

DYSON'S CRANK OF A PARTITION

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1. Introduction. In [3], F. J. Dyson defined the rank of a partition as the largest part minus the number of parts. He let $N(m, t, n)$ denote the number of partitions of n of rank congruent to m modulo t , and he conjectured

$$(1.1) \quad N(m, 5, 5n + 4) = \frac{1}{5}p(5n + 4), \quad 0 \leq m \leq 4;$$

$$(1.2) \quad N(m, 7, 7n + 5) = \frac{1}{7}p(7n + 5), \quad 0 \leq m \leq 6,$$

where $p(n)$ is the total number of partitions of n [1, Chapter 1]. These conjectures were subsequently proved by Atkin and Swinnerton-Dyer [2].

Dyson [3] went on to observe that the rank did not separate the partition of $11n + 6$ into 11 equal classes even though Ramanujan's congruence

$$(1.3) \quad p(11n + 6) \equiv 0 \pmod{11}$$

holds. He was thus led to conjecture the existence of some other partition statistic (which he called the crank); this unknown statistic should provide a combinatorial interpretation of $\frac{1}{11}p(11n + 6)$ in the same way that (1.1) and (1.2) treat the primes 5 and 7.

In [4, 5], one of us was able to find a crank relative to vector partitions as follows:

For a partition π , let $\#(\pi)$ be the number of parts of π and $\sigma(\pi)$ be the sum of the parts of π (or the number π is partitioning) with the convention $\#(\phi) = \sigma(\phi) = 0$ for the empty partition ϕ , of 0. Let

$$V = \{(\pi_1, \pi_2, \pi_3) \mid \pi_1 \text{ is a partition into distinct parts,} \\ \pi_2, \pi_3 \text{ are unrestricted partitions}\}.$$

We shall call the elements of V *vector partitions*. For $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ in V we define the sum of parts, s , a weight, ω , and a crank, r , by

$$(1.4) \quad s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3),$$

$$(1.5) \quad \omega(\vec{\pi}) = (-1)^{\#(\pi_1)},$$

$$(1.6) \quad r(\vec{\pi}) = \#(\pi_2) - \#(\pi_3).$$

We say $\vec{\pi}$ is a vector partition of n if $s(\vec{\pi}) = n$. For example, if

$$\vec{\pi} = (5 + 3 + 2, 2 + 2 + 1, 2 + 1 + 1)$$

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then $s(\vec{\pi}) = 19$, $\omega(\vec{\pi}) = -1$, $r(\vec{\pi}) = 0$ and $\vec{\pi}$ is a vector partition of 19. The number of vector partitions of n (counted according to the weight ω) with crank m is denoted by $N_V(m, n)$, so that

$$(1.7) \quad N_V(m, n) = \sum_{\substack{\vec{\pi} \in V \\ s(\vec{\pi})=n \\ r(\vec{\pi})=m}} \omega(\vec{\pi}).$$

The number of vector partitions of n (counted according to the weight ω) with crank congruent to k modulo t is denoted by $N_V(k, t, n)$, so that

$$(1.8) \quad N_V(k, t, n) = \sum_{m=-\infty}^{\infty} N_V(mt + k, n) = \sum_{\substack{\vec{\pi} \in V \\ s(\vec{\pi})=n \\ r(\vec{\pi}) \equiv k \pmod{t}}} \omega(\vec{\pi}).$$

By considering the transformation that interchanges π_2 and π_3 we have

$$(1.9) \quad N_V(m, n) = N_V(-m, n)$$

so that

$$(1.10) \quad N_V(t - m, t, n) = N_V(m, t, n).$$

We have the following generating function for $N_V(m, n)$:

$$(1.11) \quad \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - zq^n)(1 - z^{-1}q^n)}.$$

By putting $z = 1$ in (1.11) we find

$$(1.12) \quad \sum_{m=-\infty}^{\infty} N_V(m, n) = p(n).$$

VECTOR-CRANK THEOREM (GARVAN [4, 5]).

$$(1.13)$$

$$N_V(0, 5, 5n + 4) = N_V(1, 5, 5n + 4) = \dots = N_V(4, 5, 5n + 4) = \frac{p(5n + 4)}{5},$$

$$(1.14)$$

$$N_V(0, 7, 7n + 5) = N_V(1, 7, 7n + 5) = \dots = N_V(6, 7, 7n + 5) = \frac{p(7n + 5)}{7},$$

$$(1.15)$$

$$N_V(0, 11, 11n + 6) = \dots = N_V(10, 11, 11n + 6) = \frac{p(11n + 6)}{11}.$$

The above still leaves open the question of whether there is a crank for ordinary partitions. The answer is “yes” when the crank is defined as follows:

DEFINITION. For a partition π , let $l(\pi)$ denote the largest part of π , $\omega(\pi)$ denote the number of ones in π , and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The crank $c(\pi)$ is given by

$$c(\pi) = \begin{cases} l(\pi) & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

Our main result is the following.

