}_2$ are close to each other if so are $\mathbf{w}_i$ and the generating elements of Λ_i. Note also that $\widehat{\Omega}_k$ is noncompact and has finite Haar measure, and that Ω_k (the set of *true* lattices) can be identified with a subset of $\widehat{\Omega}_k$ (affine lattices containing the zero vector). Finally, g_t as in (4.1) acts on $\widehat{\Omega}_k$, and it is not hard to show that the expanding horospherical subgroup corresponding to g_1 is exactly the set of all elements of $\widehat{G}$ with linear part L_A and translation part $\binom{\mathbf{b}}{0}$, $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b}\in\mathbb{R}^m$.

The following is proved in [115]:

THEOREM 5.2.12. *Let $g_{\mathbb{R}}$ be as in* (4.1). *Then $\langle A, \mathbf{b}\rangle$ is badly approximable whenever*

$$g_{\mathbb{R}_+}\left(L_A\mathbb{Z}^k + \begin{pmatrix}\mathbf{b}\\0\end{pmatrix}\right) \quad \textit{is bounded and stays away from } \Omega_k.$$

Even though $\widehat{G}$ is not semisimple, it follows from Dani's mixing criterion (Theorem 2.2.9) that the $g_{\mathbb{R}}$-action on $\widehat{\Omega}$ is mixing. Since $\Omega \subset \widehat{\Omega}$ is closed, null and g_t-invariant, Theorem 4.1.6 applies and Theorem 5.2.11 follows.

2e. *Diophantine approximation on manifolds.* We start from the setting of §5.2b but specialize to the case $m = 1$; that is, to Diophantine approximation of just one linear form given by $\mathbf{y}\in\mathbb{R}^n$. Recall that (the trivial part of) Theorem 5.2.5 says that whenever $\sum_{l=1}^{\infty}\psi(l)$ is finite, almost every $\mathbf{y}$ is not ψ-approximable; that is, the inequality

$$|\mathbf{q}\cdot\mathbf{y} + p| \leqslant \psi(\|\mathbf{q}\|^n)$$

has at most finitely many solutions. In particular, almost all $\mathbf{y}\in\mathbb{R}^n$ are not VWA.

Now consider the following problem, raised by Mahler in [128]: is it true that for almost all $x\in\mathbb{R}$ the inequality

$$|p + q_1x + q_2x^2 + \cdots + q_nx^n| \leqslant \|\mathbf{q}\|^{-n(1+\beta)}$$

has at most finitely many solutions? In other words, for a.e. $x \in \mathbb{R}$, the n-tuple

$$\mathbf{y}(x) = \left(x, x^2, \ldots, x^n\right) \tag{5.12}$$

is not VWA. This question cannot be answered by trivial Borel–Cantelli type considerations. Its importance has several motivations: the original motivation of Mahler comes from transcendental number theory; then, in the 1960s, interest in Mahler's problem was revived due to connections with KAM theory. However, from the authors' personal viewpoint, the appeal of this branch of number theory lies in its existing and potential generalizations. In a sense, the affirmative solution to Mahler's problem shows that a certain property of $\mathbf{y} \in \mathbb{R}^n$ (being not VWA) which holds for generic $\mathbf{y} \in \mathbb{R}^n$ in fact holds for generic points on the curve (5.12). In other words, the curve inherits the above Diophantine property from the ambient space, unlike, for example, a line $\mathbf{y}(x) = (x, \ldots, x)$ – it is clear that every point on this line is VWA. This gives rise to studying other subsets of $\mathbb{R}^n$ and other Diophantine properties, and looking at whether this inheritance phenomenon takes place.

Mahler's problem remained open for more than 30 years until it was solved in 1964 by Sprindžuk [218,219]; the solution to Mahler's problem has eventually led to the development of a new branch of metric number theory, usually referred to as 'Diophantine approximation with dependent quantities' or 'Diophantine approximation on manifolds'. We invite the reader to look at Sprindžuk's monographs [219,220] and a recent book [20] for a systematic exposition of the field.

