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$E_{7(7)}$ exceptional field theory in superspace

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Abstract: We formulate the locally supersymmetric $E_{7(7)}$ exceptional field theory in a $(4+56|32)$ dimensional superspace, corresponding to a 4D $N = 8$ "external" superspace augmented with an "internal" 56-dimensional space. This entails the unification of external diffeomorphisms and local supersymmetry transformations into superdiffeomorphisms. The solutions to the superspace Bianchi identities lead to on-shell duality equations for the p-form field strengths for $p \leq 4$. The reduction to component fields provides a complete description of the on-shell supersymmetric theory. As an application of our results, we perform a generalized Scherk-Schwarz reduction and obtain the superspace formulation of maximal gauged supergravity in four dimensions parametrized by an embedding tensor.

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Contents

1 Introduction

It is well known that compactifications of 11-dimensional supergravity on an n-torus give rise to an enhancement of the manifest $\mathrm{SL}(n, \mathbb{R})$ symmetry to symmetries including the exceptional groups $E_{n(n)}$ [1, 2], and that their suitable discrete subgroups are interpreted as the U-duality symmetries of M-theory [3]. The search for a manifest origin of these symmetries in 11-dimensions prior to any toroidal compactification and without any truncation, which started in [4, 5],[1] has culminated in a series of papers [7–10] where this was achieved in a framework called exceptional field theory (ExFT). It is based on a generalization to exceptional geometry [11–15] of the double field theories (DFT) that provide a manifest realization of the T-duality group $O(n, n)$ that arises in toroidal compactification of string theory [16–18]. In that case the 10-dimensional spacetime coordinates are doubled and certain conditions on fields known as section constraints are imposed. The latter are required for the symmetries to form a closed algebra and, in effect, remove dependence on coordinates beyond ten dimensions. For a more detailed description of the ideas behind these theories, with several references to earlier works, see [7].

Exceptional field theories are well motivated for a number of reasons. Firstly, they have made it possible to derive fully nonlinear and consistent reductions to gauged supergravities in lower dimensions. For example the long standing problem of finding the nonlinear and consistent reduction of Type IIB supergravity on $AdS_5 \times S^5$ was solved in this way [19]. Second, exceptional field theory provides a convenient framework for taking into account the BPS states in the computation of loop corrections to the string low energy effective action [20]. Furthermore, higher derivative corrections to the supergravity limit of string/M-theory may be powerfully tackled by employing the DFT/ExFT in which the U-duality symmetry is manifestly realized. For the case of DFT, see [21] and several references therein for earlier work. Last but not least, the generalized geometry underlying exceptional field theories may pave the way to the construction of effective actions that genuinely go beyond 11D supergravity, thereby shedding light on important aspects of M-theory.

In this paper, we shall focus on the exceptional field theory based on $\mathrm{E}_{7(7)}$ [9], and starting from its supersymmetric extension provided in [22], we formulate the theory in $(4 + 56|32)$ dimensional superspace. One of our main motivations is the construction of actions for M-branes propagating in a target space described by the generalized geometry of exceptional field theory. This problem is still open, though progress has been made in the form of exceptional sigma models for string theory [23]. The importance of a superspace formulation of target space supergravities becomes especially clear with the realization that all known actions for branes beyond strings are feasible only as sigma models in which the target is a superspace. Another motivation for the exploration of supergeometry in exceptional field theories is to find clues in the search for an extended geometrical framework which would unify the external (spacetime) and internal space diffeomorphisms.

Our approach to the superspace formulation of the supersymmetric $\mathrm{E}_{7(7)}$ ExFT is to elevate the 4-dimensional "external" spacetime to $(4|32)$ dimensional "external" superspace[2] and to augment this with a 56-dimensional "internal" space. As such, the external diffeomorphisms and local supersymmetry transformations of ExFT are unified to external superdiffeomorphisms with structure group $\mathrm{GL}(4|32)$, with $\mathrm{E}_{7(7)}$ internal diffeomorphisms

[1]See also [2, 6] where related conjectures were made.

[2]Ungauged 4D $N = 8$ superspace was constructed in [24, 25], see also [26, 27]. Our construction will reduce to this upon discarding all dependence on internal coordinates.

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treated separately. In particular, there are separate (super)vielbeins for the two spaces. This is in contrast to early work involving so-called "central charge superspace" [28] where the vielbeins were unified into a single sehrvielbein but with all fields independent of the additional 56 coordinates,[3] as well as more recent efforts in superspace double field theory where a unified description is sought (see e.g. [30–33]).

Our approach turns out to require more than just a superspace lift of [22]. We find that it is important to redefine a constrained two-form of the theory, so that it transforms inhomogeneously under Lorentz transformations. This allows one to eliminate the internal part of the Lorentz spin connection everywhere, with the constrained two-form now playing its role. Another important step is the relaxation of the constraints imposed on the $E_{7(7)}$ connection $\Gamma_{\underline{mn}}{}^{\underline{p}}$ in [22]. Recall that these constraints amounted to (i) the elimination of non-metricity of the internal generalized vielbein postulate; (ii) the vanishing of the $E_{7(7)}$ torsion tensor; and (iii) requiring that the 4D volume form be covariantly constant, $\nabla_{\underline{m}} e = 0$. Here we will find it convenient to relax all of them, and to take a completely generic internal $E_{7(7)}$ connection. Naturally, this is consistent only if the undetermined pieces drop out of the supersymmetry transformations, which we will show.

We also probe further the sector of the theory that involves extra 3-form and 4-form potentials within the framework of the tensor hierarchy formalism. In particular, we show that the solutions to the superspace Bianchi identities lead to on-shell duality equations for the p-form field strengths for $p \leq 4$. We also show that the reduction to component fields provides a complete description of the on-shell supersymmetric theory, including the higher order fermion terms. As an application of our results, we perform a generalized Scherk-Schwarz reduction and obtain the superspace formulation of maximal gauged supergravity in four dimensions parametrized by an embedding tensor.

The paper is organized as follows. In section 2, we review the locally supersymmetric $E_{7(7)}$ exceptional field theory in components. In section 3 we lay the groundwork for the superspace formulation, in particular describing the required redefinition of a constrained 2-form potential, and its consequences. In section 4, we describe the superspace formulation, including the superspace Bianchi identities and their solutions. In section 5, we present the component results, establishing that they agree with the component formulation of [22] subject to the redefinition of the 2-form potential. In section 6, we perform a generalized Scherk-Schwarz reduction and obtain the superspace formulation of maximal gauged supergravity in four dimensions parametrized by an embedding tensor. In section 7 we comment further on our results and point out future directions. In appendix A, we give some details of our conventions. Appendix B contains some technical details of the algebra of external and internal covariant derivatives that we found useful in explicit computations.

2 Supersymmetric $E_{7(7)}$ exceptional field theory in components

Let us begin by reviewing the structure of the $E_{7(7)}$-covariant ExFT, first in its original bosonic formulation [9] and then its supersymmetrized extension [22]. The bosonic field

[3] For a discussion of how to derive the $E_{7(7)}$ section condition from a superparticle moving in central charge superspace, see [29].

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content is given by

$$\left\{e_m{}^a\,,\mathcal{V}_{\underline{m}}{}^{\underline{a}}\,,A_m{}^{\underline{m}}\,,B_{mn\,\boldsymbol{\alpha}}\,,B_{mn\,\underline{m}}\right\}\quad. \tag{2.1}$$

The vierbein $e_m{}^a$ describes the geometry of external 4D spacetime, while the 56-bein $\mathcal{V}_{\underline{m}}{}^{\underline{a}}$, parametrizing the coset $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$, describes the internal geometry. The 1-form $A_m{}^{\underline{m}}$ gauges internal diffeomorphisms on external spacetime and lies in the fundamental (**56**) of $\mathrm{E}_{7(7)}$. Requiring closure of internal diffeomorphisms on the 1-form requires the existence of 2-forms $B_{mn\,\boldsymbol{\alpha}}$ and $B_{mn\,\underline{m}}$ valued respectively in the adjoint (**133**) and fundamental (**56**) representations. The internal tangent space index $\underline{a}$ on the 56-bein decomposes under SU(8) as $\mathbf{28}+\overline{\mathbf{28}}$,

$$\mathcal{V}_{\underline{m}}{}^{\underline{a}}=\{\mathcal{V}_{\underline{m}}{}^{ij},\mathcal{V}_{\underline{m}ij}\}\,, \tag{2.2}$$

satisfying $\mathcal{V}_{\underline{m}ij}=(\mathcal{V}_{\underline{m}}{}^{ij})^*$ with SU(8) indices $i,j,\dots=1,\dots,8$.

All fields in the theory, including the symmetry transformation parameters that will be encountered below, depend on both external (x^m) and internal ($y^{\underline{m}}$) coordinates, with the dependence on the latter subject to the section conditions. We write these as

$$(t_{\boldsymbol{\alpha}})^{\underline{mn}}\,\partial_{\underline{m}}\otimes\partial_{\underline{n}}\;=\;0\,,\qquad \Omega^{\underline{mn}}\,\partial_{\underline{m}}\otimes\partial_{\underline{n}}=0 \tag{2.3}$$

where the derivatives are understood to act on any two (or the same) fields or parameters. Here $(t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}$ are the $\mathrm{E}_{7(7)}$ generators in the fundamental representation, $\Omega^{\underline{mn}}$ is the invariant symplectic form of $\mathrm{E}_{7(7)}\subset\mathrm{Sp}(56)$, and we employ the usual (NW-SE) conventions for raising and lowering **56** indices, e.g. $(t_{\boldsymbol{\alpha}})^{\underline{mn}}=\Omega^{\underline{mp}}\,(t_{\boldsymbol{\alpha}})_{\underline{p}}{}^{\underline{n}}$ and $(t_{\boldsymbol{\alpha}})_{\underline{mn}}=(t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{p}}\,\Omega_{\underline{pn}}$.

In addition, the field $B_{mn\,\underline{m}}$ is constrained on its internal index so that it obeys the section condition with respect to both $\partial_{\underline{m}}$ and itself, i.e.

$$\begin{aligned}(t_{\boldsymbol{\alpha}})^{\underline{mn}}\,B_{\underline{m}}\otimes\partial_{\underline{n}}&=0\,, &\qquad \Omega^{\underline{mn}}\,B_{\underline{m}}\otimes\partial_{\underline{n}}&=0\,,\\ (t_{\boldsymbol{\alpha}})^{\underline{mn}}\,B_{\underline{m}}B_{\underline{n}}&=0\,, &\qquad \Omega^{\underline{mn}}\,B_{\underline{m}}B_{\underline{n}}&=0\;.\end{aligned} \tag{2.4}$$

In the first set of equations, the derivative may act on another field or on $B_{mn\,\underline{m}}$ itself.

In principle, 3-forms and 4-forms are also required for a complete description of the tensor hierarchy, but these drop out of the action, and so one can usually avoid any explicit discussion of their properties. Nevertheless, we will find it useful to discuss them briefly in a few places. The 3-forms are $C_{mnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}}$ and $C_{mnp\,\underline{m}}{}^{\underline{n}}$, with the former valued in the **912** and the latter constrained on its lower index. The unconstrained 4-forms are $D_{mnpq\,\boldsymbol{\alpha}}$ and $D_{mnpq}{}^{\underline{mn}}{}_{\boldsymbol{\alpha}}$, respectively in the **133** and in the **8645**, while there appear to be as many as three constrained 4-forms $D_{mnpq\,\underline{m}}$, $D_{mnpq\,\underline{m}}{}^{\boldsymbol{\alpha}}$, and $D_{mnpq\,\underline{m}}{}^{\underline{np}}$, each obeying the section condition on their lower index $\underline{m}$, with the last field constrained in the **1539** in its upper indices.

2.1 Generalized vielbein postulates

For later purposes, we record the generalized vielbein postulates (GVP) satisfied by external and internal vielbeins:[4]

$$
\begin{aligned}
0 &= \partial_m e_n{}^a - A_m{}^{\underline{n}} \partial_{\underline{n}} e_n{}^a - \frac{1}{2}\, \partial_{\underline{n}} A_m{}^{\underline{n}}\, e_n{}^a - \omega_m{}^{ab} e_{nb} - \Gamma_{mn}{}^p e_p{}^a \\
&= D_m e_n{}^a - \omega_m{}^{ab} e_{nb} - \Gamma_{mn}{}^p e_p{}^a\,, \qquad (2.5)\\
0 &= \partial_{\underline{m}} e_n{}^a - \frac{1}{3}\, \Gamma_{\underline{km}}{}^{\underline{k}}\, e_n{}^a - \omega_{\underline{m}}{}^{ab} e_{nb} + \pi_{\underline{m}}{}^{ab} e_{nb}\,, \qquad (2.6)\\
0 &= D_m \mathcal{V}_{\underline{n}}{}^{\underline{a}} - \mathcal{V}_{\underline{n}}{}^{\underline{b}} \mathcal{Q}_{m\underline{b}}{}^{\underline{a}} - \mathcal{V}_{\underline{n}}{}^{\underline{b}} \mathcal{P}_{m\underline{b}}{}^{\underline{a}} \qquad (2.7)\\
0 &= \partial_{\underline{m}} \mathcal{V}_{\underline{n}}{}^{\underline{a}} - \mathcal{V}_{\underline{n}}{}^{\underline{b}} \mathcal{Q}_{\underline{mb}}{}^{\underline{a}} - \Gamma_{\underline{mn}}{}^{\underline{p}}\, \mathcal{V}_{\underline{p}}{}^{\underline{a}}\,. \qquad (2.8)
\end{aligned}
$$

The connections $\Gamma_{mn}{}^p$ and $\pi_{\underline{m}}{}^{ab} = \pi_{\underline{m}}{}^{(ab)}$ are defined by (2.5) and (2.6), and $\mathcal{Q}$ and $\mathcal{P}$ live in SU(8) and its orthogonal complement in $\mathrm{E}_{7(7)}$, respectively, so that

$$
\mathcal{Q}_{m\underline{b}}{}^{\underline{a}} = \begin{pmatrix} \delta_{[k}{}^{[i} \mathcal{Q}_{ml]}{}^{j]} & 0 \\ 0 & -\delta_{[i}{}^{[k} \mathcal{Q}_{mj]}{}^{l]} \end{pmatrix}, \qquad \mathcal{P}_{m\underline{b}}{}^{\underline{a}} = \begin{pmatrix} 0 & \mathcal{P}_m{}^{klij} \\ \mathcal{P}_{mklij} & 0 \end{pmatrix} \qquad (2.9)
$$

and similarly for $\mathcal{Q}_{\underline{m}}$ and $\mathcal{P}_{\underline{m}}$. The $\mathrm{E}_{7(7)}$ covariant derivative D_m is defined as

$$
D_m := \partial_m - \mathbb{L}_{A_m}\,, \qquad (2.10)
$$

where the generalized Lie derivative acts on a fundamental vector $V^{\underline{m}}$ of weight $\lambda(V^{\underline{m}})$ as

$$
\mathbb{L}_\Lambda V^{\underline{m}} \;:=\; \Lambda^{\underline{n}} \partial_{\underline{n}} V^{\underline{m}} - 12\, \big[\partial_{\underline{n}} \Lambda^{\underline{m}}\big]_{\mathrm{adj}}\, V^{\underline{n}} + \lambda(V)\, \partial_{\underline{n}} \Lambda^{\underline{n}}\, V^{\underline{m}} \qquad (2.11)
$$

where the second term is projected onto the adjoint of $\mathrm{E}_{7(7)}$, i.e.

$$
\big[\partial_{\underline{n}} \Lambda^{\underline{m}}\big]_{\mathrm{adj}} := \mathbb{P}^{\underline{m}}{}_{\underline{n}}{}^{\underline{r}}{}_{\underline{s}}\; \partial_{\underline{r}} \Lambda^{\underline{s}}\,, \qquad (2.12)
$$

with the adjoint projector given by

$$
\mathbb{P}^{\underline{m}}{}_{\underline{n}}{}^{\underline{r}}{}_{\underline{s}} = (t^{\boldsymbol{\alpha}})^{\underline{m}}{}_{\underline{n}}\, (t_{\boldsymbol{\alpha}})^{\underline{r}}{}_{\underline{s}} = \frac{1}{24} \delta^{\underline{m}}{}_{\underline{n}}\, \delta^{\underline{r}}{}_{\underline{s}} + \frac{1}{12} \delta^{\underline{m}}{}_{\underline{s}}\, \delta^{\underline{r}}{}_{\underline{n}} + (t^{\boldsymbol{\alpha}})^{\underline{mr}}\, (t_{\boldsymbol{\alpha}})_{\underline{ns}} - \frac{1}{24} \Omega^{\underline{mr}}\, \Omega_{\underline{ns}}\,. \qquad (2.13)
$$

We emphasize that the symplectic metric $\Omega^{\underline{mn}}$ is used to raise and lower the 56-plet indices as $V^{\underline{m}} = \Omega^{\underline{mn}} V_{\underline{n}}$ and $V_{\underline{n}} = V^{\underline{m}} \Omega_{\underline{mn}}$, and it is an invariant tensor of weight 0, namely $\mathbb{L}_\Lambda \Omega^{\underline{mn}} = 0$.

Defining the generalized torsion tensors as

$$
\Gamma_{[mn]}{}^p = \frac{1}{2} T_{mn}{}^p\,, \qquad (2.14)
$$

$$
\Gamma_{\underline{mn}}{}^{\underline{k}} - 12\, \mathbb{P}^{\underline{k}}{}_{\underline{n}}{}^{\underline{r}}{}_{\underline{s}}\, \Gamma_{\underline{rm}}{}^{\underline{s}} + 4\, \mathbb{P}^{\underline{k}}{}_{\underline{n}}{}^{\underline{r}}{}_{\underline{m}}\, \Gamma_{\underline{sr}}{}^{\underline{s}} = \mathcal{T}_{\underline{mn}}{}^{\underline{k}}\,, \qquad (2.15)
$$

[4] Our convention for the spin connection matches that of [34], which differs in sign from that used in the previous $\mathrm{E}_{7(7)}$ papers [9, 22].

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the following constraints are imposed in [22]

$$T_{mn}{}^{p} = 0\,, \qquad \mathcal{T}_{\underline{mn}}{}^{\underline{k}} = 0\ . \tag{2.16}$$

The definition of generalized torsion $\mathcal{T}_{\underline{mn}}{}^{\underline{k}}$ is motivated by the relation

$$\mathbb{L}^{\nabla}_{V} W^{\underline{m}} - \mathbb{L}_{V} W^{\underline{m}} = \mathcal{T}_{\underline{nk}}{}^{\underline{m}}\, V^{\underline{n}} W^{\underline{k}}\,, \tag{2.17}$$

for vectors V, W of weight $\frac{1}{2}$ where $\mathbb{L}^{\nabla}_{V}$ denotes the generalized Lie derivative with all partial derivatives replaced by covariant derivatives. Explicit evaluation of this relation gives the expression (2.15).

The connection $\Gamma_{\underline{mn}}{}^{\underline{p}}$ lives in the algebra $\mathfrak{e}_{7(7)}$ and as such we can write

$$\Gamma_{\underline{mn}}{}^{\underline{p}} = \Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}} (t_{\boldsymbol{\alpha}})_{\underline{n}}{}^{\underline{p}} \;\in\; \mathfrak{e}_{7(7)}\ . \tag{2.18}$$

Using this relation in (2.15), one finds that

$$\frac{1}{7}\mathcal{T}_{\underline{m}}{}^{\boldsymbol{\alpha}} = \mathbb{P}_{(912)\underline{m}}{}^{\boldsymbol{\alpha}\,\underline{n}}{}_{\boldsymbol{\beta}}\; \Gamma_{\underline{n}}{}^{\boldsymbol{\beta}}\,, \tag{2.19}$$

where the projector onto the **912** dimensional representation is given by [35]

$$\mathbb{P}_{(912)\underline{m}}{}^{\boldsymbol{\alpha}\,\underline{n}}{}_{\boldsymbol{\beta}} := \frac{1}{7}\left(\delta_{\underline{m}}{}^{\underline{n}}\,\delta^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} + 4(t^{\boldsymbol{\alpha}} t_{\boldsymbol{\beta}})_{\underline{m}}{}^{\underline{n}} - 12(t_{\boldsymbol{\beta}} t^{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\right)\ . \tag{2.20}$$

We have used the notation $(t_{\boldsymbol{\alpha}} t_{\boldsymbol{\beta}})_{\underline{m}}{}^{\underline{n}} \equiv (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{k}}\, (t_{\boldsymbol{\beta}})_{\underline{k}}{}^{\underline{n}}$.

