Review Article
Dynkin's Games and Israeli Options

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We start by briefly surveying a research on optimal stopping games since their introduction by Dynkin more than 40 years ago. Recent renewed interest to Dynkin's games is due, in particular, to the study of Israeli (game) options introduced in 2000. We discuss the work on these options and related derivative securities for the last decade. Among various results on game options we consider error estimates for their discrete approximations, swing game options, game options in markets with transaction costs, and other questions.

1. Introduction

Optimal stopping games were introduced in 1969 by Dynkin in [1] as an extension of the optimal stopping problem which has been already actively studied since 1950. Optimal stopping and, in particular, its game version was often discussed on Dynkin's undergraduate seminar at Moscow State University in the end of 1960 which resulted in papers [2–5].

The original setup of optimal stopping games consisted of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration of σ-algebras $\{\mathcal{F}_t\}$, $\mathcal{F}_t \subset \mathcal{F}$ with either $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$ (discrete time case) or $t \in \mathbb{R}_+ = [0, \infty)$ (continuous time case), $\{\mathcal{F}_t\}$-adapted payoff process $\{X_t\}$, and a pair of $\{\mathcal{F}_t\}$-adapted 0-1 valued "permission" processes $\varphi_t^{(i)}$, $i = 1, 2$, such that the player i is allowed to stop the game at time t if and only if $\varphi_t^{(i)} = 1$. If the game is stopped at time t then the first player pays to the second one the sum X_t. Clearly, if $\varphi_t^1 \equiv 1$ and $\varphi_t^{(2)} \equiv 0$, we arrive back at the usual optimal stopping problem. Observe that in the one-player optimal stopping problem the goal is maximization of the payoff, and the corresponding supremum always exists (may be infinite), so only optimal or almost optimal stopping times remain to be found while in the game version already existence of the game value is the question which should be resolved first, and only then we can look for optimal (saddle point) or almost optimal stopping times of the players.

Few years later Neveu suggested in [6] a very useful generalization of the above setup which turned out to be more convenient for both further study and applications. Namely, now the "permission" processes were dropped off and the players could stop whenever they want, but instead two payoff adapted processes $X_t \geq Y_t$ were introduced. It was prescribed that if the first player stops the game at time s and the second one at time t, then the former pays to the latter the amount X_s or Y_t if $s \leq t$ or $s > t$, respectively. If desired we can have virtual "permission" processes within this setup not by direct regulations but by "market economy" tools. Namely, in order to accomplish this it suffices to prescribe very high payment X_t or a very low (may be negative) payment Y_t where we "forbid" to stop the game by the first player or by the second one, respectively.

We observe that from a bit different perspective differential games with stopping times were studied in the 1970 in a series of papers (see [7, 8] and references there). Game versions of optimal stopping of a Markov process and of a diffusion were considered in [9, 10], respectively. It seems that the term "Dynkin's game" appeared first in [11].

Israeli or game options were introduced first in [12] though some special callable derivative security LION was discussed before in [13] in a kind of game framework without any rigorous justification. An option or a contingent claim is a certain contract and an American option enables its buyer (holder) to exercise it at any time up to the maturity. A game option gives additionally the right to the option seller (writer,

issuer) to cancel it early paying for this a prescribed penalty. The rationale behind this provision comes from an idea that essentially any contract stipulates conditions for a way out so that the financial market should not be an exception.

The classical approach to pricing of options is based on hedging arguments. Namely, the price is defined as a minimal initial amount of a self-financing portfolio which can provide protection (hedging) against any exercising strategy of the option buyer. So, somehow heuristically, this leads to the infimum over the seller's strategies and to the supremum over the buyer's strategies; that is, we arrive at a game-type infsup representation which still should be rigorously justified.

The structure of this paper is as follows. In Section 2 we briefly survey main results concerning Dynkin's games. In Section 3 we discuss the up-to-date research on game options and related derivative securities. In Sections 4 and 5 we exhibit more special results concerning discrete approximations of game options and game options in markets with transaction costs, respectively.

2. Dynkin's Games

The general modern setup for a Dynkin's game consists of a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a right continuous filtration of complete σ-algebras $\{\mathscr{F}_t\}$, and three $\{\mathscr{F}_t\}$-adapted stochastic processes X_t, Y_t, and Z_t so that when the first player stops the game at time s and the second one stops at time t, then the former pays to the latter the amount

$$H(s,t) = X_s \mathbb{I}_{s<t} + Y_t \mathbb{I}_{s>t} + Z_t \mathbb{I}_{s=t}, \qquad (1)$$

where $\mathbb{I}_\Gamma = 1$ if an event Γ occurs and $= 0$, otherwise. We allow the time t to run either along nonnegative integers $\mathbb{N}$ or along nonnegative reals $\mathbb{R}_+$ up to some horizon $T \leq \infty$ when the game is stopped and the first player pays to the second one the amount

$$H(T,T) = X_T = Y_T = Z_T, \qquad (2)$$

where in case $T = \infty$, we assume that

$$0 = X_\infty = \lim_{t \to \infty} X_t = Y_\infty = \lim_{t \to \infty} Y_t = Z_\infty = \lim_{t \to \infty} Z_t. \qquad (3)$$

In the continuous time case; that is, when t runs over $\mathbb{R}_+$, the processes X_t, Y_t, and Z_t are supposed to be right continuous.

Next, assume that for any $t \in [0, T]$,

$$Y_t \leq Z_t \leq X_t \quad \mathbb{P}\text{-almost surely}, \qquad (4)$$

$$\mathbb{E} \sup_{0 \leq t \leq T} \left(|Y_t| + |Z_t| + |X_t| \right) < \infty. \qquad (5)$$

Denote by $\mathscr{T}_{s,t}$, $s \leq t$ the collection of all stopping times τ with values between s and t (i.e., nonnegative random variables such that $\{\tau \leq u\} \in \mathscr{F}_u$ for all u). Introduce the upper and the lower values of the game starting at time $t \leq T$ by

$$\overline{V}_t = \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_{tT}} \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_{tT}} \mathbb{E}\left(H(\sigma, \tau) \mid \mathscr{F}_t \right),$$

$$\underline{V}_t = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_{tT}} \operatorname*{ess\,inf}_{\sigma \in \mathscr{T}_{tT}} \mathbb{E}\left(H(\sigma, \tau) \mid \mathscr{F}_t \right). \qquad (6)$$

It turns out that we can choose these processes $\{\overline{V}_t\}$ and $\{\underline{V}_t\}$ to be right upper semicontinuous which is a sufficient regularity in order to proceed here.

Theorem 1. *Under the above conditions $V_\tau \overset{\text{def}}{=} \overline{V}_\tau = \underline{V}_\tau$ almost surely for any stopping time $\tau \in \mathscr{T}_{0T}$ and, in particular, the Dynkin's game has a value*

$$V = V_0 = \overline{V}_0 = \underline{V}_0. \qquad (7)$$

Furthermore, for any $\varepsilon > 0$ the stopping times

$$\sigma_\varepsilon = \inf\left\{ t \leq T : V_t \geq X_t - \varepsilon \right\},$$

$$\tau_\varepsilon = \inf\left\{ t \leq T : V_t \leq Y_t + \varepsilon \right\}, \qquad (8)$$

are ε-optimal, that is, for any $\sigma, \tau \in \mathscr{T}_{0T}$,

$$\mathbb{E}\left(H(\sigma_\varepsilon, \tau) \right) - \varepsilon \leq \mathbb{E}\left(H(\sigma_\varepsilon, \tau_\varepsilon) \right) \leq \mathbb{E}\left(H(\sigma, \tau_\varepsilon) \right) + \varepsilon. \qquad (9)$$

Under additional regularity conditions (say, X_t, Y_t, and Z_t are continuous stochastic processes), the inequality (9) remains true for $\varepsilon = 0$ with some σ_0, τ_0; that is, there exists a saddle point for the Dynkin's game above. In the discrete time case we have also the following backward recursive (dynamical programming) relation:

$$V_n = \min\left(X_n, \max\left(Y_n, \mathbb{E}\left(V_{n+1} \mid \mathscr{F}_n \right) \right) \right). \qquad (10)$$

The theorem above follows from [6, 14–16] in the discrete time case and from [17–19] in the continuous time case. Observe that (6), (7), and (9) imply

$$\left| V_0 - \mathbb{E}\left(H(\sigma_\varepsilon, \tau_\varepsilon) \right) \right| \leq \varepsilon. \qquad (11)$$

If the condition (4) does not hold true, then the above game value may not exist (i.e., $\overline{V}_0 > \underline{V}_0$) if the players are restricted to usual (pure) stopping times, and to have the game value, they should be allowed to use randomized stopping times (see [20–24]). Other results on Dynkin's games leading to randomized stopping times can be found in [25–28].

Remark 2. We observe that randomized stopping times used in the above mentioned papers in order to obtain Dynkin's game value without the condition (4) look somewhat different from randomized stopping times we employ in Section 5 in order to study game options in markets with transaction costs. Namely, the above papers deal with randomized stopping times having (in the discrete time case) the form $\lambda(p) = \min\{n \geq 0 : A_n \leq p_n\}$, where $p = (p_0, p_1, \dots)$ is an adapted to the filtration $\{\mathscr{F}_n\}$ process with $p_n \in [0, 1]$ for all n and $A_0, A_1, A_2, \dots$ is a sequence of independent identically uniformly distributed on $[0, 1]$ random variables independent of payoff processes. Sometimes, it is assumed additionally (see [21]) that A_n is $\mathscr{F}_{n+1}$-measurable and independent of $\mathscr{F}_n$. If $W = (W_0, W_1, W_2, \dots)$ is an adapted stochastic process, then we can write

$$W_{\lambda(p)} = \sum_{n=0}^{\infty} \psi_n W_n \quad \text{where } \psi_n = \mathbb{I}_{\{A_n \leq p_n\}} \prod_{j=0}^{n-1} \mathbb{I}_{\{A_j > p_j\}}. \qquad (12)$$

On the other hand, randomized stopping times employed in Section 5 are determined by an adapted nonnegative sequence $\chi = (\chi_0, \chi_1, \chi_2, \ldots)$ such that $\sum_{j=0}^{\infty} \chi_j = 1$ and for an adapted stochastic process W as above we write $W_\chi = \sum_{n=0}^{\infty} \chi_n W_n$. Here $\{\chi_n\}$ is an adapted sequence but not necessarily indicators of events while the above sequence $\{\psi_n\}$ is not adapted (unless the filtration is properly enlarged) and it consists of indicators of events. Still, with respect to the enlarged filtration, $\lambda(p)$ is a usual (pure) stopping time while randomized stopping times of Section 5 look rather differently. Nevertheless, it turns out that these two approaches to randomized stopping times are essentially equivalent if $\prod_{n \geq 0}(1 - p_n) = 0$ (see [29] in the discrete time case and the corresponding discussion in [22] for the continuous time case).

Among other works on Dynkin's games we can mention results concerning nonzero-sum games (see [30–33]), Dynkin's games with asymmetric information (see [34]), more than 2 person optimal stopping games (see [35–37]), optimal stopping games driven by Markov processes (see [2, 9, 38, 39]), Dynkin's games via backward stochastic differential equations with reflection (see [40–42]) and via Dirichlet forms (see [43]), and some other results on Dynkin's and similar games (see [44–55]).