Note that the curve (5.12) is not contained in any affine subspace of $\mathbb{R}^n$ (in other words, constitutes an essentially n-dimensional object). The latter property, or, more precisely, its infinitesimal analogue, is formalized in the following way. Let V be an open subset of $\mathbb{R}^d$. Say that an n-tuple $\mathbf{f} = (f_1, \ldots, f_n)$ of C^l functions $V \mapsto \mathbb{R}$ is *nondegenerate at* $x \in V$ if the space $\mathbb{R}^n$ is spanned by partial derivatives of $\mathbf{f}$ at x of order up to l. If $M \subset \mathbb{R}^n$ is a d-dimensional smooth submanifold, one says that M is *nondegenerate at* $\mathbf{y} \in M$ if any (equivalently, some) diffeomorphism $\mathbf{f}$ between an open subset V of $\mathbb{R}^d$ and a neighborhood of $\mathbf{y}$ in M is nondegenerate at $\mathbf{f}^{-1}(\mathbf{y})$. We will say that $\mathbf{f}: V \to \mathbb{R}^n$ (resp. $M \subset \mathbb{R}^n$) is *nondegenerate* if it is nondegenerate at almost every point of V (resp. M, in the sense of the natural measure class on M). If the functions f_i are analytic, it is easy to see that the linear independence of $1, f_1, \ldots, f_n$ over $\mathbb{R}$ in V is equivalent to all points of $M = \mathbf{f}(V)$ being nondegenerate.

It was conjectured in 1980 by Sprindžuk [221, Conjecture H_1] that almost all points on a nondegenerate analytic submanifold of $\mathbb{R}^n$ are not VWA. This conjecture was supported before and after 1980 by a number of partial results, and the general case was settled in 1996 by Kleinbock and Margulis [117] using the dynamical approach. Namely, the following was proved:

THEOREM 5.2.13. *Let M be a nondegenerate smooth submanifold of $\mathbb{R}^n$. Then almost all points of M are not VWA.*

In another direction, Sprindžuk's solution to Mahler's problem was improved in 1966 by Baker [10] and later by Bernik [19,20]; the latter proved that whenever $\sum_{l=1}^{\infty} \psi(l)$ is finite, almost all points of the curve (5.12) are not ψ-approximable. And several years ago

Beresnevich [16] proved the divergence counterpart, thus establishing a complete analogue of the Khintchine–Groshev Theorem for the curve (5.12).

It turned out that a modification of the methods from [117] allows one to prove the convergence part of the Khintchine–Groshev Theorem for any nondegenerate manifold. In other words, the following is true:

THEOREM 5.2.14. *Let M be a nondegenerate smooth submanifold of $\mathbb{R}^n$ and let ψ be such that $\sum_{l=1}^{\infty} \psi(l)$ is finite. Then almost all points of M are not ψ-approximable.*

This is proved in [21] and also independently in [17].

Let us now sketch a proof of Theorem 5.2.13 by first restating it in the language of flows on the space of lattices. For this we set $k = n + 1$ and look at the one-parameter group

$$g_t = \operatorname{diag}\left(e^t, e^{-t/n}, \ldots, e^{-t/n}\right) \tag{5.13}$$

acting on $\Omega_k = \mathrm{SL}(k, \mathbb{R})/\mathrm{SL}(k, \mathbb{Z})$, and given $\mathbf{y} \in \mathbb{R}^n$, consider $L_{\mathbf{y}} \overset{\text{def}}{=} \left(\begin{smallmatrix} 1 & \mathbf{y}^T \\ 0 & I_n \end{smallmatrix}\right)$ (cf. (4.1) and (4.2)). Having Example 5.2.10 in mind, let us turn to the setting of Theorem 5.2.13. Namely, let V be an open subset of $\mathbb{R}^d$ and $\mathbf{f} = (f_1, \ldots, f_n)$ be an n-tuple of C^k functions $V \mapsto \mathbb{R}$ which is nondegenerate at almost every point of V. The theorem would be proved if we show that for any $\gamma > 0$ the set

$$\left\{x \in V \mid \Delta\left(g_t L_{\mathbf{f}(x)} \mathbb{Z}^k\right) \geqslant \gamma t \text{ for infinitely many } t \in \mathbb{N}\right\}$$

has measure zero. In other words, a submanifold $\mathbf{f}(V)$ of $\mathbb{R}^n$ gives rise to a submanifold $L_{\mathbf{f}(V)} \mathbb{Z}^k$ of the space of lattices, and one needs to show that growth rate of generic orbits originating from this submanifold is consistent with the growth rate of an orbit of a generic point of Ω_k (Theorem 4.1.11 gives an explanation of why lattices Λ such that $\Delta(g_t \Lambda) \geqslant \gamma t$ for infinitely many $t \in \mathbb{N}$ form a null subset of Ω_k.)

