Hindawi
Complexity
Volume 2021, Article ID 6209908, 12 pages
https://doi.org/10.1155/2021/6209908

Research Article

Dynamics on Effect of Prey Refuge Proportional to Predator in Discrete-Time Prey-Predator Model

G. S. Mahapatra [1], **P. K. Santra** [2] **and Ebenezer Bonyah** [3]

[1] *Department of Mathematics, National Institute of Technology Puducherry, Karaikal 609609, Puducherry, India*
[2] *Abada Nsup School, Howrah 711313, West Bengal, India*
[3] *Department of Mathematics Education, Akenten Appiah Menka University of Skills Training and Entrepreneurial Development, Kumasi, Ghana*

Correspondence should be addressed to Ebenezer Bonyah; ebbonya@gmail.com

Received 6 August 2021; Revised 20 September 2021; Accepted 11 October 2021; Published 25 October 2021

Academic Editor: Viorel-Puiu Paun

Prey-predator models with refuge effect have great importance in the context of ecology. Constant refuge and refuge proportional to prey are the most popular concepts of refuge in the existing literature. Now, there are new different types of refuge concepts attracting researchers. This study considers a refuge concept proportional to the predator due to the fear induced by predators. When predators increase, fears also increase and that is why prey refuges also increase. Here, we examine the influence of prey refuge proportional to predator effect in a discrete prey-predator interaction with the Holling type II functional response model. Is this refuge stabilizing or destabilizing the system? That is the central question of this study. The existence and stability of fixed points, Period-Doubling Bifurcation, Neimark–Sacker Bifurcation, the influence of prey refuge, and chaos are analyzed. This work provides the bifurcation diagrams and Lyapunov exponents to analyze the refuge parameter of the model. The proposed discrete model indicates rich dynamics as the effect of prey refuge through numerical simulations.

1. Introduction

In the current biomathematical literature, prey-predator interaction has become an exciting subject matter due to its influence on the environment. Many researchers [1–4] have devoted considerable time to explore several perspectives of the dynamical behaviour of this subject matter in ecology and the associated growth of population models [5–7]. Nature offers some amount of protection to some prey populations in the form of refuge dimensions. In some instances, refugia give a prolonged prey-predator interaction [8, 9] to reduce the probability of extinction due to predation activity. This phenomenon has been investigated extensively by many researchers in the discrete domain of refuse concepts [10–12].

In the context of populations characterized by overlapping generations, the nature of the birth process is a continuous matter; therefore, the predator-prey relationship is frequently developed through different models utilizing a deterministic approach such as ordinary differential equations. Several species, including monocarpic plants and semelparous animals, possess a discrete nonoverlapping generation character and have birth and regular breeding seasons. Their interactions are presented in the form of difference equations or other forms such as discrete-time mappings [13–17]. A discrete-time prey-predator model is usually characterized by more complicated dynamics behaviour than the associated continuous-time models [18–21]. Several researchers have focused on this concept to present a comprehensive rich dynamics of this phenomenon, including stability analysis of equilibria [22–27], Period-Doubling Bifurcation [28], Neimark–Sacker bifurcation [29], and chaos control [30]. Liu and Cai [31] explored the presence of bifurcation and chaos in a discrete-time prey-predator system. Elettreby et al. [32] used mixed functional response to investigate the dynamics of a discrete prey-predator model. AlSharawi et al. [33] studied a discrete prey-predator model with a nonmonotonic functional response and strong Allee effect in prey. Wang and Fečkan [34] studied the dynamics of a discrete nonlinear prey-predator model. Rech [35] investigated two discrete-time counterparts of a continuous-time prey-predator model. Khan and Khalique [36] discussed Neimark–Sacker bifurcation and hybrid control in a discrete-time Lotka–Volterra model. For some more dynamical investigations related to different versions of prey-predator models, we refer to Elsadany et al. [37]; Baydemir et al. [38]; Rozikov and Shoyimardonov [39]; Singh and Deolia [40]; and references therein.

In this paper, we consider a refuge term proportional to predator density due to fear induced by predators. The earlier literature has pointed out that the use of refuge by a fixed number or a small percentage of prey exerts a stabilizing effect on the dynamics of the interacting populations. This article is concerned with the above statement assuming that the quantity of prey in refugia is proportional to that of predators due to the fear induced by the predator. We analyze the dynamic properties of such a prey-predator model with prey self-limitation. The rest of the paper is partitioned as follows. A discrete-time prey-predator model with refuge is formulated in the second section. Section 3 is devoted to the local stability analysis of the model. Bifurcation analysis of the proposed model is presented in Section 4. Section 5 deals with the influence of prey refuge on the proposed model. Section 6 gives a chaos control procedure. Section 7 presents numerical simulations with support of the presented dynamical analysis of the proposed model. Section 8 presents the conclusion that is the last aspect of the findings from this study.

2. Development of a Discrete Prey-Predator System

The densities of prey and predator populations vary with time, have a uniform distribution over space, and have no stage structure for neither prey nor predators. The proposed model takes into consideration a generalised prey-predator model incorporating the logistic growth of prey with functional response $\varphi(x)$ of the predator population as $(dx/dt) = rx(1-(x/k)) - c\varphi(x)y$, and the predator as a population is epitomised by the expression $(dy/dt) = d\varphi(x)y - fy$, where x and y are the density of prey and predator populations correspondingly at any time t. The parameters r, k, c, d, and f are all taken to be positive constants and have its appropriate biological interpretations, respectively. r denotes the intrinsic per capita growth rate of prey population, k represents the environmental carrying capacity of prey population, c denotes the maximal per capita consumption rate of predators, d considers the efficiency with which predators transform consumed prey into new predators, and f represents the per capita death rate of predators. $\varphi(x)$ deals with the functional response of the predator population and fulfils the assumption $\varphi(0) = 0$ and $\varphi'(x) > 0$ for $x > 0$. There exists a quantity x_r of prey population that incorporates refuges, and therefore we have modified the functional response to incorporate prey refuges, and we consider $\varphi(x) = (x/(e+x))$, and $x_r = by$. The mathematical form of the proposed predator-prey model is as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \frac{c(x-by)y}{e+x-by}, \\ \frac{dy}{dt} &= \frac{d(x-by)y}{e+x-by} - fy. \end{aligned} \tag{1}$$

To arrive at a discrete-time model from (1), let $(dx/dt) = ((x_{t+h}-x_t)/h)$ and $(dy/dt) = ((y_{t+h}-y_t)/h)$, where x_t and y_t are the densities of the prey and predator populations, respectively, in discrete time t. Let $h \longrightarrow 1$ and $f = 1$, we obtain the equations for the $(n+1)$th generation of the prey and predator populations substituting t by n as follows:

$$\begin{aligned} x_{n+1} &= (r+1)x_n\left(1-\frac{r}{k(r+1)}x_n\right) - \frac{c(x_n-by_n)y_n}{e+x_n-by_n}, \\ y_{n+1} &= \frac{d(x_n-by_n)y_n}{e+x_n-by_n}. \end{aligned} \tag{2}$$

Let $(r/k(r+1)) = 1$ and $(r+1) = a$, we obtain the discrete-time predator-prey system from (2) as follows:

$$\begin{aligned} x_{n+1} &= ax_n(1-x_n) - \frac{c(x_n-by_n)y_n}{e+x_n-by_n}, \\ y_{n+1} &= \frac{d(x_n-by_n)y_n}{e+x_n-by_n}, \end{aligned} \tag{3}$$

where a, b, c, d, and e are all positive constants. By the biological meaning of the model variables, we only consider the system in the region $\Omega = \{(x,y)\colon x \geq 0, y \geq 0\}$ in the (x,y) – plane.

3. Fixed Points and Stability Analysis

Fixed points of the discrete system (3) are determined by solving the following nonlinear system of equations:

$$\begin{aligned} x &= ax(1-x) - \frac{c(x-by)y}{e+x-by}, \\ y &= \frac{d(x-by)y}{e+x-by}. \end{aligned} \tag{4}$$

We get three nonnegative fixed points by solving above equations: (i) $P_0 = (0,0)$, (ii) $P_1 = (((a-1)/a), 0)$, $a > 1$, and (iii) $P_2 = (x_2, y_2)$; here, x_2 is positive root of the equation $x^2 + Ax + B = 0$, where $A = ((1-a)/a) + (c/abd)$, $B = -(ce/bd(d-1))$, and $y_2 = (x_2/b) - (e/b(d-1))$.

3.1. Dynamic Behaviour of the Discrete System.

In this part, we examine the local behaviour of the discrete system (3) for every steady point of the model. The stability of system (3) is accomplished by deriving a Jacobian matrix connected to each equilibrium point. The Jacobian Matrix J for system (3) is

$$J = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad (5)$$

where $a_{11} = a(1-2x) - (cey/(e+x-by)^2)$, $a_{12} = -(c(e(x-2by) + (x-by)^2)/(e+x-by)^2)$, $a_{21} = (de\, y/(e+x-by)^2)$, and $a_{22} = (d(e(x-2by) + (x-by)^2)/(e+x-by)^2)$.

The characteristic equation of the matrix J is $\lambda^2 - \mathrm{Tr}(J)\lambda + \mathrm{Det}(J) = 0$, where $\mathrm{Tr}(J)$ = trace of matrix $= a(1-2x) + ((d(e(x-2by) + (x-by)^2) - cey)/(e+x-by)^2)$ and $\mathrm{Det}(J)$ = determinant of matrix $= a(1-2x)(d(e(x-2by) + (x-by)^2)/(e+x-by)^2)$.