3. Game Options and Their Shortfall Risk

A game (Israeli) option (or contingent claim) studied in [12] is a contract between a writer and a holder at time $t = 0$ such that both have the right to exercise at any stopping time before the expiry date T. If the holder exercises at time t, he or she receives the amount $Y_t \geq 0$ from the writer and if the writer, exercises at time t before the holder he must pay to the holder, the amount $X_t \geq Y_t$ so that $\delta_t = X_t - Y_t$ is viewed as a penalty imposed on the writer for cancellation of the contract. If both exercise at the same time t, then the holder may claim Y_t, and if neither exercised until the expiry time T, then the holder may claim the amount Y_T. In short, if the writer will exercise at a stopping time $\sigma \leq T$ and the holder at a stopping time $\tau \leq T$, then the former pays to the latter the amount $H(\sigma, \tau)$ where

$$H(s, t) = X_s \mathbb{I}_{s < t} + Y_t \mathbb{I}_{t \leq s}. \tag{13}$$

We consider such game options in a standard securities market consisting of a nonrandom component b_t representing the value of a savings account at time t with an interest rate r and of a random component S_t representing the stock price at time t. As usual, we view S_t, $t > 0$, as a stochastic process on a complete probability space $(\Omega, \mathcal{F}, P)$, and we assume that it generates a right continuous filtration $\{\mathcal{F}_t\}$ and that the payoff processes X_t and Y_t are right continuous processes adapted to this filtration and satisfying the integrability conditions (5).

The classical approach suggests that valuation of options should be based on the notions of a self-financing portfolio and on hedging. We start with a portfolio strategy

$\pi = \{\pi_t\}_{0 \leq t \leq T}$ which is a collection of pairs $\pi_t = (\beta_t, \gamma_t)$ so that the portfolio value W_t at time t equals

$$W_t^\pi = \beta_t b_t + \gamma_t S_t, \tag{14}$$

where the process (β_t, γ_t), $0 \leq t \leq T$, is supposed to be predictable in the discrete time case and progressively measurable in the continuous time case. A portfolio strategy π is called self-financing if all changes in the portfolio value are due to capital gains or losses but not due to withdrawal or infusion of funds. This can be expressed by the relations (see [56])

$$b_{t-1} (\beta_t - \beta_{t-1}) + S_{t-1} (\gamma_t - \gamma_{t-1}) = 0 \quad \text{for } t = 1, 2, \ldots, T \tag{15}$$

in the discrete time case and

$$W_t^\pi = W_0^\pi + \int_0^t \beta_u db_u + \int_0^t \gamma_u dS_u \tag{16}$$

in the continuous time case. We assume also in the continuous time case that with probability one:

$$\int_0^T |b_t \beta_t| \, dt < \infty, \qquad \int_0^T (\gamma_t S_t)^2 dt < \infty. \tag{17}$$

A pair (σ, π) of a stopping time $\sigma \leq T$ and a self-financing portfolio strategy π is called a hedge (against the game contingent claim) if $W_{\sigma \wedge t}^\pi \geq H(\sigma, t)$ with probability one for any $t \in [0, T]$. Now the fair price of the game option is defined as the infimum of capitals x for which there exists a hedge (σ, π) with $W_0^\pi = x$. In a complete market (i.e., having a unique martingale measure) this is a widely acceptable fair price of the option while in an incomplete market or in a market with transaction costs this definition leads to what is known as superhedging (see [56]).

Two popular models of complete markets were considered in [12] for pricing of game options. First, the discrete time Cox, Rox, Rubinstein (CRR) binomial model (see [57]) was treated there where the stock price S_k at time k is equal to

$$S_k = S_0 \prod_{j=1}^{k} \left(1 + \rho_j\right), \quad S_0 > 0, \tag{18}$$

where ρ_j, $j = 1, 2, \ldots$ are independent identically distributed (i.i.d.) random variables such that $\rho_j = b > 0$ with probability $p > 0$ and $\rho_j = a < 0$, $a > -1$ with probability $q = 1 - p > 0$. Secondly, [12] deals with the continuous time Black-Scholes (BS) market model where the stock price S_t at time t is given by the geometric Brownian motion:

$$S_t = S_0 \exp\left(\left(\alpha - \frac{\kappa^2}{2}\right) t + \kappa B_t\right), \quad S_0 > 0, \tag{19}$$

where $\{B_t\}_{t \geq 0}$ is the standard one-dimensional continuous-in-time Brownian motion (Wiener process) starting at zero and $\kappa > 0$, $\alpha \in (-\infty, \infty)$ are some parameters. In addition to the stock which is a risky security, the market includes in

both cases also a savings account with a deterministic growth given by the formulas

$$b_n = (1 + r)^n b_0, \quad b_t = b_0 e^{rt}, \quad b_0, r > 0 \qquad (20)$$

in the CRR model (where we assume in addition that $r < b$) and in the BS model, respectively.

Recall (see [56]) that a probability measure describing the evolution of a stock price in a stochastic financial market is called martingale (risk-neutral) if the discounted stock prices $((1 + r)^{-k} S_k$ in the CRR model and $e^{-rt} S_t$ in the BS model) become martingales. Relying on the above hedging arguments the following result was proved in [12].

Theorem 3. *The fair price V of the game option is given by the formulas*

$$V = \min_{\sigma \in \mathcal{T}_{0T}} \max_{\tau \in \mathcal{T}_{0T}} E\left((1 + r)^{-\sigma \wedge \tau} H(\sigma, \tau)\right) \qquad (21)$$

in the CRR market (with usual notations $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$) and

$$V = \inf_{\sigma \in \mathcal{T}_{0T}} \sup_{\tau \in \mathcal{T}_{0T}} E\left(e^{-r\sigma \wedge \tau} H(\sigma, \tau)\right) \qquad (22)$$

in the BS market, where the expectations are taken with respect to the corresponding martingale probabilities, which are uniquely defined since these markets are known to be complete (see [56]), T is the expiry time, and $\mathcal{T}_{st}$ is the space of corresponding stopping times with values between s and t taking into account that in the CRR model σ and τ are allowed to take only integer values.

Observe that the formulas (21) and (22) represent also the values of corresponding Dynkin's (optimal stopping) games with payoffs $(1 + r)^{-\sigma \wedge \tau} H(\sigma, \tau)$ and $e^{-r\sigma \wedge \tau} H(\sigma, \tau)$, respectively, when the first and the second players stop the game at stopping times σ and τ, respectively. The continuous time BS model is generally considered as a better description of the evolution of real stocks, in particular, since the CRR model allows only two possible values $(1 + b)S_k$ and $(1 + a)S_k$ for the stock price S_{k+1} at time $k + 1$ given its price S_k at time k. The main advantage of the CRR model is its simplicity and the possibility of easier computations of the value V in (21), in particular, by means of the dynamical programming recursive relations (see [12]),

$$V = V_{0,N}, \quad V_{N,N} = (1 + r)^{-N} Y_N,$$

$V_{k,N}$

$$= \min\left((1 + r)^{-k} X_k, \max\left((1 + r)^{-k} Y_k, E\left(V_{k+1,N} \mid \mathcal{F}_k\right)\right)\right), \qquad (23)$$

where a positive integer N is an expiry time and $\{\mathcal{F}_k\}_{k \geq 0}$ is the corresponding filtration of σ-algebras. By this reason it makes sense to study approximations of the BS model by CRR models which we describe in the next section.

Though game options do not appear explicitly yet as a trading security in contemporary financial markets, it became popular recently to employ game options as a framework for the study of convertible (callable) bonds (see [58–65]). A holder of such bond either does nothing or decides to convert it into a predetermined number of stocks which can be considered as a cash payment depending on the current stock price, especially, in a market without transaction costs. On the other hand, the firm which issued this callable convertible bond may redeem it any time at a call price or force its conversion into stocks, and so this situation can be treated within the setup of game options.

Several papers deal with computation of the fair price of game options in special situations when the underlying stock price evolves according to a Markov process which usually, as in the BS model, turns out to be the geometric Brownian motion and when the payoffs depend only on the current stock price, usually just for the put and call options' payoffs arriving at a study of the free boundary problem with buyer's and seller's exercise boundaries (see [66–73]). For other callable derivative securities which were studied within the game options framework and its generalizations, we refer the reader to [74–79].

In real market conditions an investor (seller) may not be willing for various reasons to tie in a hedging portfolio the full initial capital required for a (perfect) hedge. In this case the seller is ready to accept a risk that his portfolio value at an exercise time may be less than his obligation to pay and he will need additional funds to fulfil the contract. Thus a portfolio shortfall comes into the picture and it is important to estimate the corresponding risk. We consider here a certain type of risk called the shortfall risk which was defined for game options in [80] by

$$R(x) = \inf_{(\pi, \sigma)} R(\pi, \sigma),$$

$$\text{where } R(\pi, \sigma) = \sup_{\tau} \mathbb{E}\left(\frac{\left(H(\sigma, \tau) - b_0 W_{\sigma \wedge \tau}^{\pi}\right)^+}{b_{\sigma \wedge \tau}}\right), \qquad (24)$$

where the infimum is taken over all self-financing portfolio strategies π with an initial capital x, and in both infimum and supremum the stopping times σ and τ do not exceed the option expiration date (horizon) T. It was shown in [80] that in the discrete time case both the shortfall risk and the corresponding minimizing portfolio strategies and stopping times could be obtained by means of a backward induction (dynamical programming) algorithm. In the continuous time case the situation is more complicated. For the shortfall risk in the American options case [81] obtained existence of minimizing strategies relying on some convex analysis arguments which are not available in the game options case, and so existence of minimizing portfolio strategies and stopping times in (24) remains an open question.

The papers [82, 83] deal with the, so-called, swing game options which are, in fact, multiple exercise game options. This question was studied before for American options in [84] but the option price obtained there was not justified by classical hedging arguments. This justification was done in [82, 83] for multiple exercise game options in the discrete and continuous time cases, respectively, which by simplification yields the result for American options as well. This

investigation required the study of Dynkin's games with multiple stopping which did not appear in the literature before. Observe that multiple exercise options may appear in their own rights when an investor wants to buy or sell an underlying security in several instalments at times of his choosing and, actually, any usual American or game option can be naturally extended to the multiexercise setup so that they may emerge both in commodities, energy and in different financial markets. Suppose, for instance, that a European car producer (having most expenses in euros or pounds) plans to supply autos to USA during a year in several shipments and buys a multiple-exercise option which guarantees a favorable dollar-euro (or dollar-pound) exchange rate at time of shipments (of his choice). The seller of such option can use currencies as underlying securities for his hedging portfolio. A multiple exercise option could be cheaper than a basket of usual one-exercise options if the former stipulates certain delay time between exercises which is quite natural in the above example. Furthermore, the acting sides above may prefer to deal with game rather than American multiple-exercise options since the former is cheaper for the buyer and safer (because of cancellation clause) for the seller.