Now one can use the Borel–Cantelli Lemma to reduce Theorem 5.2.13 to the following statement:

THEOREM 5.2.15. *Let V be an open subset of $\mathbb{R}^d$ and $\mathbf{f} = (f_1, \ldots, f_n)$ an n-tuple of C^k functions $V \mapsto \mathbb{R}$ which is nondegenerate at $x_0 \subset V$. Then there exists a neighborhood B of x_0 contained in V such that for any $\gamma > 0$ one has*

$$\sum_{t=1}^{\infty} \left|\left\{x \in B \mid \Delta\left(g_t L_{\mathbf{f}(x)} \mathbb{Z}^k\right) \geqslant \gamma t\right\}\right| < \infty.$$

The latter inequality is proved by applying generalized nondivergence estimates from §3.2a, namely, Theorem 3.2.4. The key observation is that nondegeneracy of $(f_1, \ldots, f_n)$ at x_0 implies that all linear combinations of $1, f_1, \ldots, f_n$ are (C, α)-good in some neighborhood of x_0 (see Proposition 3.2.3).

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5.3. *Products of linear forms*

Let us start by stating the following conjecture, made by Littlewood in 1930:

CONJECTURE 5.3.1. *For every* $\mathbf{y} \in \mathbb{R}^n$, $n \geqslant 2$, *one has*

$$\inf_{\mathbf{q}\in\mathbb{Z}^n\setminus\{0\},\ p\in\mathbb{Z}} |\mathbf{y}\cdot\mathbf{q}+p|\cdot \Pi_+(\mathbf{q}) = 0, \tag{5.14}$$

where $\Pi_+(\mathbf{q})$ *is defined to be equal to* $\prod_{i=1}^n \max(|q_i|, 1)$ *or, equivalently,* $\prod_{q_i\neq 0}|q_i|$.

The main difference from the setting of the previous section is that here the magnitude of the integer vector $\mathbf{q}$ is measured by taking the *product* of coordinates rather than the maximal coordinate (that is the norm of the vector). Let us formalize it by saying, for ψ as before, that $A \in M_{m,n}(\mathbb{R})$ is ψ*-multiplicatively approximable* (ψ-MA) if there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\Pi(A\mathbf{q}+\mathbf{p}) \leqslant \psi\big(\Pi_+(\mathbf{q})\big) \quad \text{for some } \mathbf{p}\in\mathbb{Z}^m,$$

where for $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ one defines

$$\Pi(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^k |x_i| \quad \text{and} \quad \Pi_+(\mathbf{x}) \stackrel{\text{def}}{=} \prod_{i=1}^k \max\big(|x_i|, 1\big).$$

Clearly any ψ-approximable system of linear forms is automatically ψ-MA, but not necessarily other way around. Similarly to the standard setting, one can define *badly multiplicatively approximable* (*BMA*) and *very well multiplicatively approximable* (*VWMA*) systems. It can be easily shown that almost no $A \in M_{m,n}(\mathbb{R})$ are ψ-MA if the sum

$$\sum_{l=1}^{\infty} (\log l)^{k-2}\psi(l)$$

converges (here we again set $k = m+n$); in particular, VWMA systems form a set of measure zero. The converse (i.e., a multiplicative analogue of Theorem 5.2.5) follows from the results of Schmidt [196] and Gallagher [85].

It turns out that the problems rooted in multiplicative Diophantine approximation bring one to *higher rank actions* on the space of lattices. We illustrate this by two examples below, where for the sake of simplicity of exposition we specialize to the case $m = 1$ (one linear form $\mathbf{q} \mapsto \mathbf{y}\cdot\mathbf{q}$, $\mathbf{y} \in \mathbb{R}^n$), setting $k = n+1$.

3a. *Variations on the theme of Littlewood.* It is easy to see that (5.14) is equivalent to a vector $\mathbf{y} \in \mathbb{R}^n$ (viewed as a linear form $\mathbf{q} \mapsto \mathbf{y}\cdot\mathbf{q}$) not being BMA; in other words, Conjecture 5.3.1 states that no $\mathbf{y} \in \mathbb{R}^n$, $n \geqslant 2$, is badly multiplicatively approximable. Here one can clearly observe the similarity between the statements of the Oppenheim and Littlewood's conjectures. Indeed, (5.14) amounts to saying that 0 is the infimum of absolute values of a certain homogeneous polynomial at integer points $(p, \mathbf{q})$ with $q_i \neq 0$ for every i. Moreover, as it was the case with the Oppenheim conjecture, one can easily see that it is enough to prove Conjecture 5.3.1 for $n = 2$. The similarity is further deepened by an observation made by Cassels and Swinnerton-Dyer in 1955 [37] concerning cubic forms $Q(\mathbf{x})$ in 3 variables which are products of three linear forms. As in Section 5.1, say that Q is *rational* if it is a multiple of a polynomial with rational coefficients, and *irrational* otherwise. It is shown in [37] that the '$k = 3$'-case of the following conjecture would imply Conjecture 5.3.1:

CONJECTURE 5.3.2. *Let Q be an irrational homogeneous polynomial in k variables represented as a product of k linear forms, $k \geqslant 3$. Then given any $\varepsilon > 0$ there exists an integer vector $\mathbf{x} \in \mathbb{Z}^k \setminus \{0\}$ such that $|Q(\mathbf{x})| < \varepsilon$.*

Because of the similarity with the quadratic form case, one can attempt to understand the situation according to the scheme developed in the preceding sections. Indeed, in view of Lemma 1.4.4, a dynamical system reflecting Diophantine properties of Q as above must come from the action of the group stabilizing Q on the space of lattices in $\mathbb{R}^k$, and the latter group is a subgroup of $\mathrm{SL}(k, \mathbb{R})$ conjugate to the full diagonal subgroup D. Arguing as in §5.1a, one can show that Conjecture 5.3.2 is equivalent to the following conjecture, already mentioned in §4.4c:

CONJECTURE 5.3.3. *Let D be the subgroup of diagonal matrices in $\mathrm{SL}(k, \mathbb{R})$, $k \geqslant 3$. Then any relatively compact orbit $D\Lambda$, $\Lambda \in \Omega_k$, is compact.*

As we saw in §4.1b, the above statement does not hold if $k = 2$. This once again highlights the difference between rank-one and higher rank dynamics. Note also that one can show the above conjecture to be a special case of Conjecture 4.4.11 of Margulis.

3b. *Multiplicative approximation on manifolds.* Since every VWA vector is VWMA (that is, ψ_ε-MA for some $\varepsilon > 0$) but not conversely, it is a more difficult problem to prove that a generic point of a nondegenerate manifold is not very well multiplicatively approximable. This has been known as Conjecture H_2 of Sprindžuk [221]; the polynomial special case (that is, a multiplicative strengthening of Mahler's problem) was conjectured by Baker in [11]. Both conjectures stood open, except for low-dimensional special cases, until [117] where the following was proved:

THEOREM 5.3.4. *Let M be a nondegenerate smooth submanifold of $\mathbb{R}^n$. Then almost all points of M are not VWMA.*

The strategy of the proof of Theorem 5.2.13 applies with minor changes. For $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ let us define

$$g_{\mathbf{t}} = \mathrm{diag}\Big(\exp\Big(\sum t_i\Big), \mathrm{e}^{-t_1}, \ldots, \mathrm{e}^{-t_n}\Big) \in \mathrm{SL}(k, \mathbb{R}). \tag{5.15}$$

One can show[22] (see [117] for a partial result) that $\mathbf{y} \in \mathbb{R}^n$ is VWMA iff for some $\gamma > 0$ there are infinitely many $\mathbf{t} \in \mathbb{Z}_+^n$ such that $\Delta(g_{\mathbf{t}} L_{\mathbf{y}} \mathbb{Z}^k) \geqslant \gamma \sum t_i$. Therefore it is enough to use Theorem 3.2.4 to prove a modification of the measure estimate of Theorem 5.2.15 with g_t as in (4.1) replaced by $g_{\mathbf{t}}$ as in (5.15).

Finally let us mention a multiplicative version of Theorem 5.2.14, proved in [21] by a modification of the method described above:

THEOREM 5.3.5. *Let M be a nondegenerate smooth submanifold of $\mathbb{R}^n$ and let ψ be such that $\sum_{l=1}^{\infty} (\log l)^{n-1} \psi(l)$ is finite. Then almost all points of M are not ψ-multiplicatively approximable.*

5.4. *Counting problems*

Let $\Lambda \subset \mathbb{R}^k$ be a unimodular lattice, $P \subset \mathbb{R}^k$ a compact region, and let $N(P, \Lambda)$ be the number of points of Λ inside P. By tP we denote the region obtained from P by the uniform dilatation by a factor $t > 0$. It is a classical problem of geometry of numbers to look at the asymptotics of $N(tP, \Lambda)$ as $t \to \infty$. It is well known that the main term of the asymptotics is $t^k \operatorname{vol}(P)$ whenever P has piecewise-smooth boundary [122], and the next question is to bound the error term

$$R(tP, \Lambda) = N(tP, \Lambda) - t^k \operatorname{vol}(P)$$

(for the unit disc in $\mathbb{R}^2$ this has been known as the *circle problem*). An 'algebraic' version is to take only those integral points which lie on some algebraic subvariety $V \subset \mathbb{R}^k$ and study the asymptotics of $N(tP, \Lambda \cap V)$. Below we describe two examples that show how the technique of homogeneous flows can be useful for this class of problems.