2.2 Bosonic symmetries, duality equations and tensor hierarchy

The full bosonic theory is invariant under the generalized diffeomorphisms, vector, and tensor gauge symmetries, and shift symmetries with parameters (ξ^{m}, $\Lambda^{\underline{m}}$, $\Xi_{m\,\boldsymbol{\alpha}}$, $\Xi_{m\,\underline{m}}$, $\Omega_{mn\,\underline{m}}{}^{\underline{n}}$, $\Omega_{mn}{}^{\underline{n}}{}_{\boldsymbol{\alpha}}$), respectively. These transformations are given by

$$\begin{aligned}
\delta e_{m}{}^{a} &= \xi^{n} D_{n} e_{m}{}^{a} + D_{m}\xi^{n} e_{n}{}^{a} + \mathbb{L}_{\Lambda} e_{m}{}^{a}\,,\\
\delta \mathcal{V}_{\underline{m}}{}^{\underline{a}} &= \xi^{m} D_{m} \mathcal{V}_{\underline{m}}{}^{\underline{a}} + \mathbb{L}_{\Lambda} \mathcal{V}_{\underline{m}}{}^{\underline{a}}\,,\\
\delta A_{m}{}^{\underline{m}} &= \xi^{n}\, \mathcal{F}_{nm}{}^{\underline{m}} + \mathcal{M}^{\underline{mn}}\, g_{mn}\, \partial_{\underline{n}}\xi^{n} + D_{m}\Lambda^{\underline{m}} + 12\, (t^{\boldsymbol{\alpha}})^{\underline{mn}}\, \partial_{\underline{n}}\Xi_{m\,\boldsymbol{\alpha}} + \frac{1}{2}\,\Omega^{\underline{mn}}\,\Xi_{m\,\underline{n}}\,,\\
\Delta B_{mn\,\boldsymbol{\alpha}} &= \xi^{p}\, H_{mnp\,\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{\underline{mn}}\, \Lambda^{\underline{m}} \mathcal{F}_{mn}{}^{\underline{n}} + 2\, D_{[m}\Xi_{n]\boldsymbol{\alpha}}\\
&\quad + \partial_{\underline{m}}\Omega_{mn}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\Omega_{mn\underline{n}}{}^{\underline{m}}\,,\\
\Delta B_{mn\,\underline{m}} &= \xi^{p}\, H_{mnp\,\underline{m}} - 2i\, \varepsilon_{mnpq} g^{qr} D^{p}\left(g_{rs}\partial_{\underline{m}}\xi^{s}\right) + \mathcal{F}_{mn}{}^{\underline{n}}\partial_{\underline{m}}\Lambda_{\underline{n}} - \partial_{\underline{m}}\mathcal{F}_{mn}{}^{\underline{n}}\Lambda_{\underline{n}}\\
&\quad + 2D_{[m}\Xi_{n]\underline{m}} + 48\, (t^{\boldsymbol{\alpha}})_{\underline{n}}{}^{\underline{r}}\left(\partial_{\underline{r}}\partial_{\underline{m}} A_{[m}{}^{\underline{n}}\right)\Xi_{n]\boldsymbol{\alpha}}\\
&\quad - \partial_{\underline{m}}\Omega_{mn\underline{n}}{}^{\underline{n}} - 2\,\partial_{\underline{n}}\Omega_{mn\underline{m}}{}^{\underline{n}}\,,
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}\Delta B_{mn\,\boldsymbol{\alpha}} &:= \delta B_{mn\,\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{\underline{mn}}\, A_{[m}{}^{\underline{m}}\,\delta A_{n]}{}^{\underline{n}}\,, \\ \Delta B_{mn\,\underline{m}} &:= \delta B_{mn\,\underline{m}} + \left(A_{[m}{}^{\underline{n}}\partial_{\underline{m}}\delta A_{n]\underline{n}} - \partial_{\underline{m}}A_{[m}{}^{\underline{n}}\delta A_{n]\underline{n}}\right)\,, \end{aligned} \tag{2.22}$$

and

$$\mathcal{M}_{\underline{mn}} = \mathcal{V}_{\underline{m}\,ij}\mathcal{V}_{\underline{n}}{}^{ij} + \mathcal{V}_{\underline{n}\,ij}\mathcal{V}_{\underline{m}}{}^{ij}\,, \tag{2.23}$$

$$\mathcal{F}_{mn}{}^{\underline{m}} = 2\partial_{[m}A_{n]}{}^{\underline{m}} - [A_m, A_n]_{\mathrm{E}}^{\underline{m}} - 12\,(t^{\boldsymbol{\alpha}})^{\underline{mn}}\,\partial_{\underline{n}}B_{mn\,\boldsymbol{\alpha}} - \frac{1}{2}\,\Omega^{\underline{mn}}\,B_{mn\,\underline{n}}\quad, \tag{2.24}$$

with the E-bracket defined by

$$[A_m, A_n]_{\mathrm{E}}^{\underline{m}} := 2\,A_{[m}{}^{\underline{n}}\partial_{\underline{n}}A_{n]}{}^{\underline{m}} + \frac{1}{2}\left(24\,(t_{\boldsymbol{\alpha}})^{\underline{mn}}(t^{\boldsymbol{\alpha}})_{\underline{pq}} - \Omega^{\underline{mn}}\Omega_{\underline{pq}}\right)\,A_{[m}{}^{\underline{p}}\,\partial_{\underline{n}}A_{n]}{}^{\underline{q}}\,. \tag{2.25}$$

The field strengths $H_{mnp\,\boldsymbol{\alpha}}$ and $H_{mn\,\underline{p}}$ are defined by the Bianchi identity

$$D_{[m}\mathcal{F}_{np]}{}^{\underline{m}} = -4(t^{\boldsymbol{\alpha}})^{\underline{mn}}\,\partial_{\underline{n}}H_{mnp\,\boldsymbol{\alpha}} - \frac{1}{6}\Omega^{\underline{mn}}H_{mnp\,\underline{n}}\,. \tag{2.26}$$

Note that $\mathcal{F}, D_m\mathcal{F}$ and $H_{\boldsymbol{\alpha}}$ transform covariantly under internal diffeomorphisms, while $H_{\underline{m}}$ does not, which is evident from the presence of the non-covariant term $\partial_{\underline{n}}H_{\boldsymbol{\alpha}}$ term in (2.26). From this equation, upon using the definition of $\mathcal{F}$, one finds that

$$\begin{aligned} H_{mnp\,\boldsymbol{\alpha}} &:= 3\,D_{[m}B_{np]\,\boldsymbol{\alpha}} - 3\,(t_{\boldsymbol{\alpha}})_{\underline{mn}}\,A_{[m}{}^{\underline{m}}\partial_n A_{p]}{}^{\underline{n}} + (t_{\boldsymbol{\alpha}})_{\underline{mn}}\,A_{[m}{}^{\underline{m}}[A_n, A_{p]}]_{\mathrm{E}}^{\underline{n}} \\ &\quad - \partial_{\underline{m}}C_{mnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} - (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}C_{mnp\,\underline{n}}{}^{\underline{m}}\,, \end{aligned} \tag{2.27}$$

$$\begin{aligned} H_{mnp\,\underline{m}} &:= \Big[3\,D_mB_{np\,\underline{m}} - 3\left(A_m{}^{\underline{n}}\partial_{\underline{m}}\partial_n A_{p\underline{n}} - \partial_{\underline{m}}A_m{}^{\underline{n}}\partial_n A_{p\underline{n}}\right) + 72\,(t^{\boldsymbol{\alpha}})_{\underline{k}}{}^{\underline{p}}\,\partial_{\underline{m}}\partial_{\underline{p}}A_m]^{\underline{k}}\,B_{np\boldsymbol{\alpha}} \\ &\qquad + \left(A_m{}^{\underline{n}}\partial_{\underline{m}}[A_n, A_p]_{\mathrm{E}\,\underline{n}} - \partial_{\underline{m}}A_m{}^{\underline{n}}[A_n, A_p]_{\mathrm{E}\,\underline{n}}\right)\Big]_{[mnp]} \\ &\quad + \partial_{\underline{m}}C_{mnp\,\underline{n}}{}^{\underline{n}} + 2\,\partial_{\underline{n}}C_{mnp\,\underline{m}}{}^{\underline{n}}\,. \end{aligned} \tag{2.28}$$

The three-forms $C_{mnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}}$ and $C_{mnp\,\underline{n}}{}^{\underline{m}}$ introduced in (2.27) and (2.28) are projected out of the Bianchi identity (2.26) using the section condition.[5] They may be thought of as parametrizing the part of the field strengths $H_{mnp\boldsymbol{\alpha}}$, $H_{mnp\underline{m}}$, which is left undetermined by (2.26) with their presence being necessary for invariance of the curvatures under the higher p-form gauge transformations. The covariant derivatives read explicitly

$$\begin{aligned} D_mB_{np\,\boldsymbol{\alpha}} &= \partial_m B_{np\,\boldsymbol{\alpha}} - A_m{}^{\underline{m}}\,\partial_{\underline{m}}B_{np\,\boldsymbol{\alpha}} - 12\,(t_{\boldsymbol{\gamma}})^{\underline{r}}{}_{\underline{s}}\,f^{\boldsymbol{\gamma\beta}}{}_{\boldsymbol{\alpha}}\partial_{\underline{r}}A_m{}^{\underline{s}}\,B_{np\,\boldsymbol{\beta}} - \partial_{\underline{n}}A_m{}^{\underline{n}}\,B_{np\,\boldsymbol{\alpha}}\,, \\ D_mB_{np\,\underline{m}} &= \partial_m B_{np\,\underline{m}} - A_m{}^{\underline{n}}\,\partial_{\underline{n}}B_{np\,\underline{m}} - \partial_{\underline{m}}A_m{}^{\underline{n}}B_{np\,\underline{n}} - \partial_{\underline{n}}A_m{}^{\underline{n}}\,B_{np\,\underline{m}}\,. \end{aligned} \tag{2.29}$$

The 3-form field strengths in turn obey the Bianchi identities

$$4\,D_{[m}H_{npq]\,\boldsymbol{\alpha}} = -3\,(t_{\boldsymbol{\alpha}})_{\underline{mn}}\,\mathcal{F}_{[mn}{}^{\underline{m}}\mathcal{F}_{pq]}{}^{\underline{n}} - \partial_{\underline{m}}G_{mnpq}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} - (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\,G_{mnpq\,\underline{n}}{}^{\underline{m}}\,, \tag{2.30}$$

$$\begin{aligned} 4\,D_{[m}H_{npq]\underline{m}} &= -6\,\mathcal{F}_{[mn}{}^{\underline{n}}\,\partial_{|\underline{m}|}\mathcal{F}_{pq]\,\underline{n}} - 24(t^{\boldsymbol{\alpha}})_{\underline{p}}{}^{\underline{n}}\partial_{\underline{m}}\partial_{\underline{n}}A_{[m}{}^{\underline{p}}\,H_{npq]\,\boldsymbol{\alpha}} \\ &\quad + \partial_{\underline{m}}G_{mnpq\,\underline{n}}{}^{\underline{n}} + 2\,\partial_{\underline{n}}G_{mnpq\,\underline{m}}{}^{\underline{n}}\,, \end{aligned} \tag{2.31}$$

[5]We use the identity (2.13) and $t^{\boldsymbol{\alpha}(\underline{mn}}C_{mnp}{}^{\underline{p})}{}_{\boldsymbol{\alpha}} = 0$. The latter identity follows from the fact that $C_{mnp}{}^{\underline{p}}{}_{\boldsymbol{\alpha}}$ belongs to the **912** of $\mathrm{E}_{7(7)}$ while $(\mathbf{56}\times\mathbf{56}\times\mathbf{56})_S$ does not contain the **912**.

field	$e_m{}^a$	ξ^m	$\mathcal{V}_{\underline{m}}{}^{ij}$	$A_m{}^{\underline{m}}$, $\Lambda^{\underline{m}}$	$B_{mn\,\boldsymbol{\alpha}}$, $\Xi_{m\,\boldsymbol{\alpha}}$	$B_{mn\,\underline{m}}$, $\Xi_{m\,\underline{m}}$	χ_{ijk}	$\psi_m{}^i$, ϵ^i
λ	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$

Table 1. λ-weights for the bosonic and fermionic fields and parameters.

which serve to define the curvatures $G_{mnpq}{}^{\underline{m}}{}_{\boldsymbol{\alpha}}$ and $G_{mnpq\,\underline{m}}{}^{\underline{n}}$ associated with the 3-form potentials. This leads to the introduction of 4-form potentials in certain representations of $E_{7(7)}$ and obeying certain constraints. The transformation rules for the 3-form and 4-form potentials can be determined from the requirement of the closure of the algebra. We will not need these transformation rules, except for the behavior of the 3-form potentials under external diffeomorphisms, which we shall derive below. We will also derive the duality equations obeyed by 4-form field strengths below, and we shall comment on the occurrence of particular 4-form potentials in their definitions in section 4.5.

The curvatures associated with the p-form potentials with $p = 1, 2, 3$ obey duality equations given by [9][6]

$$
\begin{aligned}
\mathcal{F}_{mn}{}^{\underline{m}} &= \frac{i}{2}\,\varepsilon_{mnpq}\,\Omega^{\underline{mn}}\,\mathcal{M}_{\underline{nk}}\,\mathcal{F}^{pq\,\underline{k}}\,, & (2.32)\\
\varepsilon^{mnpq} H_{npq\,\boldsymbol{\alpha}} &= (t_{\boldsymbol{\alpha}})_{\underline{n}}{}^{\underline{m}}\left(P^{mijkl}\mathcal{V}^{\underline{n}}{}_{ij}\mathcal{V}_{\underline{m}kl} - P^{m}{}_{ijkl}\mathcal{V}^{\underline{n}ij}\mathcal{V}_{\underline{m}}{}^{kl}\right)\,, &\\
&= -\frac{i}{2}(t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}(D^m\mathcal{M}^{\underline{mk}})\mathcal{M}_{\underline{nk}}\,, & (2.33)\\
\frac{1}{12}\varepsilon^{mnpq} H_{npq\,\underline{m}} &= -2i\,e_a{}^m e_b{}^n\left(\partial_{\underline{m}}\omega_n{}^{ab} - D_n\omega_{\underline{m}}{}^{ab}\right) - \frac{1}{3}D^m\mathcal{V}^{\underline{n}ij}\partial_{\underline{m}}\mathcal{V}_{\underline{n}ij}\,, & (2.34)
\end{aligned}
$$

where

$$
\omega_{\underline{m}}{}^{ab} := -e^{n[a}\partial_{\underline{m}}e_n{}^{b]}\ . \qquad (2.35)
$$

The first duality equation (2.32) is required together with the second-order pseudo-action given in [9] in order to describe the correct vector field dynamics. The second order field equation for the vector fields can then be obtained by the external curl of (2.32) together with the Bianchi identity (2.26). Comparing this second order equation to the one obtained from variation of the pseudo-action gives rise to the duality equations (2.33), (2.34). As in the Bianchi identity (2.26), the duality equations (2.33) and (2.34) only follow under projection with $(t^{\boldsymbol{\alpha}})^{\underline{mn}}\,\partial_{\underline{n}}$, and their remaining parts may thus be taken as an equations for the three-form potentials introduced in (2.27) and (2.28).[7] The variations of the 3-form potentials under all the bosonic symmetries can be determined from the requirement of the invariance of the duality equations above. For later purposes we shall in particular need their variations under the external diffeomorphisms. To determine them, we consider the invariance of the duality equation (2.33) under the external diffeomorphisms. To this end,

[6]The first equation can also be written as $\mathcal{F}^{-}_{mn\,ij} \equiv \frac{1}{2}\mathcal{F}_{mn\,ij} - \frac{1}{4}\,\varepsilon_{mnpq}\mathcal{F}^{pq}{}_{ij} = 0$. The third equation (2.34) had an overall sign mistake in [9, 22] that is corrected below, keeping in mind the change in sign of the spin connection.

[7]It is worth noting that the variation of the duality equation (2.32) yields (2.33) but not (2.34) for the constrained field. The latter involves two derivative terms on the right hand side, and these are derived in [9] by employing a suitable action.

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the following formula for the general variation of $H_{\boldsymbol{\alpha}}$ is useful:

$$\begin{aligned}\delta H_{mnp\,\boldsymbol{\alpha}} = & \Big[3\, D_m(\Delta B_{np\,\boldsymbol{\alpha}}) - 3\,(t_{\boldsymbol{\alpha}})_{\underline{mn}}\,\delta A_m{}^{\underline{m}}\,\mathcal{F}_{np}{}^{\underline{n}} \\ & \qquad\qquad -\partial_{\underline{m}}\Delta C_{mnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} - (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\,\Delta C_{mnp\,\underline{n}}{}^{\underline{m}}\Big]_{[mnp]}\,,\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}\Delta C_{mnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} = & \Big[\delta C_{mnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} + 21\,(\mathbb{P}_{912})_{\boldsymbol{\alpha}}{}^{\underline{m}\boldsymbol{\beta}}{}_{\underline{n}}\,B_{mn\,\boldsymbol{\beta}}\,\delta A_p{}^{\underline{n}} \\ & \quad -7\,(\mathbb{P}_{912})_{\boldsymbol{\alpha}}{}^{\underline{m}\boldsymbol{\beta}}{}_{\underline{n}}\,(t_{\boldsymbol{\beta}})_{\underline{rs}}\,A_m{}^{\underline{n}}A_n{}^{\underline{r}}\delta A_p{}^{\underline{s}}\Big]_{[mnp]}\,, \\ \Delta C_{mnp\,\underline{n}}{}^{\underline{m}} = & \Big[\delta C_{mnp\,\underline{n}}{}^{\underline{m}} - \frac{3}{2}\,\delta A_m{}^{\underline{m}}\,B_{np\,\underline{n}} + 24\,\partial_{\underline{n}}\delta A_m{}^{\underline{r}}\,B_{np\,\boldsymbol{\alpha}}(t^{\boldsymbol{\alpha}})_{\underline{r}}{}^{\underline{m}} \\ & \quad - 12\,\delta A_m{}^{\underline{r}}\,\partial_{\underline{n}}B_{np\,\boldsymbol{\alpha}}(t^{\boldsymbol{\alpha}})_{\underline{r}}{}^{\underline{m}} - 12\,(t_{\boldsymbol{\alpha}})_{\underline{rs}}(t^{\boldsymbol{\alpha}})_{\underline{k}}{}^{\underline{m}}\,\partial_{\underline{n}}A_m{}^{\underline{k}}A_n{}^{\underline{r}}\delta A_p{}^{\underline{s}} \\ & \quad -\frac{5}{6}\,\delta A_m{}^{\underline{m}}\partial_{\underline{n}}(A_{n\underline{k}}A_p{}^{\underline{k}}) + \frac{1}{6}\,\partial_{\underline{n}}\delta A_m{}^{\underline{m}}A_{n\underline{k}}A_p{}^{\underline{k}} - \frac{2}{3}\,A_m{}^{\underline{m}}\partial_{\underline{n}}\delta A_{n\,\underline{k}}A_p{}^{\underline{k}} \\ & \quad - \frac{1}{6}\,\partial_{\underline{n}}A_m{}^{\underline{m}}\delta A_{n\,\underline{k}}A_p{}^{\underline{k}} + \frac{1}{3}\,A_m{}^{\underline{m}}\delta A_{n\,\underline{k}}\partial_{\underline{n}}A_p{}^{\underline{k}}\Big]_{[mnp]}\,,\end{aligned} \tag{2.37}$$

and we have used the following identity:

$$(t^{\boldsymbol{\beta}})^{\underline{ks}}\,(t_{\boldsymbol{\beta}})_{(\underline{mn}}\,(t^{\boldsymbol{\alpha}})_{\underline{r})\underline{s}} = \frac{1}{3}(t_{\boldsymbol{\beta}})_{\underline{s}(\underline{m}}\,(t^{\boldsymbol{\beta}})_{\underline{nr})}\,(t_{\boldsymbol{\alpha}})^{\underline{ks}} - \frac{1}{12}(t^{\boldsymbol{\alpha}})_{(\underline{mn}}\,\delta_{\underline{r})}{}^{\underline{k}}\,. \tag{2.38}$$

Under external diffeomorphisms

$$\begin{aligned}\Delta C_{mnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} = & \;\xi^r\,G_{rmnp}{}^{\underline{m}}{}_{\boldsymbol{\alpha}}\,, \\ \varepsilon_m{}^{npq}\,\Delta C_{npq\,\underline{n}}{}^{\underline{m}} = & \;\varepsilon_m{}^{npq}\,\xi^r G_{rnpq\,\underline{n}}{}^{\underline{m}} + 6\,i\,\partial_{\underline{n}}\xi^n\,F_{nm}{}^{\underline{m}} + 12\,i\,\partial_{(\underline{n}}\left(\mathcal{M}^{\underline{mk}}\,\partial_{\underline{k})}\xi^n\,g_{mn}\right) \\ & - 8i\,\partial_{\underline{k}}\mathcal{M}^{\underline{km}}\,g_{mn}\partial_{\underline{n}}\xi^n\,,\end{aligned} \tag{2.39}$$

the duality equation of $H_{\boldsymbol{\alpha}}$ transforms covariantly, provided that we also impose the following duality equation for the four-form field strength

$$\varepsilon^{mnpq}\,G_{mnpq}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} = 14i\,(\mathbb{P}_{912})^{\underline{m}}{}_{\boldsymbol{\alpha}}{}_{\underline{n}}{}^{\boldsymbol{\beta}}\,\mathcal{M}^{\underline{nr}}J_{\underline{r}\boldsymbol{\beta}}\,, \tag{2.40}$$

with the current $J_{\underline{r}\boldsymbol{\beta}}$ defined by

$$\mathcal{M}^{\underline{nk}}\partial_{\underline{r}}\mathcal{M}_{\underline{km}} = J_{\underline{r}\boldsymbol{\alpha}}\,(t^{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\,. \tag{2.41}$$

In this calculation the terms involving the field strength $G_{mnpq\,\underline{m}}{}^{\underline{n}}$ cancel, and consequently a duality equation for this field strength does not follow. However, $H_{mnp\,\boldsymbol{\alpha}}$ determined from (2.33), substituted into the Bianchi identity (2.30) gives

$$(t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\left(e\,\varepsilon^{mnpq}G_{mnpq\,\underline{n}}{}^{\underline{m}} + 48i\mathcal{M}_{\underline{nr}}\frac{\delta(eV)}{\delta\mathcal{M}_{\underline{mr}}} + 14i(t^{\boldsymbol{\beta}})_{\underline{n}}{}^{\underline{m}}(\mathbb{P}_{912})^{\underline{k}}{}_{\boldsymbol{\beta}}{}_{\underline{r}}{}^{\boldsymbol{\gamma}}\partial_{\underline{k}}\left(e\mathcal{M}^{\underline{rs}}J_{\underline{s}\boldsymbol{\gamma}}\right)\right) = 0\,, \tag{2.42}$$

from which one can derive[8]

$$\frac{1}{24} e\,\varepsilon^{mnpq}\, G_{mnpq\,\underline{n}}{}^{\underline{m}} = i\partial_{\underline{r}}\Big(e\,\partial_{\underline{n}}\mathcal{M}^{\underline{rm}}\Big) + \frac{i}{3}\partial_{\underline{n}}\Big(e\,\mathcal{M}^{\underline{rs}}\mathcal{M}^{\underline{rm}}\Big)$$
$$- \frac{i}{24} e\,\mathcal{M}^{\underline{mt}}\partial_{\underline{n}}\mathcal{M}^{\underline{rs}}\Big(\partial_{\underline{t}}\mathcal{M}_{\underline{rs}} - 12\partial_{\underline{r}}\mathcal{M}_{\underline{ts}}\Big) + \frac{i}{2}\partial_{\underline{r}}\mathcal{M}^{\underline{mt}}\mathcal{M}^{\underline{rs}}\partial_{\underline{n}}\mathcal{M}_{\underline{ts}}\,. \tag{2.43}$$

In obtaining this result, we have used (2.32) and the following scalar field equation that follows from the action given in [9]:

$$(t_\alpha)_{\underline{m}}{}^{\underline{k}}\mathcal{M}_{\underline{kn}}\left(D_m\left(eD^m\mathcal{M}^{\underline{mn}}\right) + 3e\mathcal{F}_{mn}{}^{\underline{m}}\mathcal{F}^{mn\,\underline{n}} + 24\,\frac{\delta(eV)}{\delta\mathcal{M}_{\underline{mn}}}\right) = 0\,, \tag{2.44}$$

where

$$V = -\frac{1}{48}\mathcal{M}^{\underline{mn}}\,\partial_{\underline{m}}\mathcal{M}^{\underline{rs}}\left(\partial_{\underline{n}}\mathcal{M}_{\underline{rs}} - 24\,\partial_{\underline{r}}\mathcal{M}_{\underline{sn}}\right) - \left(e^{-1}\partial_{\underline{m}}e\right)e^{-1}\partial_{\underline{n}}\left(e\mathcal{M}^{\underline{mn}}\right)$$
$$-\frac{1}{4}\mathcal{M}^{\underline{mn}}\partial_{\underline{m}}g^{mn}\partial_{\underline{n}}g_{mn}\,. \tag{2.45}$$

2.3 Supersymmetry transformation rules

The supersymmetry transformation rules are given by

$$\begin{aligned}
\delta e_m{}^a &= \bar\epsilon^i\gamma^a\psi_{mi} \;+\; \bar\epsilon_i\gamma^a\psi_m{}^i\,,\\
\delta\mathcal{V}_{\underline{m}}{}^{ij} &= 2\sqrt{2}\,\mathcal{V}_{\underline{m}kl}\left(\bar\epsilon^{[i}\chi^{jkl]} + \frac{1}{24}\varepsilon^{ijklmnpq}\,\bar\epsilon_m\chi_{npq}\right),\\
\delta A_m{}^{\underline{m}} &= -i\,\sqrt{2}\,\Omega^{\underline{mn}}\mathcal{V}_{\underline{n}}{}^{ij}\left(\bar\epsilon^k\,\gamma_\mu\,\chi_{ijk} + 2\sqrt{2}\,\bar\epsilon_i\,\psi_{\mu j}\right) + \text{c.c.}\,,\\
\Delta B_{mn\,\boldsymbol{\alpha}} &= -\frac{2}{3}\sqrt{2}\,(t_{\boldsymbol{\alpha}})^{\underline{pq}}\left(\mathcal{V}_{\underline{p}\,ij}\mathcal{V}_{\underline{q}\,kl}\,\bar\epsilon^{[i}\,\gamma_{mn}\,\chi^{jkl]} + 2\sqrt{2}\,\mathcal{V}_{\underline{p}\,jk}\mathcal{V}_{\underline{q}}{}^{ik}\,\bar\epsilon_i\,\gamma_{[m}\,\psi_{n]}{}^j + \text{c.c.}\right),\\
\Delta B_{mn\,\underline{m}} &= \frac{16}{3}\,\mathcal{V}^{\underline{n}ij}\,\mathcal{D}_{\underline{m}}\mathcal{V}_{\underline{n}jk}\,\bar\epsilon^k\gamma_{[m}\psi_{n]i} - \frac{4\sqrt{2}}{3}\,\mathcal{V}^{\underline{n}}{}_{ij}\mathcal{D}_{\underline{m}}\mathcal{V}_{\underline{n}\,kl}\,\bar\epsilon^{[i}\,\gamma_{mn}\,\chi^{jkl]}\\
&\quad - 8i\left(\bar\epsilon^i\,\gamma_{[m}\mathcal{D}_{\underline{m}}\psi_{n]i} - \mathcal{D}_{\underline{m}}\bar\epsilon^i\,\gamma_{[m}\,\psi_{n]i}\right) + 2i\varepsilon_{mnpq}\,g^{qr}\,\mathcal{D}_{\underline{m}}\left(\bar\epsilon^i\gamma^p\psi_{r\,i}\right) + \text{c.c.}\,,\\
\delta\psi_m{}^i &= 2\,\mathcal{D}_m\epsilon^i - 4i\,\mathcal{V}^{\underline{m}\,ij}\widehat\nabla_{\underline{m}}\left(\gamma_m\epsilon_j\right)\,,\\
\delta\chi^{ijk} &= -2\sqrt{2}\,\mathcal{P}_m{}^{ijkl}\gamma^\mu\epsilon_l - 12\sqrt{2}i\,\mathcal{V}^{\underline{m}[ij}\,\widehat\nabla_{\underline{m}}\epsilon^{k]}\,.
\end{aligned} \tag{2.46}$$

The coset currents $\mathcal{P}_m{}^{ijkl}$ are defined as

$$\mathcal{D}_m\mathcal{V}_{\underline{n}}{}^{ij} \equiv D_m\mathcal{V}_{\underline{n}}{}^{ij} + \mathcal{Q}_{m\,k}{}^{[i}\,\mathcal{V}_{\underline{n}}{}^{j]k} \;=\; \mathcal{P}_m{}^{ijkl}\,\mathcal{V}_{\underline{n}\,kl}\,, \tag{2.47}$$

which also defines the composite SU(8) connection

$$\mathcal{Q}_{mi}{}^j \;=\; \frac{2i}{3}\,\mathcal{V}^{\underline{n}\,jk}\,D_m\mathcal{V}_{\underline{n}\,ki}\,. \tag{2.48}$$

[8]In order to strip off the $(t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}$, we make use of the fact that $G_{mnpq\,\underline{m}}{}^{\underline{n}}$ is constrained on its $\underline{m}$ index. This condition means that G lies in a generic $\mathbf{56}\times\mathbf{56}$ representation, as any projection operation would spoil the section condition. Since a generic $\mathbf{56}\times\mathbf{56}$ always contains the adjoint representation, there is no ambiguity in solving this equation.