Next, we describe more precisely game swing (multiple-exercise) options in the CRR market where the stock price evolves according to (18). We consider a swing option of the game type which has the ith payoff, $i \geq 1$, having the form

$$H^{(i)}(m, n) = X_i(m) \, \mathbb{I}_{m < n} + Y_i(n) \, \mathbb{I}_{n \leq m}, \quad \forall m, n, \qquad (25)$$

where $X_i(n), Y_i(n)$ are $\mathscr{F}_n$-adapted and $0 \leq Y_i(n) \leq X_i(n) < \infty$. Thus for any i, n there exist functions $f_n^{(i)}, g_n^{(i)} : \{a, b\}^n \to \mathbb{R}_+$ such that

$$\begin{aligned} Y_i(n) &= f_n^{(i)}(\rho_1, \ldots, \rho_n), \\ X_i(n) &= g_n^{(i)}(\rho_1, \ldots, \rho_n). \end{aligned} \qquad (26)$$

For any $1 \leq i \leq L - 1$ let C_i be the set of all pairs $((a_1, \ldots, a_i), (d_1, \ldots, d_i)) \in \{0, \ldots, N\}^i \times \{0, 1\}^i$ such that $a_{j+1} \geq N \wedge (a_j + 1)$ for any $j < i$. Such sequences represent the history of payoffs up to the ith one in the following way. If $a_j = k$ and $d_j = 1$ then the seller cancelled the jth claim at the moment k and if $d_j = 0$ then the buyer exercised the jth claim at the moment k (may be together with the seller). For $n \geq 1$ denote Γ_n the set of all stopping times with respect to the filtration $\{\mathscr{F}_n\}_{n=0}^N$ with values from n to N and set $\Gamma = \Gamma_0$.

Definition 4. A stopping strategy is a sequence $s = (s_1, \ldots, s_L)$ such that $s_1 \in \Gamma$ is a stopping time and for $i > 1$, $s_i : C_{i-1} \to \Gamma$ is a map which satisfies $s_i((a_1, \ldots, a_{i-1}), (d_1, \ldots, d_{i-1})) \in \Gamma_{N \wedge (1 + a_{i-1})}$.

In other words for the ith payoff both the seller and the buyer choose stopping times taking into account the history of payoffs so far. Denote by $\mathscr{S}$ the set of all stopping strategies

and define the map $F : \mathscr{S} \times \mathscr{S} \to \Gamma^L \times \Gamma^L$ by $F(s, b) = ((\sigma_1, \ldots, \sigma_L), (\tau_1, \ldots, \tau_L))$ where $\sigma_1 = s_1, \tau_1 = b_1$ and for $i > 1$,

$$\begin{aligned} \sigma_i &= s_i\left((\sigma_1 \wedge \tau_1, \ldots, \sigma_{i-1} \wedge \tau_{i-1}), (\mathbb{I}_{\sigma_1 < \tau_1}, \ldots, \mathbb{I}_{\sigma_{i-1} < \tau_{i-1}})\right), \\ \tau_i &= b_i\left((\sigma_1 \wedge \tau_1, \ldots, \sigma_{i-1} \wedge \tau_{i-1}), (\mathbb{I}_{\sigma_1 < \tau_1}, \ldots, \mathbb{I}_{\sigma_{i-1} < \tau_{i-1}})\right). \end{aligned} \qquad (27)$$

Set

$$c_k(s, b) = \sum_{i=1}^{L} \mathbb{I}_{\sigma_i \wedge \tau_i \leq k} \qquad (28)$$

which is a random variable equal to the number of payoffs until the moment k.

For swing options the notion of a self-financing portfolio involves not only allocation of capital between stocks and the bank account but also payoffs at exercise times. At the time k the writer's decision how much money to invest in stocks (while depositing the remaining money into a bank account) depends not only on his present portfolio value but also on the current claim. Denote by Ξ the set of functions on the (finite) probability space Ω.

Definition 5. A portfolio strategy with an initial capital $x > 0$ is a pair $\pi = (x, \gamma)$, where $\gamma : \{0, \ldots, N-1\} \times \{1, \ldots, L\} \times \mathbb{R} \to \Xi$ is a map such that $\gamma(k, i, y)$ is an $\mathscr{F}_k$-measurable random variable which represents the number of stocks which the seller buys at the moment k provided that the current claim has the number i and the present portfolio value is y. At the same time the sum $y - \gamma(k, i, y)S_k$ is deposited to the bank account of the portfolio. One calls a portfolio strategy $\pi = (x, \gamma)$ *admissible* if for any $y \geq 0$,

$$-\frac{y}{S_k b} \leq \gamma(k, i, y) \leq -\frac{y}{S_k a}. \qquad (29)$$

For any $y \geq 0$ denote $K(y) = [-y/b, -y/a]$.

Notice that if the portfolio value at the moment k is $y \geq 0$ then the portfolio value at the moment $k + 1$ before the payoffs (if there are any payoffs at this time) is given by $y + \gamma(k, i, y)S_k(S_{k+1}/S_k - 1)$, where i is the number of the next payoff. In view of independency of $S_{k+1}/S_k - 1$ and $\gamma(k, i, y)S_k$ we conclude that the inequality (29) is equivalent to the inequality $y + \gamma(k, i, y)S_k(S_{k+1}/S_k - 1) \geq 0$, that is, the portfolio value at the moment $k + 1$ before the payoffs is nonnegative. Denote by $\mathscr{A}(x)$ the set of all *admissible* portfolio strategies with an initial capital $x > 0$. Denote $\mathscr{A} = \bigcup_{x > 0} \mathscr{A}(x)$. Let $\pi = (x, \gamma)$ be a portfolio strategy and $s, b \in \mathscr{S}$. Set $((\sigma_1, \ldots, \sigma_L), (\tau_1, \ldots, \tau_L)) = F(s, b)$ and $c_k = c_k(s, b)$. The portfolio value at the moment k after the payoffs (if there are any payoffs at this moment) is given by

$$\begin{aligned} W_0^{(\pi, s, b)} &= x - H^{(1)}(\sigma_1, \tau_1) \, \mathbb{I}_{\sigma_1 \wedge \tau_1 = 0}, \quad \text{for } k > 0, \\ W_k^{(\pi, s, b)} &= W_{k-1}^{(\pi, s, b)} + \mathbb{I}_{c_{k-1} < L} \\ &\quad \times \left[\gamma\left(k - 1, c_{k-1} + 1, W_{k-1}^{(\pi, s, b)}\right) (S_k - S_{k-1}) \right. \\ &\quad \left. - \sum_{i=1}^{L} H^{(i)}(\sigma_i, \tau_i) \, \mathbb{I}_{\sigma_i \wedge \tau_i = k} \right]. \end{aligned}$$

$$(30)$$

Definition 6. A (perfect) hedge is a pair (π, s) which consists of a portfolio strategy and a stopping strategy such that $W_k^{(\pi,s,b)} \geq 0$ for any $b \in \mathcal{S}$ and $k \leq N$.

As usual, the option price V^* is defined as the infimum of $W \geq 0$ such that there exists a hedge with an initial capital W. The following result from [82] provides a dynamical programming algorithm for computation of both the option price and the corresponding hedge.

Theorem 7. *For any $n \leq N$ set*

$$X_n^{(1)} = X_L(n), \quad Y_n^{(1)} = Y_L(n),$$
$$V_n^{(1)} = \min_{\sigma \in \Gamma_n} \max_{\tau \in \Gamma_n} \widetilde{E}\left(H^{(L)}(\sigma, \tau) \mid \mathscr{F}_n\right), \tag{31}$$

and for $1 < k \leq L$,

$$X_n^{(k)} = X_{L-k+1}(n) + \widetilde{E}\left(V_{(n+1)\wedge N}^{(k-1)} \mid \mathscr{F}_n\right),$$
$$Y_n^{(k)} = Y_{L-k+1}(n) + \widetilde{E}\left(V_{(n+1)\wedge N}^{(k-1)} \mid \mathscr{F}_n\right), \tag{32}$$
$$V_n^{(k)} = \min_{\sigma \in \Gamma_n} \max_{\tau \in \Gamma_n} \widetilde{E}\left(X_\sigma^{(k)} \mathbb{1}_{\sigma<\tau} + Y_\tau^{(k)} \mathbb{1}_{\sigma \geq \tau} \mid \mathscr{F}_n\right),$$

where $\widetilde{E}$ is the expectation with respect to the unique martingale measure. Then

$$V^* = V_0^{(L)} = \min_{s \in \mathcal{S}} \max_{b \in \mathcal{S}} G(s, b), \tag{33}$$

where $G(s,b) = \widetilde{E}\sum_{i=1}^{L} H^{(i)}(\sigma_i, \tau_i)$ and $((\sigma_1, \ldots, \sigma_L), (\tau_1, \ldots, \tau_L)) = F(s, b)$. Furthermore, the stopping strategies $s^* = (s_1^*, \ldots, s_L^*) \in S$ and $b = (b_1^*, \ldots, b_L^*)$ given by

$$s_1^* = N \wedge \min\left\{k \mid X_k^{(L)} = V_k^{(L)}\right\},$$
$$b_1^* = \min\left\{k \mid Y_k^{(L)} = V_k^{(L)}\right\},$$
$$s_i^*\left((a_1, \ldots, a_{i-1}), (d_1, \ldots, d_{i-1})\right)$$
$$= N \wedge \min\left\{k > a_{i-1} \mid X_k^{(L-i+1)} = V_k^{(L-i+1)}\right\},$$
$$b_i^*\left((a_1, \ldots, a_{i-1}), (d_1, \ldots, d_{i-1})\right)$$
$$= N \wedge \min\left\{k > a_{i-1} \mid Y_k^{(L-i+1)} = V_k^{(L-i+1)}\right\}, \quad i > 1, \tag{34}$$

satisfy the saddle point inequalities

$$G\left(s^*, b\right) \leq G\left(s^*, b^*\right) \leq G\left(s, b^*\right) \quad \forall s, b, \tag{35}$$

and there exists a portfolio strategy $\pi^* \in \mathscr{A}(V_0^{(L)})$ such that (π^*, s^*) is a hedge.

4. Approximations of Game Options and of Their Shortfall Risk

Following [85] we will consider here approximations of the BS model by a sequence of CRR models with the interest rates

$r = r^{(n)}$ from (20) and with random variables $\rho_k = \rho_k^{(n)}$ from (18) given by

$$r = r^{(n)} = \exp\left(\frac{rT}{n}\right) - 1,$$
$$\rho_k = \rho_k^{(n)} = \exp\left(\frac{rT}{n} + \kappa\left(\frac{T}{n}\right)^{1/2}\xi_k\right) - 1, \tag{36}$$

where $\xi_j = \xi_j^{(n)}$, $j = 1, 2, \ldots$ are i.i.d. random variables taking on the values 1 and -1 with probabilities $p^{(n)} = (\exp(\kappa\sqrt{T/n}) + 1)^{-1}$ and $1 - p^{(n)} = (\exp(-\kappa\sqrt{T/n}) + 1)^{-1}$, respectively. This choice of random variables ξ_i, $i \in \mathbb{N}$, determines already the probability measures $P_n^\xi = \{p^{(n)}, 1 - p^{(n)}\}^\infty$ for the above sequence of CRR models and since $E_n^\xi \rho_k^{(n)} = r^{(n)}$, where E_n^ξ is the expectation with respect to P_n^ξ, we conclude that P_n^ξ is the martingale measure for the corresponding CRR market and the fair price $V = V^{(n)}$ of a game option in this market is given by the formula (21) with $E = E_n^\xi$.