4a. *Counting integral points on homogeneous affine varieties.* Let V be a Zariski closed real subvariety of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Let G be a reductive real algebraic group defined over $\mathbb{Q}$ and with no nontrivial $\mathbb{Q}$-characters. Suppose that G acts on $\mathbb{R}^k$ via a $\mathbb{Q}$-representation $\rho : G \to \mathrm{GL}(k, \mathbb{R})$, and that the action of G on V is transitive. One is interested in the asymptotics of $N(tB, \Lambda \cap V)$ where B is the unit ball in $\mathbb{R}^k$ and $\Lambda = \mathbb{Z}^k$. In what follows, we will simply denote $N(tB, \cdot)$ by $N(t, \cdot)$.

Let Γ be a subgroup of finite index in $G(\mathbb{Z})$ such that $\Gamma \mathbb{Z}^k \subset \mathbb{Z}^k$. By a theorem of Borel and Harish-Chandra [28], $V(\mathbb{Z})$ is a union of finitely many Γ-orbits. Therefore to compute the asymptotics of $N(t, V(\mathbb{Z}))$ it is enough to consider the Γ-orbit of each point $p \in V(\mathbb{Z})$ separately, and compute the asymptotics of

$$N(t, \Gamma p) = \{\text{the number of points in } \Gamma p \text{ with norm } < t\}$$

as $t \to \infty$.

[22] More generally, one can state a multiplicative version of generalized Dani's correspondence (Theorem 5.2.9), relating multiplicative Diophantine properties of $A \in M_{m,n}(\mathbb{R})$ to orbits of the form $\{g L_A \mathbb{Z}^{m+n} \mid g \in D_+\}$ where D_+ is a certain open chamber in D. For a version of such a correspondence see [118, Theorem 9.2].

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Let H denote the stabilizer of p in G. Then H is a real reductive group defined over $\mathbb{Q}$. For $t > 0$ we define

$$R_t = \{gH \in G/H \mid \|\rho(g)p\| < t\}.$$

Let $\bar{e}$ denote the coset of identity in G/H. Then $N(t, \Gamma p) =$ cardinality of $\Gamma\bar{e} \cap R_t$ in G/H.

Motivated by the approach of Duke, Rudnick and Sarnak [73], the following result was proved by Eskin, Mozes and Shah [78].

THEOREM 5.4.1. *In the counting problem, suppose further that*

(1) *H is not contained in any proper $\mathbb{Q}$-parabolic subgroup of G; and*
(2) *any proper $\mathbb{Q}$-subgroup L of G containing H and any compact set $C \subset G$ satisfy the following 'nonfocusing' condition*:

$$\limsup_{t\to\infty} \lambda(CL \cap R_t)/\lambda(R_t) = 0, \tag{5.16}$$

here λ denotes a G-invariant measure on G/H.

Then

$$\lim_{t\to\infty} N(t, \Gamma p)/\lambda(R_t) = 1, \tag{5.17}$$

where λ is determined by the normalizations of Haar measures on G and H such that $\mathrm{vol}(G/\Gamma) = \mathrm{vol}(H/H \cap \Gamma) = 1$.

The following approach to the counting problem was noted in [73]. Let χ_t denote the characteristic function of the ball of radius t in $\mathbb{R}^k$. For $g \in G$, define

$$F_t(g) = \sum_{\bar{\gamma}\in\Gamma/H\cap\Gamma} \chi_t(g\gamma p).$$

Then F_t is a function on G/Γ, and $F_t(e) = N(t, \Gamma p)$. Put $\widehat{F_t}(g) = F_t(g)/\lambda(R_t)$. As in [73], one shows that in order to prove that $\widehat{F_t}(e) \to 1$ as $t \to \infty$ (which is equivalent to (5.17)), it is enough to show that