The covariant derivatives of a spinor $X_{\underline{n}i}$ with and without Christoffel connection are defined as

$$\mathcal{D}_m X_{\underline{n}\,i} = D_m X_{\underline{n}\,i} - \frac{1}{4}\,\omega_m{}^{ab}\gamma_{ab} X_{\underline{n}\,i} + \frac{1}{2}\mathcal{Q}_{m\,i}{}^{j} X_{\underline{n}\,j}\,, \tag{2.49}$$

$$\mathcal{D}_{\underline{m}} X_{\underline{n}\,i} = \partial_{\underline{m}} X_{\underline{n}\,i} - \frac{1}{4}\,\omega_{\underline{m}}{}^{ab}\gamma_{ab} X_{\underline{n}\,i} + \frac{1}{2}\mathcal{Q}_{\underline{m}\,i}{}^{j} X_{\underline{n}\,j}\,, \tag{2.50}$$

$$\nabla_{\underline{m}} X_{\underline{n}\,i} = \mathcal{D}_{\underline{m}} X_{\underline{n}\,i} - \Gamma_{\underline{mn}}{}^{\underline{r}} X_{\underline{r}\,i} - \frac{2}{3}\lambda(X)\Gamma_{\underline{rm}}{}^{\underline{r}} X_{\underline{n}\,i}\,. \tag{2.51}$$

The hatted covariant derivative $\widehat{\nabla}_{\underline{m}}$ is obtained from $\nabla_{\underline{m}}$ by replacing $\omega_{\underline{m}}{}^{ab}$ with

$$\widehat{\omega}_{\underline{m}}{}^{ab} \equiv \omega_{\underline{m}}{}^{ab} + \frac{1}{4}\,\mathcal{M}_{\underline{mn}}\,\mathcal{F}_{mn}{}^{\underline{n}}\,e^{ma}e^{nb}\,. \tag{2.52}$$

Finally, let us note the simplification

$$\begin{aligned}\Delta B_{mn\,\underline{m}} + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}}\Delta B_{mn\boldsymbol{\alpha}} &= -8i\left(\bar{\epsilon}^i\,\gamma_{[m}\mathcal{D}_{\underline{m}}\psi_{n]i} - \mathcal{D}_{\underline{m}}\bar{\epsilon}^i\,\gamma_{[m}\,\psi_{n]i}\right)\\ &\quad +2i\,\varepsilon_{mnpq}\,g^{qr}\,\mathcal{D}_{\underline{m}}\left(\bar{\epsilon}^i\gamma^p\psi_{r\,i}\right) + \text{c.c.}\,,\end{aligned} \tag{2.53}$$

and that writing out the covariant derivatives in the variations of the fermionic fields gives

$$\begin{aligned}\delta\psi_m{}^i &= 2\mathcal{D}_m\epsilon^i + \frac{1}{4}\mathcal{F}_{ab}{}^{ij}\gamma^{ab}\gamma_m\epsilon_j + i(e_n{}^a\partial_{\underline{n}}e_{pa})\mathcal{V}^{\underline{n}ij}\gamma^{np}\gamma_m\epsilon_j - 4i\mathcal{V}^{\underline{n}ij}\partial_{\underline{n}}(\gamma_m\epsilon_j)\\ &\quad -2i\mathcal{V}^{\underline{m}ij}\,q_{\underline{m}\,j}{}^{k}\gamma_m\epsilon_k - 2i\mathcal{V}^{\underline{m}}{}_{kl}\,p_{\underline{m}}{}^{ijkl}\gamma_m\epsilon_j\,,\end{aligned} \tag{2.54}$$

$$\begin{aligned}\delta\chi^{ijk} &= -2\sqrt{2}\mathcal{P}_m{}^{ijkl}\gamma^m\epsilon_l + \frac{3\sqrt{2}}{4}\mathcal{F}_{ab}{}^{[ij}\gamma^{ab}\epsilon^{k]} + 3\sqrt{2}i(e_m{}^a\partial_{\underline{m}}e_{na})\mathcal{V}^{\underline{m}[ij}\gamma^{mn}\epsilon^{k]}\\ &\quad -12\sqrt{2}i\mathcal{V}^{\underline{m}[ij}\partial_{\underline{m}}\epsilon^{k]} + 6\sqrt{2}i\mathcal{V}^{\underline{m}[ij}\,q_{\underline{m}l}{}^{k]}\epsilon^l - 8\sqrt{2}i\mathcal{V}^{\underline{m}}{}_{rs}\,p_{\underline{m}}{}^{ijkr}\epsilon^s\\ &\quad -6\sqrt{2}i\mathcal{V}^{\underline{m}}{}_{rs}\,p_{\underline{m}}{}^{rs[ij}\epsilon^{k]}\,.\end{aligned} \tag{2.55}$$

where

$$q_{\underline{m}\,i}{}^{j} \equiv \frac{2}{3}\mathcal{V}^{\underline{n}jk}\partial_{\underline{m}}\mathcal{V}_{\underline{n}ki}\,, \qquad p_{\underline{m}}{}^{ijkl} \equiv i\mathcal{V}^{\underline{n}ij}\partial_{\underline{m}}\mathcal{V}_{\underline{n}}{}^{kl}\,. \tag{2.56}$$

3 Laying the groundwork for superspace

The supersymmetry transformations described in the previous section do not readily admit a lift to a conventional superspace due to a number of obstacles. Some of these, for example, the term involving the internal derivative of the supersymmetry parameter in the gravitino transformation (2.46), are rectified by understanding the structure of external superdiffeomorphisms in superspace. Other issues, such as the nature of the last term in the transformation of $B_{mn\,\underline{m}}$, require that we first make some redefinitions of fields appearing at the component level before considering their superspace analogues.

In this section, we will elaborate upon a few modifications of the component theory that shed light on its superspace lift. First, we describe the action of the external diffeomorphisms on the fermions. Then we proceed to describe a redefinition of the two-form $B_{mn\,\underline{m}}$, which appears necessary to make sense of its superspace analogue. The redefined two-form

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turns out to have a simpler transformation under both external diffeomorphisms and supersymmetry transformations. Finally, we compute the algebra of external diffeomorphisms. Afterwards, we will describe the fate of the four generalized vielbein postulates, (2.5)–(2.8), in superspace. As we shall see, only the latter two of these, involving the 56-bein, continue to play any role. We will further argue that a more democratic form of the internal GVP (2.8), which includes non-metricity and leaves the generalized torsion unfixed, is more natural from a superspace perspective. This will prove useful both for understanding some of the structure of the supersymmetry transformations and for connecting with the superspace of gauged supergravity after a generalized Scherk-Schwarz reduction [19], as we shall see in section 6.

3.1 External diffeomorphisms of the fermions

Let us first summarize some details about external diffeomorphisms in the supersymmetric theory that have not previously appeared in the literature. As in the bosonic theory (see (2.21)), the vierbein $e_m{}^a$ and the 56-bein transform as tensors under external diffeomorphisms. It turns out the same is not true of their superpartners ψ^i and χ^{ijk}. Rather, these fields transform under external diffeomorphisms as

$$\begin{aligned}
\delta_\xi \psi_m{}^i &= D_m \xi^n \psi_n{}^i + \xi^n D_n \psi_m{}^i + 4i\, \mathcal{V}^{\underline{m}ij} \partial_{\underline{m}} \xi^n\, \gamma_{[n} \psi_{m]j} - \frac{i\sqrt{2}}{2} \mathcal{V}^{\underline{m}}{}_{jk} \partial_{\underline{m}} \xi^n\, \gamma_n \gamma_m \chi^{ijk}\,, \\
\delta_\xi \chi^{ijk} &= \xi^n D_n \chi^{ijk} - 6i\sqrt{2}\, \mathcal{V}^{\underline{n}\,[ij} \partial_{\underline{n}} \xi^m \psi_m{}^{k]} - \frac{i}{6} \varepsilon^{ijkpqrst} \mathcal{V}^{\underline{n}}{}_{pq} \partial_{\underline{n}} \xi^m \gamma_m \chi_{rst}\,.
\end{aligned} \tag{3.1}$$

The non-tensorial terms involving internal derivatives of ξ^m, which we will refer to as anomalous terms, can be justified in a few different ways. Perhaps the simplest (which we followed) is to compute them directly in $D = 11$ supergravity after reformulating it to make the local $\mathrm{SO}(1,3) \times \mathrm{SU}(8)$ tangent space symmetry manifest [5]. This corresponds to an explicit solution of the section condition in ExFT, and so the results can be lifted to ExFT exactly along the lines followed in [22, 36, 37]. Another approach, formulated entirely within ExFT, would be to require closure of the algebra of external diffeomorphisms. It will be convenient to work out this algebra after performing a redefinition of the two-form $B_{\underline{m}}$ as we shall do below.

3.2 Redefinition of the two-form and the algebra of external diffeomorphisms

While, as we will see, the external diffeomorphisms of most of the fields can be directly lifted to superspace, the transformation of $B_{mn\,\underline{m}}$ — specifically the second term in (2.21) — proves to be problematic. This is due to the presence of the inverse vierbein. Whereas we will be identifying the component vierbein $e_m{}^a$ as the element $E_m{}^a$ of the supervielbein $E_M{}^A$, the inverse vierbein $e_a{}^m$ has no simple interpretation in superspace, as it does *not* correspond to the element $E_a{}^m$ of the inverse supervielbein. This is why, typically in supersymmetric theories, one can formulate supersymmetry transformations without explicit use of the inverse vierbein.

It turns out that there is a redefinition of the two-form $B_{mn\,\underline{m}}$ that resolves this issue.[9] We will take

$$\begin{aligned} B'_{mn\,\underline{m}} &= B_{mn\,\underline{m}} - 2i\,\varepsilon_{mn}{}^{pq} e_p{}^a \partial_{\underline{m}} e_{qa} \\ &= B_{mn\,\underline{m}} - 2i\,\varepsilon_{mnab}\,\omega_{\underline{m}}{}^{ab} \ . \end{aligned} \tag{3.2}$$

It follows that the symmetry transformations of $(A_m{}^{\underline{m}}, B_{mn\boldsymbol{\alpha}})$ given in (2.21) preserve their form with $B_{mn\,\underline{m}}$ replaced by $B'_{mn\,\underline{m}}$ in $\mathcal{F}_{mn}{}^{\underline{m}}$ and $H_{mnp\,\boldsymbol{\alpha}}$, provided one makes the compensating Ξ and Ω transformations with parameters

$$\Xi_{m\,\underline{m}} = -2i\,\xi^n \varepsilon_{nm\,ab}\,\omega_{\underline{n}}{}^{ab} \,, \qquad \Omega_{mn\,\underline{n}}{}^{\underline{m}} = i\Lambda^{\underline{m}} \varepsilon_{mnab}\,\omega_{\underline{n}}{}^{ab} \ . \tag{3.3}$$

Thus we have

$$\begin{aligned} \delta_{\xi,\Lambda} A_m{}^{\underline{m}} &= \xi^n\,\mathcal{F}_{nm}{}^{\underline{m}}(B') + \mathcal{M}^{\underline{mn}}\,g_{mn}\,\partial_{\underline{n}}\xi^n + D_m\Lambda^{\underline{m}} \,, \\ \Delta_{\xi,\Lambda} B_{mn\,\boldsymbol{\alpha}} &= \xi^p\,H_{mnp\,\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{\underline{mn}}\,\Lambda^{\underline{m}} \mathcal{F}_{mn}{}^{\underline{n}}(B') \ . \end{aligned} \tag{3.4}$$

Now noting that

$$\begin{aligned} \delta_\xi \omega_{\underline{m}}{}^{ab} &= \xi^n D_n \omega_{\underline{m}}{}^{ab} - \partial_{\underline{m}}\xi^n e^{m[a} D_n e_m{}^{b]} - e^{m[a} e_n{}^{b]} D_m \partial_{\underline{m}}\xi^n \,, \\ \varepsilon_{mn}{}^{pq} D_p (g_{qr}\partial_{\underline{m}}\xi^r) &= \varepsilon_{mnab}\,\Big(e^{ap} e_q{}^b D_p \partial_{\underline{m}}\xi^q + \partial_{\underline{m}}\xi^n e^{am}\mathcal{D}_m e_n{}^b \Big) \ , \end{aligned} \tag{3.5}$$

where we have used $\mathcal{D}_{[m} e_{n]}{}^a = 0$, one finds from (2.21) that

$$\Delta_{\xi,\Lambda} B'_{mn\underline{m}} = \xi^p H_{pmn\underline{m}}(B') - 2i\,\varepsilon_{mnab}\,\partial_{\underline{m}}\xi^p \omega_p{}^{ab} + \mathcal{F}_{mn}{}^{\underline{n}}(B')\partial_{\underline{m}}\Lambda_{\underline{n}} - \partial_{\underline{m}}\mathcal{F}_{mn}{}^{\underline{n}}(B')\Lambda_{\underline{n}} \ . \tag{3.6}$$

The second term in this transformation can be readily lifted to superspace as it involves only forms. Here we interpret $\omega_p{}^{ab}$ as a one-form that can be lifted to superspace, as opposed to expressing it as a composite in terms of the vierbein and its inverse. In achieving this simplification, we have paid a price. The field $B'_{mn\,\underline{m}}$ now transforms under local Lorentz transformations as

$$\delta B'_{mn\,\underline{m}} = -2i\,\varepsilon_{mnab}\,\partial_{\underline{m}}\lambda^{ab} \ . \tag{3.7}$$

We will find soon find that the internal spin connection $\omega_{\underline{m}}{}^{ab}$ no longer appears in any expressions and covariance under y-dependent Lorentz transformations is now ensured by the field B' and the field strength $\mathcal{F}(B')$ in which it appears.

Before moving on, there are a number of features of the field strength $\mathcal{F}(B')$ we should discuss. Because we have essentially redefined it as

$$\mathcal{F}_{mn\,ij}(B') = \mathcal{F}_{mn\,ij}(B) - i\,\mathcal{V}^{\underline{m}}{}_{ij}\,\varepsilon_{mnpq}\,e^{pa}\,\partial_{\underline{m}} e^q{}_a \ , \tag{3.8}$$

the self-duality equation (2.32) now takes the form

$$\widehat{\mathcal{F}}_{mn\,ij} = \frac{1}{2}\,e\,\varepsilon_{mnpq}\,\widehat{\mathcal{F}}^{pq}{}_{ij} \,, \tag{3.9}$$

[9] This redefinition is naturally related to one recently made in the so-called "topological phase" of E_8 ExFT that allowed its reinterpretation as a Chern-Simons theory [38]. There it was the constrained one-form $B_{m\underline{m}}$ that admitted a redefinition.

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in terms of the modified field strength

$$\widehat{\mathcal{F}}_{mn\,ij} \equiv \mathcal{F}_{mn\,ij}(B') - 2i\,\mathcal{V}^{\underline{n}}{}_{ij}\,e_{[m}{}^{a}\,\partial_{\underline{n}}e_{n]a}\;, \tag{3.10}$$

or equivalently

$$\widehat{\mathcal{F}}_{mn}{}^{\underline{m}} \equiv \mathcal{F}_{mn}{}^{\underline{m}}(B') + 2\,\mathcal{M}^{\underline{mn}}\,e_{[m}{}^{a}\,\partial_{\underline{n}}e_{n]a}\;. \tag{3.11}$$

The additional term in $\widehat{\mathcal{F}}$ can be understood as the twisted dual of the term we have added to B, which is necessary so that $\widehat{\mathcal{F}}$ continues to be twisted self-dual. It transforms under Lorentz transformations as

$$\delta\widehat{\mathcal{F}}_{ab\,ij} = -2i\,\mathcal{V}^{\underline{m}}{}_{ij}\left(\delta_a{}^c\delta_b{}^d + \frac{1}{2}\varepsilon_{ab}{}^{cd}\right)\partial_{\underline{m}}\lambda_{cd} = -4i\,\mathcal{V}^{\underline{m}}{}_{ij}\partial_{\underline{m}}\lambda^+_{ab}\;. \tag{3.12}$$

As a consequence of the additional term in its definition, $\widehat{\mathcal{F}}$ satisfies a modified Bianchi identity

$$\begin{aligned} D_{[p}\widehat{\mathcal{F}}_{mn]}{}^{\underline{m}} = &-4\,(t^{\boldsymbol{\alpha}})^{\underline{mn}}\partial_{\underline{n}}H_{mnp\,\boldsymbol{\alpha}} - \frac{1}{6}\,\Omega^{\underline{mn}}H_{mnp\,\underline{n}}(B') \\ &+2\left(D_{[p}\mathcal{M}^{\underline{mn}}\right)e_m{}^a\partial_{\underline{n}}e_{n]a} - 2\,\mathcal{M}^{\underline{mn}}e_{[m}{}^ae_n{}^b\partial_{\underline{n}}\omega_{p]ab}\,, \end{aligned} \tag{3.13}$$

where we have used the vanishing of the external torsion $T_{mn}{}^p = 0$.

Later on, it will be convenient to rewrite this expression in a form that is manifestly covariant under internal diffeomorphisms. To this end, we note that

$$\nabla_{\underline{n}}H_{\boldsymbol{\alpha}} = \partial_{\underline{n}}H_{\boldsymbol{\alpha}} + \Gamma_{\underline{n}}{}^{\boldsymbol{\beta}}\,f_{\boldsymbol{\beta\alpha}}{}^{\boldsymbol{\gamma}}\,H_{\boldsymbol{\alpha}} - \frac{2}{3}\Gamma_{\underline{kn}}{}^{\underline{k}}\,H_{\boldsymbol{\alpha}}\,, \tag{3.14}$$

and introduce a modified three-form field strength

$$\begin{aligned} \mathscr{H}_{\underline{m}} &= H_{\underline{m}}(B') - 24\,(t^{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\,\Gamma_{\underline{n}}{}^{\boldsymbol{\beta}}\,f_{\boldsymbol{\beta\alpha}}{}^{\boldsymbol{\gamma}}\,H_{\boldsymbol{\gamma}} + 16\,(t^{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}\,\Gamma_{\underline{kn}}{}^{\underline{k}}\,H_{\boldsymbol{\alpha}} + 2\,\mathcal{T}_{\underline{m}}{}^{\boldsymbol{\alpha}}\,H_{\boldsymbol{\alpha}} \\ &= H_{\underline{m}}(B') + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}}\,H_{\boldsymbol{\alpha}}\;. \end{aligned} \tag{3.15}$$

The field strength $\mathscr{H}_{\underline{m}}$ is a tensor under internal diffeomorphisms, whereas $H_{\underline{m}}(B')$ is not. Now the Bianchi identity (3.13) takes the form

$$\begin{aligned} D_{[p}\widehat{\mathcal{F}}_{mn]}{}^{\underline{m}} = &-4(t^{\boldsymbol{\alpha}})^{\underline{mn}}\nabla_{\underline{n}}H_{mnp\,\boldsymbol{\alpha}} + \frac{1}{3}\,\Omega^{\underline{mn}}\mathcal{T}_{\underline{n}}{}^{\boldsymbol{\alpha}}\,H_{mnp\,\boldsymbol{\alpha}} - \frac{1}{6}\Omega^{\underline{mn}}\mathscr{H}_{mnp\,\underline{n}} \\ &+2\left(D_{[p}\mathcal{M}^{\underline{mn}}\right)e_m{}^a\nabla_{\underline{n}}e_{n]a} - 2\,\mathcal{M}^{\underline{mn}}e_{[m}{}^ae_n{}^b\nabla_{\underline{n}}\omega_{p]ab}\,, \end{aligned} \tag{3.16}$$

In the above expressions, we have kept explicit the generalized torsion tensor $\mathcal{T}_{\underline{m}}{}^{\boldsymbol{\alpha}}$, even though it was constrained to vanish in [22]. We will soon see that it is convenient to relax this requirement and allow a non-vanishing $\mathcal{T}_{\underline{m}}{}^{\boldsymbol{\alpha}}$. The modified field strength $\mathscr{H}_{\underline{m}}$ similarly appears in the covariantized variation of $B'_{mn\,\underline{m}}$, which is given by

$$\begin{aligned} \Delta_{\xi,\Lambda}B'_{mn\,\underline{m}} + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}}\Delta_{\xi,\Lambda}B_{mn\,\boldsymbol{\alpha}} = \xi^p\mathscr{H}_{pmn\,\underline{m}} &- 2i\,\varepsilon_{mnab}\,\partial_{\underline{m}}\xi^p\omega_p{}^{ab} \\ &+ \mathcal{F}_{mn}{}^{\underline{n}}(B')\,\nabla_{\underline{m}}\Lambda_{\underline{n}} - \nabla_{\underline{m}}\mathcal{F}_{mn}{}^{\underline{n}}(B')\,\Lambda_{\underline{n}}\;. \end{aligned} \tag{3.17}$$

While the on-shell duality equation (2.33) of $H_{mnp\,\boldsymbol{\alpha}}$ is unchanged by the redefinition of B, the duality equation (2.34) for $H_{mnp\,\underline{m}}$ now reads

$$\frac{1}{12}\varepsilon^{mnpq}H_{npq\,\underline{m}}(B') = -2i\, e_a{}^m e_b{}^n \partial_{\underline{m}}\omega_n{}^{ab} - \frac{1}{3}D^m\mathcal{V}^{\underline{n}ij}\partial_{\underline{m}}\mathcal{V}_{\underline{n}ij}\ . \tag{3.18}$$

Note that the duality equation for the modified field strength can be written

$$\frac{1}{12}\varepsilon^{mnpq}\mathscr{H}_{npq\,\underline{m}}(B') = -2i\, e_a{}^m e_b{}^n \partial_{\underline{m}}\omega_n{}^{ab} + \frac{i}{24}\mathcal{D}^m\mathcal{M}^{\underline{pq}}\,\nabla_{\underline{m}}\mathcal{M}_{\underline{pq}}\,, \tag{3.19}$$

which is manifestly covariant under internal diffeomorphisms.