Let V be the fair price of the game option in the BS market. It turns out that for a certain natural class of payoffs X_t and Y_t which may depend on the whole path (history) of the stock price evolution (as in integral or Russian-type options), the error $|V - V^{(n)}|$ does not exceed $Cn^{-1/4}(\ln n)^{3/4}$ where $C > 0$ does not depend on n and it can be estimated explicitly. Moreover, we will see that the rational exercise times of our CRR binomial approximations yield near rational ($Cn^{-1/4}(\ln n)^{3/4}$-optimal stopping times for the corresponding Dynkin's games) exercise times for game options in the BS market. Since the values $V^{(n)}$ and the optimal stopping times of the corresponding discrete time Dynkin's games can be obtained directly via the dynamical programming recursive procedure (23) our results provide a justification of a rather effective method of computation of fair prices and exercise times of game options with path-dependent payoffs. The standard construction of a self-financing hedging portfolio involves usually the Doob-Meyer decomposition of supermartingales which is explicit only in the discrete but not in the continuous time case. We will see how to construct a self-financing portfolio in the BS market with a small average (maximal) shortfall and an initial capital close to the fair price of a game option using hedging self-financing portfolios for the approximating binomial CRR markets. The latter problem does not seem to have been addressed before [85] in the literature on this subject. Having in mind that hedging self-financing portfolio strategies can be computed only approximately, their possible shortfalls come naturally into the picture and they should be taken into account in option pricing even if a perfect hedging exists theoretically. Note that these results require not only an approximation of stock prices and the corresponding payoffs, but also we have to take care of the different nature of stopping times in (21) and (22).

The main tool here is the Skorokhod-type embedding (see [86]) of sums of i.i.d. random variables into a Brownian motion (with a constant drift, in this case). This tool was

already employed for similar purposes in [87, 88]. The first paper treats an optimal stopping problem which can be applied to an American style option with a payoff function depending only on the current stock price and, more importantly, this function must be bounded and have two bounded derivatives which excludes usual put and call options cases. The second paper deals only with European options and, again, only payoffs (though with some discontinuities) determined by the current stock price are allowed. We observe that the Skorokhod embedding does not provide optimal error estimates in strong approximation theorems and it would be interesting to understand whether other approaches such as the quantile method (see [89–91]) and Stein's method (see [92]) can be employed for approximation of optimal stopping game values with better estimates of errors. Skorokhod embedding does not work also in the multidimensional situation, and for this case another method from [93] was employed in [94] where, actually, more general and not only binomial approximations were considered. More general approximation results for game options were obtained in [95] where only continuity of payoffs was assumed, but as a result no error estimates could be obtained there.

For each $t > 0$ denote by $M[0, t]$ the space of Borel measurable functions on $[0, t]$ with the uniform metric $d_{0t}(v, \tilde{v}) = \sup_{0 \leq s \leq t} |v_s - \tilde{v}_s|$. For each $t > 0$, let F_t and Δ_t be nonnegative functions on $M[0, t]$ such that for some constant $L \geq 1$ and for any $t \geq s \geq 0$ and $v, \tilde{v} \in M[0, t]$,

$$\left| F_s(v) - F_s(\tilde{v}) \right| + \left| \Delta_s(v) - \Delta_s(\tilde{v}) \right| \leq L(s+1) d_{0s}(v, \tilde{v}), \tag{37}$$

$$\left| F_t(v) - F_s(v) \right| + \left| \Delta_t(v) - \Delta_s(v) \right|$$
$$\leq L \left(|t - s| \left(1 + \sup_{u \in [0, t]} |v_u| \right) \right.$$
$$\left. + \sup_{u \in [s, t]} |v_u - v_s| \right). \tag{38}$$

By (37), $F_0(v) = F_0(v_0)$ and $\Delta_0(v) = \Delta_0(v_0)$ are functions of v_0 only. By (38),

$$F_t(v) + \Delta_t(v) \leq F_0(v_0) + \Delta_0(v_0)$$
$$+ L(t+2) \left(1 + \sup_{0 \leq s \leq t} |v_s| \right). \tag{39}$$

Next, we consider the BS market on a complete probability space together with its martingale measure P^B which exists and is unique as a corollary of the Girsanov theorem (see [56]). Let B_t, $t \geq 0$ be the standard one-dimensional continuous in time Brownian motion with respect to the martingale measure P^B. Set

$$B_t^* = -\frac{\kappa}{2} t + B_t, \quad t \geq 0. \tag{40}$$

Then the stock price $S_t^B(z)$ at time t in the BS market can be written in the form

$$S_t^B(z) = z \exp\left(rt + \kappa B_t^*\right), \quad S_0^B(z) = z > 0, \tag{41}$$

where $r > 0$ is the interest rate and $\kappa > 0$ is the, so called, volatility. We will consider game options in the BS market with payoff processes in the form

$$Y_t = F_t\left(S^B(z)\right), \quad X_t = G_t\left(S^B(z)\right), \quad t \in [0, T], \ T > 0, \tag{42}$$

where $G_t = F_t + \Delta_t$, F, Δ satisfies (37) and (38), $S^B(z) = S^B(z, \omega) \in M[0, T]$ is a random function taking the value $S_t^B(z) = S_t^B(z, \omega)$ at $t \in [0, T]$, and in the notations $F_t(S^B(z)) \, G_t(S^B(z))$ for $t < T$ we take the restriction of $S^B(z)$ to the interval $[0, t]$. The fair price $V = V(z)$ of this option with an initial value $z > 0$ of the stock is given by (22).

Next, we consider a sequence of CRR markets on a complete probability space such that for each $n = 1, 2, \ldots$ the stock prices $S_t^{(n)}(z)$ at time t are given by the formula

$$S_t^{(n)}(z) = z \exp\left(\sum_{k=1}^{[nt/T]} \left(\frac{rT}{n} + \kappa\left(\frac{T}{n}\right)^{1/2} \xi_k \right) \right), \quad t \geq \frac{T}{n},$$
$$S_t^{(n)}(z) = S_0^{(n)}(z) = z > 0, \quad t \in \left[0, \frac{T}{n}\right), \tag{43}$$

where, recall, $\xi_1, \xi_2, \ldots$ are i.i.d. random variables taking the values 1 and -1 with probabilities $p^{(n)} = (\exp(\kappa\sqrt{T/n}) + 1)^{-1}$ and $1 - p^{(n)} = (\exp(-\kappa\sqrt{T/n}) + 1)^{-1}$, respectively. Namely, we consider CRR markets where stock prices $S_m = S_{m/n}^{(n)}(z)$, $m = 0, 1, 2, \ldots$ satisfy (18) with $\rho_k = \rho_k^n$ given by (36), and, in addition, in place of the interest rate r in the first formula in (20) we take the sequence of interest rates $r_n = \exp(rT/n) - 1$, where r is the interest rate of the BS market appearing in the second formula of (20) and in (22). We consider $S^{(n)}(z) = S^{(n)}(z, \omega)$ as a random function on $[0, T]$, so that $S^{(n)}(z, \omega) \in M[0, T]$ takes the value $S_t^{(n)}(z) = S_t^{(n)}(z, \omega)$ at $t \in [0, T]$. For $k = 0, 1, 2, \ldots, n$ put

$$Y_k = Y_k^{(n)}(z) = F_{kT/n}\left(S^{(n)}(z)\right),$$
$$X_k = X_k^{(n)}(z) = G_{kT/n}\left(S^{(n)}(z)\right). \tag{44}$$

Then for each n the fair price $V = V^{(n)}(z)$ of the game option in the corresponding CRR market with an initial value $z > 0$ of the stock is given by (21).

Set

$$H_z^B(s, t) = F_t\left(S^B(z)\right) \mathbb{I}_{s \geq t} + G_s\left(S^B(z)\right) \mathbb{I}_{s < t},$$
$$Q_z^B(s, t) = e^{-rs \wedge t} R_z^B(s, t), \tag{45}$$

$$H_z^{(n)}(s, t) = F_t\left(S^{(n)}(z)\right) \mathbb{I}_{s \geq t} + G_s\left(S^{(n)}(z)\right) \mathbb{I}_{s < t},$$
$$Q_z^{(n)}(s, t) = e^{-rs \wedge t} R_z^{(n)}(s, t). \tag{46}$$

Denote by $\mathcal{T}_{0T}^B$ and $\mathcal{T}_{0n}^{\xi}$ the sets of stopping times with respect to the Brownian filtration $\mathcal{F}_t^B$, $t \geq 0$, with values in

$[0, T]$ and with respect to the filtration $\mathscr{F}_k^\xi = \sigma\{\xi_1, \dots, \xi_k\}$ with values in $\{0, 1, \dots, n\}$. Set

$$V(z) = \inf_{\sigma \in \mathscr{T}_{0T}^B} \sup_{\tau \in \mathscr{T}_{0T}^B} E^B Q_z^B(\sigma, \tau), \tag{47}$$

$$V^{(n)}(z) = \min_{\zeta \in \mathscr{T}_{0n}^\xi} \max_{\eta \in \mathscr{T}_{0n}^\xi} E_n^\xi Q_z^{(n)}\left(\frac{\zeta T}{n}, \frac{\eta T}{n}\right), \tag{48}$$

where E^B and E_n^ξ are the expectations with respect to the probability measures P^B and P_n^ξ, respectively, and we observe that $\mathscr{T}_{0n}^\xi$ is a finite set so that we can use min and max in (48).

Recall, that we choose P^B to be the martingale measure for the BS market and observe that P_n^ξ is the martingale measure for the corresponding CRR market since a direct computation shows that $E_n^\xi \rho_k^{(n)} = r_n$. Thus, (47) and (48) give fair prices of the game options in the corresponding markets. We note also that all our formulas involving the expectations E^B, in particular, (47) giving the fair price V of a game option, do not depend on a particular choice of a continuous-in-time version of the Brownian motion since all of them induce the same probability measure on the space of continuous sample paths which already determines all expressions with the expectations E^B appearing in this paper.

The following result from [96] provides an estimate for the error term in approximation of the fair price of a game option in the BS market by fair prices of the sequence of game options and prices of Dynkin's games defined above.

Theorem 8. *Suppose that $V(z)$ and $V^{(n)}(z)$ are defined by (46)–(48) with functions F and $G = F + \Delta$ satisfying (37) and (38). Then there exists a constant $C > 0$ (which can be explicitly estimated) such that*

$$\left|V(z) - V^{(n)}(z)\right| \leq C\left(F_0(z) + \Delta_0(z) + z + 1\right) n^{-1/4} (\ln n)^{3/4} \tag{49}$$

for all $z, n > 0$.

We can choose more general i.i.d. random variables $\xi_1, \xi_2, \dots$ appearing in the definition of $V^{(n)}$ as well, but these generalizations do not seem to have a financial mathematics motivation since we want to approximate game options in the BS market by the simplest possible models which are, of course, game options in the CRR market.