Therefore, the discrete-time system (3) possesses the following:

(i) A dissipative dynamical system if $|a(1-2x)(d(e(x-2by) + (x-by)^2)/(e+x-by)^2)| < 1$
(ii) A conservative dynamical system if and only if $|a(1-2x)(d(e(x-2by) + (x-by)^2)/(e+x-by)^2)| = 1$
(iii) An undissipated dynamical system otherwise

3.1.1. Stability and Dynamic Behaviour of the Model at P_0. The dynamical behaviour of the system is studied utilizing the Jacobian matrix at the fixed point $P_0(0,0)$. The Jacobian matrix at $P_0(0,0)$ is $J = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. The equilibrium point $P_0(0,0)$ is called (i) sink if $|a| < 1$; (ii) saddle if $|a| > 1$; and (iii) nonhyperbolic if $|a| = 1$.

3.1.2. Stability and Dynamic Behaviour at P_1. The Jacobian matrix at the fixed point $P_1(((a-1)/a), 0)$ is $J = \begin{bmatrix} 2-a & -(c(a-1)/(ae+a-1)) \\ 0 & d(a-1)/(ae+a-1) \end{bmatrix}$. The equilibrium point P_1 is said to be

(i) Sink if $|2-a| < 1$ and $|d(a-1)/(ae+a-1)| < 1$
(ii) Source if $|2-a| > 1$ and $|d(a-1)/(ae+a-1)| > 1$
(iii) Saddle if $|2-a| > 1$ and $|d(a-1)/(ae+a-1)| < 1$ or $|2-a| < 1$ and $|d(a-1)/(ae+a-1)| > 1$
(iv) Nonhyperbolic if $|2-a| = 1$ or $|d(a-1)/(ae+a-1)| = 1$

3.1.3. Dynamic Behaviour around the Interior Fixed Point. Following the Jacobian matrix at the interior fixed point $P_2(x_2, y_2)$, we get

$$\begin{aligned} 1 - \mathrm{Tr}(L) + \mathrm{Det}(L) &= 1 - a(1-2x_2) - \frac{d\left(e(x_2-2by_2) + (x_2-by_2)^2\right) - cey_2}{(e+x_2-by_2)^2} + a(1-2x_2)\frac{d\left(e(x_2-2by_2) + (x_2-by_2)^2\right)}{(e+x_2-by_2)^2}, \\ 1 + \mathrm{Tr}(L) + \mathrm{Det}(L) &= 1 + a(1-2x_2) + \frac{d\left(e(x_2-2by_2) + (x_2-by_2)^2\right) - cey_2}{(e+x_2-by_2)^2} + a(1-2x_2)\frac{d\left(e(x_2-2by_2) + (x_2-by_2)^2\right)}{(e+x_2-by_2)^2}. \end{aligned} \quad (6)$$

If $1 - \mathrm{Tr}(L) + \mathrm{Det}(L) > 0$, then interior equilibrium point $P_2(x_2, y_2)$ is as follows:

(i) Sink if $1 + \mathrm{Tr}(L) + \mathrm{Det}(L) > 0$ and $\mathrm{Det}(L) < 1$
(ii) Source if $1 + \mathrm{Tr}(L) + \mathrm{Det}(L) > 0$ and $\mathrm{Det}(L) > 1$
(iii) Saddle if $1 + \mathrm{Tr}(L) + \mathrm{Det}(L) < 0$
(iv) Nonhyperbolic if $1 + \mathrm{Tr}(L) + \mathrm{Det}(L) = 0$ and $\mathrm{Tr}(L) \neq 0, 2$ or $[\mathrm{Tr}(L)]^2 - 4\mathrm{Det}(L) < 0$ and $\mathrm{Det}(L) = 1$

If $1 - \mathrm{Tr}(L) + \mathrm{Det}(L) > 0$, $1 + \mathrm{Tr}(L) + \mathrm{Det}(L) = 0$, and $\mathrm{Tr}(L) \neq 0, 2$, then $P_2(x_2, y_2)$ can undergo Period-Doubling Bifurcation (PDB).

If $1 - \mathrm{Tr}(L) + \mathrm{Det}(L) > 0$, $(\mathrm{Tr}(L))^2 - 4\mathrm{Det}(L) < 0$, and $\mathrm{Det}(L) = 1$, then $P_2(x_2, y_2)$ can undergo Neimark–Sacker Bifurcation (NSB).

4. Bifurcation Analysis

In this section, we investigate the existence of NSB and PDB at the positive fixed point $P_2(x_2, y_2)$ of the discrete-time of (3). Countless numbers of dynamical properties of the system can be analyzed with regard to the emergence of NSB and PDB. Bifurcation frequently occurs when there is a change in the stability of a fixed point. Thus, the qualitative properties of a dynamical system vary. The NSB and PDB are explored in the positive fixed point direction $P_2(x_2, y_2)$ of system (3) considering refuge parameter b as a bifurcation parameter. Utilizing the emergence of NSB, closed invariant circles are obtained. The bifurcation can be classified as supercritical or subcritical, leading to a stable or unstable closed invariant curve, in that order. A PDB in a discrete dynamical system is a bifurcation where a small perturbation in a parameter value brings about the system moving to a new behaviour with the double period of the original system.

4.1. Neimark–Sacker Bifurcation. Here, we examine the NSB of the discrete prey-predator model (3) at $P_2(x_2, y_2)$ for the parameters lying in the following set $A_{NSB} = \{(a, b = b_{NSB}, c, d, e): 1 - \mathrm{Tr}(L) + \mathrm{Det}(L) > 0,\ (\mathrm{Tr}(L))^2 - 4\mathrm{Det}(L) < 0,\ \mathrm{Det}(L) = 1\}$.

In exploring the NSB, b is utilised as the bifurcation parameter. Further, b^* is the perturbation of b where $|b^*| \lll 1$, and perturbation of the model is considered as follows:

$$\begin{aligned} x_{n+1} &= a x_n (1 - x_n) - \frac{c(x_n - (b + b^*) y_n) y_n}{e + x_n - (b + b^*) y_n} \equiv f(x_n, y_n, b^*), \\ y_{n+1} &= \frac{d(x_n - (b + b^*) y_n) y_n}{e + x_n - (b + b^*) y_n} \equiv g(x_n, y_n, b^*). \end{aligned} \tag{7}$$

Let $u_n = x_n - x_2$ and $v_n = y_n - y_2$, then equilibrium $P_2(x_2, y_2)$ is converted into the origin, and further making expansion f and g as a Taylor series at $(u_n, v_n) = (0, 0)$ to the third order, model (7) turns to

$$\begin{aligned} u_{n+1} &= \alpha_1 u_n + \alpha_2 v_n + \alpha_{11} u_n^2 + \alpha_{12} u_n v_n + \alpha_{22} v_n^2 + \alpha_{111} u_n^3 \\ &\quad + \alpha_{112} u_n^2 v_n + \alpha_{122} u_n v_n^2 + \alpha_{222} v_n^3 + O\left(\left(|u_n| + |v_n| \right)^4 \right), \\ v_{n+1} &= \beta_1 u_n + \beta_2 v_n + \beta_{11} u_n^2 + \beta_{12} u_n v_n + \beta_{22} v_n^2 + \beta_{111} u_n^3 \\ &\quad + \beta_{112} u_n^2 v_n + \beta_{122} u_n v_n^2 + \beta_{222} v_n^3 + O\left(\left(|u_n| + |v_n| \right)^4 \right), \end{aligned} \tag{8}$$

where $\alpha_1 = f_x(x_2, y_2, 0)$, $\alpha_2 = f_y(x_2, y_2, 0)$, $\alpha_{11} = f_{xx}(x_2, y_2, 0)$, $\alpha_{12} = f_{xy}(x_2, y_2, 0)$, $\alpha_{22} = f_{yy}(x_2, y_2, 0)$, $\alpha_{111} = f_{xxx}(x_2, y_2, 0)$, $\alpha_{112} = f_{xxy}(x_2, y_2, 0)$, $\alpha_{122} = f_{xyy}(x_2, y_2, 0)$, $\alpha_{222} = f_{yyy}(x_2, y_2, 0)$, $\beta_1 = g_x(x_2, y_2, 0)$, $\beta_2 = g_y(x_2, y_2, 0)$, $\beta_{11} = g_{xx}(x_2, y_2, 0)$, $\beta_{12} = g_{xy}(x_2, y_2, 0)$, $\beta_{22} = g_{yy}(x_2, y_2, 0)$, $\beta_{111} = g_{xxx}(x_2, y_2, 0)$, $\beta_{112} = g_{xxy}(x_2, y_2, 0)$, $\beta_{122} = g_{xyy}(x_2, y_2, 0)$, and $\beta_{222} = g_{yyy}(x_2, y_2, 0)$.

Note that the characteristic equation associated with the linearization of model (8) at $(u_n, v_n) = (0, 0)$ is given by $\lambda^2 - \mathrm{Tr}(L_1(b^*))\lambda + \mathrm{Det}(L_1(b^*)) = 0$. The roots of the characteristic equation are $\lambda_{1,2}(b^*) = ((\mathrm{Tr}(L_1(b^*)) \pm i\sqrt{4\mathrm{Det}(L_1(b^*)) - (\mathrm{Tr}(L_1(b^*)))^2}/2)$.