Turning to the supersymmetry transformations, the redefinition of the two form $B_{mn\,\underline{m}}$ clearly affects only those for $B_{mn\,\underline{m}}$, $\psi_m{}^i$, and χ^{ijk}, which now take the form

$$\begin{aligned}
\Delta B'_{mn\,\underline{m}} &= \frac{16}{3}\,\mathcal{V}^{\underline{n}ij}\,\mathcal{D}_{\underline{m}}\mathcal{V}_{\underline{n}jk}\,\bar{\epsilon}^k\gamma_{[m}\psi_{n]i} - \frac{4\sqrt{2}}{3}\,\mathcal{V}^{\underline{n}}{}_{ij}\mathcal{D}_{\underline{m}}\mathcal{V}_{\underline{n}\,kl}\,\bar{\epsilon}^{[i}\,\gamma_{mn}\,\chi^{jkl]}\\
&\quad - 8i\left(\bar{\epsilon}^i\,\gamma_{[m}\mathcal{D}'_{\underline{m}}\psi_{n]i} - \mathcal{D}'_{\underline{m}}\bar{\epsilon}^i\,\gamma_{[m}\,\psi_{n]i}\right) + \text{c.c.}\,,\\
\delta\psi_m{}^i &= 2\,\mathcal{D}_m\epsilon^i - 4i\mathcal{V}^{\underline{m}ij}\nabla'_{\underline{m}}(\gamma_m\epsilon_j) + \frac{1}{4}\gamma^{np}\gamma_m\widehat{\mathcal{F}}_{np}{}^{ij}\epsilon_j\\
\delta\chi^{ijk} &= -2\sqrt{2}\,\mathcal{P}_m{}^{ijkl}\gamma^\mu\epsilon_l - 12\sqrt{2}i\,\mathcal{V}^{\underline{m}[ij}\,\nabla'_{\underline{m}}\epsilon^{k]} + \frac{3\sqrt{2}}{4}\gamma^{mn}\epsilon^{[k}\widehat{\mathcal{F}}_{mn}{}^{ij]}\,,
\end{aligned} \tag{3.20}$$

where $\mathcal{D}'_{\underline{m}}$ is obtained from $\mathcal{D}_{\underline{m}}$ defined in (2.50) by dropping the internal connection $\omega_{\underline{m}}{}^{ab}$, and $\nabla'_{\underline{m}}$ is obtained from $\nabla_{\underline{m}}$ defined in (2.51) by replacing $\mathcal{D}_{\underline{m}}$ with $\mathcal{D}'_{\underline{m}}$. We also give the transformation rule

$$\Delta B'_{mn\,\underline{m}} + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}}\Delta B_{mn\,\boldsymbol{\alpha}} = -8i\left(\bar{\epsilon}^i\,\gamma_{[m}\mathcal{D}'_{\underline{m}}\psi_{n]i} - \mathcal{D}'_{\underline{m}}\bar{\epsilon}^i\,\gamma_{[m}\,\psi_{n]i}\right) + \text{c.c.} \tag{3.21}$$

Note that the last term in the supersymmetry variation of $B_{mn\,\underline{m}}$ has vanished, and the internal connection $\omega_{\underline{m}}{}^{ab}$ has dropped out everywhere, thereby making the superspace lift of these formulae possible, as we shall see later.

Finally, computing the commutators of the external diffeomorphism, with the two-form field redefinition performed, we find the following soft algebra[10]

$$[\delta_{\xi_1},\delta_{\xi_2}] = \delta_{\text{ext.diff.}}(\xi_{12}) + \delta_{\text{int.diff.}}(\Lambda_{12}) + \delta_{\text{Lorentz}}(\lambda_{12}) + \delta_{\text{SU}(8)}(\lambda_{12}) + \delta(\Xi_{12}) + \delta(\Omega_{12})\,, \tag{3.22}$$

with the composite parameters given by

$$\begin{aligned}
\xi_{12}{}^m &= -2\xi_{[1}{}^n D_n\xi^m_{2]}\,,\\
\Lambda_{12}{}^{\underline{m}} &= \xi_{[1}{}^n\xi_{2]}{}^p\mathcal{F}_{pn}{}^{\underline{m}}(B') + 2\xi_{[1}{}^n g_{np}\mathcal{M}^{\underline{mn}}\partial_{\underline{n}}\xi_{2]}{}^p\,,\\
\lambda_{12}{}^{ab} &= 2\mathcal{M}^{\underline{mn}}\partial_{\underline{m}}\xi_{[1}{}^n\partial_{\underline{n}}\xi_{2]}{}^p\,e_n{}^{[a}e_p{}^{b]}\,,\\
\lambda_{12j}{}^i &= -16g_{mn}\mathcal{V}^{\underline{m}}{}_{jk}\mathcal{V}^{\underline{n}ik}\partial_{\underline{m}}\xi_{[1}{}^m\partial_{\underline{n}}\xi_{2]}{}^n\,,\\
\Xi^{12}_{m\,\boldsymbol{\alpha}} &= -\xi_1^n\xi_2^p\,H_{mnp\,\boldsymbol{\alpha}}\,,\\
\Xi^{12}_{m\,\underline{k}} &= \xi_1^n\xi_2^p H_{mnp\,\underline{k}}(B') - 4i\,\varepsilon_{mnab}\,\partial_{\underline{k}}\xi^p_{[1}\,\xi^n_{2]}\,\omega_p{}^{ab}\ .
\end{aligned} \tag{3.23}$$

[10]The appearance of non-trivial $\mathrm{SO}(1,3)$ and $\mathrm{SU}(8)$ parameters is less surprising if one recalls that in the 4+7 reformulation of $D = 11$ supergravity, the external diffeomorphism corresponds to an 11D diffeomorphism plus a local $\mathrm{SO}(1,10)$ transformation; the commutator of two such transformations gives an $\mathrm{SO}(1,3)\times\mathrm{SO}(7)$ transformation.

These can be deduced by working out the commutator algebra on $e_m{}^a, \mathcal{V}_{\underline{m}}{}^a$ and $A_m{}^{\underline{m}}$. As for the composite Ω-transformations, they can be computed from the closure of external diffeomorphisms on $B_{mn\boldsymbol{\alpha}}$ and they will involve the 4-form field strengths. We shall skip the derivation of these field strengths and the resulting composite Ω parameters, as they are not needed here.

3.3 Generalized vielbein postulates

Now let us address the generalized vielbein postulates. Two of them, (2.5) and (2.6), involve the external vielbein. Neither of these turn out to have natural superspace analogues. A straightforward superspace generalization of the first equation (2.5) by extending the coordinate index to a supercoordinate index, $m \to M$, runs into the problem that its ${}_{mn}{}^a$ component differs from the ${}_{mn}{}^a$ component of the original bosonic equation (2.5) due to the presence of a term $\Gamma_{mn}{}^\rho E_\rho{}^a$ (and its complex conjugate), where ρ ($\dot\rho$) is the 16-component index of the chiral (antichiral) Grassmann coordinate θ^ρ ($\theta_{\dot\rho}$). This is problematic because $E_\rho{}^a$ has no geometric meaning at the component level.[11]

One cannot circumvent this issue by setting $\Gamma_{mn}{}^\rho$ to zero by hand, as this violates general supercovariance. Conventional superspace avoids this because the affine connection is actually unnecessary for describing supergravity; it appears in no supersymmetry transformation, nor is it included in the gravitino kinetic term. Instead, one uses the spin connection, which can be fixed to its usual expression by requiring the torsion tensor $T^a = \mathcal{D}e^a$ to vanish. This condition in turn has a natural lift to superspace. Thus, we shall abandon (2.5) in superspace and instead define the vector torsion tensor

$$T^a := DE^a + E^b \wedge \Omega_b{}^a = \mathcal{D}E^a = \frac{1}{2} E^B E^C T_{CB}{}^a \,, \tag{3.24}$$

where $\Omega_b{}^a$ is the Lorentz-valued superconnection. (A similar torsion tensor can be defined in terms of the gravitino one-form $E^{\alpha i}$, but we will postpone its discussion to the next section.) The physics originally encoded in the vanishing torsion condition will now be encoded in constraints placed upon $T_{CB}{}^a$. We will discuss these in due course. The point is that one avoids ever introducing an affine connection $\Gamma_{MN}{}^P$ in superspace and so there is no analogue to (2.5).

Similar statements pertain to (2.6), although here the situation is somewhat different. This equation can be interpreted as a definition of a field $\pi_{\underline{m}n}{}^a$,

$$\partial_{\underline{m}} e_n{}^a - \frac{1}{3}\Gamma_{\underline{km}}{}^{\underline{k}} e_n{}^a - \omega_{\underline{m}}{}^{ab} e_{nb} =: -\pi_{\underline{m}n}{}^a \,. \tag{3.25}$$

The constraint amounts to requiring $\pi_{\underline{m}}{}^a{}_b := \pi_{\underline{m}n}{}^a e_b{}^n$ to be symmetric in ab, which allows one to determine the internal spin connection. Equivalently, a choice of internal spin connection permits one to set the antisymmetric part of $\pi_{\underline{m}}{}^a{}_b$ to zero. However, this

[11] Component fields and forms are derived from superfields and superforms by projecting $\theta = 0$ and $d\theta = 0$. Geometrically, this is the pullback of the inclusion map that embeds spacetime into superspace. For the vector vielbein one-form E^a, the only component that survives this projection is $e_m{}^a = E_m{}^a|_{\theta=0}$. The components $E_\rho{}^a|_{\theta=0}$ and $E^{\dot\rho a}|_{\theta=0}$ turn out to be pure gauge degrees of freedom. While they can be set to zero as a Wess-Zumino type gauge fixing condition, this is not necessary.

has no natural superspace lift. One would need to introduce $\pi_{\underline{m}N}{}^a = \pi_{\underline{m}}{}^a{}_B E_N{}^B$, but this involves also $\pi_{\underline{m}}{}^a{}_{\beta j}$ and $\pi_{\underline{m}}{}^{a\dot\beta j}$. A choice of internal spin connection leaves these unaffected. Moreover, they cannot be set to zero without constraining the internal derivative of $E_N{}^a$ itself. Therefore, we must dispense with (2.6) as well.

At first glance, this is problematic because it forces us to drop the internal spin connection as there is no longer any ability to define it. But as mentioned above, the role of the internal spin connection will turn out to be played by the constrained two-form. The remaining two vielbein postulates (2.7) and (2.8) involve only the 56-bein and these pose no obstacles to a superspace interpretation. The external derivative of the 56-bein (2.7) we will lift to superspace simply by replacing m with M.

However, for the purely internal GVP (2.8), we find that it is useful to choose a more general form. It was already observed in [22, 36] that the internal GVP derived from the SU(8) reformulation of $D = 11$ supergravity does not take the restricted form (2.8), but rather includes so-called non-metricity. The most general form of the internal GVP is

$$\begin{aligned}\nabla_{\underline{n}} \mathcal{V}_{\underline{m}}{}^{ij} &:= \partial_{\underline{n}} \mathcal{V}_{\underline{m}}{}^{ij} + \mathcal{Q}_{\underline{n}k}{}^{[i} \mathcal{V}_{\underline{m}}{}^{j]k} - \Gamma_{\underline{nm}}{}^{\underline{p}} \mathcal{V}_{\underline{p}}{}^{ij} = \mathcal{P}_{\underline{n}}{}^{ijkl} \mathcal{V}_{\underline{m}kl} \\ &= \mathcal{D}_{\underline{n}} \mathcal{V}_{\underline{m}}{}^{ij} - \Gamma_{\underline{nm}}{}^{\underline{p}} \mathcal{V}_{\underline{p}}{}^{ij} \,. \end{aligned} \tag{3.26}$$

Here $\mathcal{Q}_{\underline{m}i}{}^j$ is the internal SU(8) connection and the non-metricity $\mathcal{P}_{\underline{m}}{}^{ijkl}$ is a pseudo-real expression in the **70** of SU(8). There is significant ambiguity in this expression because both the $\mathrm{E}_{7(7)}$ connection Γ and the set $\{\mathcal{Q}, \mathcal{P}\}$ are describing the same 56×133 degrees of freedom (up to the section condition) encoded in $\partial_{\underline{n}} \mathcal{V}_{\underline{m}}{}^{ij}$. This can be clarified as follows. Using (2.18), the internal GVP can be rewritten as

$$\partial_{\underline{n}} \mathcal{V}_{\underline{m}}{}^{ij} + (\mathcal{Q}_{\underline{n}k}{}^{[i} + \Gamma_{\underline{n}}{}^{\boldsymbol{\alpha}} \mathcal{Q}_{\boldsymbol{\alpha}k}{}^{[i}) \mathcal{V}_{\underline{m}}{}^{j]k} = \Big(\mathcal{P}_{\underline{n}}{}^{ijkl} + \Gamma_{\underline{n}}{}^{\boldsymbol{\alpha}} \mathcal{P}_{\boldsymbol{\alpha}}{}^{ijkl}\Big) \mathcal{V}_{\underline{m}kl} \,, \tag{3.27}$$

where $\mathcal{Q}_{\boldsymbol{\alpha}i}{}^j$ and $\mathcal{P}_{\boldsymbol{\alpha}}{}^{ijkl}$ correspond to the “flattened” components of the $\mathrm{E}_{7(7)}$ generator, living in the **63** and **70** of SU(8), defined by

$$(t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}} \mathcal{V}_{\underline{n}}{}^{ij} = -Q_{\boldsymbol{\alpha}k}{}^{[i} \mathcal{V}_{\underline{m}}{}^{j]k} + P_{\boldsymbol{\alpha}}{}^{ijkl} \mathcal{V}_{\underline{m}kl} \,. \tag{3.28}$$

Evidently one can solve (3.27) either for $\mathcal{Q}_{\underline{m}i}{}^j$ and $\mathcal{P}_{\underline{m}}{}^{ijkl}$ in terms of $\Gamma_{\underline{mn}}{}^{\underline{p}}$, or $\Gamma_{\underline{mn}}{}^{\underline{p}}$ in terms of $\mathcal{Q}_{\underline{m}i}{}^j$ and $\mathcal{P}_{\underline{m}}{}^{ijkl}$. In the latter case, one finds that

$$\Gamma_{\underline{nm}}{}^{\underline{p}} = -i \mathcal{D}_{\underline{n}} \mathcal{V}_{\underline{m}}{}^{ij} \mathcal{V}^{\underline{p}}{}_{ij} + i \mathcal{P}_{\underline{n}}{}^{ijkl} \mathcal{V}_{\underline{m}kl} \mathcal{V}^{\underline{p}}{}_{ij} + \text{c.c.} \tag{3.29}$$

There are a myriad of ways to reduce the ambiguity. One particular way is to set $\mathcal{Q}$ and $\mathcal{P}$ to zero, eliminating it entirely. This is the Weitzenböck connection, and we denote it by $\mathring{}$ over the various symbols:

$$\text{Weitzenböck connection:} \quad \mathring{\mathcal{Q}} = \mathring{\mathcal{P}} = 0 \,, \quad \mathring{\Gamma}_{\underline{mn}}{}^{\underline{p}} = -i\, \partial_{\underline{m}} \mathcal{V}_{\underline{n}}{}^{ij} \mathcal{V}_{ij}{}^{\underline{p}} + \text{c.c.} \tag{3.30}$$

The conventional choice in ExFT is different [22]. It involves taking $\mathcal{P} = 0$, and then eliminating as much (but not all) of the ambiguity by imposing constraints on Γ, through the vanishing torsion condition (2.16). Let's review some group theory briefly to remember how the various parts of Γ are usually constrained in ExFT. Decomposing under $\mathrm{E}_{7(7)}$ and then SU(8) gives

$$\begin{aligned}\Gamma: \qquad \mathbf{56}\times\mathbf{133} &= \mathbf{56}+\mathbf{912}+\mathbf{6480}\,,\\ \mathbf{56} &\to \mathbf{28}+\overline{\mathbf{28}}\,,\\ \mathbf{912} &\to \mathbf{36}+\mathbf{420}+\text{c.c.}\,,\\ \mathbf{6480} &\to \mathbf{28}+\mathbf{420}+\mathbf{1280}+\mathbf{1512}+\text{c.c.}\,,\end{aligned}$$

$\mathcal{Q}$ and $\mathcal{P}$ decompose under SU(8) as

$$\begin{aligned}\mathcal{Q}: &\qquad (\mathbf{28}+\overline{\mathbf{28}})\times\mathbf{63} = \mathbf{28}+\mathbf{36}+\mathbf{420}+\mathbf{1280}+\text{c.c.}\,,\\ \mathcal{P}: &\qquad (\mathbf{28}+\overline{\mathbf{28}})\times\mathbf{70} = \mathbf{28}+\mathbf{420}+\mathbf{1512}+\text{c.c.}\end{aligned}$$

In conventional ExFT, after setting $\mathcal{P}$ to zero, Γ and $\mathcal{Q}$ have residual ambiguity: the representations appearing in $\mathcal{Q}$ are counted twice. Killing the **912** part of Γ (corresponding to torsion) removes $\mathbf{36}+\mathbf{420}$ and their conjugates. Constraining the **56** part of Γ to be related to $e^{-1}\partial_{\underline{m}}e$ removes further ambiguity. The remaining ambiguity is the **1280** and its conjugate that appear in both $\mathcal{Q}$ and the **6480** of Γ. There is no $\mathrm{E}_{7(7)}$ covariant way to eliminate this piece. However, as shown in [22], this undetermined piece always drops out of the SUSY transformations.

There is already a reason to reconsider this approach when generalizing to superspace. The determinant $e = \det e_m{}^a$ is, like the inverse vielbein, an unnatural object to encounter in superspace as it violates general supercovariance, so the constraint imposed on the **56** part of Γ is difficult to lift to superspace. The superdeterminant $E = \operatorname{sdet} E_M{}^A$ is more natural, but does not reduce naturally to e when returning to components. This suggests that one should leave the **56** part unfixed and hope for it to drop out of the SUSY transformations as well. Actually, as we will demonstrate, there is no need to fix *any* of the ambiguity in Γ, $\mathcal{Q}$, and $\mathcal{P}$. We will allow both the non-metricity $\mathcal{P}$ and the torsion tensor $\mathcal{T}_{\underline{mn}}{}^{\underline{p}}$, defined in (2.15), to be nonzero. This requires that the SUSY transformations be modified to include contributions of these tensors, but in the result, all of the undetermined pieces drop out, not just the one in the **1280**.

Because the torsion tensor lies in the **912**, we can employ the same representation theory as for the embedding tensor [34]. Defining the tangent space components $\mathcal{T}_{\underline{ca}}{}^{\underline{b}} = \mathcal{V}_{\underline{c}}{}^{\underline{m}}\mathcal{V}_{\underline{a}}{}^{\underline{n}}\mathcal{T}_{\underline{mn}}{}^{\underline{p}}\mathcal{V}_{\underline{p}}{}^{\underline{b}}$ as for the embedding tensor,

$$\mathcal{T}_{ij\,\underline{a}}{}^{\underline{b}} = \begin{pmatrix} -\frac{2}{3}\delta_{[k}{}^{[p}\,\mathcal{T}^{q]}{}_{l]ij} & \frac{1}{24}\varepsilon_{klpqrstu}\,\mathcal{T}^{rstu}{}_{ij} \\[2ex] \mathcal{T}^{klpq}{}_{ij} & \frac{2}{3}\delta_{[p}{}^{[k}\,\mathcal{T}^{l]}{}_{q]ij}\end{pmatrix} \tag{3.31}$$

one finds

$$\begin{aligned}\mathcal{T}^{klmn}{}_{ij} &= -\frac{4}{3}\delta_{[i}{}^{[k}\,\mathcal{T}_{j]}{}^{lmn]}\,,\\ \mathcal{T}_i{}^{jkl} &= 2\times\left(-\frac{3}{4}A_{2i}{}^{jkl} - \frac{3}{2}A_1{}^{j[k}\,\delta^{l]}{}_i\right),\end{aligned} \tag{3.32}$$

where $A_1{}^{ij} = A_1{}^{(ij)}$ is in the **36** and $A_{2i}{}^{jkl} = A_{2i}{}^{[jkl]}$ (and traceless) is in the **420**. Here we have inserted an additional factor of 2 in the last relation to match the historical conventions for the so-called T-tensor.[12]

Later on, it will be useful to extract the undetermined pieces in the internal connections to ensure that they cancel. We will do this by converting to the Weitzenböck connection, and writing expressions in terms of $\mathring{\Gamma}$, isolating the undetermined pieces into $\mathcal{Q}$ and $\mathcal{P}$. In terms of the Weitzenböck connection, one can show that

$$A_1{}^{ij} = \mathring{A}_1{}^{ij} - i\mathcal{V}^{\underline{m}k(i}\mathcal{Q}_{\underline{m}k}{}^{j)}\,, \tag{3.33a}$$

$$\begin{aligned} A_{2i}{}^{jkl} &= \mathring{A}_{2i}{}^{jkl} + \Big[4i\,\mathcal{P}_{\underline{m}}{}^{jklp}\mathcal{V}^{\underline{m}}{}_{ip} + 3i\,\mathcal{Q}_{\underline{m}i}{}^{j}\mathcal{V}^{\underline{m}kl}\Big]_{\mathbf{420}} \\ &= \mathring{A}_{2i}{}^{jkl} + 4i\,\mathcal{P}_{\underline{m}}{}^{jklp}\mathcal{V}^{\underline{m}}{}_{ip} - 2i\,\delta_i{}^{[j}\mathcal{P}_{\underline{m}}{}^{kl]pq}\mathcal{V}^{\underline{m}}{}_{pq} \\ &\quad + 3i\mathcal{Q}_{\underline{m}i}{}^{[j}\mathcal{V}^{\underline{m}kl]} - i\,\delta_i{}^{[j}\mathcal{Q}_{\underline{m}p}{}^{k}\mathcal{V}^{\underline{m}l]p}\,, \end{aligned} \tag{3.33b}$$

$$\Gamma_{\underline{nm}}{}^{\underline{p}} = \mathring{\Gamma}_{\underline{nm}}{}^{\underline{p}} + i\mathcal{Q}_{\underline{n}kl}{}^{ij}\mathcal{V}_{\underline{m}}{}^{kl}\mathcal{V}^{\underline{p}}{}_{ij} + (i\mathcal{P}_{\underline{n}}{}^{ijkl}\mathcal{V}_{\underline{m}kl}\mathcal{V}^{\underline{p}}{}_{ij} + \text{c.c.}) \tag{3.33c}$$

We have included above the corresponding formula for the $E_{7(7)}$ connection.