Among main examples of options with path-dependent payoff we have in mind integral options where

$$F_t(v) = \left(\int_0^t f_u(v_u) du - L\right)^+ \text{ (call option case)}, \tag{50}$$

or

$$F_t(v) = \left(L - \int_0^t f_u(v_u) du\right)^+ \text{ (put option case)}, \tag{51}$$

where, as usual, $a^+ = \max(a, 0)$. The penalty functional may also have here the integral form

$$\Delta_t(v) = \int_0^t \delta_u(v_u) du. \tag{52}$$

In order to satisfy the conditions (37) and (38), we can assume that for some $K > 0$ and all x, y, u,

$$\left|f_u(x) - f_u(y)\right| + \left|\delta_u(x) - \delta_u(y)\right| \leq K|x - y|,$$
$$\left|f_u(x)\right| + \left|\delta_u(x)\right| \leq K|x|. \tag{53}$$

Observe also that the Asian-type (averaged integral) payoffs of the form

$$F_t(v) = \left(\frac{1}{t}\int_0^t f_u(v_u) du - L\right)^+ \text{ or } = \left(L - \frac{1}{t}\int_0^t f_u(v_u) du\right)^+ \tag{54}$$

do not satisfy the condition (38) if arbitrarily small exercise times are allowed though the latter seems to have only some theoretical interest as it hardly happens in reality. Still, also in this case, the binomial approximation errors can be estimated in a similar way considering separately estimates for small stopping times and for stopping times bounded away from zero. Namely, define $V_\varepsilon(z)$ and $V_\varepsilon^{(n)}(z)$ for $\varepsilon \geq 0$ by (47) and (48), where $Q_z^{(B)}(\sigma, \tau)$ and $Q_z^{(n)}(\zeta T/n, \eta T/n)$ are replaced by $Q_z^{(B)}(\sigma \vee \varepsilon, \tau \vee \varepsilon)$ and $Q_z^{(n)}(\zeta T/n \vee \varepsilon, \eta T/n \vee \varepsilon)$, respectively. Assuming that f_u and δ_u are Lipschitz continuous also in u (at least for u close to 0) in the form $|f_s(x) - f_u(x)| + |\delta_s(x) - \delta_u(x)| \leq K(x+1)|s-u|$ for some $K > 0$ and all $s, u, x \geq 0$, we obtain that if $v_0 = z$ and $F_0(v) = (f_0(z) - L)^+$ or $= (L - f_0(z))^+$, then

$$\left|F_s(v) - F_0(z)\right| \leq Ks\left(1 + \sup_{0 \leq u \leq s}|v_u|\right) + K\sup_{0 \leq u \leq s}|v_u - z|. \tag{55}$$

It is not difficult to see from here that $|V(z) - V_\varepsilon(z)|$ and $|V^{(n)}(z) - V_\varepsilon^{(n)}(z)|$ do not exceed $C(1+z)\sqrt{\varepsilon}$ for all small ε and some constant C. On the other hand, similar to Theorem 8, we see that for some constant $C > 0$ and all $n, \varepsilon > 0$,

$$\left|V_\varepsilon(z) - V_\varepsilon^{(n)}(z)\right| \leq C(1+z)\varepsilon^{-1} n^{-1/4}(\ln n)^{3/4}. \tag{56}$$

Choosing $\varepsilon = n^{-1/6}\sqrt{\ln n}$, we obtain that under the above conditions in the case of Asian options, $|V(z) - V^{(n)}(z)|$ can be estimated by $3C(1+z)n^{-1/12}(\ln n)^{1/4}$.

Another important example of path-dependent payoffs are the, so-called, Russian options where, for instance,

$$F_t(v) = \max\left(m, \sup_{u \in [0,t]} v_u\right), \qquad \Delta_t(v) = \delta v_t. \tag{57}$$

Such payoffs satisfy the conditions of Theorem 8. Indeed, (37) is clear in this case and (38) follows since for $t \geq s$,

$$\max\left(m, \sup_{u \in [0,t]} v_u\right) - \max\left(m, \sup_{u \in [0,s]} v_u\right) \leq \sup_{u \in [0,t]} v_u - \sup_{u \in [0,s]} v_u$$
$$\leq \sup_{u \in [s,t]} v_u - v_s$$
$$\leq \sup_{u \in [s,t]}|v_u - v_s|. \tag{58}$$

In order to compare $V(z)$ and $V^{(n)}(z)$ in the case of path-dependent payoffs, we have to consider both BS and CRR markets on one probability space in an appropriate way, and the main tool in achieving this goal will be here the Skorokhod-type embedding (see, for instance, [86], Section 37). In fact, for the binomial i.i.d. random variables $\xi_1, \xi_2, \ldots$ appearing in the setup of the CRR market models above, the embedding is explicit and no general theorems are required, but if we want to extend the result for other sequences of i.i.d. random variables, we have to rely upon the general result. Namely, define recursively

$$\theta_0^{(n)} = 0, \quad \theta_{k+1}^{(n)} = \inf\left\{ t > \theta_k^{(n)} : \left| B_t^* - B_{\theta_k^{(n)}}^* \right| = \sqrt{\frac{T}{n}} \right\}, \quad (59)$$

where, recall, $B_t^* = -(\kappa/2)t + B_t$. The standard strong Markov property-based arguments (cf. [86], Section 37) show that $\theta_k^{(n)} - \theta_{k-1}^{(n)}$, $k = 1, 2, \ldots$ are i.i.d. sequences of random variables such that $(\theta_{k+1}^{(n)} - \theta_k^{(n)}, B_{\theta_{k+1}^{(n)}}^* - B_{\theta_k^{(n)}}^*)$ are independent of $\mathscr{F}_{\theta_k^{(n)}}^B$ (where, recall, $\mathscr{F}_t^B = \sigma\{B_s, s \le t\}$).

It turns out (see [85, 88]) that $B_{\theta_1^{(n)}}^*$ has the same distribution as $\sqrt{T/n}\xi_1$. Set

$$\Xi_k^{(n)} = \left(\frac{T}{n}\right)^{1/2} \sum_{j=1}^{k} \xi_j, \quad (60)$$

then $\Xi_k^{(n)}$ has the same distribution as $B_{\theta_k^{(n)}}^*$.

Theorem 8 provides an approximation of the fair price of game options in the BS market by means of fair prices of game options in the CRR market which becomes especially useful if we can provide also a simple description of rational (or δ-rational) exercise times of these options in the BS market via exercise times of their CRR market approximations which are, by the definition, optimal (or δ-optimal) stopping times for the Dynkin's game whose price is given by (48). For each $k = 1, 2, \ldots$ introduce the finite σ-algebra $\mathscr{G}_k^{B,n} = \sigma\{B_{\theta_1^{(n)}}^*, B_{\theta_2^{(n)}}^* - B_{\theta_1^{(n)}}^*, \ldots, B_{\theta_k^{(n)}}^* - B_{\theta_{k-1}^{(n)}}^*\}$ which is, clearly, isomorphic to $\mathscr{F}_k^\xi = \sigma\{\xi_i, \ i \le k\}$ considered before since each element of $\mathscr{G}_k^{B,n}$ and of $\mathscr{F}_k^\xi$ is an event of the form

$$A_{\iota^{(k)}}^{B,n} = \left\{ B_{\theta_j^{(n)}}^* - B_{\theta_{j-1}^{(n)}}^* = \iota_j \sqrt{\frac{T}{n}}, \ j = 1, \ldots, k \right\},$$

$$A_{\iota^{(k)}}^\xi = \left\{ \xi_j = \iota_j, \ j = 1, \ldots k \right\}, \quad (61)$$

respectively, where $\iota^{(k)} = (\iota_1, \ldots, \iota_k) \in \{-1, 1\}^k$, $\theta_0^{(n)} = 0$, and $B_0 = 0$. Let $\mathscr{S}^{B,n}$ be the set of stopping times with respect to the filtration $\mathscr{G}_k^{B,n}$, $k = 0, 1, 2, \ldots$, where $\mathscr{G}_0^{B,n} = \{\emptyset, \Omega_B\}$ is the trivial σ-algebra, and Ω_B is the sample space of the Brownian motion. The subset of these stopping times with values in $\{0, 1, \ldots, n\}$ will be denoted by $\mathscr{S}_{0,n}^{B,n}$. For each $\iota^{(n)} = (\iota_1, \ldots, \iota_n) \in \{-1, 1\}^n$ and $k < n$ we set $\iota^{(k)} = (\iota_1, \ldots, \iota_k) \in \{-1, 1\}^k$. Denote by $\mathscr{J}_{0,n}$ the set of functions $\nu : \{-1, 1\}^n \to \{0, 1, \ldots, n\}$ such that if $\nu(\iota^{(n)}) = k \le n$ and $\tilde{\iota}^{(k)} = \iota^{(k)}$ for

some $\tilde{\iota}^{(n)} \in \{-1, 1\}^n$, then $\nu(\tilde{\iota}^{(n)}) = k$, as well. Define the functions $\lambda_\xi^{(n)} : \Omega_\xi \to \{-1, 1\}^n$ and $\lambda_B^{(n)} : \Omega_B \to \{-1, 1\}^n$ by $\lambda_\xi^{(n)}(\omega) = (\xi_1(\omega), \ldots, \xi_n(\omega))$ and

$$\lambda_B^{(n)}(\omega) = \sqrt{\frac{n}{T}} \left(B_{\theta_1^{(n)}(\omega)}^*(\omega), B_{\theta_2^{(n)}(\omega)}^*(\omega) \right.$$

$$\left. - B_{\theta_1^{(n)}(\omega)}^*(\omega), \ldots, B_{\theta_n^{(n)}(\omega)}^*(\omega) - B_{\theta_{n-1}^{(n)}(\omega)}^*(\omega) \right), \quad (62)$$

where Ω_ξ and Ω_B are sample spaces on which the sequence $\xi_1, \xi_2, \ldots$ and the Brownian motion B_t are defined, respectively. It is clear that any $\zeta \in \mathscr{T}_{0n}^\xi$ and $\eta \in \mathscr{S}_{0,n}^{B,n}$ can be represented uniquely in the form $\zeta = \mu \circ \lambda_\xi^{(n)}$ and $\eta = \nu \circ \lambda_B^{(n)}$ for some $\mu, \nu \in \mathscr{J}_{0,n}$.

Theorem 9. *There exists a constant $C > 0$ (which can be estimated explicitly) such that if $\zeta_n^* = \mu_n^* \circ \lambda_\xi^{(n)}$ and $\eta_n^* = \nu_n^* \circ \lambda_\xi^{(n)}$, $\mu_n^*, \nu_n^* \in \mathscr{J}_{0n}$ are rational exercise times for the game option in the CRR market defined by (43); that is,*

$$V^{(n)}(z) = \min_{\zeta \in \mathscr{T}_{0n}^\xi} E^\xi Q_z^{(n)}\left(\zeta \frac{T}{n}, \eta_n^* \frac{T}{n}\right)$$

$$= \max_{\eta \in \mathscr{T}_{0n}^\xi} E^\xi Q_z^{(n)}\left(\zeta_n^* \frac{T}{n}, \eta \frac{T}{n}\right), \quad (63)$$

then $\varphi_n^* = \theta_{\mu_n^* \circ \lambda_B^{(n)}}^{(n)}$ and $\psi_n^* = \theta_{\nu_n^* \circ \lambda_B^{(n)}}^{(n)}$ are $\delta_n(z)$-rational exercise times for the game option in the BS market defined by (39) and (41); that is,

$$\sup_{\tau \in \mathscr{T}_{0T}^B} E^B Q_z^B(\varphi_n^*, \tau) - \delta_n(z)$$

$$\le V(z) \le \inf_{\sigma \in \mathscr{T}_{0T}^B} E^B Q_z^B(\sigma, \psi_n^*) + \delta_n(z), \quad (64)$$

where $\delta_n(z) = C(F_0(z) + \Delta_0(z) + z + 1)n^{-1/4}(\ln n)^{3/4}$.