$$\widehat{F_t} \to 1 \quad \text{weakly in } L^2(G/\Gamma). \tag{5.18}$$

Using Fubini's theorem, one shows that for any $\psi \in C_c(G/\Gamma)$,

$$\langle \widehat{F_t}, \psi \rangle = \frac{1}{\lambda(R_t)} \int_{R_t} \overline{\psi^H} \,\mathrm{d}\lambda, \tag{5.19}$$

where

$$\psi^H(g) = \int_{H/H\cap\Gamma} \psi(gh\Gamma)\,\mathrm{d}\mu_H(\bar{h}) = \int_{G/\Gamma} \psi\,\mathrm{d}(g\mu_H) \tag{5.20}$$

is a function on G/H (here μ_H denotes the H-invariant probability measure on $H\Gamma/\Gamma \cong H/H\cap\Gamma$).

Using Theorem 3.7.4, Theorem 3.7.5 and the definition of unfocused sequences, we get the following: given any sequence $t_n \to \infty$ and an $\varepsilon > 0$, there exists an open set $\mathcal{A} \subset G/H$ such that

$$\liminf_{n\to\infty} \frac{\lambda(\mathcal{A}\cap R_{t_n})}{\lambda(R_{t_n})} > 1-\varepsilon; \tag{5.21}$$

and given any sequence $\{\bar{g}_i\} \subset \mathcal{A}$ which is divergent in G/H, the sequence $\{g_i\mu_H\}$ converges to μ_G, and hence $\psi^H(\bar{g}_i) \to \langle\psi, 1\rangle$. Thus by (5.19) and (5.20), one obtains $\lim_{t\to\infty}\langle\widehat{F_t}, \psi\rangle = \langle 1, \psi\rangle$, which proves (5.18).

First we give a special case of Theorem 5.4.1.

COROLLARY 5.4.2. *In the main counting problem, further suppose that H^0 is a maximal $\mathbb{Q}$-subgroup of G and admits no nontrivial $\mathbb{Q}$-characters. Then the conclusion* (5.17) *holds.*

In [78], Theorem 5.4.1 was applied to prove the following:

Let P be a monic $\mathbb{Q}$-irreducible polynomial of degree $n \geqslant 2$ with integral coefficients. Let

$$V_P = \{X \in M_n(\mathbb{R}) \colon \det(\lambda I - X) = P(\lambda)\}.$$

Since P has n distinct roots, V_P is the set of real $n\times n$-matrices X such that roots of P are the eigenvalues of X. Let $V_P(\mathbb{Z})$ denote the set of matrices in V_P with integral entries. Let B_t denote the ball in $M_n(\mathbb{R})$ centered at 0 and of radius t with respect to the Euclidean norm: $\|(x_{ij})\| = (\sum_{i,j} x_{ij}^2)^{1/2}$. We are interested in estimating, for large T, the number of integer matrices in B_t with characteristic polynomial P.

THEOREM 5.4.3. *There exists a constant $C_P > 0$ such that*

$$\lim_{t\to\infty} \frac{\#(V_P(\mathbb{Z})\cap B_t)}{t^{n(n-1)/2}} = C_P.$$

THEOREM 5.4.4. *Let α be a root of P and $K = \mathbb{Q}(\alpha)$. Suppose that $\mathbb{Z}[\alpha]$ is the integral closure of $\mathbb{Z}$ in K. Then*

$$C_P = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{D}} \cdot \frac{\pi^{m/2}/\Gamma(1+(m/2))}{\prod_{s=2}^{n} \pi^{-s/2}\Gamma(s/2)\zeta(s)},$$

where h = ideal class number of K, R = regulator of K, w = order of the group of roots of unity in K, D = discriminant of K, r_1 (*resp*. *r_2*) *= number of real* (*resp*. *complex*) *places of K, $m = n(n-1)/2$, and Γ denotes the usual Γ-function.*

REMARK 5.4.5. The hypothesis of Theorem 5.4.4 is satisfied if α is a root of the unity (see [119, Theorem 1.61]).

In the general case, the formula for C_P is a little more involved.