From now on, unless we comment otherwise, $\nabla_{\underline{m}}$ will correspond to an internal covariant derivative carrying an $E_{7(7)}$ connection and SU(8) connection with arbitrary non-vanishing torsion and non-metricity. It will not carry any internal spin connection.

4 $\mathbf{E_{7(7)}}$ exceptional field theory in $(4+56|32)$ superspace

Now we turn to the construction of $E_{7(7)}$ exceptional field theory in superspace. In addition to the four external coordinates x^m describing spacetime and the 56 internal coordinates $y^{\underline{m}}$ describing the exceptional structure, there will be 32 anticommuting (Grassmann) coordinates, which we split into chiral and antichiral coordinates θ^μ and $\theta_{\dot\mu}$.[13] Supersymmetry will be associated with diffeomorphisms in the fermionic direction, in a manner to be described in due course. The full set of coordinates are collectively denoted $Z^{\underline{M}}$,

$$Z^{\underline{M}} = \{Z^M, y^{\underline{m}}\} = \{x^m, \theta^\mu, \theta_{\dot\mu}, y^{\underline{m}}\}\ . \tag{4.1}$$

We reserve Z^M to denote the $(4|32)$ coordinates $(x^m, \theta^\mu, \theta_{\dot\mu})$ parametrizing an "external" 4D $N=8$ superspace, with the additional 56 "internal" coordinates $y^{\underline{m}}$ describing the exceptional structure.

The supervielbein on the $(4|32)$ superspace is denoted $E_M{}^A$ whose tangent space index A decomposes as $(a, \alpha\, i, \dot\alpha\, i)$ so that

$$E_M{}^A = (E_M{}^a, E_M{}^{\alpha i}, E_{M\dot\alpha i})\ . \tag{4.2}$$

We will refer to $E_M{}^a$ and $E_M{}^{\alpha i}$ as the vielbein and gravitino super one-forms, respectively, as they are the superfield analogues of $e_m{}^a$ and $\psi_m{}^{\alpha i}$. The internal exceptional space is

[12] In other words, the tangent space components of $\mathcal{T}$ differ from the T-tensor by a factor of 2.

[13] Here we use the chiral (antichiral) Grassmann coordinate θ^μ ($\theta_{\dot\mu}$) with $\mu = 1, \cdots, 16$. Typically in 4D $N=8$ superspace, one writes $\theta^{\mu I}$ with $\mu = 1,2$ and $I = 1, \cdots, 8$ as curved analogues of the two-component spinor and SU(8) indices. For compactness, we use μ collectively for μI.

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equipped with a 56-bein,

$$\mathcal{V}_{\underline{m}}{}^{\underline{a}} = (\mathcal{V}_{\underline{m}}{}^{ij}, \mathcal{V}_{\underline{m}ij}) \ . \tag{4.3}$$

While it would be natural to encode $E_M{}^A$ and $\mathcal{V}_{\underline{m}}{}^{\underline{a}}$ as components of an even larger sehrvielbein $\mathcal{E}_{\underline{M}}{}^{\underline{A}}$ (as in central charge superspace [28]), we will leave discussion of this to future work. From the point of view of the external 4D $N = 8$ superspace, the 56-bein is a scalar superfield while $E_M{}^A$ are components of a super one-form $E^A = \mathrm{d}Z^M E_M{}^A$.

In addition to the external and internal vielbeins, there is a tensor hierarchy of p-forms. These include the one-form vector field

$$A^{\underline{m}} = \mathrm{d}Z^M A_M{}^{\underline{m}} , \tag{4.4}$$

and two super two-forms

$$B_{\boldsymbol{\alpha}} = \frac{1}{2}\mathrm{d}Z^M \mathrm{d}Z^N B_{NM\,\boldsymbol{\alpha}} , \qquad B'_{\underline{m}} = \frac{1}{2}\mathrm{d}Z^M \mathrm{d}Z^N B'_{NM\,\underline{m}} \ . \tag{4.5}$$

As in components, the second two-form is a constrained tensor on its fundamental $\mathrm{E}_{7(7)}$ index. Just as in ExFT, there are additional 3-forms and 4-forms making up the tensor hierarchy, but we will stop our analysis at the two-forms. *Because the unprimed B does not naturally occur in superspace, henceforth, we will drop the prime.*

Finally, the superspace is also equipped with a pair of one-forms that gauge the local tangent group $\mathrm{SO}(1,3) \times \mathrm{SU}(8)$. These are the Lorentz connection $\Omega_a{}^b$ and the SU(8) connection $\mathcal{Q}_i{}^j$,

$$\Omega_a{}^b = \mathrm{d}Z^M \Omega_{M a}{}^b , \qquad \mathcal{Q}_i{}^j = \mathrm{d}Z^M \mathcal{Q}_{M i}{}^j \ . \tag{4.6}$$

Constraints in superspace will be chosen so that these connections become composite, describing no independent degrees of freedom of their own.

These superfields each have natural analogues in the component theory. The only component field we have not mentioned in superspace yet is the spin-1/2 fermion. And indeed, there is also a fermionic superfield, which we denote $\chi_\alpha{}^{ijk}$, whose lowest component is the field of the same name. In conventional 4D $N = 8$ superspace [24, 25], this superfield actually appears in the curvature super-forms, and so can be treated as a derived quantity. In exceptional superspace, it plays a somewhat more fundamental role, as the gravitino one-form $E_M{}^{\alpha i}$ turns out to directly transform into it under external superdiffeomorphisms.

4.1 Symmetry transformations

Under internal diffeomorphisms ($\Lambda^{\underline{m}}$) and the tensor hierarchy transformations ($\Xi_{M\,\boldsymbol{\alpha}}$, $\Xi_{M\,\underline{m}}$, $\Omega_{MN}{}^{\underline{m}}{}_{\boldsymbol{\alpha}}$,$\Omega_{MN\underline{m}}{}^{\underline{n}}$) the various superfields transform exactly as their component analogues, with

$$
\begin{aligned}
\delta E_M{}^A &= \mathbb{L}_\Lambda E_M{}^A\,,\\
\delta \mathcal{V}_{\underline{m}}{}^{\underline{a}} &= \mathbb{L}_\Lambda \mathcal{V}_{\underline{m}}{}^{\underline{a}}\,,\\
\delta \chi_\alpha{}^{ijk} &= \mathbb{L}_\Lambda \chi_\alpha{}^{ijk}\,,\\
\delta A_M{}^{\underline{m}} &= D_M \Lambda^{\underline{m}} + 12\,(t^{\boldsymbol{\alpha}})^{\underline{mn}} \partial_{\underline{n}} \Xi_{M\,\boldsymbol{\alpha}} + \frac{1}{2}\,\Omega^{\underline{mn}}\,\Xi_{M\,\underline{n}}\,,\\
\Delta B_{NM\,\boldsymbol{\alpha}} &= (t_{\boldsymbol{\alpha}})_{\underline{mn}}\,\Lambda^{\underline{m}} \mathcal{F}_{NM}{}^{\underline{n}} + 2\,D_{[N}\Xi_{M]\boldsymbol{\alpha}} + \partial_{\underline{m}} \Omega_{NM}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}} \Omega_{NM\,\underline{n}}{}^{\underline{m}}\,,\\
\Delta B_{NM\,\underline{m}} &= \mathcal{F}_{NM}{}^{\underline{n}} \partial_{\underline{m}} \Lambda_{\underline{n}} - \partial_{\underline{m}} \mathcal{F}_{NM}{}^{\underline{n}} \Lambda_{\underline{n}} + 2 D_{[N}\Xi_{M]\underline{m}} + 48\,(t^{\boldsymbol{\alpha}})_{\underline{n}}{}^{\underline{r}} \left(\partial_{\underline{r}} \partial_{\underline{m}} A_{[N}{}^{\underline{n}}\right) \Xi_{M]\boldsymbol{\alpha}}\\
&\quad - \partial_{\underline{m}} \Omega_{NM\,\underline{n}}{}^{\underline{n}} - 2\,\partial_{\underline{n}} \Omega_{NM\,\underline{m}}{}^{\underline{n}}\,,
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
\Delta B_{NM\,\boldsymbol{\alpha}} &:= \delta B_{NM\,\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{\underline{mn}}\, A_{[N}{}^{\underline{m}}\, \delta A_{M]}{}^{\underline{n}}\,,\\
\Delta B_{NM\,\underline{m}} &:= \delta B_{NM\,\underline{m}} + A_{[N}{}^{\underline{n}} \partial_{\underline{m}} \delta A_{M]\underline{n}} - \partial_{\underline{m}} A_{[N}{}^{\underline{n}} \delta A_{M]\underline{n}}\,.
\end{aligned}
\tag{4.8}
$$

Here the λ weights of the various superfields match their component cousins. The curvature two-form $\mathcal{F}_{NM}{}^{\underline{m}}$ (and later, the three-forms $H_{PNM\,\boldsymbol{\alpha}}$ and $H_{PNM\,\underline{m}}$) are defined exactly as in components, replacing $m \to M$. We collect their superspace definitions in the next subsection.

There is a minor technical subtlety that the superindices M and A come equipped with a $\mathbb{Z}_2$ grading, which causes certain signs to appear when their relative ordering changes. For example $[NM]$ above should be understood as a graded commutator. This is common in superspace and we briefly review it in appendix A. To keep formulae as legible as possible, we suppress such grading factors.

The transformations under external diffeomorphisms are somewhat more involved. We list first the ones whose transformations can be directly compared to (2.21):

$$
\begin{aligned}
\delta E_M{}^a &= D_M \xi^N\, E_N{}^a + \xi^N D_N E_M{}^a\,,\\
\delta \mathcal{V}_{\underline{m}}{}^{\underline{a}} &= \xi^M D_M \mathcal{V}_{\underline{m}}{}^{\underline{a}}\,,\\
\delta A_M{}^{\underline{m}} &= \xi^N \mathcal{F}_{NM}{}^{\underline{m}} + \mathcal{M}^{\underline{mn}} \partial_{\underline{n}} \xi^N\, G_{NM}\,, \qquad\qquad G_{NM} := E_N{}^a E_{Ma}\\
\Delta B_{NM\,\boldsymbol{\alpha}} &= \xi^P H_{PNM\,\boldsymbol{\alpha}}\,.
\end{aligned}
\tag{4.9}
$$

A relevant feature is, as in components, the appearance of an anomalous term in $\delta A_M{}^{\underline{m}}$ involving $\partial_{\underline{m}} \xi^N$. The constrained two-form $B_{NM\,\underline{m}}$ transforms not only with an additional explicit spin connection term (matching the redefined component two-form (3.6)), but with a few additional terms involving the gravitino one-forms,

$$
\begin{aligned}
\Delta B_{NM\,\underline{m}} = \xi^P H_{PNM\underline{m}} &- 2i\, E_N{}^a E_M{}^b \varepsilon_{abcd}\, \partial_{\underline{m}} \xi^P \Omega_P{}^{cd}\\
&- 16i\, \partial_{\underline{m}} \xi^P E_{P\dot{\alpha} i}\, E_{[N}{}^{\alpha i} E_{M]}{}^a (\gamma_a)_\alpha{}^{\dot{\alpha}} + 16i\, \partial_{\underline{m}} \xi^P E_P{}^{\alpha i}\, E_{[N\dot{\alpha} i} E_{M]}{}^a (\gamma_a)_\alpha{}^{\dot{\alpha}}\,.
\end{aligned}
\tag{4.10}
$$

The gravitino one-form $E_M{}^{\alpha i}$ and the spin-1/2 fermion $\chi^{\alpha\, ijk}$ have even more involved anomalous terms,

$$\begin{aligned}\delta E_M{}^{\alpha i} &= D_M\xi^N E_N{}^{\alpha i} + \xi^N D_N E_M{}^{\alpha i} + 2i\,\mathcal{V}^{\underline{m}ij}\partial_{\underline{m}}\xi^N\Big(E_{N\,\dot\beta j}E_M{}^c - E_N{}^c E_{M\,\dot\beta j}\Big)(\gamma_c)^{\dot\beta\alpha}\\ &\quad - \frac{i}{2\sqrt{2}}\mathcal{V}^{\underline{m}}{}_{jk}\partial_{\underline{m}}\xi^N E_N{}^d E_M{}^c\chi^{\beta jki}(\gamma_c\gamma_d)_\beta{}^\alpha\,,\\ \delta\chi^{\alpha ijk} &= \xi^N D_N\chi^{\alpha ijk} - 12i\sqrt{2}\,\mathcal{V}^{\underline{n}\,[ij}\partial_{\underline{n}}\xi^N E_N{}^{\alpha k]} + \frac{i}{6}\varepsilon^{ijkpqrst}\mathcal{V}^{\underline{n}}{}_{pq}\partial_{\underline{n}}\xi^N E_N{}^b\bar\chi_{\dot\beta rst}(\gamma_b)^{\dot\beta\alpha}\,.\end{aligned} \tag{4.11}$$

While the exact relation between the supervielbein and the component fields has not been specified yet, a natural definition, which will be provided in section 5, motivates the above form of the external superdiffeomorphisms in view of the supersymmetry transformation rules (3.1). Note also that, as already mentioned above, because the gravitino transforms directly into χ, we are led to treat χ on the same level as the gravitino and the other fundamental superfields rather than as a derived curvature superfield.

Finally, we should mention that just as in components, the constrained two-form $B_{NM\underline{m}}$ possesses an anomalous Lorentz transformation,

$$\delta B_{NM\,\underline{m}} = -2i\,E_N{}^a E_M{}^b\varepsilon_{abcd}\,\partial_{\underline{m}}\lambda^{cd}\,. \tag{4.12}$$

4.2 Covariant external superdiffeomorphisms and modified curvature tensors

The curvature super-forms of the tensor hierarchy have already appeared above in the symmetry transformations of the tensor hierarchy fields. They are defined as at the component level[14]

$$\mathcal{F}^{\underline{m}} = \mathrm{d}A^{\underline{m}} + \frac{1}{2}[A,A]_{\mathrm{E}}^{\underline{m}} - 12\,(t^{\boldsymbol\alpha})^{\underline{mn}}\,\partial_{\underline{n}}B_{\boldsymbol\alpha} - \frac{1}{2}\,\Omega^{\underline{mn}}\,B_{\underline{n}}\,, \tag{4.13}$$

$$H_{\boldsymbol\alpha} = DB_{\boldsymbol\alpha} - \frac{1}{2}(t_{\boldsymbol\alpha})_{\underline{mn}}A^{\underline{m}}\mathrm{d}A^{\underline{n}} + \frac{1}{12}(t_{\boldsymbol\alpha})_{\underline{mn}}A^{\underline{m}}[A,A]_{\mathrm{E}}^{\underline{n}} - \partial_{\underline{m}}C^{\underline{m}}{}_{\boldsymbol\alpha} - (t_{\boldsymbol\alpha})_{\underline{m}}{}^{\underline{n}}C_{\underline{n}}{}^{\underline{m}}\,, \tag{4.14}$$

$$\begin{aligned}H_{\underline{m}} &= DB_{\underline{m}} + 24\,(t_{\boldsymbol\alpha})_{\underline{r}}{}^{\underline{s}}\,\partial_{\underline{m}}\partial_{\underline{s}}A^{\underline{r}}\,B_{\boldsymbol\alpha} - \frac{1}{2}\Omega_{\underline{rs}}\,\partial_{\underline{m}}A^{\underline{r}}\,\mathrm{d}A^{\underline{s}} + \frac{1}{2}\Omega_{\underline{rs}}\,A^{\underline{r}}\,\partial_{\underline{m}}\mathrm{d}A^{\underline{s}}\\ &\quad - \frac{1}{3}\Omega_{\underline{rs}}\Big(A^{\underline{r}}\,\partial_{\underline{m}}[A,A]_E^{\underline{s}} + [A,A]_E^{\underline{s}}\,\partial_{\underline{m}}A^{\underline{r}}\Big) + \partial_{\underline{m}}C_{\underline{n}}{}^{\underline{n}} + 2\,\partial_{\underline{n}}C_{\underline{m}}{}^{\underline{n}}\,.\end{aligned} \tag{4.15}$$

We have included the super 3-forms $C^{\underline{m}}{}_{\boldsymbol\alpha}$ and $C_{\underline{m}}{}^{\underline{n}}$ in their definitions for completeness, but they will not play a major role in the subsequent discussion. We emphasize that because $B_{\underline{m}}$ transforms anomalously under Lorentz transformations, the same is true of its curvature $H_{\underline{m}}$. In fact, the curvature $H_{\underline{m}}$ is not even a tensor under internal diffeomorphisms, a fact that we will return to soon.

In addition to the tensor hierarchy curvatures, there are curvature super-forms associated with the supervielbein $E_M{}^A$ and the Lorentz and SU(8) connections, Ω and $\mathcal{Q}$. The former define the super torsion tensor,

$$T^A := \mathcal{D}E^A = DE^A + E^B\Omega_B{}^A + E^B\mathcal{Q}_B{}^A \tag{4.16}$$

[14]The reader is cautioned that we employ superspace conventions for differential forms, see e.g. [39] and appendix A.

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where

$$\Omega_B{}^A = \begin{pmatrix} \Omega_b{}^a & 0 & 0 \\ 0 & \frac{1}{4}\Omega^{cd}(\gamma_{cd})_\beta{}^\alpha & 0 \\ 0 & 0 & \frac{1}{4}\Omega^{cd}(\gamma_{cd})^{\dot\beta}{}_{\dot\alpha} \end{pmatrix}, \qquad \mathcal{Q}_B{}^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}\mathcal{Q}_j{}^i\delta_\beta{}^\alpha & 0 \\ 0 & 0 & \frac{1}{2}\mathcal{Q}_i{}^j\delta^{\dot\beta}{}_{\dot\alpha} \end{pmatrix}, \tag{4.17}$$

as well as the Lorentz and SU(8) curvatures,

$$R_a{}^b := D\Omega_a{}^b + \Omega_a{}^c\Omega_c{}^b\,, \qquad R_i{}^j := D\mathcal{Q}_i{}^j - \frac{1}{2}\mathcal{Q}_i{}^k\mathcal{Q}_k{}^j\ . \tag{4.18}$$

Typically, the superspace Bianchi identities determine the latter curvatures in terms of the torsion tensor. One then finds that imposing suitable constraints on the tangent space components $T_{CB}{}^A$ of the torsion tensor prove to define the supergeometry. However, the situation is more subtle in exceptional superspace. The main reason is that the curvature tensors we have introduced above are not actually the natural curvature tensors from the point of view of superspace. By this, we mean that some of them do not possess natural expansions in terms of the superspace frame E^A.

It turns out to be more illuminating to first consider the curvature two-form $\mathcal{F}^{\underline{m}}$. Recall that under external diffeomorphisms

$$\delta A_M{}^{\underline{m}} = \xi^N\mathcal{F}_{NM}{}^{\underline{m}} + \mathcal{M}^{\underline{mn}}\nabla_{\underline{n}}\xi^N E_N{}^a E_{Ma}\,,$$

where we have used $\partial_{\underline{n}}\xi^N = \nabla_{\underline{n}}\xi^N$. We would like to rewrite this expression as a covariant external diffeomorphism. A covariant external diffeomorphism is defined in terms of the tangent space parameter $\xi^A = \xi^M E_M{}^A$ as

$$\delta_{\rm cov}(\xi^A) = \delta(\xi^M) - \delta_{\rm Lorentz}(\xi^N\Omega_N{}^{ab}) - \delta_{\rm SU(8)}(\xi^N\mathcal{Q}_{Ni}{}^j)\ . \tag{4.19}$$

For the vector fields, the additional transformations do not contribute, but nevertheless the transformation rule takes a different form when rewritten in terms of ξ^A:

$$\delta_{\rm cov}(\xi)A_M{}^{\underline{m}} = \xi^N\widehat{\mathcal{F}}_{NM}{}^{\underline{m}} + \mathcal{M}^{\underline{mn}}\nabla_{\underline{n}}\xi^a E_{Ma} - \mathcal{M}^{\underline{mn}}\xi^a\,\nabla_{\underline{n}}E_{Ma}\,, \tag{4.20}$$

where we have defined

$$\widehat{\mathcal{F}}_{NM}{}^{\underline{m}} := \mathcal{F}_{NM}{}^{\underline{m}} + 2\mathcal{M}^{\underline{mn}}\,E_{[N}{}^a\,\nabla_{\underline{n}}E_{M]a}\ . \tag{4.21}$$

This expression for $\widehat{\mathcal{F}}^{\underline{m}}$ proves to be the superspace analogue of the component modified field strength introduced in (3.11).[15] Recall that the motivation for introducing this modified field strength in components was that, upon redefining the constrained two-form, it was this modified field strength that possessed the twisted self-duality relation. The analogous statement in superspace is that $\widehat{\mathcal{F}}^{\underline{m}}$ will be the tensor that is constrained in order to define the supergeometry. That is, it will be chosen to possess a sensible expansion[16]

$$\widehat{\mathcal{F}}_{NM}{}^{\underline{m}} = E_N{}^B E_M{}^A\widehat{\mathcal{F}}_{BA}{}^{\underline{m}}\,, \tag{4.22}$$

[15] One can trade $\partial_{\underline{n}}$ for $\nabla_{\underline{n}}$ in (4.21) as the connection terms drop out.

[16] Of course, because the supervielbein is assumed to be invertible, one can always define $\mathcal{F}_{BA}{}^{\underline{m}} := E_B{}^N E_A{}^M\mathcal{F}_{NM}{}^{\underline{m}}$. The problem is that one finds a contribution to $\mathcal{F}_{Ba}{}^{\underline{m}}$ of the form $\mathcal{M}^{\underline{mn}}E_B{}^N\nabla_{\underline{n}}E_{Na}$ that is difficult to make sense of upon reducing to components. No such contribution to $\widehat{\mathcal{F}}_{BA}{}^{\underline{m}}$ occurs.