It is well known (see, for instance, [6]) that when payoffs depend only on the current stock price (a Markov case), δ-optimal stopping times of Dynkin's games can be obtained as first arrival times to domains where the payoff is δ-close to the value of the game (as a function of the initial stock price). For path-dependent payoffs the situation is more complicated, and, in general, in order to construct δ-optimal stopping times, we have to know the stochastic process of values of the games starting at each time $t \in [0, T]$ conditioned to the information up to t. It is not clear what kind of approximation of this process can provide some information about δ-rational exercise times, and the convenient alternative method of their construction exhibited in Theorem 9 seems to be important for both the theory and applications. Moreover, this construction is effective and can be employed in practice since μ_n^* and ν_n^* are functions on sequences of 1's and -1's which can be computed (and stored in a computer) using the recursive formulas (23) even before the

stock evolution begins. In order to compute $\lambda_B^{(n)}$, we have to watch the discounted stock price $\check{S}_t^B(z) = e^{-rt}S_t^B(z)$ evolution of a real stock at moments $\theta_k^{(n)}$ which are obtained recursively by $\theta_0^{(n)} = 0$ and

$$\theta_{k+1}^{(n)} = \inf\left\{t > \theta_k^{(n)} : \check{S}_t^B(z) = e^{\pm\kappa(T/n)^{1/2}}\check{S}_{\theta_k^{(n)}}^B(z)\right\} \quad (65)$$

and to construct the $\{1, -1\}$ sequence $\lambda_B^{(n)}(\omega)$ by writing 1 or -1 on kth place depending on whether $\check{S}_{\theta_k^{(n)}}^B(z) = e^{\kappa(T/n)^{1/2}}\check{S}_{\theta_{k-1}^{(n)}}^B(z)$ or $\check{S}_{\theta_k^{(n)}}^B(z) = e^{-\kappa(T/n)^{1/2}}\check{S}_{\theta_{k-1}^{(n)}}^B(z)$, respectively.

Recall (see [56]) that a sequence $\pi = (\pi_1, \ldots, \pi_n)$ of pairs $\pi_k = (\beta_k, \gamma_k)$ of $\mathscr{F}_{k-1}^\xi$-measurable random variables β_k, γ_k, $k = 1, \ldots, n$, is called a self-financing portfolio strategy in the CRR market determined by (18), (20), (36), and (43) if the price of the portfolio at time k is given by the formula

$$W_k^{\pi,n} = \beta_k b_k + \gamma_k S_{kT/n}^{(n)}(z) = \beta_{k+1}b_k + \gamma_{k+1}S_{kT/n}^{(n)}(z), \quad (66)$$

and the latter equality means that all changes in the portfolio value are due to capital gains and losses but not due to withdrawal or infusion of funds. A pair (ζ, π) of a stopping time $\zeta \in \mathscr{T}_{0n}^\xi$ and a self-financing portfolio strategy π is called a hedge for (against) the game option with the payoff $R_z^{(n)}$ given by (46) if (see [12])

$$W_{\zeta \wedge k}^{\pi,n} \geq H_z^{(n)}\left(\frac{\zeta T}{n}, \frac{kT}{n}\right), \quad \forall k = 0, 1, \ldots, n. \quad (67)$$

It follows from [12] that for any $\zeta \in \mathscr{T}_{0n}^\xi$ there exists a self-financing portfolio strategy π^ζ so that (ζ, π^ζ) is a hedge. In particular, if we take the rational exercise time $\zeta = \zeta_n^*$ of the writer, then such π^ζ exists with the initial portfolio capital $V^{(n)}(z)$. The construction of π^ζ goes directly via the Doob-Meyer decomposition of supermartingales and a martingale representation lemma (see [12, 56]), both being explicit in the CRR market case. In the continuous time BS market we cannot write the corresponding portfolio strategies in an explicit way, and so some approximations are necessary.

Theorem 10. *Let $\zeta \in \mathscr{T}_{0n}^\xi$, $\pi = \pi^\zeta$, and (66) together with (67) hold true with $\mathscr{F}_k^\xi$-measurable $\beta_k = \beta_k^\zeta$ and $\gamma_k = \gamma_k^\xi$, so that (ζ, π^ζ) is a hedge. Then $\beta_k^\zeta = f_k \circ \lambda_\xi^{(k-1)}$, $\gamma_k^\zeta = g_k \circ \lambda_\xi^{(k-1)}$, and $\zeta = \mu \circ \lambda_\xi^{(n)}$ for some uniquely defined functions f_k, g_k on $\{1, -1\}^{k-1}$ and some $\mu \in \mathscr{J}_{0n}$. Let $\varphi = \mu \circ \lambda_B^{(n)}$ and set $\beta_t^\varphi = f_k \circ \lambda_B^{(k-1)}$ and $\gamma_t^\varphi = g_k \circ \lambda_B^{(k-1)}$ whenever $t \in (\theta_{k-1}^{(n)}, \theta_k^{(n)}]$. Then*

$$W_t^B = \beta_t^\varphi b_t + \gamma_t^\varphi S_t^B(z) \quad (68)$$

is a self-financing portfolio in the BS market and there exists a constant $C > 0$ such that

$$E^B \sup_{0 \leq t \leq T}\left(H_z^B\left(\theta_\varphi^{(n)}, t\right) - W_{\theta_\varphi^{(n)} \wedge t}^B\right)^+ \\ \leq C\left(F_0(z) + \Delta_0(z) + z + 1\right)n^{-1/4}(\ln n)^{3/4}, \quad (69)$$

where $a^+ = \max(a, 0)$. *In particular, there exists a self-financing portfolio of this form satisfying* (69) *with the initial value $V^{(n)}(z)$ (which according to* (49) *is close to the fair price $V(z)$ of the game option) taking $\varphi^* = \mu^* \circ \lambda_B^{(n)}$ if $\zeta^* = \mu^* \circ \lambda_\xi^{(n)}$ is the rational exercise time and $\pi = \pi^{\zeta^*}$ is the corresponding optimal self-financing hedging portfolio strategy for the CRR market.*

The inequality (69) estimates the expectation of the maximal shortfall (risk) of certain (nearly hedging) portfolio strategy which can be constructed effectively in applications since the functions f_l, g_l, and μ are determined by a self-financing hedging strategy in the CRR market which can be computed directly and stored in a computer even before the real stock evolution begins or in case of computer memory limitations we can compute these functions each time when needed using corresponding algorithms for the CRR market. The functions $\lambda_B^{(n)}$ or, in other words, the sequences from $\{-1, 1\}^n$ which should be plugged into the functions f_l, g_l, and μ should be obtained in practice by watching the evolution of the discounted stock price $e^{-rt}S_t^B$ at moments $\theta_k^{(n)}$ as described after Theorem 9.

The paper [96] studied approximations of the shortfall risk $R(x)$ given by (24) for game options in the BS market by the shortfall risks $R_n(x)$ of game options in the sequence of CRR markets defined above where the initial capital x of all portfolios under consideration is kept the same and the payoffs satisfy the same conditions as above. The convergence $\lim_{n \to \infty}R_n(x) = R(x)$ was proved in [96] but only the one sided error estimate

$$R(x) \leq R_n(x) + Cn^{-1/4}(\ln n)^{3/4} \quad (70)$$

was obtained there for game options. On the other hand, relying on some convexity arguments, it was possible to obtain for American options two-sided estimates with the same error term.

In [97] similar approximation results as above were extended to barrier game options. Namely, [97] deals with double knock-out barrier option with two constant barriers L, R such that $0 \leq L < S_0 < R \leq \infty$ which means that the option becomes worthless to its holder (buyer) at the first time τ_I the stock price S_t exits the open interval $I = (L, R)$. Thus for $t \geq \tau_{(L,R)}$ the payoff is $X_t = Y_t = 0$. For $t < \tau_{(L,R)}$ path-dependent payoffs satisfying (38) and (39) are considered. Such a contract is of potential value to a buyer who believes that the stock price will not exit the interval I up to a maturity date and to a seller who does not want to worry about hedging if the stock price will reach one of the barriers L, R. Such an option is equivalent to the usual game option when the payoffs X_t and Y_t are replaced by $X_t^I = X_t \mathbb{1}_{t < \tau_I}$ and $Y_t^I = Y_t \mathbb{1}_{t < \tau_I}$, respectively. Now, these new payoffs lose regularity conditions (38) and (39), but still it turns out that the error estimates in (49) remain true when we approximate the price of the above barrier game options in the BS market by the prices of corresponding barrier game options in the CRR markets as in Theorem 8 above. The results concerning approximation of the shortfall risk turn out to be very similar

for barrier game options to the corresponding results for usual game options described above.

When payoffs depend only on the current stock price (and not path-dependent as in (45) and (46)) then in some special cases it is possible to obtain better error estimates for binomial approximations of prices of game options relying on partial differential equations methods in the free boundary problem. In [98] this was done for American put options in the BS market, and in [99] this was extended to game put options with error estimates of order $n^{-1/2}$ in comparison to $n^{-1/4}(\ln n)^{3/4}$ obtained in Theorem 8.

5. Incomplete Markets and Transaction Costs

Both in incomplete markets and in markets with transaction costs there is no one arbitrage free price of each derivative which can be considered as its fair price, and one of approaches in these circumstances is to study superhedging. Game options in incomplete markets were studied in several papers; in particular, in [100] they were studied from the point of view of utility maximization which leads to non-zero-sum Dynkin's games while in [101] they were studied from the point of view of superhedging and arbitrage free prices.

Next, we concentrate in this section on superhedging pricing of game options in discrete markets with transaction costs. The market model here will consist of a finite probability space Ω with the σ-field $\mathcal{F} = 2^\Omega$ of all subspaces of Ω and a probability measure $\mathbb{P}$ on $\mathcal{F}$ giving a positive weight $\mathbb{P}(\omega)$ to each $\omega \in \Omega$. The setup includes also a filtration $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$ where T is a positive integer called the time horizon. It is convenient to denote by Ω_t the set of atoms in $\mathcal{F}_t$ so that any $\mathcal{F}_t$-measurable random variable (vector) Z can be identified with a function (vector function) defined on Ω_t and its value at $\mu \in \Omega_t$ will be denoted either by $Z(\mu)$ or by Z^μ.

The market model consists of a risk-free bond and a risky stock. Without loss of generality, we can assume that all prices are discounted so that the bond price equals 1 all the time, and a position in bonds is identified with cash holding. On the other hand, the shares of the stock can be traded which involves proportional transaction costs. This will be represented by bid-ask spreads; that is, shares can be bought at an ask price S_t^a or sold at the bid price S_t^b, where $S_t^s \geq S_t^b > 0$, $t = 0, 1, \dots, T$ are processes adapted to the filtration $\{\mathcal{F}_t\}_{t=o}^T$.