From $|\lambda_{1,2}(b^*)| = 1$ and $b^* = 0$, we have $|\lambda_{1,2}(b^*)| = [\mathrm{Det}(L_1(b^*))]^{1/2}$ and $l = [\mathrm{d}|\lambda_{1,2}(b^*)|/\mathrm{d}b^*]_{b^*=0} \neq 0$.

In addition, it is required that when $b^* = 0$, $\lambda_{1,2}^i \neq 1$, $i = 1, 2, 3, 4$, which is equivalent to $\mathrm{Tr}(L_1(0)) \neq -2, -1, 1, 2$.

To study the normal form, let $\gamma = \mathrm{Im}(\lambda_{1,2})$ and $\delta = \mathrm{Re}(\lambda_{1,2})$. We define $T = \begin{bmatrix} 0 & 1 \\ \gamma & \delta \end{bmatrix}$, and using the transformation $\begin{bmatrix} u_n \\ v_n \end{bmatrix} = T \begin{bmatrix} \overline{x}_n \\ \overline{y}_n \end{bmatrix}$, model (8) becomes

$$\begin{aligned} \overline{x}_{n+1} &= \delta \overline{x}_n - \gamma \overline{y}_n + f_1(\overline{x}_n, \overline{y}_n), \\ \overline{y}_{n+1} &= \gamma \overline{x}_n + \delta \overline{y}_n + g_1(\overline{x}_n, \overline{y}_n), \end{aligned} \tag{9}$$

where the functions f_1 and g_1 denote the terms in model (9) in variables $(\overline{x}_n, \overline{y}_n)$ with the order at least two.

To undergo NSB, we require that the following discriminatory quantity Ω be nonzero:

$$\Omega = -\mathrm{Re}\left[\frac{(1 - 2\overline{\lambda})\overline{\lambda}^2}{1 - \lambda} \xi_{11} \xi_{20} \right] - \frac{1}{2} |\xi_{11}|^2 - |\xi_{02}|^2 + \mathrm{Re}\left(\overline{\lambda} \xi_{21} \right), \tag{10}$$

where

$$\begin{aligned} \xi_{20} &= \frac{\delta}{8}(2\beta_{22} - \delta\alpha_{22} - \alpha_{12} + 4\gamma\alpha_{22} + i(4\gamma\alpha_{22} - 2\alpha_{22} - 2\delta\alpha_{22})) + \frac{\gamma}{4}\alpha_{12} + i\frac{1}{8}\left(4\gamma\beta_{22} + 2\gamma^2\alpha_{22} - 2\alpha_{11}\right) \\ &\quad + \frac{\beta_{12}}{8} + \frac{\delta\alpha_{11} - 2\beta_{11} + \delta^3\alpha_{22} - \delta^2\beta_{22} - \delta^2\alpha_{12} + \delta\beta_{12}}{4\gamma}, \\ \xi_{11} &= \frac{\gamma}{2}(\beta_{22} - \delta\alpha_{22}) + i\frac{1}{2}\left(\gamma^2\alpha_{22} + \alpha_{11} + \delta\alpha_{12} + \delta^2\alpha_{22}\right) + \frac{\beta_{11} - \delta\alpha_{11} + \delta\beta_{12} - \delta^2\alpha_{12} - 2\delta^2\beta_{22} + 2\delta^3\alpha_{22}}{2\gamma}, \\ \xi_{02} &= \frac{1}{4}\gamma(2\delta\alpha_{22} + \alpha_{12} + \beta_{22}) + i\frac{1}{4}(\beta_{12} + 2\delta\beta_{22} - 2\delta\alpha_{12} - \alpha_{11}) - \frac{\beta_{11} - \delta\alpha_{11} + \delta\beta_{12} - \delta^2\alpha_{12} + \delta^2\beta_{22} - \delta^3\alpha_{22}}{4\gamma} + \frac{1}{4}\alpha_{22} i\left(\gamma^2 - 3\delta^2\right), \\ \xi_{21} &= \frac{3}{8}\beta_{222}\left(\gamma^2 + \delta^2\right) + \frac{\beta_{112}}{8} + \frac{\delta}{4}\alpha_{112} + \frac{\delta}{4}\beta_{122} + \alpha_{122}\left(\frac{\gamma^2}{8} + \frac{3\delta^2}{8} - \frac{\delta}{4}\right) + \frac{3}{8}\alpha_{111} + i\frac{3}{8}\alpha_{222}\left(\gamma^2 + 2\delta^2\right) + i\frac{3\gamma\delta}{8}\alpha_{122} - \frac{1}{8}\beta_{122}\gamma i \\ &\quad - i\frac{3\gamma\delta}{8}\beta_{222} - i\frac{3\beta_{111} - 3\delta\alpha_{111}}{8\gamma} - i\frac{3\delta\beta_{112} - 3\delta^2\alpha_{112}}{8\gamma} - i\frac{3\delta^2\beta_{122} - 3\delta^3\alpha_{122}}{8\gamma} - i\frac{3\delta^3\beta_{222} - 3\delta^4\alpha_{222}}{8\gamma}. \end{aligned} \tag{11}$$

Finally, the above analysis leads to the following associated results.

Theorem 1. *If* $\Omega \neq 0$, *then model (3) undergoes NSB at* $P_2(x_2, y_2)$ *when the parameter b changes in a small neighbourhood of* b_{NSB}. *If* $\Omega < 0\ (\Omega > 0)$, *then an attracting (repelling) invariant closed curve bifurcates from* $P_2(x_2, y_2)$.

4.2. Period-Doubling Bifurcation. Here, we consider one of the eigenvalues of the positive fixed point that $P_2(x_2, y_2)$ is $\lambda_1 = -1$, and the other (λ_2) is neither 1 nor -1, whenever parameters of the model are located in the following set $A_{PDB} = \{(a, b = b_{PDB}, c, d, e)\colon\ 1 - \mathrm{Tr}(J) + \mathrm{Det}(J) > 0,\ 1 + \mathrm{Tr}(J) + \mathrm{Det}(J) = 0, \mathrm{Tr}(J) \neq 0, 2\}$. Here, we discuss PDB of model (3) at $P_2(x_2, y_2)$ when parameters change in a small neighbourhood of A_{PDB}. In studying the flip bifurcation, b is made use as the bifurcation parameter. Furthermore, b^* is the perturbation of b where $|b^*| \lll 1$, and perturbation of the model is considered as follows:

$$
\begin{aligned}
x_{n+1} &= a x_n (1 - x_n) - \frac{c(x_n - (b + b^*) y_n) y_n}{e + x_n - (b + b^*) y_n} \equiv f(x_n, y_n, b^*),\\
y_{n+1} &= \frac{d(x_n - (b + b^*) y_n) y_n}{e + x_n - (b + b^*) y_n} \equiv g(x_n, y_n, b^*).
\end{aligned}
\tag{12}
$$

Let $u_n = x_n - x_2$ and $v_n = y_n - y_2$; then, the equilibrium $P_2(x_2, y_2)$ is transformed into the origin, and further expanding f and g as a Taylor series at $(u_n, v_n, b^*) = (0, 0, 0)$ up to the third order, system (12) becomes

$$
\begin{aligned}
u_{n+1} &= \alpha_1 u_n + \alpha_2 v_n + \alpha_{11} u_n^2 + \alpha_{12} u_n v_n + \alpha_{13} u_n b^* + \alpha_{23} v_n b^*\\
&\quad + \alpha_{111} u_n^3 + \alpha_{112} u_n^2 v_n + \alpha_{113} u_n^2 b^* + \alpha_{123} u_n v_n b^* + O\left(\left(|u_n| + |v_n| + |b^*|\right)^4\right),\\
v_{n+1} &= \beta_1 u_n + \beta_2 v_n + \beta_{11} u_n^2 + \beta_{12} u_n v_n + \beta_{22} v_n^2 + \beta_{13} u_n b^*\\
&\quad + \beta_{23} v_n b^* + \beta_{111} u_n^3 + \beta_{112} u_n^2 v_n + \beta_{113} u_n^2 b^* + \beta_{123} u_n v_n b^* + \beta_{223} v_n^2 b^* + O\left(\left(|u_n| + |v_n| + |b^*|\right)^4\right),
\end{aligned}
\tag{13}
$$

where $\alpha_1 = f_x(x_2, y_2, 0)$, $\alpha_2 = f_y(x_2, y_2, 0)$, $\alpha_{11} = f_{xx}(x_2, y_2, 0)$, $\alpha_{12} = f_{xy}(x_2, y_2, 0)$, $\alpha_{13} = f_{xb^*}(x_2, y_2, 0)$, $\alpha_{23} = f_{yb^*}(x_2, y_2, 0)$, $\alpha_{111} = f_{xxx}(x_2, y_2, 0)$, $\alpha_{112} = f_{xxy}(x_2, y_2, 0)$, $\alpha_{113} = f_{xxb^*}(x_2, y_2, 0)$, $\alpha_{123} = f_{xyb^*}(x_2, y_2, 0)$, $\beta_1 = g_x(x_2, y_2, 0)$, $\beta_2 = g_y(x_2, y_2, 0)$, $\beta_{11} = g_{xx}(x_2, y_2, 0)$, $\beta_{12} = g_{xy}(x_2, y_2, 0)$, $\beta_{22} = g_{yy}(x_2, y_2, 0)$, $\beta_{13} = g_{xb^*}(x_2, y_2, 0)$, $\beta_{23} = g_{yb^*}(x_2, y_2, 0)$, $\beta_{111} = g_{xxx}(x_2, y_2, 0)$, $\beta_{112} = g_{xxy}(x_2, y_2, 0)$, $\beta_{113} = g_{xxb^*}(x_2, y_2, 0)$, $\beta_{123} = g_{xyb^*}(x_2, y_2, 0)$, and $\beta_{223} = g_{yyb^*}(x_2, y_2, 0)$.