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where (as we will discuss in the next section) the tangent space components $\widehat{\mathcal{F}}_{BA}{}^{\underline{m}}$ are set equal to other covariant superfields (such as $\chi^{\alpha ijk}$) or are constrained in some other way, e.g. twisted self-duality (up to fermions) in the case of $\widehat{\mathcal{F}}_{ba}{}^{\underline{m}}$. As at the component level, the term we have added in (4.21) is not Lorentz invariant because the internal derivative carries no internal spin connection. But $\mathcal{F}^{\underline{m}}$ is itself not Lorentz invariant due to the anomalous Lorentz transformation (4.12) of the constrained two-form. This leads (only) the top component $\widehat{\mathcal{F}}_{ab}{}^{\underline{m}}$ in (4.22) to transform. As in components, we find, for the inhomogeneous part of the Lorentz transformation,

$$\delta_{\text{anom}}\widehat{\mathcal{F}}_{ab}{}^{ij} = 2i\,\mathcal{V}^{\underline{m}ij}\left(\delta_a{}^c\delta_b{}^d - \frac{1}{2}\varepsilon_{ab}{}^{cd}\right)\partial_{\underline{m}}\lambda_{cd} = 4i\,\mathcal{V}^{\underline{m}ij}\partial_{\underline{m}}\lambda^-_{ab}\,. \tag{4.23}$$

What about the supervielbein? For the vierbein one-form $E_M{}^a$, it turns out that a covariant external diffeomorphism leads to the usual expression

$$\delta_{\text{cov}}(\xi)E_M{}^a = \mathcal{D}_M\xi^a + \xi^N T_{NM}{}^a\,, \tag{4.24}$$

which suggests that the vector torsion tensor possesses a sensible tangent space expansion

$$T_{NM}{}^a = E_N{}^C E_M{}^B T_{CB}{}^a\,, \tag{4.25}$$

without modification. For the gravitino $E_M{}^{\alpha i}$, the situation is more subtle. We find

$$\begin{aligned}\delta_{\text{cov}}(\xi)E_M{}^{\alpha i} &= \mathcal{D}_M\xi^{\alpha i} + \xi^N T_{NM}{}^{\alpha i} + 2i\,\mathcal{V}^{\underline{m}ij}\nabla_{\underline{m}}\xi^N\left(E_{N\,\dot\beta j}E_M{}^c - E_N{}^c E_{M\,\dot\beta j}\right)(\gamma_c)^{\dot\beta\alpha}\\ &\quad - \frac{i}{2\sqrt{2}}\mathcal{V}^{\underline{m}}{}_{jk}\nabla_{\underline{m}}\xi^N E_N{}^d E_M{}^c\chi^{\beta jki}(\gamma_c\gamma_d)_\beta{}^\alpha\,.\end{aligned} \tag{4.26}$$

This can be rewritten as

$$\begin{aligned}\delta_{\text{cov}}(\xi)E_M{}^{\alpha i} &= \mathcal{D}_M\xi^{\alpha i} + \xi^N\widehat{T}_{NM}{}^{\alpha i}\\ &\quad + 2i\,\mathcal{V}^{\underline{m}ij}\nabla_{\underline{m}}\left(\bar\xi_{\dot\beta j}E_M{}^c(\gamma_c)^{\dot\beta\alpha}\right) - 2i\,\mathcal{V}^{\underline{m}ij}\nabla_{\underline{m}}\left(\xi^c E_{M\dot\beta j}(\gamma_c)^{\dot\beta\alpha}\right)\\ &\quad - \frac{i}{2\sqrt{2}}\left(\nabla_{\underline{m}}\xi^d\,E_M{}^c - \xi^c\nabla_{\underline{m}}E_M{}^d\right)\mathcal{V}^{\underline{m}}{}_{jk}\chi^{\beta jki}(\gamma_c\gamma_d)_\beta{}^\alpha\,,\end{aligned} \tag{4.27}$$

where we have exchanged ξ^M in the additional terms for $\xi^A = \xi^M E_M{}^A$. The modified gravitino torsion tensor in this expression is

$$\widehat{T}^{\alpha i} = T^{\alpha i} + 2i\,\mathcal{V}^{\underline{m}ij}\nabla_{\underline{m}}(E_{\dot\beta j}\wedge E^c)(\gamma_c)^{\dot\beta\alpha} - \frac{i}{2\sqrt{2}}\mathcal{V}^{\underline{m}}{}_{jk}\,\chi^{\beta jki}(\gamma_c\gamma_d)_\beta{}^\alpha\nabla_{\underline{m}}E^d\wedge E^c\quad. \tag{4.28}$$

As we will see, this leads to a sensible tangent space expansion $\widehat{T}_{NM}{}^{\alpha i} = E_N{}^C E_M{}^B\widehat{T}_{CB}{}^{\alpha i}$.

The internal counterparts to the supervielbein are the two superfields $\mathcal{V}_{\underline{m}}{}^{ij}$ and $\chi^{\alpha ijk}$. While these are not gauge superfields, they also have curvatures naturally associated with them: their covariant derivatives. Because the 56-bein transforms under external covariant diffeomorphisms as

$$\delta_{\text{cov}}(\xi)\mathcal{V}_{\underline{m}}{}^{ij} = \xi^M\mathcal{D}_M\mathcal{V}_{\underline{m}}{}^{ij}\,, \tag{4.29}$$

its covariant derivative should possess a sensible tangent space expansion. As in components we take

$$\mathcal{D}\mathcal{V}_{\underline{m}}{}^{ij} = \mathcal{P}^{ijkl}\mathcal{V}_{\underline{m}kl}\,, \qquad \mathcal{P}^{ijkl} = E^A\mathcal{P}_A{}^{ijkl}\,, \tag{4.30}$$

to both define the one-form $\mathcal{P}^{ijkl}$ valued in the **70** of SU(8) and the SU(8) connection $\mathcal{Q}_i{}^j$. However, the fermion superfield $\chi^{\alpha ijk}$ has additional terms in its transformation rule, suggesting we define the one-form

$$\widehat{\tau}_M{}^{\alpha ijk} := \mathcal{D}_M\chi^{\alpha ijk} + 12i\sqrt{2}\,\mathcal{V}^{\underline{n}[ij}\nabla_{\underline{n}}E_M{}^{\alpha k]} - \frac{i}{6}\varepsilon^{ijklpqrs}\,\mathcal{V}^{\underline{n}}{}_{lp}\nabla_{\underline{n}}E_{Mb}\chi_{\dot\beta qrs}(\gamma_b)^{\dot\beta\alpha}\,, \tag{4.31}$$

so that

$$\delta_{\rm cov}(\xi)\chi^{\alpha ijk} = \xi^M\widehat{\tau}_M{}^{\alpha ijk} - 12i\sqrt{2}\,\mathcal{V}^{\underline{n}[ij}\nabla_{\underline{n}}\xi^{\alpha k]} + \frac{i}{6}\varepsilon^{ijklpqrs}\,\mathcal{V}^{\underline{n}}{}_{lp}\nabla_{\underline{n}}\xi^b\chi_{\dot\beta qrs}(\gamma_b)^{\dot\beta\alpha}\,. \tag{4.32}$$

This suggests that the one-form $\widehat{\tau}$ possesses a natural tangent space expansion, $\widehat{\tau}_M{}^{\alpha ijk} = E_M{}^B\widehat{\tau}_B{}^{\alpha ijk}$.

For the two-forms, the situation is again somewhat subtle. The transformation for $B_{NM\,\boldsymbol{\alpha}}$ is unchanged,

$$\Delta_{\rm cov}(\xi)B_{NM\,\boldsymbol{\alpha}} = \xi^P H_{PNM\,\boldsymbol{\alpha}}\,, \tag{4.33}$$

suggesting that $H_{PNM\,\boldsymbol{\alpha}}$ possesses a sensible tangent space expansion. The covariant transformation of $B_{NM\,\underline{m}}$ is more involved. Keeping in mind the anomalous Lorentz transformation (4.12) and the Lorentz connection contribution to covariant diffeomorphisms (4.19), one finds that

$$\begin{aligned}\Delta_{\rm cov}(\xi)B_{NM\,\underline{m}} &= \xi^P H_{PNM\underline{m}} + 2i\,E_N{}^aE_M{}^b\varepsilon_{abcd}\,\xi^P\nabla_{\underline{m}}\Omega_P{}^{cd}\\ &\quad - 16i\,\nabla_{\underline{m}}\xi^P E_{P\dot\alpha i}\,E_{[N}{}^{\alpha i}E_{M]}{}^a(\gamma_a)_\alpha{}^{\dot\alpha}\\ &\quad + 16i\,\nabla_{\underline{m}}\xi^P E_P{}^{\alpha i}\,E_{[N\dot\alpha i}E_{M]}{}^a(\gamma_a)_\alpha{}^{\dot\alpha}\,.\end{aligned} \tag{4.34}$$

In light of the comments in section 3.2, a more natural form for the covariant variation of $B_{NM\,\underline{m}}$ is

$$\begin{aligned}\Delta_{\rm cov}B_{NM\,\underline{m}} + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}}\Delta_{\rm cov}B_{NM\,\boldsymbol{\alpha}} &= \xi^P\mathscr{H}_{PNM\,\underline{m}} + 2i\,E_N{}^aE_M{}^b\varepsilon_{abcd}\,\xi^P\nabla_{\underline{m}}\Omega_P{}^{cd}\\ &\quad - 16i\,\nabla_{\underline{m}}\xi^P E_{P\dot\alpha i}\,E_{[N}{}^{\alpha i}E_{M]}{}^a(\gamma_a)_\alpha{}^{\dot\alpha}\\ &\quad + 16i\,\nabla_{\underline{m}}\xi^P E_P{}^{\alpha i}\,E_{[N\dot\alpha i}E_{M]}{}^a(\gamma_a)_\alpha{}^{\dot\alpha}\,,\end{aligned} \tag{4.35}$$

where $\mathscr{H}_{\underline{m}}$ is the modification of the field strength $H_{\underline{m}}$,

$$\mathscr{H}_{\underline{m}} := H_{\underline{m}} + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}}\,H_{\boldsymbol{\alpha}}\,, \tag{4.36}$$

which transforms covariantly under internal diffeomorphisms, in contrast to $H_{\underline{m}}$ itself. The form of (4.35) suggests the definition

$$\widehat{\mathscr{H}}_{\underline{m}} := \mathscr{H}_{\underline{m}} - iE^aE^b\,\varepsilon_{abcd}\nabla_{\underline{m}}\Omega^{cd} + 8i\,E_a\,\nabla_{\underline{m}}E^{\alpha i}E_{\dot\alpha i}(\gamma^a)_\alpha{}^{\dot\alpha} - 8i\,E_a\,\nabla_{\underline{m}}E_{\dot\alpha i}E^{\alpha i}(\gamma^a)_\alpha{}^{\dot\alpha}\,, \tag{4.37}$$

so that

$$
\begin{aligned}
\Delta_{\rm cov} B_{NM\,\underline{m}} &+ 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}} \Delta_{\rm cov} B_{NM\,\boldsymbol{\alpha}} \\
&= \xi^P\, \widehat{\mathscr{H}}_{PNM\,\underline{m}} \\
&\quad - 16i\, \xi^{\alpha i}\, \overleftrightarrow{\nabla}_{\underline{m}} E_{[N \dot\alpha i}\, E_{M]}{}^a (\gamma_a)_\alpha{}^{\dot\alpha} + 16i\, \xi_{\dot\alpha i}\, \overleftrightarrow{\nabla}_{\underline{m}} E_{[N}{}^{\alpha i}\, E_{M]}{}^a (\gamma_a)_\alpha{}^{\dot\alpha} \\
&\quad - 4i\, \xi^a \Big(\varepsilon_{abcd}\, E_{[N}{}^b \nabla_{\underline{m}} \Omega_{M]}{}^{cd} + 4\, E_{[N}{}^{\alpha i}\, \overleftrightarrow{\nabla}_{\underline{m}} E_{M]\dot\alpha i} (\gamma_a)_\alpha{}^{\dot\alpha} \Big)\,. \qquad (4.38)
\end{aligned}
$$

Again, the suggestion is that $\widehat{\mathscr{H}}_{PNM\,\underline{m}}$ should possess a sensible tangent space expansion.

4.3 Superspace Bianchi identities

Having now some idea of the relevant superspace curvatures and what combinations of them should involve sensible tangent space expansions, we turn to a brief discussion of the Bianchi identities that need to be solved.

We begin with the fields of the p-form hierarchy. As at the component level, the field strength $\mathcal{F}^{\underline{m}}$ must obey the Bianchi identity

$$
\begin{aligned}
\mathcal{D}\mathcal{F}^{\underline{m}} &= -12\, (t^{\boldsymbol{\alpha}})^{\underline{mn}} \partial_{\underline{n}} H_{\boldsymbol{\alpha}} - \frac{1}{2} \Omega^{\underline{mn}} H_{\underline{n}} \\
&= -12\, (t^{\boldsymbol{\alpha}})^{\underline{mn}} \nabla_{\underline{n}} H_{\boldsymbol{\alpha}} + \Omega^{\underline{mn}}\, \mathcal{T}_{\underline{n}}{}^{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}} - \frac{1}{2} \Omega^{\underline{mn}} \mathscr{H}_{\underline{n}}\,, \qquad (4.39)
\end{aligned}
$$

where we have used the definition (4.37) for $\mathscr{H}_{\underline{m}}$. The above form of the Bianchi makes it apparent that $H_{\underline{m}}$ cannot be covariant under internal diffeomorphisms (because $\partial_{\underline{n}} H_{\boldsymbol{\alpha}}$ is not), whereas $\mathscr{H}_{\underline{m}}$ is. Keeping in mind that it is $\widehat{\mathcal{F}}^{\underline{m}}$ rather than $\mathcal{F}^{\underline{m}}$ that will possess a conventional tangent space expansion, one can rewrite this Bianchi identity as in (3.16)

$$
\begin{aligned}
\mathcal{D}\widehat{\mathcal{F}}^{\underline{m}} = &-12 (t^{\boldsymbol{\alpha}})^{\underline{mn}} \nabla_{\underline{n}} H_{\boldsymbol{\alpha}} + \Omega^{\underline{mn}}\, \mathcal{T}_{\underline{n}}{}^{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}} - \frac{1}{2} \Omega^{\underline{mn}} \mathscr{H}_{\underline{n}} - \mathcal{M}^{\underline{mn}} E^a \nabla_{\underline{n}} T_a + \mathcal{M}^{\underline{mn}} T^a \nabla_{\underline{n}} E_a \\
&- (D\mathcal{M}^{\underline{mn}})\, E^a \nabla_{\underline{n}} E_a + \mathcal{M}^{\underline{mn}} E^a E^b \nabla_{\underline{n}} \Omega_{ab}\,. \qquad (4.40)
\end{aligned}
$$

The 3-form field strength $H_{\boldsymbol{\alpha}}$ in turn obeys the Bianchi identity

$$
\begin{aligned}
\mathcal{D} H_{\boldsymbol{\alpha}} &= -\frac{1}{2} (t_{\boldsymbol{\alpha}})_{\underline{mn}}\, \mathcal{F}^{\underline{m}} \mathcal{F}^{\underline{n}} - \partial_{\underline{m}} G^{\underline{m}}{}_{\boldsymbol{\alpha}} - (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}} G_{\underline{n}}{}^{\underline{m}} \\
&= -\frac{1}{2} (t_{\boldsymbol{\alpha}})_{\underline{mn}}\, \mathcal{F}^{\underline{m}} \mathcal{F}^{\underline{n}} - \nabla_{\underline{m}} G^{\underline{m}}{}_{\boldsymbol{\alpha}} - \frac{1}{3} (t_{\boldsymbol{\alpha}})_{\underline{n}}{}^{\underline{m}}\, \mathcal{T}_{\underline{m}}{}^{\boldsymbol{\beta}} G^{\underline{n}}{}_{\boldsymbol{\beta}} - (t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}} \mathscr{G}_{\underline{n}}{}^{\underline{m}}\,. \qquad (4.41)
\end{aligned}
$$

In the second line, we have introduced

$$
\mathscr{G}_{\underline{m}}{}^{\underline{n}} := G_{\underline{m}}{}^{\underline{n}} - \Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}} G^{\underline{n}}{}_{\boldsymbol{\alpha}}\,, \qquad (4.42)
$$

which unlike $G_{\underline{m}}{}^{\underline{n}}$ is a tensor under internal diffeomorphisms. These 4-form curvatures are further discussed in section 4.5.

The constrained 3-form $H_{\underline{m}}$ obeys the Bianchi identity

$$
\mathcal{D} H_{\underline{m}} = \Omega_{\underline{rs}} \mathcal{F}^{\underline{r}} \partial_{\underline{m}} \mathcal{F}^{\underline{s}} + 24 (t^{\boldsymbol{\alpha}})_{\underline{r}}{}^{\underline{n}} \partial_{\underline{m}} \partial_{\underline{n}} A^{\underline{r}} H_{\boldsymbol{\alpha}} + \partial_{\underline{m}} G_{\underline{n}}{}^{\underline{n}} + 2\, \partial_{\underline{n}} G_{\underline{m}}{}^{\underline{n}}\,. \qquad (4.43)
$$

Its covariant version $\mathscr{H}_{\underline{m}}$ in turn obeys

$$\mathcal{D}\mathscr{H}_{\underline{m}} = \Omega_{\underline{pq}}\mathcal{F}^{\underline{p}}\nabla_{\underline{m}}\mathcal{F}^{\underline{q}} - 2\,R_{\underline{m}}{}^{\boldsymbol{\alpha}}\,H_{\boldsymbol{\alpha}} + \nabla_{\underline{m}}\mathscr{G}_{\underline{n}}{}^{\underline{n}} + 2\,\nabla_{\underline{n}}\mathscr{G}_{\underline{m}}{}^{\underline{n}} - 2\,\mathbf{T}_{\underline{mn}}{}^{\underline{k}}\mathscr{G}_{\underline{k}}{}^{\underline{n}} \\ +\frac{1}{7}\nabla_{\underline{m}}G^{\underline{n}}{}_{\boldsymbol{\alpha}}\,\mathcal{T}_{\underline{n}}{}^{\boldsymbol{\alpha}} + \frac{3}{7}G^{\underline{n}}{}_{\boldsymbol{\alpha}}\,\nabla_{\underline{m}}\mathcal{T}_{\underline{n}}{}^{\boldsymbol{\alpha}} - 2R_{\underline{mn}}{}^{\boldsymbol{\alpha}}G^{\underline{n}}{}_{\boldsymbol{\alpha}}\,. \tag{4.44}$$

The tensors $\mathbf{T}_{\underline{mn}}{}^{\underline{k}}$, $R_{\underline{m}}{}^{\boldsymbol{\alpha}} = \mathrm{d}Z^N R_{N\underline{m}}{}^{\boldsymbol{\alpha}}$, and $R_{\underline{mn}}{}^{\boldsymbol{\alpha}}$ correspond to objects that appear in the commutators between internal covariant derivatives. They are collected in appendix B.

The Bianchi identities for the supervielbein are a bit more complicated. From the definition (4.16), one concludes that

$$\mathcal{D}T^a = E^b R_b{}^a - \mathbb{L}_{\mathcal{F}}E^a\,, \tag{4.45}$$

$$\mathcal{D}T^{\alpha i} = \frac{1}{4}E^{\beta i}R^{cd}(\gamma_{cd})_\beta{}^\alpha - \frac{1}{2}E^{\alpha j}R_j{}^i - \mathbb{L}_{\mathcal{F}}E^{\alpha i}\,. \tag{4.46}$$

Typically in superspace, the torsion Bianchi identity allows one to determine the curvature tensors $R_b{}^a$ and $R_j{}^i$ in terms of the torsion tensor. This is somewhat more subtle in exceptional superspace because these curvature tensors may now involve terms with internal derivatives of the supervielbein. This is apparent when considering the Bianchi identities for the 56-bein, which read (using $R_{kl}{}^{ij} = \delta_{[k}{}^{[i}R_{l]}{}^{j]}$)

$$\mathcal{D}^2\mathcal{V}_{\underline{m}}{}^{ij} = -R_{kl}{}^{ij}\mathcal{V}_{\underline{m}}{}^{kl} - \mathbb{L}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij} \quad\Longrightarrow\quad \begin{cases} \mathcal{D}\mathcal{P}^{ijkl} = -i\mathcal{V}^{\underline{m}kl}\mathbb{L}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij} \\ R_{kl}{}^{ij} = i\mathcal{V}^{\underline{m}}{}_{kl}\mathbb{L}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij} - \mathcal{P}^{ijrs}\mathcal{P}_{rskl} \end{cases}. \tag{4.47}$$

The second identity defines the SU(8) curvature and involves terms with internal derivatives on the field strength tensor $\mathcal{F}$ (which itself involves internal derivatives of the vielbein). Finally, we mention the Bianchi identities for the χ curvature, which we leave in the form

$$\mathcal{D}^2\chi^{\alpha ijk} = \frac{1}{4}\chi^{\beta ijk}R^{cd}(\gamma_{cd})_\beta{}^\alpha - \frac{3}{2}\chi^{\alpha l[ij}R_l{}^{k]} - \mathbb{L}_{\mathcal{F}}\chi^{\alpha ijk}\,. \tag{4.48}$$

4.4 Constraints and solution of the Bianchi identities

We present here the set of constraints on the various curvatures that provide the solution to the Bianchi identities. While we have not explicitly checked the higher dimension components of (4.46) or (4.48), which provide the explicit form of the Riemann tensor (and the superspace version of Einstein's equation), the other Bianchi identities are sufficient to determine the other curvatures. We leave its full characterization to future work, where a unified exceptional geometry would be expected to shed light on some of the structure encountered. While the identities that we need to solve are a good bit more involved than in conventional superspace, luckily, most of the relations correspond exactly to results expected from $N = 8$ superspace [24–26]. We summarize them below. Most of the computations were achieved using *Cadabra* [40, 41].

The 56-bein curvature $\mathcal{P}^{ijkl}$. We impose the following constraints on the supercovariant derivative of the 56-bein (4.30):

$$\mathcal{P}_{\alpha k}{}^{ijpq} = 2\sqrt{2}\,\delta_k^{[i}\chi_\alpha{}^{jpq]}\,, \qquad \mathcal{P}^{\dot\alpha k\,ijpq} = \frac{\sqrt{2}}{12}\,\varepsilon^{ijpqkrst}\chi^{\dot\alpha}{}_{rst}\,. \tag{4.49}$$

The vector torsion tensor. The vector torsion tensor is constrained so that its non-vanishing components in tangent space are

$$T_{\alpha i}{}^{\dot\beta j c} = 2\,\delta_i{}^j\,(\gamma^c)_\alpha{}^{\dot\beta}\,, \qquad T_{abc} = \frac{1}{12}\,\varepsilon_{abcd}\chi^{\alpha ijk}\,(\gamma^d)_{\alpha\dot\alpha}\,\chi^{\dot\alpha}{}_{ijk}\,. \tag{4.50}$$

The choice of $T_{ab}{}^c$ is a matter of convention and can be altered by a covariant redefinition of the spin connection. The choice we expect here is to match the convention used in 4D gauged supergravity [34], although it is easy to change this. Therefore, the full constraint on the covariant derivative of E^a can be written as

$$T^a := \mathcal{D}E^a = -\frac{1}{24}\,\varepsilon^{abcd}\,E_b\wedge E_c\,\chi^{\alpha ijk}\,(\gamma_d)_{\alpha\dot\alpha}\,\chi^{\dot\alpha}{}_{ijk} + 2\,E^{\alpha i}\wedge E_{\dot\beta}\,(\gamma^a)_\alpha{}^{\dot\beta}\,. \tag{4.51}$$

The two-form curvature $\mathcal{F}^{\underline{m}}$. The two-form curvature $\mathcal{F}^{\underline{m}}$ is constrained through the modified field strength $\widehat{\mathcal{F}}^{\underline{m}}$ given in (4.21). The lower dimension parts of $\widehat{\mathcal{F}}_{BA}{}^{\underline{m}}$ are constrained as

$$\widehat{\mathcal{F}}_{\beta j\,\alpha i}{}^{\underline{m}} = -8i\,\mathcal{V}^{\underline{m}}{}_{ji}\,\epsilon_{\beta\alpha}\,, \qquad \widehat{\mathcal{F}}^{\dot\beta j\,\dot\alpha i\underline{m}} = +8i\,\mathcal{V}^{\underline{m}ji}\,\epsilon^{\dot\beta\dot\alpha}\,, \tag{4.52a}$$

$$\widehat{\mathcal{F}}_{\beta j\,a}{}^{\underline{m}} = -\sqrt{2}\,i\,\mathcal{V}^{\underline{m}kl}(\gamma_a)_{\beta\dot\beta}\chi^{\dot\beta}{}_{jkl}\,, \qquad \widehat{\mathcal{F}}^{\dot\beta j}{}_a{}^{\underline{m}} = \sqrt{2}\,i\,\mathcal{V}^{\underline{m}}{}_{kl}(\gamma_a)^{\dot\beta\beta}\chi_\beta{}^{jkl}\,. \tag{4.52b}$$

The vector-vector component $\widehat{\mathcal{F}}_{ab}{}^{\underline{m}}$ is also constrained so that

$$\widehat{\mathcal{F}}^{+\,ij}_{ab} := \widehat{\mathcal{F}}^{+\,\underline{m}}_{ab}\mathcal{V}_{\underline{m}}{}^{ij} = \frac{1}{144}\varepsilon^{ijklpqrs}\bar\chi_{klp}\gamma_{ab}\chi_{qrs}\,, \tag{4.53}$$

as in gauged supergravity. This is the twisted self-duality constraint in ExFT. Note that the self-dual part of $\widehat{\mathcal{F}}_{ab}{}^{ij}$ is actually Lorentz covariant, whereas the anti-self-dual part transforms as (4.23).