The liquidation value at time t of a portfolio (γ, δ) consisting of an amount γ of cash (or bond) and δ shares of the stock equals

$$\theta_t(\gamma, \delta) = \gamma + S_t^b \delta^+ - S_t^a \delta^- \qquad (71)$$

which in case $\delta < 0$ means that a portfolio owner should spend the amount $S_t^a \delta^-$ in order to close his short position. Observe that fractional numbers of shares are allowed here so that both γ and δ in a portfolio (γ, δ) could be, in principle, any real numbers. By definition, a self-financing portfolio strategy is a predictable process (α_t, β_t) representing positions

in cash (or bonds) and stock at time t, $t = 0, 1, \dots, T$ such that

$$\theta_t(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1}) \geq 0 \qquad \forall t = 0, 1, \dots, T-1, \quad (72)$$

and the set of all such portfolio strategies will be denoted by Φ.

As before, we consider here a game option which is a contract between its seller and buyer such that both have the right to exercise it at any time up to a maturity date (horizon) T. In the presence of transaction costs there is a difference whether we stipulate that the option to be settled in cash or both in cash and shares of stock while in the former case an assumption concerning transaction costs in the process of portfolio liquidation should be made. We adopt here the setup where the payments X_t and Y_t are made both in cash and shares of the stock, and transaction costs take place always when a portfolio adjustment occurs. Thus, the payments are, in fact, adapted random 2-vectors $X_t = (X_t^{(1)}, X_t^{(2)})$ and $Y_t = (Y_t^{(1)}, Y_t^{(2)})$ where the first and the second coordinates represent, respectively, a cash amount to be paid and a number of stock shares to be delivered, and as we allow also fractional numbers of shares, both coordinates can take on any nonnegative real value. The inequality $X_t \geq Y_t$ in the zero transaction costs case is replaced in our present setup by

$$\Delta_t = \theta_t\left(X_t^{(1)} - Y_t^{(1)}, X_t^{(2)} - Y_t^{(2)}\right) \geq 0, \qquad (73)$$

and Δ_t is interpreted as a cancellation penalty. We impose also a natural assumption that $X_T^{(1)} = Y_T^{(1)}$ and $X_T^{(2)} = Y_T^{(2)}$; that is, on the maturity date there is no penalty. Therefore, if the seller cancels the contract at time s while the buyer exercises at time t, the former delivers to the latter a package of cash and stock shares which can be represented as a 2-vector in the form

$$H(s, t) = \left(H^{(1)}(s, t), H^{(2)}(s, t)\right) = X_s \mathbb{I}_{s<t} + Y_t \mathbb{I}_{t\leq s}, \quad (74)$$

where $\mathbb{I}_A = 1$ if an event A occurs and $\mathbb{I}_A = 0$ if not. It will be convenient to allow the payment components $X_t^{(1)}$, $X_t^{(2)}$ and $Y_t^{(1)}$, $Y_t^{(2)}$ to take on any real (and not only nonnegative) values which will enable us to demonstrate complete duality (symmetry) between the seller's and the buyer's positions.

A pair (σ, π) of a stopping time $\sigma \leq T$ and of a self-financing strategy $\pi = (\alpha_t, \beta_t)_{t=0}^T$ will be called a superhedging strategy for the seller of the game option with a payoff given by (74) if for all $t \leq T$,

$$\theta_{\sigma \wedge t}\left(\alpha_{\sigma \wedge t} - H^{(1)}(\sigma, t), \ \beta_{\sigma \wedge t} - H^{(2)}(\sigma, t)\right) \geq 0, \qquad (75)$$

where, as usual, $c \wedge d = \min(c, d)$ and $c \vee d = \max(c, d)$. The seller's (ask or upper hedging) price V^a of a game option is defined as the infimum of initial amounts required to start a superhedging strategy for the seller. Since in order to get α_0 amount of cash and β_0 shares of stock at time 0, the seller should spend

$$-\theta_0(-\alpha_0, -\beta_0) = \alpha_0 + \beta_0^+ S_0^a - \beta_0^- S_0^b \qquad (76)$$

in cash, we can write

$$V^a = \inf_{\sigma,\pi} \left\{ -\theta_0 \left(-\alpha_0, -\beta_0 \right) : (\sigma, \pi) \text{ with} \right.$$

$$\pi = \left(\alpha_t, \beta_t \right)_{t=0}^{T} \text{ being a superhedging} \qquad (77)$$

$$\left. \text{strategy for the seller} \right\}.$$

On the other hand, the buyer may borrow from a bank an amount $\theta_0(-\alpha_0, -\beta_0)$ to purchase a game option with the payoff (74) and begining with the negative valued portfolio (α_0, β_0) to manage a self-financing strategy $\pi = (\alpha_t, \beta_t)_{t=0}^{T}$ so that for a given stopping time $\tau \leq T$ and all $s \leq T$,

$$\theta_{s \wedge \tau} \left(\alpha_{s \wedge \tau} + H^{(1)}(s, \tau), \ \beta_{s \wedge \tau} + H^{(2)}(s, \tau) \right) \geq 0. \qquad (78)$$

In this case the pair (τ, π) will be called a superhedging strategy for the buyer. The buyer's (bid or lower hedging) price V^b of the game option above is defined as the supremum of initial bank loan required to purchase this game option and to manage a superhedging strategy for the buyer. Thus,

$$V^b = \sup_{\tau,\pi} \left\{ \theta_0 \left(-\alpha_0, -\beta_0 \right) : (\tau, \pi) \text{ with} \right.$$

$$\pi = \left(\alpha_t, \beta_t \right)_{t=0}^{T} \text{ being a superhedging} \qquad (79)$$

$$\left. \text{strategy for the buyer} \right\}.$$

It follows from the representations of Theorem 11 that $V^a \geq V^b$.

First, we recall the notion of a randomized stopping time (see [102–104] and references there) which is defined as a nonnegative adapted process χ such that $\sum_{t=0}^{T} \chi_t = 1$. The set of all randomized stopping times will be denoted by $\mathcal{X}$ while the set of all usual or pure stopping times will be denoted by $\mathcal{T}$. It will be convenient to identify each pure stopping time τ with a randomized stopping time χ^τ such that $\chi_t^\tau = \mathbb{I}_{\{\tau=t\}}$ for any $t = 0, 1, \ldots, T$, so that we could write $\mathcal{T} \subset \mathcal{X}$. For any adapted process Z and each randomized stopping time χ the time-χ value of Z is defined by

$$(Z)_\chi = Z_\chi = \sum_{t=0}^{T} \chi_t Z_t. \qquad (80)$$

Considering a game option with a payoff given by (74) we write also

$$H(\chi, \tilde{\chi}) = \sum_{s,t=0}^{T} \chi_s \tilde{\chi}_t H(s, t) \qquad (81)$$

which is the seller's payment to the buyer when the former cancels and the latter exercises at randomized stopping times χ and $\tilde{\chi}$, respectively. In particular, if σ and τ are pure stopping times, then

$$H(\chi, \chi^\tau) = \sum_{s=0}^{T} \chi_s H(s, \tau),$$

$$\qquad (82)$$

$$H(\chi^\sigma, \chi) = \sum_{t=0}^{T} \chi_t H(\sigma, t).$$

Next, we introduce the notion of an approximate martingale which is defined for any randomized stopping time χ as a pair (P, S) of a probability measure P on Ω and of an adapted process S such that for each $t = 0, 1, \ldots, T$,

$$S_t^b \leq S_t \leq S_t^a, \quad \chi_{t+1}^* S_t^b \leq \mathbb{E}_P \left(S_{t+1}^{\chi^*} \mid \mathcal{F}_t \right) \leq \chi_{t+1}^* S_t^a, \qquad (83)$$

where $\mathbb{E}_P$ is the expectation with respect to P,

$$\chi_t^* = \sum_{s=t}^{T} \chi_s, \quad Z_t^{\chi^*} = \sum_{s=t}^{T} \chi_s Z_s, \quad \chi_{T+1}^* = 0, \ Z_{T+1}^{\chi^*} = 0. \qquad (84)$$

Given a randomized stopping time χ the space of corresponding approximate martingales (P, S) will be denoted by $\overline{\mathcal{P}}(\chi)$ and we denote by $\mathcal{P}(\chi)$ the subspace of $\overline{\mathcal{P}}(\chi)$ consisting of pairs (P, S) with P being equivalent to the original (market) probability $\mathbb{P}$.

Next, we introduce some convex analysis notions and notations (see [104, 105] for more details). Denote by Θ the family of functions $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that either $f \equiv -\infty$ or f is a (finite) real valued polyhedral (continuous piecewise linear with finite number of segments) function. If $f, g \in \Theta$, then, clearly, $f \wedge g, f \vee g \in \Theta$. The epigraph of $f \in \Theta$ is defined by $\text{epi}(f) = \{(x, y) \in \mathbb{R}^2 : x \geq f(y)\}$. For any $c \geq d$ the function $h_{[d,c]}(y) = cy^- - dy^+$, clearly, belongs to Θ. Observe that the self-financing condition (72) can be rewritten in the form

$$\left(\alpha_t - \alpha_{t+1}, \ \beta_t - \beta_{t+1} \right) \in \text{epi}\left(h_{[S_t^b, S_t^a]} \right). \qquad (85)$$

For each $f \in \Theta$ and $c \geq d$ there exists a unique function $\text{gr}_{[d,c]}(f) \in \Theta$ such that

$$\text{epi}\left(\text{gr}_{[d,c]}(f) \right) = \text{epi}\left(h_{[d,c]} \right) + \text{epi}(f). \qquad (86)$$

For any $y \in \mathbb{R}$, $\mu \in \Omega_t$, and $t = 0, 1, \ldots, T$ define $q_t^a(y) = q_t^a(\mu, y), q_t^b(y) = q_t^b(\mu, y), \ r_t^a(y) = r_t^a(\mu, y), \ r_t^b(y) = r_t^b(\mu, y)$ by

$$q_t^a(y) = X_t^{(1)} + h_{[S_t^b, S_t^a]}\left(y - X_t^{(2)} \right),$$

$$r_t^a(y) = Y_t^{(1)} + h_{[S_t^b, S_t^a]}\left(y - Y_t^{(2)} \right),$$

$$\qquad (87)$$

$$q_t^b(y) = -X_t^{(1)} + h_{[S_t^b, S_t^a]}\left(y + X_t^{(2)} \right),$$

$$r_t^b(y) = -Y_t^{(1)} + h_{[S_t^b, S_t^a]}\left(y + Y_t^{(2)} \right)$$

with $h_{[d,c]}$ the same as in (85) and (86). Observe that if $c \geq d \geq 0$, then either $h_{[d,c]} \equiv 0$ or $h_{[d,c]}$ is a monotone decreasing function, and so

$$q_t^a \geq r_t^a, \quad q_t^b \leq r_t^b. \qquad (88)$$

Introduce also

$$G_{s,t}^a(y) = H^{(1)}(s, t) + h_{[S_{s \wedge t}^b, S_{s \wedge t}^a]}\left(y - H^{(2)}(s, t) \right)$$

$$= q_s^a(y) \mathbb{I}_{s < t} + r_t^a(y) \mathbb{I}_{t \leq s}, \ G_{s,t}^b(y)$$

$$\qquad (89)$$

$$= -H^{(1)}(s, t) + h_{[S_{s \wedge t}^b, S_{s \wedge t}^a]}\left(y + H^{(2)}(s, t) \right)$$

$$= q_s^b(y) \mathbb{I}_{s < t} + r_t^b(y) \mathbb{I}_{t \leq s}.$$

Clearly, the superhedging conditions (75) of the seller and (78) of the buyer are equivalent to

$$(\alpha_{\sigma \wedge t}, \beta_{\sigma \wedge t}) \in \mathrm{epi}\left(G_{\sigma,t}^a\right) \quad \forall t = 0, 1, \ldots, T,$$

$$(\alpha_{s \wedge \tau}, \beta_{s \wedge \tau}) \in \mathrm{epi}\left(G_{s,\tau}^b\right) \quad \forall s = 0, 1, \ldots, T, \tag{90}$$

respectively. Observe also that

$$q_t^a(0) = -q_t^b(0) = \theta_t\left(X_t^{(1)}, X_t^{(2)}\right),$$

$$r_t^a(0) = -r_t^b(0) = \theta_t\left(Y_t^{(1)}, Y_t^{(2)}\right). \tag{91}$$

We recall that $X_T^{(1)} = Y_T^{(1)}$ and $X_T^{(2)} = Y_T^{(2)}$, and so $q_T^a = r_T^a$ and $q_T^b = r_T^b$. In [106] the following results were obtained.