THEOREM 5.4.6. *Let the notation be as in Theorem* 5.4.3. *Then*

$$C_P = \sum_{\mathcal{O} \supset \mathbb{Z}[\alpha]} \frac{2^{r_1}(2\pi)^{r_2} h_{\mathcal{O}} R_{\mathcal{O}}}{w_{\mathcal{O}}\sqrt{D}} \cdot \frac{\pi^{m/2}/\Gamma(1+(m/2))}{\prod_{s=2}^{n} \pi^{-s/2}\Gamma(s/2)\zeta(s)},$$

where $\mathcal{O}$ denotes an order in $\mathbb{Z}[\alpha]$, $h_{\mathcal{O}}$ the class number of the order, $R_{\mathcal{O}}$ the regulator of the order, and $w_{\mathcal{O}}$ the order of the group of roots of unity contained in $\mathcal{O}$.

We note that the orders in K containing $\mathbb{Z}[\alpha]$ are precisely the subrings of K which contain $\mathbb{Z}[\alpha]$ and are contained in the integral closure of $\mathbb{Z}$ in K. The reader is referred to [119, Chapter 1, Section 1] for further details about orders in algebraic number fields.

Note also that in [205], Theorems 5.4.3, 5.4.4 and 5.4.6 are obtained as direct consequences of a slightly modified version of Theorem 3.6.3.

4b. *Counting lattice points in polyhedra.* Here we discuss the version of the 'circle problem' where P is a compact polyhedron in $\mathbb{R}^k$. It is easy to show that the estimate $R(tP, \Lambda) = \mathrm{O}(t^{k-1})$, $t \to \infty$, is valid for any polyhedron P and any lattice Λ. Moreover, it is best possible if $\Lambda = \mathbb{Z}^k$ and P is a parallelepiped with edges parallel to the coordinate axes. However, the error term may be logarithmically small (and, presumably, logarithmically small errors may appear only for polyhedra).

First results for $k = 2$ with the help of continued fractions were obtained in the 20-s by Hardy and Littlewood [93] and Khintchine [111].

THEOREM 5.4.7. *Suppose that a polygon $P \subset \mathbb{R}^2$ with m sides is such that the ith side is parallel to a vector $(1, a_i)$, $i = 1, \ldots, m$. If all the numbers a_i, $1 \leqslant i \leqslant m$, are badly approximable, then $R(tP, \mathbb{Z}^2) = \mathrm{O}(\ln t)$. If all the numbers are algebraic irrationals, then $R(tP, \mathbb{Z}^2) = \mathrm{O}(t^{\varepsilon})$ with arbitrarily small $\varepsilon > 0$. Finally, for almost all collections $(a_1, \ldots, a_m)$, $R(tP, \mathbb{Z}^2) = \mathrm{O}((\ln t)^{1+\varepsilon})$ with arbitrarily small $\varepsilon > 0$.*

Note that the first assertion was implicitly contained in [93]. The second one was proved by Skriganov in [215] with the help of Roth's theorem, and the third one in [111] and [214].

Subsequently Skriganov [214] generalized the result of Hardy and Littlewood to higher dimensions as follows. For $\Lambda \in \Omega_k$ let

$$\mathrm{Nm}(\Lambda) = \inf\bigl\{\bigl|\mathrm{Nm}(\mathbf{x})\bigr|,\ \mathbf{x} \in \Lambda - \{0\}\bigr\}$$

be the homogeneous minimum with respect to the norm form

$$\mathrm{Nm}(\mathbf{x}) = \prod_{i=1}^{k} x_i, \quad \mathbf{x} \in \mathbb{R}^k.$$

The lattice Λ is said to be *admissible* if $\mathrm{Nm}(\Lambda) > 0$.

THEOREM 5.4.8. *Let Π be a parallelepiped with sides parallel to coordinate axes, and let Λ be an admissible unimodular lattice in $\mathbb{R}^k$. Then $R(t\Pi, \Lambda) = \mathrm{O}((\ln t)^{k-1})$.*

Notice that the exponent $k-1$ here is, apparently, the best possible. Apart from parallelepipeds, no other examples in higher dimensions are known with such an asymptotics of the error term.

Recently Skriganov [216] obtained remarkable results about 'typical' error for a given polyhedron $P \subset \mathbb{R}^k$.