The 3-forms $H_{\boldsymbol\alpha}$ and $H_{\underline{m}}$. Analyzing the Bianchi identity (4.40), and comparing terms with explicit internal derivatives, one determines the tangent space components of $H_{\boldsymbol\alpha} = \frac{1}{3!}E^AE^BE^C H_{CBA\boldsymbol\alpha}$ to be

$$H^{\dot\gamma k}{}_{\beta j\,a\,\boldsymbol\alpha} = \frac{8}{3}(t_{\boldsymbol\alpha})^{\underline{mn}}\mathcal{V}_{\underline{m}jl}\mathcal{V}_{\underline{n}}{}^{kl}(\gamma_a)^{\dot\gamma}{}_\beta\,, \tag{4.54a}$$

$$H_{\gamma i\,ba\,\boldsymbol\alpha} = -\frac{2}{3}\sqrt{2}\,(t_{\boldsymbol\alpha})^{\underline{mn}}\mathcal{V}_{\underline{m}ij}\mathcal{V}_{\underline{n}kl}(\gamma_{ba})_\gamma{}^\beta\chi_\beta{}^{jkl}\,, \tag{4.54b}$$

$$H^{\dot\gamma i}{}_{ba\,\boldsymbol\alpha} = -\frac{2}{3}\sqrt{2}\,(t_{\boldsymbol\alpha})^{\underline{mn}}\mathcal{V}_{\underline{m}}{}^{ij}\mathcal{V}_{\underline{n}}{}^{kl}(\gamma_{ba})^{\dot\gamma}{}_{\dot\beta}\chi^{\dot\beta}{}_{jkl}\,, \tag{4.54c}$$

$$\begin{aligned} H_{cba\,\boldsymbol\alpha} = -\frac{1}{3}\varepsilon_{cbad}\,(t_{\boldsymbol\alpha})^{\underline{mn}}\bigg(&\frac{1}{2}\mathcal{V}_{\underline{m}}{}^{ij}\mathcal{V}_{\underline{n}}{}^{kl}\mathcal{P}^d{}_{ijkl} - \frac{1}{2}\mathcal{V}_{\underline{m}ij}\mathcal{V}_{\underline{n}kl}\mathcal{P}^{dijkl} \\ &+ \mathcal{V}_{\underline{m}}{}^{ik}\mathcal{V}_{\underline{n}jk}\,\chi^{\alpha jpq}(\gamma^d)_\alpha{}^{\dot\alpha}\chi_{\dot\alpha ipq}\bigg)\,, \end{aligned} \tag{4.54d}$$

with all other components vanishing. From the component perspective, the last equality is an equation of motion on the three-form field strength and corresponds to its on-shell duality condition. These expressions agree with those from ungauged 4D $N = 8$ superspace [26].

Determining the constrained 3-form field strength is somewhat more involved. From the Bianchi identity (4.40), one can directly show that

$$\begin{aligned}\mathscr{H}_{\underline{m}} &= iE^aE^b\,\varepsilon_{abcd}\nabla_{\underline{m}}\Omega^{cd} - 8i\,E_a\,\nabla_{\underline{m}}E^{\alpha i}E_{\dot\alpha i}(\gamma^a)_\alpha{}^{\dot\alpha} + 8i\,E_a\,\nabla_{\underline{m}}E_{\dot\alpha i}E^{\alpha i}(\gamma^a)_\alpha{}^{\dot\alpha}\\
&\quad - \frac{2i}{3}\sqrt{2}E^aE^bE^{\beta i}(\gamma_{ab})_{\beta\alpha}\chi^{\alpha jkl}\mathcal{P}_{\underline{m}ijkl} + \frac{2i}{3}\sqrt{2}E^aE^bE_{\dot\beta i}(\gamma_{ab})^{\dot\beta\dot\alpha}\chi_{\dot\alpha jkl}\mathcal{P}_{\underline{m}}{}^{ijkl}\\
&\quad - \frac{i}{9}\,E^aE^bE^c\,\varepsilon_{abcd}\left(\mathcal{P}_{\underline{m}}{}^{ijkl}\mathcal{P}^d{}_{ijkl} + \frac{1}{2}\nabla_{\underline{m}}\chi^{\alpha ijk}\chi_{\dot\alpha ijk}(\gamma^d)_\alpha{}^{\dot\alpha}\right.\\
&\qquad\qquad\qquad\qquad\qquad\left. + \frac{1}{2}\nabla_{\underline{m}}\chi_{\dot\alpha ijk}\chi^{\alpha ijk}(\gamma^d)_\alpha{}^{\dot\alpha}\right)\,. \end{aligned} \tag{4.55}$$

This explicit expression for the field strength $\mathscr{H}_{\underline{m}}$ is covariant under internal diffeomorphisms as required by the Bianchi identity (4.40). However, the presence of the explicit spin connection means it cannot be covariant under Lorentz transformations, which is as expected. Note that the definition (4.37) for $\widehat{\mathscr{H}_{\underline{m}}}$, which we motivated in the hope of it having a conventional tangent space expansion, indeed leads to such a result:

$$\begin{aligned}\widehat{\mathscr{H}_{\underline{m}}} &:= \frac{1}{3!}E^AE^BE^C\widehat{\mathscr{H}}_{CBA\,\underline{m}}\\
&= -\frac{2i}{3}\sqrt{2}E^aE^bE^{\beta i}(\gamma_{ab})_{\beta\alpha}\chi^{\alpha jkl}\mathcal{P}_{\underline{m}ijkl} + \frac{2i}{3}\sqrt{2}E^aE^bE_{\dot\beta i}(\gamma_{ab})^{\dot\beta\dot\alpha}\chi_{\dot\alpha jkl}\mathcal{P}_{\underline{m}}{}^{ijkl}\\
&\quad - \frac{i}{9}\,E^aE^bE^c\,\varepsilon_{abcd}\left(\mathcal{P}_{\underline{m}}{}^{ijkl}\mathcal{P}^d{}_{ijkl} + \frac{1}{2}\nabla_{\underline{m}}\chi^{\alpha ijk}\chi_{\dot\alpha ijk}(\gamma^d)_\alpha{}^{\dot\alpha}\right.\\
&\qquad\qquad\qquad\qquad\qquad\left. + \frac{1}{2}\nabla_{\underline{m}}\chi_{\dot\alpha ijk}\chi^{\alpha ijk}(\gamma^d)_\alpha{}^{\dot\alpha}\right)\,. \end{aligned} \tag{4.56}$$

From the component perspective, the constraint on $\widehat{\mathscr{H}}_{cba\,\underline{m}}$ corresponds to an on-shell duality condition.

The actual (non-covariant) field strength $H_{\underline{m}}$ can be found by inverting (4.36):

$$\begin{aligned}H_{\underline{m}} &= iE^aE^b\,\varepsilon_{abcd}\partial_{\underline{m}}\Omega^{cd} - 8i\,E^a\left(\partial_{\underline{m}}E^{\alpha i} - \frac{1}{2}q_{\underline{m}j}{}^iE^{\alpha j}\right)E_{\dot\alpha i}(\gamma_a)_\alpha{}^{\dot\alpha}\\
&\quad + 8i\,E^a\left(\partial_{\underline{m}}E_{\dot\alpha i} + \frac{1}{2}q_{\underline{m}i}{}^jE_{\dot\alpha j}\right)E^{\alpha i}(\gamma_a)_\alpha{}^{\dot\alpha} - \frac{2i}{3}\sqrt{2}E^aE^bE^{\beta i}(\gamma_{ab})_{\beta\alpha}\chi^{\alpha jkl}p_{\underline{m}ijkl}\\
&\quad + \frac{2i}{3}\sqrt{2}E^aE^bE_{\dot\beta i}(\gamma_{ab})^{\dot\beta\dot\alpha}\chi_{\dot\alpha jkl}p_{\underline{m}}{}^{ijkl} - \frac{i}{9}\,E^aE^bE^c\,\varepsilon_{abcd}\left(p_{\underline{m}}{}^{ijkl}\mathcal{P}^d{}_{ijkl}\right.\\
&\quad\left. + \frac{1}{2}\partial_{\underline{m}}\chi^{\alpha ijk}\chi_{\dot\alpha ijk}(\gamma^d)_\alpha{}^{\dot\alpha} + \frac{1}{2}\partial_{\underline{m}}\chi_{\dot\alpha ijk}\chi^{\alpha ijk}(\gamma^d)_\alpha{}^{\dot\alpha} - \frac{3}{2}q_{\underline{m}l}{}^i\chi^{\alpha ljk}\chi_{\dot\alpha ijk}(\gamma^d)_\alpha{}^{\dot\alpha}\right), \end{aligned} \tag{4.57}$$

where $q_{\underline{m}i}{}^j$ and $p_{\underline{m}}{}^{ijkl}$ are given by (2.56).

There are several consistency checks which the expressions for $H_{\underline{m}}$ and $\mathscr{H}_{\underline{m}}$ satisfy:

- The definition of the covariant $\mathscr{H}_{\underline{m}}$ (4.36) involves the $\mathrm{E}_{7(7)}$ connection, which as we have discussed contains undetermined pieces. These drop out from the Bianchi identity (4.40) when the explicit expressions for $\mathscr{H}_{\underline{m}}$ and $H_{\boldsymbol{\alpha}}$ are used, as well as the expression (3.29) for the $\mathrm{E}_{7(7)}$ connection.

- $H_{\underline{m}}$ is a constrained tensor on its $\underline{m}$ index, obeying the conditions (2.4). However, $\mathscr{H}_{\underline{m}}$ does not unless the $\mathrm{E}_{7(7)}$ connection is also constrained. As this involves undetermined pieces, this may or may not be the case.

- Because of the underlying non-Lorentz invariance of the two-form field strength, the curvature $H_{\underline{m}}$ has the appropriate anomalous Lorentz transformation, consistent with (4.12),

$$\delta H_{\underline{m}} = D\Big(i\varepsilon_{abcd} E^a E^b \partial_{\underline{m}} \lambda^{cd}\Big) \,. \tag{4.58}$$

The gravitino torsion tensor. The modified gravitino torsion tensor $\widehat{T}^{\alpha i}$, defined in (4.28), is constrained so that its lower tangent space components are

$$\widehat{T}_{\gamma k\, \beta j}{}^{\alpha i} = 0 \,, \tag{4.59a}$$

$$\widehat{T}^{\dot\gamma k}{}_{\beta j}{}^{\alpha i} = 0 \,, \tag{4.59b}$$

$$\widehat{T}^{\dot\gamma k\, \dot\beta j\, \alpha i} = \sqrt{2}\, \epsilon^{\dot\gamma\dot\beta} \chi^{\alpha kji} \,, \tag{4.59c}$$

$$\widehat{T}_{\beta j c}{}^{\alpha i} = \frac{1}{8} (\bar\chi^{ikl} \gamma^a \chi_{jkl}) (\gamma_c \gamma_a)_\beta{}^\alpha \,, \tag{4.59d}$$

$$\begin{aligned}
\widehat{T}^{\dot\beta j}{}_c{}^{\alpha i} &= \frac{1}{8} (\gamma_c \gamma^{ab})^{\dot\beta\alpha} \widehat{\mathcal{F}}_{ab}{}^{ij} - \frac{1}{1152} \varepsilon^{ijklpqrs} (\bar\chi_{klp} \gamma^{ab} \chi_{qrs}) (\gamma_{ab} \gamma_c)^{\dot\beta\alpha} \\
&\quad + i (\gamma_c)^{\dot\beta\alpha} \mathcal{V}^{\underline{m}}{}_{kl} \mathcal{P}_{\underline{m}}{}^{ijkl} - (\gamma_c)^{\dot\beta\alpha} A_1{}^{ij} \\
&= \frac{1}{8} (\gamma_c \gamma^{ab})^{\dot\beta\alpha}\, \widehat{\mathcal{F}}_{ab}{}^{ij} - \frac{1}{8} (\gamma^{ab} \gamma_c)^{\dot\beta\alpha}\, \widehat{\mathcal{F}}_{ab}{}^{ij} + i (\gamma_c)^{\dot\beta\alpha} \mathcal{V}^{\underline{m}}{}_{kl} \mathcal{P}_{\underline{m}}{}^{ijkl} - (\gamma_c)^{\dot\beta\alpha} A_1{}^{ij} \,.
\end{aligned} \tag{4.59e}$$

Because the internal covariant derivative $\nabla_{\underline{m}}$ does not carry any internal spin connection, the modified tensor $\widehat{T}^{\alpha i}$ has an anomalous Lorentz transformation; this is reproduced by the constraints above due to the field strength $\widehat{\mathcal{F}}_{ab}{}^{ij}$. Also, because $\nabla_{\underline{m}}$ *does* depend on the internal $\mathrm{E}_{7(7)}$ connection, $\widehat{T}^{\alpha i}$ depends on the precise choice of internal connections even though $T^{\alpha i} := \mathcal{D} E^{\alpha i}$ does not. This is apparent above in the appearance of both $\mathcal{P}_{\underline{m}}{}^{ijkl}$ and component $A_1{}^{ij}$ of the $\mathrm{E}_{7(7)}$ torsion tensor. However, one can check that the undetermined pieces of the internal GVP drop out of $T^{\alpha i}$ itself. This is the superspace version of the observation in [22] that the **1280** component of the SU(8) connection drops out of the SUSY transformation of the gravitino.

As a consequence of the Bianchi identities, in particular the $\mathcal{F}^{\underline{m}}$ Bianchi identity (4.40), the top component $\widehat{T}_{cb}{}^{\alpha i}$ itself obeys several constraints. Its self-dual component is fixed as

$$\widehat{T}^{+\alpha i}_{ab} = \frac{\sqrt{2}}{12} \mathcal{P}_{[a}{}^{ijkl} (\gamma_{b]})^{\dot\alpha\alpha} \chi_{\dot\alpha jkl} + \frac{\sqrt{2}}{24} \varepsilon_{abcd}\, \mathcal{P}^{cijkl} \chi_{\dot\alpha jkl} (\gamma^d)^{\dot\alpha\alpha} + \frac{\sqrt{2}}{8} \chi^{\alpha ijk} \widehat{\mathcal{F}}^{+}_{ab\, jk} \,, \tag{4.60}$$

whereas the spin-1/2 part of its anti-self-dual component is

$$\begin{aligned}
\widehat{T}^{-\alpha i}_{ab} (\gamma^b)_{\alpha\dot\alpha} &= \frac{\sqrt{2}}{12} A_2{}^i{}_{jkl} \chi^{\alpha jkl} (\gamma_a)_{\alpha\dot\alpha} + \frac{\sqrt{2}}{24} \mathcal{P}^{bijkl} \chi_{\dot\beta jkl} (\gamma_b \gamma_a)^{\dot\beta}{}_{\dot\alpha} + i \frac{\sqrt{2}}{2} \mathcal{V}^{\underline{m}}{}_{jk}\, \nabla_{\underline{m}} \chi^{\alpha ijk}\, (\gamma_a)_{\alpha\dot\alpha} \\
&\quad - i \frac{\sqrt{2}}{3} \mathcal{V}^{\underline{m} ij} \mathcal{P}_{\underline{m} jklp} \chi^{\alpha klp} (\gamma_a)_{\alpha\dot\alpha} + i \frac{\sqrt{2}}{4} \mathcal{V}^{\underline{m} jk} \mathcal{P}_{\underline{m} jklp} \chi^{\alpha ilp} (\gamma_a)_{\alpha\dot\alpha} \\
&\quad + \frac{\sqrt{2}}{1728} \bar\chi^{ijk} \chi^{lpq}\, \chi^{\alpha rst} (\gamma_a)_{\alpha\dot\alpha}\, \varepsilon_{jklpqrst} \,.
\end{aligned} \tag{4.61}$$

These correspond to the gravitino equations of motion in the underlying component theory. Note that $\widehat{T}_{ab}{}^{\alpha i}$ itself is not Lorentz covariant, although $T_{nm}{}^{\alpha i}$ is. This is because it is $T_{nm}{}^{\alpha i}$ itself that is directly related to the gravitino equations of motion.

The χ curvature. The curvature $\widehat{\tau}_M{}^{\alpha i}$ is defined in (4.31), and it takes values in tangent space as

$$\begin{aligned}\widehat{\tau}_{\beta l}{}^{\alpha ijk} &= -\frac{3}{4}\sqrt{2}\,\delta_l^{[i}(\gamma_{ab})_\beta{}^\alpha \widehat{\mathcal{F}}_{ab}{}^{jk]} - \frac{\sqrt{2}}{24}\delta_\beta{}^\alpha \varepsilon^{ijkpqrst}\bar{\chi}_{pqr}\chi_{stl} \\ &\quad - 6i\sqrt{2}\,\delta_\beta{}^\alpha \mathcal{V}^{\underline{n}}{}_{pq}\mathcal{P}_{\underline{n}}{}^{pq[ij}\delta_l{}^{k]} - 8i\sqrt{2}\,\delta_\beta{}^\alpha \mathcal{V}^{\underline{n}}{}_{pl}\mathcal{P}_{\underline{n}}{}^{ijkp} - 2\sqrt{2}\,\delta_\beta{}^\alpha A_{2l}{}^{ijk}\,, \qquad (4.62a)\\ \widehat{\tau}^{\dot\beta l\,\alpha ijk} &= 2\sqrt{2}(\gamma_a)^{\dot\beta\alpha}\mathcal{P}_a{}^{ijkl}\;. \qquad (4.62b)\end{aligned}$$

The spin-1/2 part of its top component $\widehat{\tau}_a{}^{\alpha ijk}$ is constrained as

$$\begin{aligned}\widehat{\tau}_a{}^{\alpha ijk}(\gamma_a)_{\alpha\dot\alpha} &= \frac{1}{6}A_2{}^{[i}{}_{lpq}\,\varepsilon^{jk]lpqrst}\,\chi_{\dot\alpha rst} + \frac{i}{3}\varepsilon^{ijklpqrs}\mathcal{V}^{\underline{m}}{}_{rs}\nabla_{\underline{m}}\chi_{\dot\alpha lpq} \\ &\quad - 2i\mathcal{V}^{\underline{m}rs}\mathcal{P}_{\underline{m}}{}^{ijkt}\chi_{\dot\alpha rst} - 6i\mathcal{V}^{\underline{m}r[i}\mathcal{P}_{\underline{m}}{}^{jk]st}\chi_{\dot\alpha rst} + 6i\mathcal{V}^{\underline{m}[ij}\mathcal{P}_{\underline{m}}{}^{k]rst}\chi_{\dot\alpha rst} \\ &\quad + \frac{1}{4}\bar{\chi}^{ijk}\chi^{rst}\,\chi_{\dot\alpha rst} - \frac{3}{4}\bar{\chi}^{r[ij}\chi^{k]st}\,\chi_{\dot\alpha rst} + \frac{1}{48}\varepsilon^{ijklpqrs}\widehat{\mathcal{F}}_{abrs}\,\chi_{\dot\beta lpq}(\gamma_{ab})^{\dot\beta}{}_{\dot\alpha} \qquad (4.63)\end{aligned}$$

corresponding to the χ equation of motion. As with the gravitino torsion components, these constraints arise most directly by analyzing the $\mathcal{F}^{\underline{m}}$ Bianchi identity (4.40). Here as well the specific choice of connection terms in $\nabla_{\underline{n}}$ influences the χ curvature. The absence of an internal Lorentz connection is reflected in the appearance of $\widehat{\mathcal{F}}_{ab}{}^{ij}$, and the dependence on the precise $\mathrm{E}_{7(7)}$ connection is reflected by the appearance of $\mathcal{P}_{\underline{m}}{}^{ijkl}$ and the generalized torsion component $A_{2l}{}^{ijk}$. As with the gravitino curvature, one can check that the undetermined pieces of the internal GVP drop out of $\mathcal{D}\chi^{\alpha ijk}$.

4.5 The G curvatures

For the sake of completeness, we record here a number of results related to the 4-form field strengths $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$ and $G_{\underline{m}}{}^{\underline{n}}$ in superspace. These arise by solving the Bianchi identities (4.41) and (4.44), which provided for us a consistency check on our solutions for $H_{\boldsymbol{\alpha}}$ and $\mathscr{H}_{\underline{m}}$. As when one solves for the H field strengths using the $\mathcal{F}$ Bianchi identities, there is ambiguity in these solutions having to do with the kernel of the projector appearing on the right-hand side of the Bianchi identity. To put it more simply, to solve for the 4-form curvatures G, we must implicitly make a choice for the 4-form potentials of the tensor hierarchy, as these have not yet appeared in any curvatures. It is interesting that the superspace versions of the G curvatures that we will give below possess on-shell duality conditions that *do not* reduce to the ones given in (2.40) and (2.43), and thus must correspond to a redefinition of one or more of the 4-form potentials.