We define $T = \begin{bmatrix} \alpha_2 & \alpha_2 \\ -1 - \alpha_1 & \lambda_2 - \alpha_1 \end{bmatrix}$, and it is obvious that T is invertible. Using the transformation $\begin{bmatrix} u_n \\ v_n \end{bmatrix} = T \begin{bmatrix} \overline{x}_n \\ \overline{y}_n \end{bmatrix}$, system (13) becomes

$$
\begin{aligned}
\overline{x}_{n+1} &= -\overline{x}_n + f_1(u_n, v_n, b^*),\\
\overline{y}_{n+1} &= \lambda_2 \overline{y}_n + g_1(u_n, v_n, b^*),
\end{aligned}
\tag{14}
$$

where the functions f_1 and g_1 denote the terms in model (14) in variables (u_n, v_n, b^*) with order at least two.

Using the centre manifold theorem, we know that there exists a centre manifold $W^c(0, 0, 0)$ of system (14) at $(0, 0)$ in a small neighbourhood of $b^* = 0$, which can be approximately described as follows:

$$
\begin{aligned}
W^c(0, 0, 0) = \Big\{&(\overline{x}_n, \overline{y}_n, b^*) \varepsilon R^3\colon\ \overline{y}_{n+1}\\
&= \overline{\alpha}_1 \overline{x}_n^2 + \overline{\alpha}_2 \overline{x}_n b^* + O\left(\left(|\overline{x}_n| + |b^*|\right)^3\right)\Big\},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
\overline{\alpha}_1 &= \frac{\alpha_2 [(1 + \alpha_1)\alpha_{11} + \alpha_2 \beta_{11}]}{1 - \lambda_2^2} + \frac{\beta_{22}(1 + \alpha_1)^2}{1 - \lambda_2^2} - \frac{(1 + \alpha_1)[\alpha_{12}(1 + \alpha_1) + \alpha_2 \beta_{12}]}{1 - \lambda_2^2},\\
\overline{\alpha}_2 &= \frac{(1 + \alpha_1)[\alpha_{23}(1 + \alpha_1) + \alpha_2 \beta_{23}]}{\alpha_2 (1 + \lambda_2)^2} - \frac{(1 + \alpha_1)\alpha_{13} + \alpha_2 \beta_{13}]}{(1 + \lambda_2)^2}.
\end{aligned}
\tag{16}
$$

System (14) which is restricted to the centre manifold $W^c(0,0,0)$ has the following form:

$$\begin{aligned}\overline{x}_{n+1} &= -\overline{x}_n + h_1\overline{x}_n^2 + h_2\overline{x}_n b^* + h_3\overline{x}_n^2 b^* + h_4\overline{x}_n b^{*2} + h_5\overline{x}_n^3 \\ &\quad + O\left(\left(\left|\overline{x}_n\right| + \left|b^*\right|\right)^3\right) \equiv F\left(\overline{x}_n, b^*\right),\end{aligned} \tag{17}$$

where

$$\begin{aligned}
h_1 &= \frac{\overline{\alpha}_2\left[(\lambda_2-\overline{\alpha}_1)\alpha_{11}-\overline{\alpha}_2\beta_{11}\right]}{1+\lambda_2} - \frac{\beta_{22}\left(1+\overline{\alpha}_1\right)^2}{1+\lambda_2} - \frac{\left(1+\overline{\alpha}_1\right)\left[(\lambda_2-\overline{\alpha}_1)\alpha_{12}-\overline{\alpha}_2\beta_{12}\right]}{1+\lambda_2},\\
h_2 &= \frac{(\lambda_2-\overline{\alpha}_1)\alpha_{13}-\overline{\alpha}_2\beta_{13}}{1+\lambda_2} - \frac{\left(1+\overline{\alpha}_1\right)\left[(\lambda_2-\overline{\alpha}_1)\alpha_{23}-\overline{\alpha}_2\beta_{23}\right]}{\overline{\alpha}_2\left(1+\lambda_2\right)},\\
h_3 &= \frac{(\lambda_2-\alpha_1)\overline{\alpha}_1\alpha_{13}-\alpha_2\beta_{13}}{1+\lambda_2} + \frac{\left[(\lambda_2-\alpha_1)\alpha_{23}-\alpha_2\beta_{23}\right](\lambda_2-\alpha_1)\overline{\alpha}_1}{\alpha_2\left(1+\lambda_2\right)} - \frac{\left(1+\alpha_1\right)\left[(\lambda_2-\alpha_1)\alpha_{123}-\alpha_2\beta_{123}\right]}{1+\lambda_2}\\
&\quad + \frac{\alpha_2\left[(\lambda_2-\alpha_1)\alpha_{113}-\alpha_2\beta_{113}\right]}{1+\lambda_2} - \frac{\beta_{223}\left(1+\alpha_1\right)^2}{1+\lambda_2} + \frac{2\alpha_2\overline{\alpha}_2\left[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}\right]}{1+\lambda_2} - \frac{2\beta_{22}\overline{\alpha}_2\left(1+\alpha_1\right)(\lambda_2-\alpha_1)}{1+\lambda_2}\\
&\quad + \frac{\overline{\alpha}_2\left[(\lambda_2-\alpha_1)\alpha_{12}-\alpha_2\beta_{12}\right](\lambda_2-1-2\alpha_1)}{1+\lambda_2},\\
h_4 &= \frac{\overline{\alpha}_2\left[(\lambda_2-\alpha_1)\alpha_{13}-\alpha_2\beta_{13}\right]}{1+\lambda_2} + \frac{\left[(\lambda_2-\alpha_1)\alpha_{23}-\alpha_2\beta_{23}\right](\lambda_2-\alpha_1)\overline{\alpha}_2}{\alpha_2\left(1+\lambda_2\right)} + \frac{2\alpha_2\overline{\alpha}_2\left[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}\right]}{1+\lambda_2}\\
&\quad + \frac{2\beta_{22}\overline{\alpha}_2\left(1+\alpha_1\right)(\lambda_2-\alpha_1)}{1+\lambda_2} + \frac{\overline{\alpha}_2\left[(\lambda_2-\alpha_1)\alpha_{12}-\alpha_2\beta_{12}\right](\lambda_2-1-2\alpha_1)}{1+\lambda_2},\\
h_5 &= \frac{2\alpha_2\overline{\alpha}_1\left[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}\right]}{1+\lambda_2} + \frac{2\beta_{22}\overline{\alpha}_1(\lambda_2-\alpha_1)(1+\alpha_1)}{1+\lambda_2} + \frac{\left[(\lambda_2-\alpha_1)\alpha_{11}-\alpha_2\beta_{11}\right](\lambda_2-1-2\alpha_1)\overline{\alpha}_1}{1+\lambda_2}\\
&\quad + \frac{\overline{\alpha}_2^2\left[(\lambda_2-\alpha_1)\alpha_{111}-\alpha_2\beta_{111}\right]}{1+\lambda_2} - \frac{\overline{\alpha}_2\left(1+\alpha_1\right)\left[(\lambda_2-\alpha_1)\alpha_{112}-\alpha_2\beta_{112}\right]}{1+\lambda_2}.
\end{aligned} \tag{18}$$

For PDB, the two discriminatory quantities ξ_1 and ξ_2 be nonzero,

$$\begin{aligned}\xi_1 &= \left(\frac{\partial^2 F}{\partial\overline{x}\partial b^*} + \frac{1}{2}\frac{\partial F}{\partial b^*}\frac{\partial^2 F}{\partial\overline{x}^2}\right)\Bigg|_{(0,0)},\\ \xi_2 &= \left(\frac{1}{6}\frac{\partial^3 F}{\partial\overline{x}^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial\overline{x}^2}\right)^2\right)\Bigg|_{(0,0)}.\end{aligned} \tag{19}$$

Finally, from the above analysis, we have the following result.

Theorem 2. *If $\xi_1 \neq 0$ and $\xi_2 \neq 0$, then model (3) undergoes PDB at $P_2(x_2, y_2)$ when the parameter b varies in a small neighbourhood of b_{PDB}. If $\xi_2 > 0\,(\xi_2 < 0)$, then the period-two orbits that bifurcate from $P_2(x_2, y_2)$ are stable (unstable).*

5. Influence of Refuge

This section presents the impact of prey refuge on the model when the coexistence steady-state point $P_2(x_2, y_2)$ exists and is stable. It is simple to observe that x and y are continuous differential functions of parameter b.

Here, x_2 is the positive root of the equation $x^2 + Ax + B = 0$ where $A = ((1-a)/a) + (c/abd)$, $B = -(ce/bd(d-1))$, and $y_2 = (x_2/b) - (e/b(d-1))$.

If $(dx_2/db) > y_2$, then $(dy_2/db) > 0$; hence, y_2 is strictly increasing function of the parameter b.

If $(dx_2/db) < y_2$, then $(dy_2/db) < 0$; hence, y_2 is strictly decreasing function of the parameter b.

