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Dynamics of the Cosmological Constant in Two-Dimensional Universe

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We consider a two-dimensional model of gravity with the cosmological constant as a dynamical variable. The effective cosmological constant is derived when the universe has no initial boundary. It turns out to be extremely small if the universe is sufficiently large.

§ 1. Introduction

It is a profound mystery about the Universe that the observational bounds for the cosmological constant are incredibly small. This has motivated various ideas[1] on the subject of gravitation and cosmology.

Recent interest in two-dimensional gravity might be largely rooted in stringy approach to unified theory. However, we hope that two-dimensional theories of gravity may also serve as toy models for investigating qualitative features of realistic gravity in four dimensions.

In this paper, we consider a simple model of two-dimensional gravity where the cosmological constant appears as a constant of integration. Namely, the cosmological constant is determined by an initial conditon for a dynamical variable, whose expectation value will be computed when the state of the universe is of the Hartle-Hawking type.[2] The effective cosmological constant turns out to be extremely small if the universe is sufficiently large.

§ 2. The model

Let us consider the following model Lagrangian in two dimensions:

$$\mathcal{L}=\mathcal{L}_c(g_{\mu\nu})+\frac{1}{2\pi}(\rho\surd+\lambda\surd+\lambda\epsilon^{\mu\nu}\partial_\mu A_\nu)\,, \tag{1}$$

where $\mathcal{L}_c(g_{\mu\nu})$ denotes the effective Lagrangian for conformal matter with central charge c coupled to gravity,[3),4)] ρ is a renormalized cosmological constant, $\surd$ represents the invariant volume density[5] in terms of the metric tensor $g_{\mu\nu}$, λ is a scalar field which contributes to the effective cosmological constant, A_μ is an abelian gauge field, and $\epsilon^{\mu\nu}$ denotes the Levi-Civita tensor.

The model (1) may be regarded as a two-dimensional analogue of the covariant form of unimodular gravity in four dimensions given by Henneaux and Teitelboim.[6] Classically its physical contents are almost the same as those of the conventional gravity. A major difference results from equations of motion $\partial_\mu\lambda=0$, which indicate that the effective cosmological constant appears as a constant of integration. Thus it is determined by an initial condition for the universe.

In the following sections, we will calculate the expectation value of the observable λ in the case of no-boundary universe, which involves an initial condition we need. That is, we adopt as our universe a hemisphere with w wormholes attached. Then the desired expectation value will be obtained as a one-point function $\langle\lambda\rangle$ on a closed Riemann surface M with $h=2w$ handles.

§ 3. Partition function

In this section, we estimate the partition function for the model (1) exposed in the previous section:

$$Z=\int\mathcal{D}g\mathcal{D}\lambda\mathcal{D}A\, e^{-S}, \tag{2}$$

where

$$S=\int_M d^2x\mathcal{L}. \tag{3}$$

We note that integration over the multiplier field λ should be performed along the direction of the imaginary axis[7] so as to put the theory (1) properly in the Euclidean path integral (2).

Let us first define the zero mode λ_0 of the field λ as follows:

$$\lambda=\lambda_0+\lambda_1, \quad \partial_\mu\lambda_0=0, \tag{4}$$

where λ_1 satisfies a condition

$$\partial_\mu\lambda_1=0 \Longleftrightarrow \lambda_1=0. \tag{5}$$

Then the action (3) is written as

$$S=S_c+\lambda_0\left(\frac{1}{2\pi}\int_M d^2x\sqrt{\ } - T\right)+\frac{1}{2\pi}\int_M d^2x(\lambda_1\sqrt{\ } +\lambda_1\epsilon^{\mu\nu}\partial_\mu A_\nu). \tag{6}$$

Here we have introduced

$$S_c=\int_M d^2x\left(\mathcal{L}_c+\frac{1}{2\pi}\rho\sqrt{\ }\right), \quad T=-\frac{1}{2\pi}\int_M d^2x\epsilon^{\mu\nu}\partial_\mu A_\nu, \tag{7}$$

where T comes out to be a number which is independent of fluctuation in the field A_μ.

The form (6) of the action S allows us to perform successive path integration in (2) over the fields A_μ and λ_1 to obtain

$$Z=\int\mathcal{D}g\mathcal{D}\lambda_0\, e^{-S'}, \tag{8}$$

where

$$S'=S_c+\lambda_0\left(\frac{1}{2\pi}\int_M d^2x\sqrt{\ } - T\right). \tag{9}$$

Further integration over the variable λ_0 results in the expression

$$Z=\int \mathcal{D}g\, e^{-S_c}\delta\left(\frac{1}{2\pi}\int_M d^2x\sqrt{\ } -T\right), \tag{10}$$

which implies that T characterizes the size of the universe.

This expression of the partition function Z makes its T dependence apparent[4] through scaling behavior:

$$Z\sim T^X e^{-\rho T}, \tag{11}$$

where we have defined a constant

$$X=\frac{1}{12}(h-1)(25-c+\sqrt{(25-c)(1-c)})-1\,. \tag{12}$$

Note that the form (11) is universal in the sense that it appears independent of the detailed content of matter-gravity action S_c when the volume T is large.

§ 4. Cosmological constant

Now we proceed to compute the desired one-point function $\langle\lambda\rangle$. With the aid of Eqs. (8) and (9), we see

$$Z^{-1}\frac{\partial}{\partial T}Z=\langle\lambda_0\rangle=\langle\lambda\rangle\,, \tag{13}$$

where the last equality follows from the definition (4) and (5). Thus, by means of (11), we obtain

$$\langle\lambda\rangle=\frac{X}{T}-\rho\,. \tag{14}$$

Quantum fluctuation $\tilde{\lambda}$ is defined by

$$\lambda=\langle\lambda\rangle+\tilde{\lambda}\,, \tag{15}$$

which satisfies $\langle\tilde{\lambda}\rangle=0$. Substituting the above expressions into the Lagrangian (1), we immediately get

$$\mathcal{L}=\mathcal{L}_c+\frac{1}{2\pi}(\Lambda\sqrt{\ }+\tilde{\lambda}\sqrt{\ }+\tilde{\lambda}\epsilon^{\mu\nu}\partial_\mu A_\nu+\langle\lambda\rangle\epsilon^{\mu\nu}\partial_\mu A_\nu)\,, \tag{16}$$

where we have written

$$\Lambda=\frac{X}{T}. \tag{17}$$

As a conceivable interpretation, these results imply that the effective cosmological constant, which directly affects the motion of the metric $g_{\mu\nu}$, is given by Λ with the fluctuation $\tilde{\lambda}$ contributing to it no more. In view of (17), we conclude that the effective cosmological constant is expected to be extremely small when the universe is sufficiently large.

§ 5. Discussion

We have computed the effective cosmological constant (17) in the theory (1) of two-dimensional gravity when the state of the universe is of the Hartle-Hawking type. Two remarks are in order:

i) Although the value Λ is tiny for large T, it turned out to be non-zero. Observational cosmology suggests that this feature might be realized in the Universe.

ii) The effective cosmological constant Λ is not necessarily small when the size of the universe T is not so large. This might be adequate for inflationary scenarios which need dominance of the cosmological-constant effect in an early epoch of the Universe.

It seems interesting to ask whether these features will be attained in realistic four-dimensional quantum gravity yet to come.

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