Theorem 11. *(I) Price Representations. In the above notations,*

$$V^a = \min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \max_{(P,S) \in \overline{\mathcal{P}}(\chi)} \mathbb{E}_P\left(H^{(1)}(\sigma, \cdot) + SH^{(2)}(\sigma, \cdot)\right)_\chi$$

$$= \min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \sup_{(P,S) \in \mathcal{P}(\chi)} \mathbb{E}_P\left(H^{(1)}(\sigma, \cdot) + SH^{(2)}(\sigma, \cdot)\right)_\chi, \tag{92}$$

$$V^b = \max_{\tau \in \mathcal{T}} \min_{\chi \in \mathcal{X}} \min_{(P,S) \in \overline{\mathcal{P}}(\chi)} \mathbb{E}_P\left(H^{(1)}(\cdot, \tau) + SH^{(2)}(\cdot, \tau)\right)_\chi$$

$$= \max_{\tau \in \mathcal{T}} \min_{\chi \in \mathcal{X}} \inf_{(P,S) \in \mathcal{P}(\chi)} \mathbb{E}_P\left(H^{(1)}(\cdot, \tau) + SH^{(2)}(\cdot, \tau)\right)_\chi, \tag{93}$$

where $H^{(1)}(\sigma, \cdot)$, $H^{(2)}(\sigma, \cdot)$ and $H^{(1)}(\cdot, \tau)$, $H^{(2)}(\cdot, \tau)$ denote functions on $\{0, 1, \ldots, T\}$ whose values at t are obtained by replacing $\cdot$ by t.

(II) Recurrent Price Computations

(i) *For any $x \in \mathbb{R}$, $\mu \in \Omega_T$ and $\sigma \in \mathcal{T}$ define*

$$z_T^\mu(x) = w_T^\mu(x) = r_T^a(\mu, x). \tag{94}$$

Next, for $t = 1, 2, \ldots, T$ and each $\mu \in \Omega_{t-1}$ define by backward induction

$$\mathbf{z}_{t-1}^\mu = \max_{\nu \subset \mu, \nu \in \Omega_t} z_t^\nu,$$

$$w_{t-1}^\mu = \mathrm{gr}_{[S_{t-1}^b(\mu), S_{t-1}^a(\mu)]}\left(\mathbf{z}_{t-1}^\mu\right),$$

$$z_{t-1}^\mu(x) = \min\left(q_{t-1}^a(\mu, x), \max\left(r_{t-1}^a(\mu, x), w_{t-1}^\mu(x)\right)\right). \tag{95}$$

Then $z_0(0) = V^a$.

(ii) *For any $x \in \mathbb{R}$, $\mu \in \Omega_T$ and $\tau \in \mathcal{T}$ define*

$$u_T^\mu(x) = v_T^\mu(x) = r_T^b(\mu, x). \tag{96}$$

Next, for $t = 1, 2, \ldots, T$ and each $\mu \in \Omega_{t-1}$ define by the backward induction

$$\mathbf{u}_{t-1}^\mu = \max_{\nu \subset \mu, \nu \in \Omega_t} u_t^\nu, \qquad v_{t-1}^\mu = \mathrm{gr}_{[S_{t-1}^b(\mu), S_{t-1}^a(\mu)]}\left(\mathbf{u}_{t-1}^\mu\right),$$

$$u_{t-1}^\mu(x) = \min\left(r_{t-1}^b(\mu, x), \max\left(q_{t-1}^b(\mu, x), v_{t-1}^\mu(x)\right)\right). \tag{97}$$

Then $u_0(0) = -V^b$.

(III) Superhedging Strategies

(i) *Construct by induction a sequence of (pure) stopping times $\sigma_t \in \mathcal{T}$ and a self-financing strategy (α, β) such that*

$$(\alpha_t, \beta_t) \in \mathrm{epi}(z_t) \setminus \mathrm{epi}(q_t^a) \quad on \quad \{t < \sigma_t\} \tag{98}$$

for each $t = 0, 1, \ldots, T$ in the following way. First, take any $\mathcal{F}_0$-measurable portfolio $(\alpha_0, \beta_0) \in \mathrm{epi}(z_0)$ and set

$$\sigma_0 = \begin{cases} 0 & if \;\; (\alpha_0, \beta_0) \in \mathrm{epi}(q_0^a), \\ T & if \;\; (\alpha_0, \beta_0) \notin \mathrm{epi}(q_0^a). \end{cases} \tag{99}$$

Suppose that an $\mathcal{F}_t$-measurable portfolio $(\alpha_t, \beta_t) \in \mathrm{epi}(z_t)$ and a stopping time $\sigma_t \in \mathcal{T}$ have already been constructed for some $t = 0, 1, \ldots, T - 1$ so that (98) holds true. By (86) and (96),

$$(\alpha_t, \beta_t) \in \mathrm{epi}(w_t) = \mathrm{epi}\left(h_{[S_t^b, S_t^a]}\right) + \mathrm{epi}(\mathbf{z}_t) \quad on \quad \{t < \sigma_t\} \tag{100}$$

and so there exists an $\mathcal{F}_t$-measurable portfolio $(\alpha_{t+1}, \beta_{t+1})$ such that

$$(\alpha_{t+1}, \beta_{t+1}) \in \mathrm{epi}(\mathbf{z}_t),$$

$$(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1}) \in \mathrm{epi}\left(h_{[S_t^b, S_t^a]}\right) \quad on \quad \{t < \sigma_t\}, \tag{101}$$

and $(\alpha_{t+1}, \beta_{t+1}) = (\alpha_t, \beta_t)$ on $\{t \geq \sigma_t\}$ which provides the self-financing condition (85) both on $\{t < \sigma_t\}$ and on $\{t \geq \sigma_t\}$. By (96) it follows also that $(\alpha_{t+1}, \beta_{t+1}) \in \mathrm{epi}(z_{t+1})$ on $\{t < \sigma_t\} \supset \{t + 1 < \sigma_{t+1}\}$. Set

$$\sigma_{t+1} = \begin{cases} \sigma_t & if \; t \geq \sigma_t, \\ t+1 & if \; t < \sigma_t \;\; and \;\; (\alpha_{t+1}, \beta_{t+1}) \in \mathrm{epi}(q_{t+1}^a), \\ T & if \; t < \sigma_t \;\; and \;\; (\alpha_{t+1}, \beta_{t+1}) \notin \mathrm{epi}(q_{t+1}^a). \end{cases} \tag{102}$$

Finally, set $\sigma = \sigma_T \in \mathcal{T}$. Then the pair (σ, π) with $\pi = (\alpha, \beta)$ constructed by the above algorithm with $(\alpha_0, \beta_0) = (V^a, 0)$ is a superhedging strategy for the seller.

(ii) *Construct by induction a sequence of (pure) stopping times $\tau_t \in \mathcal{T}$ and a self-financing strategy (α, β) such that*

$$(\alpha_t, \beta_t) \in \mathrm{epi}(u_t) \setminus \mathrm{epi}\left(r_t^b\right) \quad on \quad \{t < \tau_t\} \tag{103}$$

for each $t = 0, 1, \ldots, T$ in the following way. First, take any $\mathcal{F}_0$-measurable portfolio $(\alpha_0, \beta_0) \in \mathrm{epi}(u_0)$ and set

$$\tau_0 = \begin{cases} 0 & if \;\; (\alpha_0, \beta_0) \in \mathrm{epi}\left(r_0^b\right), \\ T & if \;\; (\alpha_0, \beta_0) \notin \mathrm{epi}\left(r_0^b\right). \end{cases} \tag{104}$$

Suppose that an $\mathscr{F}_t$-measurable portfolio $(\alpha_t, \beta_t) \in$ epi (u_t) and a stopping time $\tau_t \in \mathscr{T}$ have already been constructed for some $t = 0, 1, \ldots, T - 1$ so that (98) holds true. By (86) and (98),

$$(\alpha_t, \beta_t) \in \text{epi}(v_t) = \text{epi}\left(h_{[S_t^b, S_t^a]}\right) + \text{epi}(\mathbf{u}_t) \ \text{ on } \ \{t < \tau_t\},$$
(105)

and so there exists an $\mathscr{F}_t$-measurable portfolio $(\alpha_{t+1}, \beta_{t+1})$ such that

$$(\alpha_{t+1}, \beta_{t+1}) \in \text{epi}(\mathbf{u}_t),$$

$$(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1}) \in \text{epi}\left(h_{[S_t^b, S_t^a]}\right) \ \text{ on } \ \{t < \tau_t\},$$
(106)

and $(\alpha_{t+1}, \beta_{t+1}) = (\alpha_t, \beta_t)$ on $\{t \geq \tau_t\}$ which provides the self-financing condition (85) both on $\{t < \tau_t\}$ and on $\{t \geq \tau_t\}$. By (98) it follows also that $(\alpha_{t+1}, \beta_{t+1}) \in$ epi(u_{t+1}) on $\{t < \tau_t\} \supset \{t + 1 < \tau_{t+1}\}$. Set

$$\tau_{t+1} = \begin{cases} \tau_t & \text{if } t \geq \tau_t, \\ t + 1 & \text{if } t < \tau_t \text{ and } (\alpha_{t+1}, \beta_{t+1}) \in \text{epi}\left(r_{t+1}^b\right), \\ T & \text{if } t < \tau_t \text{ and } (\alpha_{t+1}, \beta_{t+1}) \notin \text{epi}\left(r_{t+1}^b\right). \end{cases}$$
(107)

Finally, set $\tau = \tau_T \in \mathscr{T}$. Then the pair (τ, π) with $\pi = (\alpha, \beta)$ constructed by the above algorithm with $(\alpha_0, \beta_0) = (-V^b, 0)$ is a superhedging strategy for the buyer.

There are by now very few papers on game options with transaction costs. In [107] it is shown that the cheapest superhedging strategy for a game option in a Black-Scholes market with transaction costs is the buy-and-hold portfolio strategy together with a hitting time of a Borel set. The shortfall risk for a game option in a Black-Scholes market with transaction costs is obtained in [108] as a limit of corresponding expressions for a sequence of binomial models in the spirit of Section 4.

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