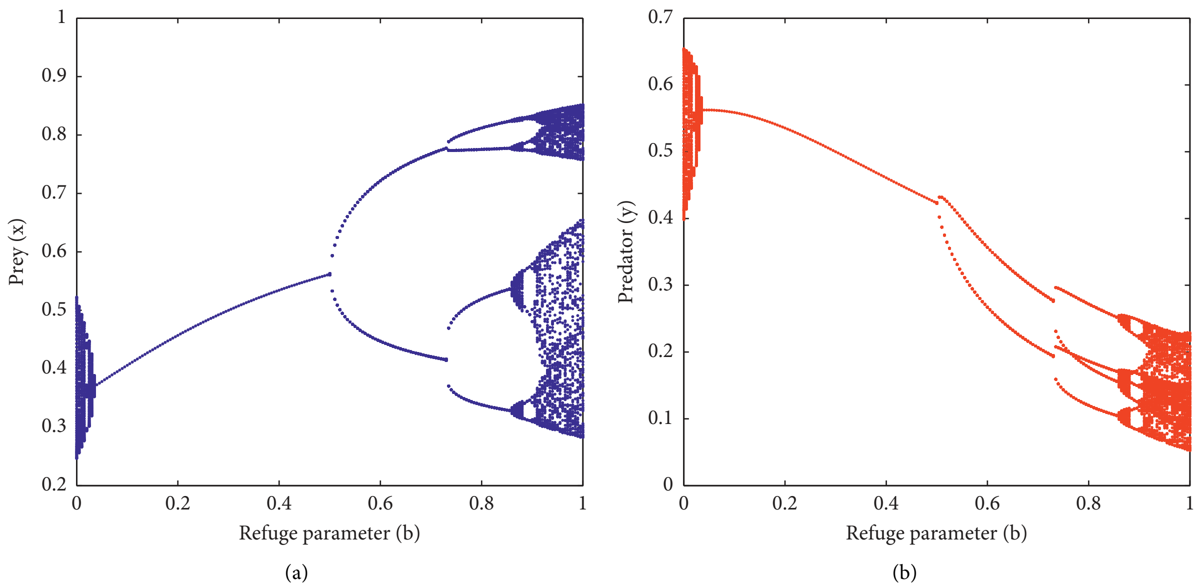

FIGURE 1: Bifurcation diagrams for $a = 4, b \in [0,1], c = 3, d = 3, e = 0.7$, and initial population $(0.6, 0.3)$. (a) Bifurcation diagram for prey (x). (b) Bifurcation diagram for predator (y).

Clearly, if $0 < (dx_2/db) < y_2$, then x_2 is strictly increasing function of the parameter b and y_2 is strictly decreasing function of the parameter b.

If $(dx_2/db) < 0$, then x_2 and y_2 are strictly decreasing function of the parameter b.

If $(dx_2/db) > y_2$, then $(dy_2/db) > 0$; hence, x_2 and y_2 are strictly increasing function of the parameter b.

6. Chaos Control

In discrete-time models, several approaches can be used in chaos control. These include the following: hybrid control method, pole-placement technique, and the state feedback method [9, 14, 18, 30]. In this regard, a feedback control method is employed to stabilize the chaotic orbits at an unstable positive fixed point of model (3) as follows:

$$x_{n+1} = ax_n(1 - x_n) - \frac{c(x_n - by_n)y_n}{e + x_n - by_n} + S,$$
$$y_{n+1} = \frac{d(x_n - by_n)y_n}{e + x_n - by_n}. \tag{20}$$

With the feedback control law as the control force, $S = -q_1(x_n - x_2) - q_2(y_n - y_2)$, where q_1 and q_2 are the feedback gain and (x_2, y_2) is a positive fixed point of the model.

The Jacobian matrix J for system (20) at (x_2, y_2) is $J = \begin{bmatrix} a_{11} - q_1 & a_{12} - q_2 \\ a_{21} & a_{22} \end{bmatrix}$, where $a_{11} = a(1 - 2x) - (cey/(e + (x - by))^2)$, $a_{12} = -(c(e(x - 2by) + (x - by)^2)/(e + (x - by))^2)$, $a_{21} = (dey/(e + (x - by))^2)$, and $a_{22} = (d(e(x - 2by) + (x - by)^2)/(e + (x - by))^2)$.

The characteristic equation of the matrix J is given by $\lambda^2 - (a_{11} + a_{22} - q_1)\lambda + a_{22}(a_{11} - q_1) - a_{21}(a_{12} - q_2) = 0$.

Let λ_1 and λ_2 be the eigenvalues, then

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} - q_1,$$
$$\lambda_1\lambda_2 = a_{22}(a_{11} - q_1) - a_{21}(a_{12} - q_2). \tag{21}$$

The lines of marginal stability are determined by solving the equation $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These conditions guarantee that the eigenvalues λ_1 and λ_2 have modulus less than 1.

Suppose $\lambda_1\lambda_2 = 1$, we have a line l_1 from (21) as $a_{22}q_1 - a_{21}q_2 = a_{22}a_{11} - a_{21}a_{12} - 1$.

Suppose $\lambda_1 = \pm 1$, we have the line l_2 and l_3 from (21) as $(1 - a_{22})q_1 + a_{21}q_2 = a_{11} + a_{22} - 1 - a_{22}a_{11} + a_{21}a_{12}$ and $(1 + a_{22})q_1 - a_{21}q_2 = a_{11} + a_{22} + 1 + a_{22}a_{11} - a_{21}a_{12}$.

The stable eigenvalues lie within a triangular region by the lines l_1, l_2, and l_3.

7. Numerical Simulation

To support the analytical results, this section presents the numerical simulation based on system (3) through the following examples.

Example 1. The equilibrium points and their nature for $a = 4$, $b = 0.4$, $c = 3$, $d = 3$, and $e = 0.7$ of system (3) are derived.

The discrete prey-predator system (3) has three equilibrium points. One is the trivial equilibrium point $(0, 0)$, another one is axial equilibrium point $(0.75, 0)$, and the last one is unique positive equilibrium point $(0.53, 0.46)$. The characteristic equation of the Jacobian matrix of system (3) evaluated at the trivial equilibrium point $(0, 0)$ is given by

$$F(\lambda) = \lambda^2 - 4\lambda = 0. \tag{22}$$

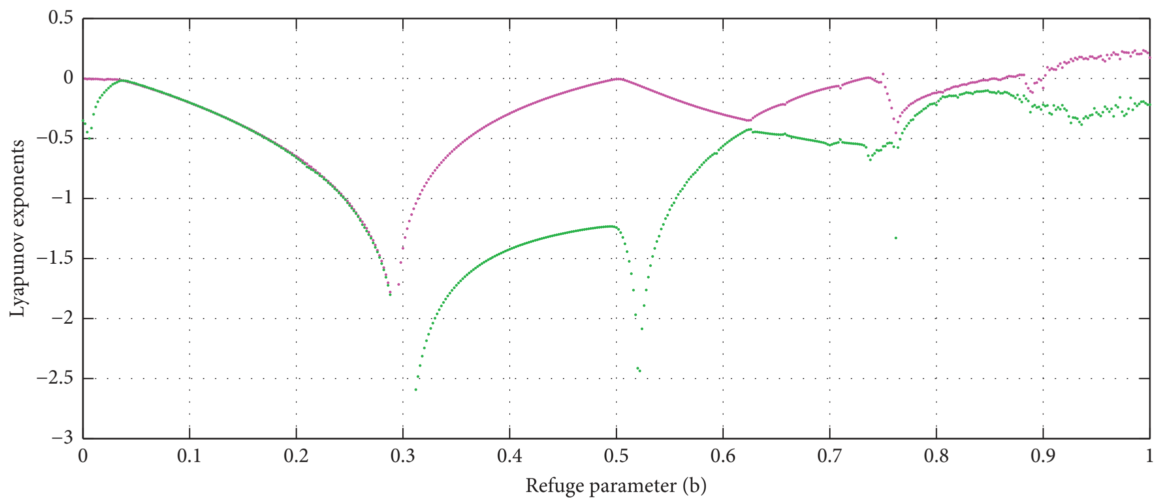

FIGURE 2: Lyapunov exponents with respect to refuge parameter b.