THEOREM 5.4.9. *Let P be a compact polyhedron. Then for almost all unimodular lattices $\Lambda \subset \mathbb{R}^k$, $R(tP, \Lambda) = \mathrm{O}((\ln t)^{k-1+\varepsilon})$ with arbitrarily small $\varepsilon > 0$.*

The result was obtained via studying the dynamics of Cartan subgroup action on the space Ω_k. In short, the idea is as follows. To each flag

$$f = \left\{P = P_f^k \supset P_f^{k-1} \supset \cdots \supset P_f^0,\ \dim P_f^j = j\right\}$$

of faces of P one associates an orthogonal matrix $g_f \in \mathrm{SO}(k)$ whose j-th row represents the unit vector parallel to P_f^{k-j+1} and orthogonal to P_f^{k-j}. Given a lattice $\Lambda \in \Omega_k$, let $\Lambda^\perp$ be the *dual lattice* determined by the relation $(\Lambda, \Lambda^\perp) \subset \mathbb{Z}$. One considers the orbits $Dg_f\Lambda^\perp \subset \Omega_k$, where $D \subset \mathrm{SL}(k, \mathbb{R})$ is the Cartan subgroup of all diagonal matrices. Notice that D keeps $\mathrm{Nm}(\Lambda)$ invariant. It turns out that the rate of approach of these orbits to infinity (= the cusp of Ω_k) determines the behavior of the corresponding error $R(tP, \Lambda)$ as $t \to \infty$.

The connection requires rather refined Fourier analysis on $\mathbb{R}^k$; see [216] for the details. It turns out that the asymptotics of $R(tP, \Lambda)$ depends on the behavior of the ergodic sums $S(g_f\Lambda^\perp, r)$. Here

$$S(\Lambda, r) = \sum_{d \in \Delta_r} \delta(d\Lambda)^{-k},$$

where

$$\Delta = \left\{\mathrm{diag}\left(2^{m_1}, \ldots, 2^{m_k}\right),\ m_i \in \mathbb{Z},\ m_1 + \cdots + m_k = 0\right\}$$

is a discrete lattice in D,

$$\Delta_r = \left\{\mathrm{diag}\left(2^{m_1}, \ldots, 2^{m_k}\right),\ |m_i| \leqslant r,\ m_1 + \cdots + m_k = 0\right\}$$

is a 'ball' in Δ, and $\delta(\cdot)$ is as defined in (1.2). Note that $\Delta \simeq \mathbb{Z}^{k-1}$ and $\mathrm{card}(\Delta_r) \sim r^{k-1}$.

Clearly, $\delta(\Lambda) \geqslant (\mathrm{Nm}(\Lambda))^{1/k}$ and hence $S(\Lambda, r) = \mathrm{O}(r^{k-1})$ for any admissible lattice Λ. Also, by Mahler's criterion, if Λ is admissible then the orbit $D\Lambda \subset \Omega_k$ is bounded. If Π is a parallelepiped with edges parallel to coordinate axes then the matrices g_f form the group of permutations and reorientations of coordinate axes. Hence $S(g_f\Lambda^\perp, r) = S(\Lambda^\perp, r)$. If

Λ is admissible then so is $\Lambda^\perp$ and hence $S(g_f\Lambda^\perp, r) = \mathrm{O}(r^{k-1})$ for any flag of faces of Π. This leads to the bound $R(t\Pi, \Lambda) = \mathrm{O}((\ln t)^{k-1})$ and proves Theorem 5.4.8.

Notice that the set of admissible lattices is a null set in Ω_k because the action of Δ on Ω_k is ergodic and hence almost all Δ-orbits are everywhere dense in Ω_k. Typical asymptotics for $S(\Lambda, r)$ can be obtained using the Pointwise Ergodic Theorem applied to the action of Δ. Unfortunately, the function $\delta(\Lambda)^{-k}$ is not integrable over Ω_k. However, the function $\delta(\Lambda)^{\varepsilon-k}$ with arbitrary $\varepsilon > 0$ is already integrable and one easily derives that for almost all $\Lambda \in \Omega_k$, we have $S(\Lambda, r) = \mathrm{O}(r^{k-1+\varepsilon})$ with arbitrarily small $\varepsilon > 0$. One treats this saying that for a typical Λ, the 'balls' $\Delta_r\Lambda$ approach infinity very slowly. This leads to the typical asymptotics $S(tP, \Lambda) = \mathrm{O}((\ln t)^{k-1+\varepsilon})$.

The main result of [216] was sharpened by Skriganov and Starkov [217] as follows:

THEOREM 5.4.10. *Given a compact polyhedron P and a lattice $\Lambda \in \Omega_k$, for almost all orthogonal rotations $g \in \mathrm{SO}(k)$ one has the bound $R(tP, g\Lambda) = \mathrm{O}((\ln t)^{k-1+\varepsilon})$ with arbitrarily small $\varepsilon > 0$.*

This result is derived from Theorem 5.4.9 using the symmetries generated by the Weyl group.

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