THEOREM 1. *The number of partitions π of n with $c(\pi) = m$ is $N_V(m, n)$ for all $n > 1$.*

Obviously, in light of the Vector-Crank Theorem, we see that Theorem 1 supplies the crank asked for by Dyson.

2. PROOF OF THEOREM 1. We shall require the standard notation of q -series:

$$(2.1) \quad (A; q)_n = (A)_n = \prod_{j=0}^{\infty} \frac{(1 - Aq^j)}{(1 - Aq^{j+n})}$$

(= $(1 - A)(1 - Aq) \cdots (1 - Aq^{n-1})$ when n is a positive integer), and

$$(2.2) \quad (A; q)_\infty = (A)_\infty = \prod_{j=0}^{\infty} (1 - Aq^j).$$

We now transform (1.11):

$$(2.3) \quad \begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) z^m q^n &= \frac{(1 - q)}{(zq)_\infty} \cdot \frac{(q^2; q)_\infty}{(q/z)_\infty} \\ &= \frac{(1 - q)}{(zq)_\infty} \sum_{j=0}^{\infty} \frac{(zq)_j (q/z)^j}{(q)_j} \quad (\text{by [1, p. 17]}) \\ &= \frac{(1 - q)}{(zq)_\infty} + \sum_{j=1}^{\infty} \frac{q^j z^{-j}}{(q^2; q)_{j-1} (zq^{j+1})_\infty}. \end{aligned}$$

As was noted in (1.12), when we set $z = 1$ the series on the left of (2.3) reduces to the generating function for $p(n)$. For $j > 0$, the j th term in the sum on the right is

$$\frac{z^{-j} \overbrace{q^{1+1+\cdots+1}}^{j \text{ times}}}{(1 - q^2)(1 - q^3) \cdots (1 - q^j)(1 - zq^{j+1})(1 - zq^{j+2}) \cdots}.$$

The standard techniques of partition theory [1, Chapter 1] show that this expression generates partitions with $\omega(\pi) = j$ and the exponent on z is clearly $\mu(\pi) - \omega(\pi)$, i.e. $c(\pi)$, since $j > 0$.

Thus we must interpret

$$\frac{(1 - q)}{(1 - zq)(1 - zq^2)(1 - zq^3) \cdots}$$

as the generating function for partitions without ones. By considering conjugate partitions, we note that

$$\frac{1}{(1 - zq)(1 - zq^2)(1 - zq^3) \cdots}$$

generates all partitions with the exponent on z counting the largest part, and for integers larger than 1

$$\frac{q}{(1-zq)(1-zq^2)(1-zq^3)\cdots}$$

generates partitions with at least one 1 appearing again with the exponent on z counting the largest part. Note that this interpretation fails for 1 because this is the unique instance in which introducing a 1 into the partitions of $n-1$ alters the largest part. Hence

$$\frac{1-q}{(zq)_\infty}$$

counts (for $n > 1$) the number of partitions with no ones and with the exponent on z being the largest part of the partition $l(\pi) = c(\pi)$. Thus in the double series expansion of

$$\frac{(1-q)}{(zq)_\infty} + \sum_{j=1}^{\infty} \frac{q^j z^{-j}}{(q^2; q)_{j-1} (zq^{j+1})_\infty}$$

we see that the coefficient of $z^m q^n$ ($n > 1$) is the number of ordinary partitions of n in which $c(\pi) = m$. Therefore by (2.3), we have Theorem 1. \square

3. Conclusion. We can't resist exhibiting $c(\pi)$ for the first instance of (1.3).

partitions of 6	$l(\pi)$	$\omega(\pi)$	$\mu(\pi)$	$c(\pi)$
6	6	0	1	6
5 + 1	5	1	1	0
4 + 2	4	0	2	4
4 + 1 + 1	4	2	1	-1
3 + 3	3	0	2	3
3 + 2 + 1	3	1	2	1
3 + 1 + 1 + 1	3	3	0	-3
2 + 2 + 2	2	0	3	2
2 + 2 + 1 + 1	2	2	0	-2
2 + 1 + 1 + 1 + 1	2	4	0	-4
1 + 1 + 1 + 1 + 1 + 1	1	6	0	-6

As Theorem 1 together with (1.15) predicts, $c(\pi)$ provides eleven different residue classes modulo 11.

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