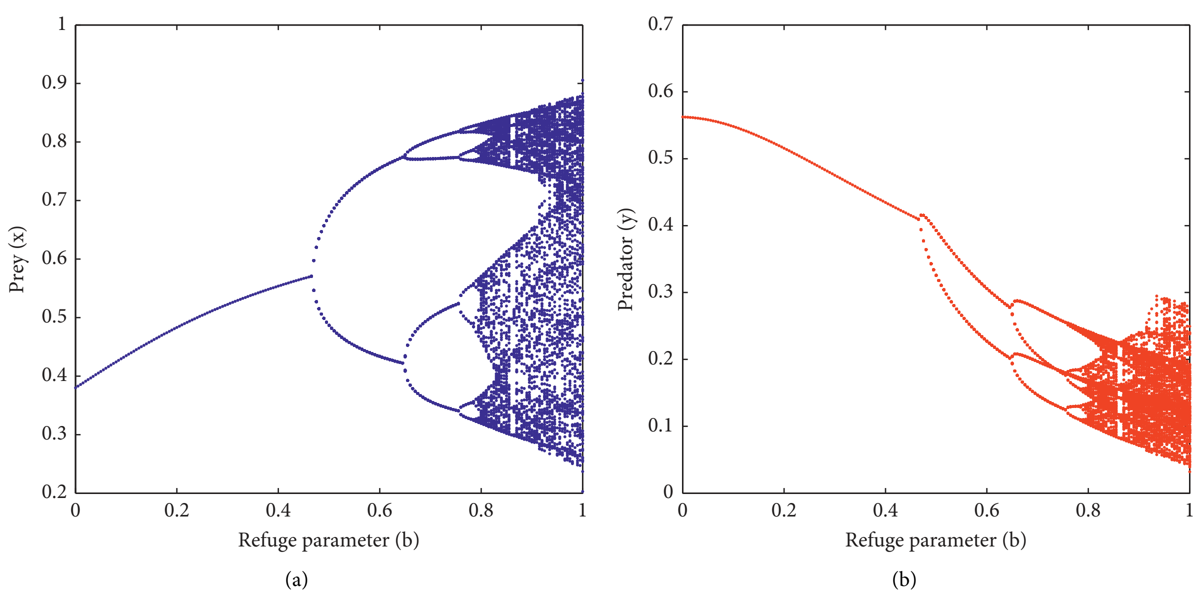

FIGURE 3: Bifurcation diagrams for $a = 4, b \in [0, 1], c = 3, d = 3, e = 0.76$, and initial population $(0.6, 0.3)$. (a) Bifurcation diagram for prey (x). (b) Bifurcation diagram for predator (y).

Now, $F(1) = -3 < 0$, $F(-1) = 5 > 0$, and $F(0) = 0 < 1$. The roots of (22) are given by $\lambda_1 = 0$ and $\lambda_2 = 4$. Therefore, the trivial equilibrium point $(0, 0)$ is saddle.

The characteristic equation of the Jacobian matrix of system (3) evaluated at the axial equilibrium point $(0.75, 0)$ is given by

$$F(\lambda) = \lambda^2 + 0.4483\lambda - 3.1034 = 0. \tag{23}$$

Now, $F(1) = -1.6551 < 0$, $F(-1) = -2.5517 < 0$, and $F(0) = -3.1034 < 1$. The roots of (23) are given by $\lambda_1 = -2$ and $\lambda_2 = 1.5517$. Therefore, the axial equilibrium point $(0.75, 0)$ is source. The characteristic equation of the Jacobian matrix of system (3) evaluated at the positive equilibrium point $(0.53, 0.46)$ is given by

$$F(\lambda) = \lambda^2 + 0.4837\lambda - 0.1534 = 0. \tag{24}$$

Now, $F(1) = 1.3303 > 0$, $F(-1) = 0.3629 > 0$, and $F(0) = -0.1534 < 1$, and then $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The roots of (24) are given by $\lambda_1 = -0.7022$ and $\lambda_2 = 0.2185$. Therefore, the positive equilibrium point $(0.53, 0.46)$ is sink.

Example 2. We consider the system parameters $a = 4$, $b \in [0, 1]$, $c = 3$, $d = 3$, $e = 0.7$, and initial population $(0.6, 0.3)$. Then, system (3) undergoes the Neimark–Sacker bifurcation as b varies in a small neighbourhood of $b = 0.0355$. The corresponding bifurcation diagrams and Lyapunov exponents are shown in Figures 1 and 2, respectively. For $a = 4$, $b = 0.0355$, $c = 3$, $d = 3$, and $e = 0.7$,

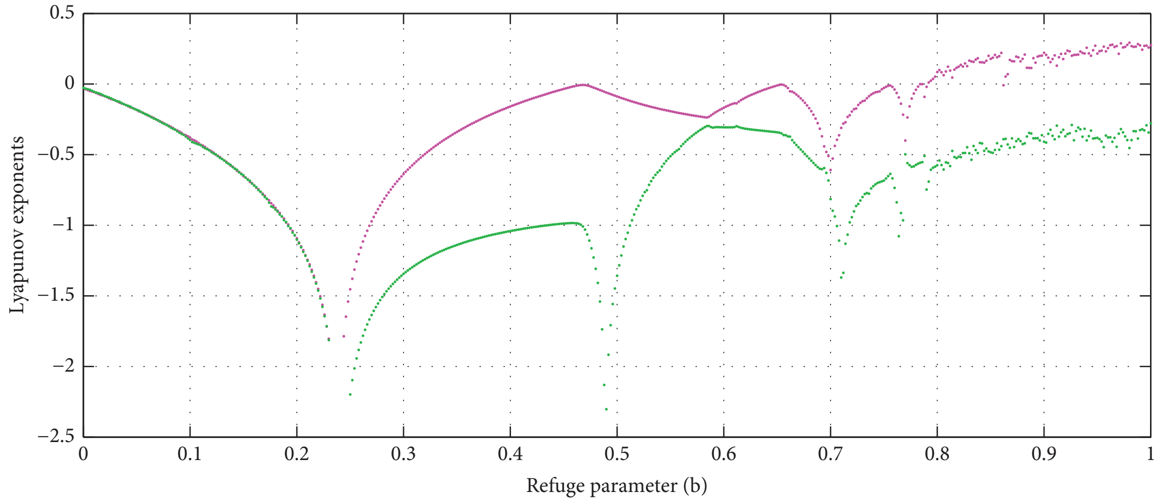

Figure 4: Lyapunov exponents with respect to refuge parameter b for $a = 4, b \in [0,1], c = 3, d = 3, e = 0.76$, and initial population $(0.6, 0.3)$.

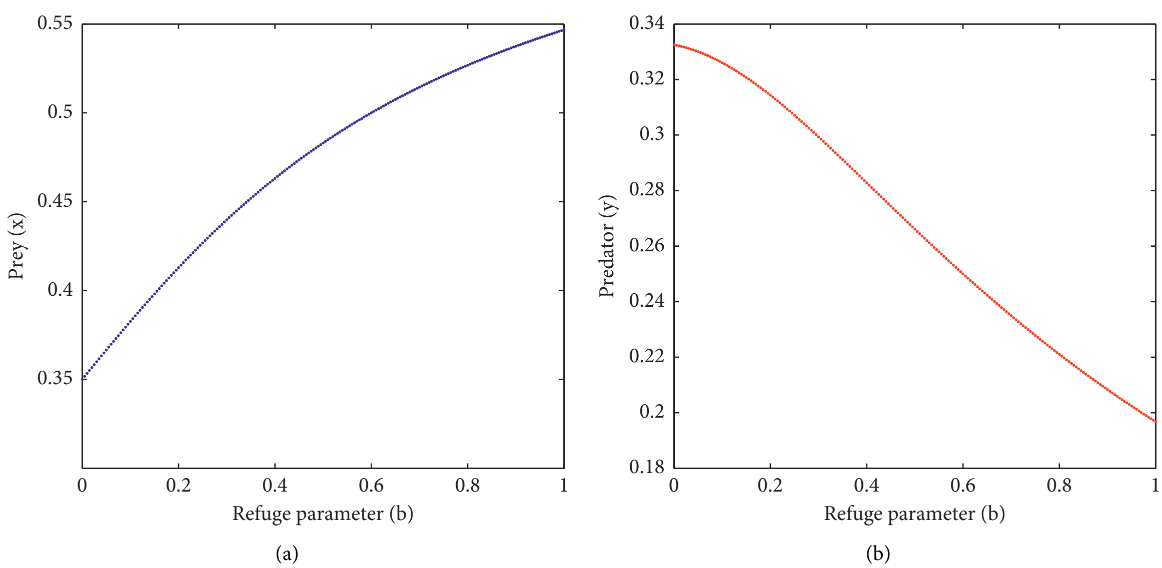

Figure 5: Bifurcation diagrams for $a = 3, b \in [0,1], c = 3, d = 3, e = 0.7$, and initial population $(0.6, 0.3)$. (a) Bifurcation diagram for prey (x). (b) Bifurcation diagram for predator (y).

system (3) has three equilibrium points. One is the trivial equilibrium point $(0, 0)$, another one is axial equilibrium point $(0.75, 0)$, and the last one is unique positive equilibrium point $(0.37, 0.56)$. Now, we will study the dynamics of the interior equilibrium point $(0.37, 0.56)$. Furthermore, the characteristic equation of the Jacobian matrix of system (3) for $a = 4$, $b = 0.0355$, $c = 3$, $d = 3$, and $e = 0.7$ evaluated at positive steady-state $(0.37, 0.56)$ is given by

$$\lambda^2 - 0.9359\lambda + 1.0009 = 0. \tag{25}$$

The roots of (25) are given by $\lambda_{1,2} = 0.4680 \pm 0.8842i$ with $|\lambda_{1,2}| = 1$. Thus, $A_{\text{NSB}} = (a = 4, b = 0.0355, c = 3, d = 3, e = 0.7)$.

Example 3. We consider parameters $a = 4$, $b \in [0,1]$, $c = 3$, $d = 3$, $e = 0.7$, and initial population $(0.6, 0.3)$. Then, system (3) undergoes the Period-Doubling Bifurcation as b varies in a small neighbourhood of $b = 0.52$. The corresponding bifurcation diagram and Lyapunov exponents are shown in Figures 1 and 2, respectively. For $a = 4$, $b = 0.52$, $c = 3$, $d = 3$, and $e = 0.7$, system (3) has three equilibrium points. One is the trivial equilibrium point $(0, 0)$, another one is axial equilibrium point $(0.75, 0)$, and the last one is unique positive equilibrium point $(0.56, 0.42)$. Now, we will study the dynamics of the interior equilibrium point $(0.56, 0.42)$. Furthermore, the characteristic equation of the Jacobian matrix of system (3) for $a = 4$, $b = 0.52$, $c = 3$, $d = 3$, and $e = 0.7$ evaluated at positive steady-state $(0.56, 0.42)$ is given by

$$\lambda^2 + 0.7318\lambda - 0.2693 = 0. \tag{26}$$

The roots of (26) are given by $\lambda_1 = -1$ and $\lambda_2 = 0.269$ with $|\lambda_2| \neq 1$. Thus, $A_{\text{PDB}} = (a = 4, b = 0.52, c = 3, d = 3, e = 0.7)$.

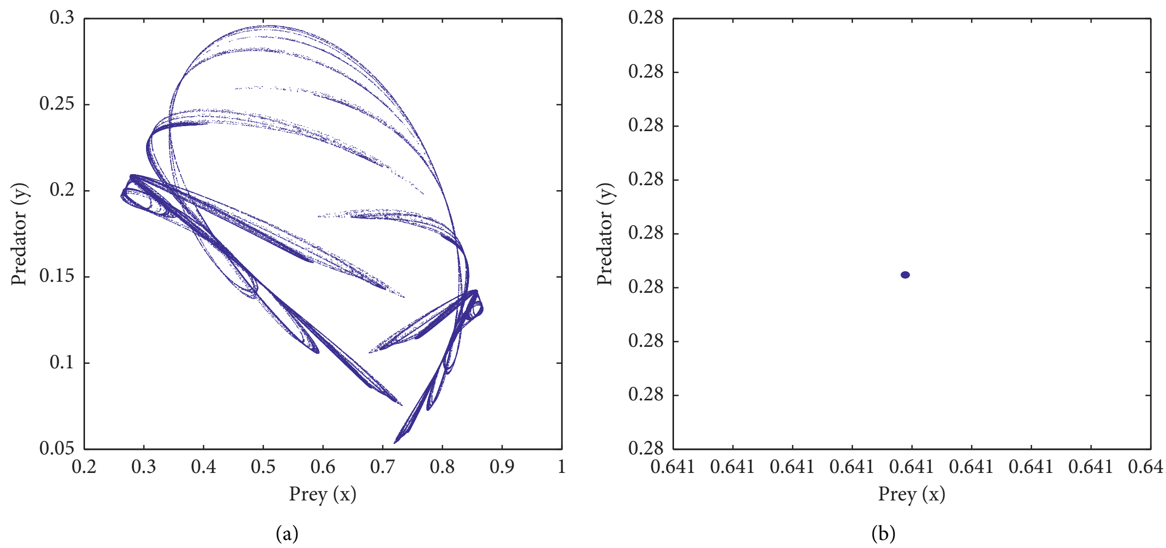

FIGURE 6: Chaos control of the system for the parameter set $a = 4$, $b = 0.95$, $c = 3$, $d = 3$, and $e = 0.75$. (a) Chaotic behaviour of system (3). (b) Stable behaviour of system (27) for feedback gain $q_1 = -0.5$ and $q_2 = 0.5$.

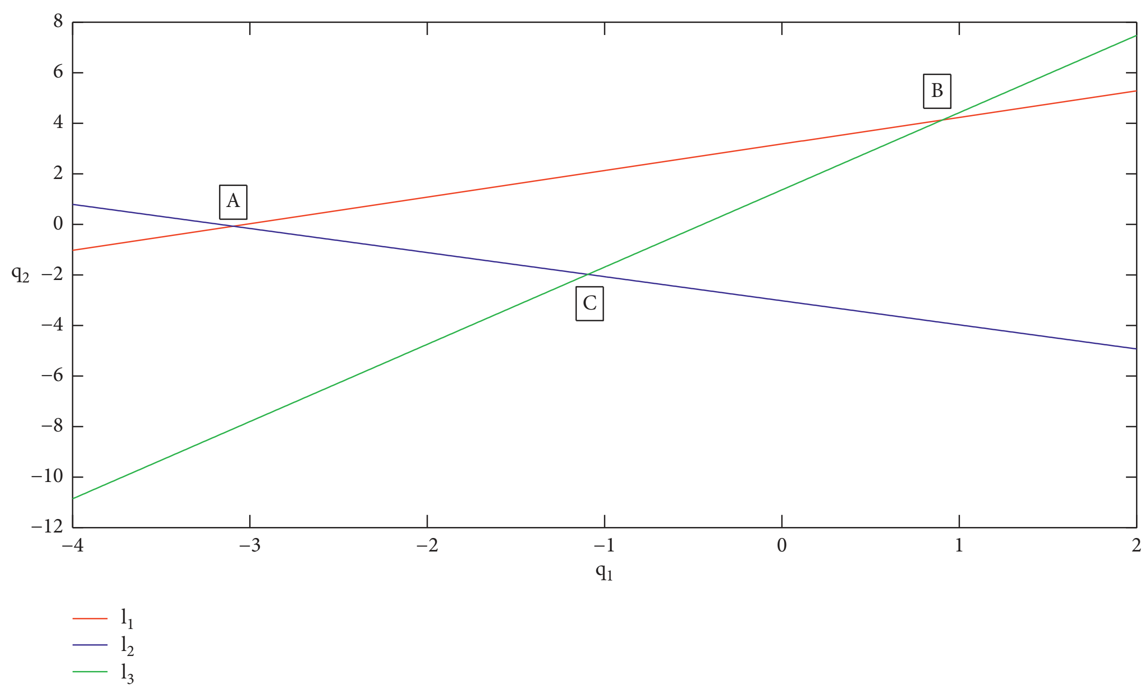

FIGURE 7: Stability region of the controlled system (27) for the parameter set $a = 4$, $b = 0.95$, $c = 3$, $d = 3$, $e = 0.75$, and feedback gain $q_1 = -0.5$ and $q_2 = 0.5$.

With the comparison of bifurcation analysis through the pictorial representation in Figure 3 with Figure 1 for the same parameter value with only the difference in functional response parameter e, we observed that Neimark–Sacker bifurcation vanishes from Figure 3, but Period-Doubling Bifurcation occurs. Figure 4 is the corresponding Lyapunov exponent diagram of Figure 3. Furthermore, the bifurcation diagram by changing the parameter value a is shown in Figure 5, and compared with Figures 1 and 3, here both bifurcations are vanished.

Example 4. For the parameter set $a = 4$, $b = 0.95$, $c = 3$, $d = 3$, $e = 0.75$, and the initial population $(0.6, 0.3)$, we observe the chaotic behaviour of prey predator shown in Figure 6(a). In this case, the fixed point $(0.64, 0.28)$ is unstable. For feedback gain $q_1 = -0.5$ and $q_2 = 0.5$, we observe that fixed point $(0.64, 0.28)$ is stable as shown in Figure 6(b). Model (3) with the feedback control law as the control force $S = 0.5(x_n - 0.64) - 0.5(y_n - 0.28)$ is as follows:

$$x_{n+1} = ax_n(1 - x_n) - \frac{c(x_n - by_n)y_n}{e + x_n - by_n} + S,$$
$$y_{n+1} = \frac{d(x_n - by_n)y_n}{e + x_n - by_n}. \tag{27}$$

The Jacobian matrix J for system (27) at $(0.64, 0.28)$ is $J = \begin{bmatrix} -1.1187 & -1.0245 \\ 0.4987 & 0.5245 \end{bmatrix}$. The characteristic equation of the matrix J is given by

$$\lambda^2 + 0.5942\lambda - 0.0759 = 0. \tag{28}$$

The roots of (28) are given by $\lambda_1 = -0.7023$ and $\lambda_2 = 0.1081$. Now, line l_1 is $0.5245q_1 - 0.4987q_2 = -1.5874$, line l_2 is $0.4755q_1 + 0.4987q_2 = -1.5068$, and line l_3 is $1.5245q_1 - 0.4987q_2 = -0.6816$.

The stable eigenvalues lie within a triangular region by the lines l_1, l_2, and l_3 shown in Figure 7. The system is stable for the triangular region ABC bounded by the marginal lines l_1, l_2, and l_3.

8. Conclusion

In this paper, the impacts of refuges used by prey on a prey-predator interaction are studied through the analytical approach and graphical representation. This study considered the prey refuge proportional to predator density due to fear induced by a predator on prey. The detailed effects of the local stability of the interior steady-state point and the long-term dynamics of the interacting populations on the model have been investigated. The results show that the impact of refuges can stabilize and destabilize the interior equilibrium point of the proposed prey-predator model for different parameter sets.

Different observations have been made on the types of refuge and their impacts on the proposed system. Generally, in the existing literature, we always found the stabilizing effect of refuge. However, here, we observed an unstable to stable to unstable behaviour in Figure 1. For a different parameter set in Figure 3, we have also found a stable to unstable behaviour of the system. In Figure 5, another interesting observation is that refuge is profitable for prey and adverse effect on the predator. We have studied the bifurcations of the proposed discrete prey-predator model with refuge. It is shown that the model exhibits various bifurcations of codimension one, including PDB and NSB, as the values of the parameters vary.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

G. S. Mahapatra conceptualized the study, provided software, and supervised and visualized the study. P. K. Santra conceptualized the study, developed methodology, provided software, and investigated the study. Ebenezer Bonyah visualized the study and reviewed and edited the manuscript.

References

[1] Y. Shi, Q. Ma, and X. Ding, "Dynamical behaviors in a discrete fractional-order predator-prey system," *Filomat*, vol. 32, no. 17, pp. 5857–5874, 2018.

[2] K. S. Al-Basyouni and A. Q. Khan, "Discrete-time predator-prey model with bifurcations and chaos," *Mathematical Problems in Engineering*, vol. 2020, Article ID 8845926, 14 pages, 2020.

[3] F. Kangalgil and S. Isik, "Controlling chaos and Neimark-Sacker bifurcation in a discrete-time predator-prey system," *Hacettepe Journal of Mathematics and Statistics*, vol. 49, no. 5, pp. 1761–1776, 2020.

[4] W. Liu and Y. Jiang, "Flip bifurcation and Neimark-Sacker bifurcation in a discrete predator-prey model with harvesting," *International Journal of Biomathematics*, vol. 13, no. 1, Article ID 1950093, 2020.

[5] G. S. Mahapatra and P. Santra, "Prey-predator model for optimal harvesting with functional response incorporating prey refuge," *International Journal of Biomathematics*, vol. 9, no. 1, Article ID 1650014, 2016.

[6] D. Pal, P. Santra, and G. S. Mahapatra, "Dynamical behavior of three species predator-prey system with mutual support between non refuge prey," *Ecological Genetics and Genomics*, vol. 3–5, pp. 1–6, 2017.

[7] P. K. Santra, G. S. Mahapatra, and G. R. Phaijoo, "Bifurcation analysis and chaos control of discrete prey-predator model incorporating novel prey-refuge concept," *Computational and Mathematical Methods*, Article ID e1185, 2021.

[8] R. Ma, Y. Bai, Y. Bai, and F. Wang, "Dynamical behavior analysis of a two-dimensional discrete predator-prey model with prey refuge and fear factor," *Journal of Applied Analysis & Computation*, vol. 10, no. 4, pp. 1683–1697, 2020.

[9] P. K. Santra, G. S. Mahapatra, and G. R. Phaijoo, "Bifurcation and chaos of a discrete predator-prey model with Crowley-Martin functional response incorporating proportional prey refuge," *Mathematical Problems in Engineering*, vol. 2020, Article ID 5309814, 18 pages, 2020.

[10] S. Rana, A. R. Bhowmick, and S. Bhattacharya, "Impact of prey refuge on a discrete time predator-prey system with Allee effect," *International Journal of Bifurcation and Chaos*, vol. 24, no. 9, Article ID 1450106, 2014.

[11] E. Sebastian and P. Victor, "The dynamics of a discrete-time ratio-dependent prey-predator model incorporating prey refuge and harvesting on prey," *International Journal of Applied Engineering Research*, vol. 10, no. 55, pp. 2385–2388, 2015.

[12] P. K. Santra and G. S. Mahapatra, "Dynamical study of discrete-time prey predator model with constant prey refuge under imprecise biological parameters," *Journal of Biological Systems*, vol. 28, no. 3, pp. 681–699, 2020.

[13] H. Cao, Z. Yue, and Y. Zhou, "The stability and bifurcation analysis of a discrete Holling-Tanner model," *Advances in Difference Equations*, vol. 2013, no. 1, p. 330, 2013.

[14] A. Atabaigi, "Bifurcation and chaos in a discrete time predator-prey system of Leslie type with generalized Holling type

III functional response," *Journal of Applied Analysis & Computation*, vol. 7, no. 2, pp. 411–426, 2017.
[15] R. Banerjee, P. Das, and D. Mukherjee, "Stability and permanence of a discrete-time two-prey one-predator system with Holling type-III functional response," *Chaos, Solitons & Fractals*, vol. 117, pp. 240–248, 2018.
[16] J. Zhang, T. Deng, Y. Chu, S. Qin, W. Du, and H. Luo, "Stability and bifurcation analysis of a discrete predator-prey model with Holling type III functional response," *Journal of Nonlinear Sciences and Applications*, vol. 09, no. 12, pp. 6228–6243, 2016.
[17] Q. Cui, Q. Zhang, Z. Qiu, and Z. Hu, "Complex dynamics of a discrete-time predator-prey system with Holling IV functional response," *Chaos, Solitons & Fractals*, vol. 87, pp. 158–171, 2016.
[18] S. M. S. Rana and U. Kulsum, "Bifurcation analysis and chaos control in a discrete-time predator-prey system of Leslie type with simplified Holling type IV functional response," *Discrete Dynamics in Nature and Society*, vol. 2017, Article ID 9705985, 11 pages, 2017.
[19] S. M. Salman, A. M. Yousef, and A. A. Elsadany, "Stability, bifurcation analysis and chaos control of a discrete predator-prey system with square root functional response," *Chaos, Solitons & Fractals*, vol. 93, pp. 20–31, 2016.
[20] P. Chakraborty, U. Ghosh, and S. Sarkar, "Stability and bifurcation analysis of a discrete prey-predator model with square-root functional response and optimal harvesting," *Journal of Biological Systems*, vol. 28, no. 1, pp. 91–110, 2020.
[21] J. Huang, S. Liu, S. Ruan, and D. Xiao, "Bifurcations in a discrete predator-prey model with nonmonotonic functional response," *Journal of Mathematical Analysis and Applications*, vol. 464, no. 1, pp. 201–230, 2018.
[22] H. Baek, "Complex dynamics of a discrete-time predator-prey system with Ivlev functional response," *Mathematical Problems in Engineering*, vol. 2018, Article ID 8635937, 15 pages, 2018.
[23] X.-L. Zhuo and F.-X. Zhang, "Stability for a new discrete ratio-dependent predator-prey system," *Qualitative Theory of Dynamical Systems*, vol. 17, no. 1, pp. 189–202, 2018.
[24] X. P. Wu and L. Wang, "Analysis of oscillatory patterns of a discrete-time Rosenzweig-MacArthur model," *International Journal of Bifurcation and Chaos*, vol. 28, no. 14, Article ID 1899001, 2018.
[25] J. Zhao and Y. Yan, "Stability and bifurcation analysis of a discrete predator-prey system with modified Holling-Tanner functional response," *Advances in Difference Equations*, vol. 2018, no. 1, p. 402, 2018.
[26] S. Isik, "A study of stability and bifurcation analysis in discrete-time predator-prey system involving the Allee effect," *International Journal of Biomathematics*, vol. 12, no. 1, Article ID 1950011, 2019.
[27] A. Singh, A. A. Elsadany, and A. Elsonbaty, "Complex dynamics of a discrete fractional-order Leslie-Gower predator-prey model," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 11, pp. 3992–4007, 2019.
[28] C. Duque, "Existence of unique and global asymptotically stable almost periodic solution of a discrete predator-prey system with Beddington-DeAngelis functional response and density dependent," *Revista Colombiana de Matemáticas*, vol. 52, no. 1, pp. 87–105, 2018.
[29] F. Kangalgil, "Neimark-Sacker bifurcation and stability analysis of a discrete-time prey-predator model with Allee effect in prey," *Advances in Difference Equations*, vol. 2019, no. 1, p. 92, 2019.
[30] J. Ren, L. Yu, and S. Siegmund, "Bifurcations and chaos in a discrete predator-prey model with Crowley-Martin functional response," *Nonlinear Dynamics*, vol. 90, no. 1, pp. 19–41, 2017.
[31] W. Liu and D. Cai, "Bifurcation, chaos analysis and control in a discrete-time predator-prey system," *Advances in Difference Equations*, vol. 2019, no. 1, p. 11, 2019.
[32] M. F. Elettreby, A. Khawagi, and T. Nabil, "Dynamics of a discrete prey-predator model with mixed functional response," *International Journal of Bifurcation and Chaos*, vol. 29, no. 14, Article ID 1950199, 2019.
[33] Z. AlSharawi, S. Pal, N. Pal, and J. Chattopadhyay, "A discrete-time model with non-monotonic functional response and strong Allee effect in prey," *Journal of Difference Equations and Applications*, vol. 26, no. 3, pp. 404–431, 2020.
[34] J. Wang and M. Fečkan, "Dynamics of a discrete nonlinear prey-predator model," *International Journal of Bifurcation and Chaos*, vol. 30, no. 4, Article ID 2050055, 2020.
[35] P. C. Rech, "On two discrete-time counterparts of a continuous-time prey-predator model," *Brazilian Journal of Physics*, vol. 50, no. 2, pp. 119–123, 2020.
[36] A. Q. Khan and T. Khalique, "Neimark-Sacker bifurcation and hybrid control in a discrete-time Lotka-Volterra model," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 9, pp. 5887–5904, 2020.
[37] A. A. Elsadany, Q. Din, and S. M. Salman, "Qualitative properties and bifurcations of discrete-time Bazykin-Berezovskaya predator-prey model," *International Journal of Biomathematics*, vol. 13, no. 6, Article ID 2050040, 2020.
[38] P. Baydemir, H. Merdan, E. Karaoglu, and G. Sucu, "Complex dynamics of a discrete-time prey-predator system with Leslie type: stability, bifurcation analyses and chaos," *International Journal of Bifurcation and Chaos*, vol. 30, no. 10, Article ID 2050149, 2020.
[39] U. A. Rozikov and S. K. Shoyimardonov, "Leslie's prey-predator model in discrete time," *International Journal of Biomathematics*, vol. 13, no. 6, Article ID 2050053, 2020.
[40] A. Singh and P. Deolia, "Dynamical analysis and chaos control in discrete-time prey-predator model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 90, Article ID 105313, 2020.