The superspace curvature $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$, which is in the **912**, enjoys like $H_{\boldsymbol{\alpha}}$ a standard tangent space expansion with components

$$G_{\beta j\,\alpha i\,ba}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} = -\frac{32i}{3}(\gamma_{ab})_{\alpha\beta}\,\mathcal{V}^{\underline{m}}{}_{ik}\,\mathcal{V}^{\underline{n}}{}_{jl}\mathcal{V}^{\underline{p}kl}\,(t_{\boldsymbol{\alpha}})_{\underline{np}}\,, \qquad G^{\dot\beta j}{}_{\alpha i\,ba}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} = 0\,, \qquad (4.64a)$$

$$G_{\alpha i\,cba}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} = \frac{14i}{3}\sqrt{2}\,\varepsilon_{abcd}(\gamma_d)_\alpha{}^{\dot\alpha}\,(\chi_{\dot\alpha jkl}\,\mathcal{V}^{\underline{n}}{}_{iq}\,\mathcal{V}_{\underline{r}}{}^{qj}\,\mathcal{V}_{\underline{s}}{}^{kl})\,t_{\boldsymbol{\beta}}{}^{\underline{rs}}\,\mathbb{P}_{912\underline{n}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\alpha}}{}^{\underline{m}}\,, \tag{4.64b}$$

$$\begin{aligned} G_{dcba}{}^{\underline{m}}{}_{\boldsymbol{\alpha}} = \frac{7i}{3}\varepsilon_{abcd}\Big(&\chi^{\alpha ijk}\epsilon_{\alpha\beta}\chi^{\beta rst}\mathcal{V}_{\underline{r}kr}\mathcal{V}_{\underline{s}st}\mathcal{V}^{\underline{n}}{}_{ij} \\ &+\chi_{\dot\alpha ijk}\epsilon^{\dot\alpha\dot\beta}\chi_{\dot\beta rst}\mathcal{V}_{\underline{r}}{}^{kr}\mathcal{V}_{\underline{s}}{}^{st}\mathcal{V}^{\underline{n}ij}\Big)\,t_{\boldsymbol{\beta}}{}^{\underline{rs}}\,\mathbb{P}_{912\underline{n}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\alpha}}{}^{\underline{m}} + 24i\,\varepsilon_{abcd}\,\mathcal{Z}^{\underline{m}}{}_{\boldsymbol{\alpha}} \end{aligned} \tag{4.64c}$$

where $\mathcal{Z}^{\underline{m}}{}_{\boldsymbol{\alpha}}$ is a purely scalar expression determined only by derivatives of the coset fields. In terms of the Weitzenböck connection, it can be written most simply as

$$\mathcal{Z}^{\underline{m}}{}_{\boldsymbol{\alpha}} = \frac{1}{288}\Big(t_{\boldsymbol{\beta}\underline{k}}{}^{\underline{r}}\mathring{\mathcal{T}}_{\underline{pq}}{}^{\underline{s}}\mathcal{M}^{\underline{np}}\mathcal{M}^{\underline{kq}}\mathcal{M}_{\underline{rs}} + 7\,t_{\boldsymbol{\beta}\underline{k}}{}^{\underline{q}}\mathring{\mathcal{T}}_{\underline{pq}}{}^{\underline{k}}\mathcal{M}^{\underline{np}}\Big)\mathbb{P}_{912\underline{n}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\alpha}}{}^{\underline{m}}\,, \tag{4.65}$$

entirely in terms of the Weitzenböck torsion and the internal metric. Note that this *does not* coincide with the bosonic expression (2.40) given in section 2. This suggests that these two bosonic results for $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$ must differ by a redefinition of a 4-form potential. We will show this below. The fact that this form of the expression seems to more naturally arise in superspace is quite remarkable for the following reason. In a Scherk-Schwarz reduction of the type we will discuss in section 6, the Weitzenböck torsion is replaced by the embedding tensor, and the above result is then proportional to the variation of the scalar potential of gauged supergravity with respect to the embedding tensor. It is expected that the D-form field strengths of gauged supergravities should be equal to this quantity, see e.g. the $D=3$ discussion of [42].

The compact expression (4.65) can be rewritten as

$$\begin{aligned} \mathcal{Z}^{\underline{m}}{}_{\boldsymbol{\alpha}} = \Big[&\mathcal{Q}_{\boldsymbol{\beta}i}{}^j\Big(\frac{i}{8}\mathcal{V}^{\underline{n}ki}A_{1jk} + \frac{i}{8}\mathcal{V}^{\underline{n}}{}_{kj}A_1{}^{ik} + \frac{1}{8}\mathcal{V}^{\underline{n}ki}\mathcal{V}^{\underline{m}}{}_{jl}\,\mathcal{Q}_{\underline{m}k}{}^l + \frac{1}{8}\mathcal{V}^{\underline{n}}{}_{kj}\mathcal{V}^{\underline{m}il}\,\mathcal{Q}_{\underline{m}l}{}^k\Big) \\ &+\mathcal{P}_{\boldsymbol{\beta}ijkl}\Big(-\frac{7i}{144}\mathcal{V}^{\underline{n}lp}A_{2p}{}^{ijk} - \frac{7}{36}\mathcal{V}^{\underline{n}lp}\mathcal{V}^{\underline{m}}{}_{pq}\mathcal{P}_{\underline{m}}{}^{ijkq} - \frac{7}{48}\mathcal{V}^{\underline{n}lp}\mathcal{V}^{\underline{m}ij}\,\mathcal{Q}_{\underline{m}p}{}^k\Big) \\ &+\mathcal{P}_{\boldsymbol{\beta}}{}^{ijkl}\Big(\frac{7i}{144}\mathcal{V}^{\underline{n}}{}_{lp}A_2{}^p{}_{ijk} - \frac{7}{36}\mathcal{V}^{\underline{n}}{}_{lp}\mathcal{V}^{\underline{m}pq}\mathcal{P}_{\underline{m}ijkq} + \frac{7}{48}\mathcal{V}^{\underline{n}}{}_{lp}\mathcal{V}^{\underline{m}}{}_{ij}\,\mathcal{Q}_{\underline{m}k}{}^p\Big)\Big]\,\mathbb{P}_{912\underline{n}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\alpha}}{}^{\underline{m}}\,, \end{aligned} \tag{4.66}$$

where $\mathcal{Q}_{\boldsymbol{\alpha}}$ and $\mathcal{P}_{\boldsymbol{\alpha}}$ are the SU(8) projections of $(t_{\boldsymbol{\alpha}})_{\underline{m}}{}^{\underline{n}}$, see (3.28). In the latter expression, we have done two things. First, we have exchanged $\mathring{\mathcal{T}}$ built from the Weitzenböck connection for $\mathcal{T}$ built from the generic $\mathrm{E}_{7(7)}$ connection Γ. Remember this carries undetermined pieces, corresponding to freedom to redefine $\mathcal{Q}_{\underline{m}}$ and $\mathcal{P}_{\underline{m}}$, which now appear explicitly. We have subsequently rewritten $\mathcal{T}$ in terms of the SU(8) tensors A_1 and A_2.

The reason for rewriting $Z^{\underline{m}}{}_{\boldsymbol{\alpha}}$ in this way is to emphasize that it is not SU(8) invariant, with the internal connection $\mathcal{Q}_{\underline{m}i}{}^j$ appearing explicitly. The SU(8) transformation of $Z^{\underline{m}}{}_{\boldsymbol{\alpha}}$ leads to a transformation of $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$ itself,

$$\begin{aligned} \delta G^{\underline{m}}{}_{\boldsymbol{\alpha}} = iE^aE^bE^cE^d\varepsilon_{abcd}\Big[&\frac{1}{36}\mathcal{P}_{\boldsymbol{\alpha}ijkl}\mathcal{V}^{\underline{m}ij}\mathcal{V}^{\underline{n}kp}\partial_{\underline{n}}\lambda_p{}^l + \frac{1}{12}\mathcal{P}_{\boldsymbol{\alpha}ijkl}\mathcal{V}^{\underline{m}ip}\mathcal{V}^{\underline{n}jk}\partial_{\underline{n}}\lambda_p{}^l \\ &-\mathcal{Q}_{\boldsymbol{\alpha}i}{}^j\Big(\frac{1}{18}\mathcal{V}^{\underline{m}}{}_{jk}\mathcal{V}^{\underline{n}il}\partial_{\underline{n}}\lambda_l{}^k + \frac{5}{72}\mathcal{V}^{\underline{m}}{}_{jk}\mathcal{V}^{\underline{n}kl}\partial_{\underline{n}}\lambda_l{}^i + \frac{1}{24}\mathcal{V}^{\underline{m}}{}_{kl}\mathcal{V}^{\underline{n}ik}\partial_{\underline{n}}\lambda_j{}^l \\ &+\frac{1}{48}\mathcal{V}^{\underline{m}}{}_{kl}\mathcal{V}^{\underline{n}kl}\partial_{\underline{n}}\lambda_j{}^i\Big) + \text{c.c.}\Big] \end{aligned} \tag{4.67}$$

Because there is no internal derivative on a frame field, this can only arise from a non-covariant SU(8) transformation of one of the constrained 4-form potentials $D_{QPNM\underline{m}}{}^{\underline{pq}}$. Extrapolating from the pattern of the 2-form and 3-form potentials, this field should obey the section condition on $\underline{m}$ and the upper pair of indices $\underline{pq}$ should live in some particular representation of $\mathrm{E}_{7(7)}$ in the product $\mathbf{56}\times\mathbf{56}=\mathbf{1}+\mathbf{133}+\mathbf{1463}+\mathbf{1539}$. It is not hard to show that all but the $\mathbf{1463}$ are in principle present, meaning that they are projected out from the right-hand side of the H Bianchi identities. (It may be that they are not actually required to ensure gauge invariance of the field strengths.) Writing these three fields as $D_{\underline{m}}$, $D_{\underline{m}}{}^{\boldsymbol{\alpha}}$, and $D_{\underline{m}}{}^{\underline{np}}$, their full contributions to the field strengths $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$ and $G_{\underline{m}}{}^{\underline{n}}$ are

$$G^{\underline{m}}{}_{\boldsymbol{\alpha}} = \cdots + 7\,\mathbb{P}_{912}{}^{\boldsymbol{\beta}}{}_{\underline{n}}{}^{\underline{m}}{}_{\boldsymbol{\alpha}}\Big(\kappa_{\boldsymbol{\beta\gamma}}\,\Omega^{\underline{np}}\,D_{\underline{p}}{}^{\gamma} + D_{\underline{p}}{}^{\underline{qn}}\,(t_{\boldsymbol{\beta}})_{\underline{q}}{}^{\underline{p}}\Big)\,, \tag{4.68a}$$

$$\begin{aligned} G_{\underline{m}}{}^{\underline{n}} = \cdots &+ \Omega^{\underline{np}}(\partial_{\underline{m}}D_{\underline{p}} - \partial_{\underline{p}}D_{\underline{m}}) + 12\,\partial_{\underline{p}}D_{\underline{m}}{}^{\boldsymbol{\beta}}\,(t_{\boldsymbol{\beta}})^{\underline{pn}} + 4\,\partial_{\underline{m}}D_{\underline{p}}{}^{\boldsymbol{\beta}}\,(t_{\boldsymbol{\beta}})^{\underline{pn}} \\ &+ 2\,\partial_{\underline{p}}D_{\underline{m}}{}^{\underline{pn}} + \frac{2}{3}\partial_{\underline{m}}D_{\underline{p}}{}^{\underline{pn}}\,. \end{aligned} \tag{4.68b}$$

In order to generate the anomalous SU(8) transformation (4.67), one should assign the following anomalous transformation to $D_{\underline{m}}{}^{\underline{np}}$,

$$\delta D_{\underline{n}}{}^{\underline{pq}} = \frac{i}{4!}E^aE^bE^cE^d\,\varepsilon_{abcd}\left(-\,\partial_{\underline{n}}\lambda_i{}^j\mathcal{V}^{\underline{p}ik}\mathcal{V}^{\underline{q}}{}_{kj} + \partial_{\underline{n}}\lambda_j{}^i\mathcal{V}^{\underline{p}}{}_{ik}\mathcal{V}^{\underline{q}kj}\right)\,. \tag{4.69}$$

This is an intriguing result, because it seems very similar to what we found for the constrained 2-form, where it seemed necessary to assign an anomalous Lorentz transformation (4.12).

This anomalous SU(8) behavior seems to be at the root of the difference between the bosonic part (4.65) of the superform $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$ and the purely bosonic expression (2.40) for $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$. Namely, there seems to be a tension between maintaining SU(8) invariance and maintaining covariance under internal diffeomorphisms. We derived the expression (2.40) in a formulation with only an internal metric and no explicit 56-bein; it was not possible to violate SU(8) invariance, and we were led to an expression that transforms anomalously under internal diffeomorphisms. In deriving the superform $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$ above, manifest $\mathrm{E}_{7(7)}$ diffeomorphism covariance was assumed everywhere and led to an expression that violates SU(8) invariance. As it turns out, one can write down the bosonic shift in $D_{\underline{m}}{}^{\underline{pq}}$ that exchanges (2.40) for (4.65):

$$\begin{aligned}\Delta D_{\underline{m}}{}^{\underline{pq}} &= \frac{i}{4!}E^aE^bE^cE^d\,\varepsilon_{abcd} \\ &\times\left(\partial_{\underline{m}}\mathcal{V}^{[\underline{p}ij}\mathcal{V}^{\underline{q}]}{}_{ij} + \partial_{\underline{m}}\mathcal{V}^{[\underline{p}}{}_{ij}\mathcal{V}^{\underline{q}]ij} + \frac{1}{56}\Omega^{\underline{pq}}(\partial_{\underline{m}}\mathcal{V}^{\underline{n}ij}\mathcal{V}_{\underline{n}ij} + \partial_{\underline{m}}\mathcal{V}^{\underline{n}}{}_{ij}\mathcal{V}_{\underline{n}}{}^{ij})\right)\,. \end{aligned} \tag{4.70}$$

It is easy to see that this induces the anomalous SU(8) transformation discussed above.

For completeness, we also give the rather complicated expression for $\mathscr{G}_{\underline{m}}{}^{\underline{n}}$ that we found in superspace. As with $\mathscr{H}_{\underline{m}}$, it is useful to separate out a part $\widehat{\mathscr{G}}_{\underline{m}}{}^{\underline{n}}$ that possesses a conventional tangent space expression from the rest:

$$
\begin{aligned}
\mathscr{G}_{\underline{m}}{}^{\underline{n}} - \widehat{\mathscr{G}}_{\underline{m}}{}^{\underline{n}} = {} & 8\, E^a E^b E^{\alpha i} \nabla_{\underline{m}} E^{\beta j} (\gamma_{ab})_{\alpha\beta} \mathcal{V}^{\underline{n}}{}_{ij} - 4\, E^{\alpha i} E^{\beta j} E^a \nabla_{\underline{m}} E_a\, \epsilon_{\alpha\beta} \mathcal{V}^{\underline{n}}{}_{ij} \qquad (4.71)\\
& + \frac{\sqrt{2}}{3} E^a E^b \nabla_{\underline{m}} E^c E^{\alpha i} \varepsilon_{abcd} (\gamma^d)_\alpha{}^{\dot\alpha} \chi_{\dot\alpha ijk} \mathcal{V}^{\underline{n}jk} \\
& + \sqrt{2}\, E^a E^b \nabla_{\underline{m}} E_b E^{\alpha i} (\gamma_a)_\alpha{}^{\dot\alpha} \chi_{\dot\alpha ijk} \mathcal{V}^{\underline{n}jk} \\
& - \frac{5\sqrt{2}}{9} E^a E^b E^c \nabla_{\underline{m}} E^{\alpha i} \varepsilon_{abcd} (\gamma^d)_\alpha{}^{\dot\alpha} \chi_{\dot\alpha ijk} \mathcal{V}^{\underline{n}jk} \\
& + E^a E^b E^c \nabla_{\underline{m}} E_c \left(\frac{i}{8} \varepsilon_{abde} \widehat{\mathcal{F}}^{de\underline{n}} - \frac{1}{144} (\gamma_{ab})_{\alpha\beta} \chi^{\alpha ijk} \chi^{\beta pqr} \varepsilon_{ijkpqrst} \mathcal{V}^{\underline{n}st} \right) \\
& + \frac{i}{6} \varepsilon_{abcd}\, E^a E^b E^c \nabla_{\underline{m}} \nabla_{\underline{p}} E^d\, \mathcal{M}^{\underline{pn}} \\
& + \frac{i}{9} \varepsilon_{abcd} E^a E^b E^c \nabla_{\underline{m}} E^d \left(\mathcal{P}_{\underline{p}}{}^{ijkl} \mathcal{V}^{\underline{p}}{}_{ij} \mathcal{V}^{\underline{n}}{}_{kl} + \mathcal{Q}_{\underline{p}i}{}^j \mathcal{V}^{\underline{p}ik} \mathcal{V}^{\underline{n}}{}_{jk} \right) \\
& + \frac{i}{3} \varepsilon_{abcd} E^a E^b E^c \nabla_{\underline{p}} E^d \left(\mathcal{P}_{\underline{m}}{}^{ijkl} \mathcal{V}^{\underline{p}}{}_{ij} \mathcal{V}^{\underline{n}}{}_{kl} + \mathcal{Q}_{\underline{m}i}{}^j \mathcal{V}^{\underline{p}ik} \mathcal{V}^{\underline{n}}{}_{jk} \right) \\
& + \text{c.c.} \qquad (4.72)
\end{aligned}
$$

The conventional part $\widehat{\mathscr{G}}_{\underline{m}}{}^{\underline{n}} = \frac{1}{4!} E^A E^B E^C E^D \widehat{\mathcal{G}}_{DCBA\,\underline{m}}{}^{\underline{n}}$ has non-vanishing pieces

$$
\begin{aligned}
\widehat{\mathscr{G}}_{cba\,\alpha i\,\underline{m}}{}^{\underline{n}} = {} & 6\sqrt{2}\, \varepsilon_{abcd} (\gamma_d)_\alpha{}^{\dot\alpha} \bigg(\frac{2}{9} \chi_{\dot\alpha ijk} \mathcal{P}_{\underline{m}}{}^{jklp} \mathcal{V}^{\underline{n}}{}_{lp} \\
& \qquad - \frac{4}{9} \chi_{\dot\alpha jkl} \mathcal{P}_{\underline{m}}{}^{jklp} \mathcal{V}^{\underline{n}}{}_{ip} - \frac{2}{18} \nabla_{\underline{m}} \chi_{\dot\alpha ijk}\, \mathcal{V}^{\underline{n}jk} \bigg), \qquad (4.73a)\\
\widehat{\mathscr{G}}_{cba}{}^{\dot\alpha i}{}_{\underline{m}}{}^{\underline{n}} = {} & -6\sqrt{2}\, \varepsilon_{abcd} (\gamma_d)_\alpha{}^{\dot\alpha} \bigg(\frac{2}{9} \chi^{\alpha ijk} \mathcal{P}_{\underline{m}jklp} \mathcal{V}^{\underline{n}lp} \\
& \qquad - \frac{4}{9} \chi^{\alpha jkl} \mathcal{P}_{\underline{m}jklp} \mathcal{V}^{\underline{n}ip} - \frac{2}{18} \nabla_{\underline{m}} \chi^{\alpha ijk}\, \mathcal{V}^{\underline{n}}{}_{jk} \bigg), \qquad (4.73b)\\
\widehat{\mathscr{G}}_{dcba\,\underline{m}}{}^{\underline{n}} = {} & 24i\, \varepsilon_{abcd} \bigg(\frac{i}{18} \chi^{\alpha ijk} \epsilon_{\alpha\beta} \chi^{\beta lrs}\, \mathcal{P}_{\underline{m}ijkl}\, \mathcal{V}^{\underline{n}}{}_{rs} \\
& \qquad - \frac{i}{432} \chi^{\alpha ijk} \epsilon_{\alpha\beta} \nabla_{\underline{m}} \chi^{\beta lpq} \varepsilon_{ijklpqrs} \mathcal{V}^{\underline{n}rs} + \text{c.c.} \bigg) + 24i\, \varepsilon_{abcd}\, \mathcal{Z}_{\underline{m}}{}^{\underline{n}}\,. \qquad (4.73c)
\end{aligned}
$$

The last term $\mathcal{Z}_{\underline{m}}{}^{\underline{n}}$ gives the purely bosonic part of $\widehat{\mathscr{G}}_{\underline{m}}{}^{\underline{n}}$ and involves the rather unwieldy expression[17]

$$
\begin{aligned}
\mathcal{Z}_{\underline{m}}{}^{\underline{n}} = {} & -\frac{1}{96} \mathcal{Q}_{\underline{m}i}{}^j \mathcal{Q}_{\underline{p}j}{}^i \mathcal{M}^{\underline{pn}} + \frac{1}{9} \mathcal{V}^{\underline{n}}{}_{ij} \mathcal{V}^{\underline{p}ik} \nabla_{\underline{m}} \mathcal{Q}_{\underline{p}k}{}^j + \frac{1}{9} \mathcal{V}^{\underline{n}}{}_{ij} \mathcal{V}^{\underline{p}}{}_{kl} \nabla_{\underline{m}} \mathcal{P}_{\underline{p}}{}^{ijkl} \\
& + \frac{i}{8} A_{1ij} \mathcal{Q}_{\underline{m}k}{}^i \mathcal{V}^{\underline{n}jk} - \frac{i}{48} A_{2i}{}^{jkl} \mathcal{Q}_{\underline{m}j}{}^i \mathcal{V}^{\underline{n}}{}_{kl} + \frac{i}{36} A_{2i}{}^{jkl} \mathcal{P}_{\underline{m}jklp} \mathcal{V}^{\underline{n}ip} \\
& + \mathcal{P}_{\underline{m}ijkl} \bigg(\frac{1}{9} \mathcal{P}_{\underline{p}}{}^{ijkp} \mathcal{V}^{\underline{p}}{}_{pq} \mathcal{V}^{\underline{n}lq} + \frac{1}{9} \mathcal{P}_{\underline{p}}{}^{ijpq} \mathcal{V}^{\underline{n}}{}_{pq} \mathcal{V}^{\underline{p}kl} - \frac{1}{18} \mathcal{P}_{\underline{p}}{}^{ijpq} \mathcal{V}^{\underline{p}}{}_{pq} \mathcal{V}^{\underline{n}kl} \\
& \qquad - \frac{1}{36} \mathcal{Q}_{\underline{pp}}{}^i \mathcal{V}^{\underline{n}jk} \mathcal{V}^{\underline{p}lp} + \frac{1}{36} \mathcal{Q}_{\underline{pp}}{}^i \mathcal{V}^{\underline{n}jp} \mathcal{V}^{\underline{p}kl} \bigg)
\end{aligned}
$$

[17]The $\nabla_{\underline{m}}$ in $\nabla_{\underline{m}} \mathcal{Q}_{\underline{n}i}{}^j$ is to be understood to carry the same SU(8) connection as if $\mathcal{Q}$ were a tensor.

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$$+\frac{1}{12}\mathcal{Q}_{\underline{m}p}{}^{i}\mathcal{P}_{\underline{p}ijkl}\left(\mathcal{V}^{\underline{n}jp}\mathcal{V}^{\underline{p}kl}-\mathcal{V}^{\underline{n}jk}\mathcal{V}^{\underline{p}lp}\right)+\mathcal{Q}_{\underline{m}i}{}^{j}\mathcal{Q}_{\underline{p}j}{}^{k}\left(\frac{1}{24}\mathcal{V}^{\underline{n}}{}_{kl}\mathcal{V}^{\underline{p}il}+\frac{1}{8}\mathcal{V}^{\underline{p}}{}_{kl}\mathcal{V}^{\underline{n}il}\right)$$
$$+\text{c.c.} \tag{4.74}$$

As with $\mathscr{H}_{\underline{m}}$, much of the structure is determined by requiring that the undetermined parts of the various connections cancel when one computes $G_{\underline{m}}{}^{\underline{n}}$ from $\mathscr{G}_{\underline{m}}{}^{\underline{n}}$. The expression for $G_{\underline{m}}{}^{\underline{n}}$ can be recovered by setting $\Gamma = 0$, $\mathcal{Q} = q$, and $\mathcal{P} = p$ in the expression for $\mathscr{G}_{\underline{m}}{}^{\underline{n}}$. We should also add that the expression for $\mathring{\mathscr{G}}_{\underline{m}}{}^{\underline{n}}$ (that is, with the Weitzenböck connection) is quite simple as $\mathring{\mathcal{Q}}_{\underline{m}}$ and $\mathring{\mathcal{P}}_{\underline{m}}$ both vanish.

The bosonic part of $G_{\underline{m}}{}^{\underline{n}}$ must coincide with the one given in (2.43) after some redefinition of the 4-forms. We have already seen for $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$ that the redefinition is restricted to the constrained 4-forms, in particular (4.70) for $D_{\underline{m}}{}^{\underline{np}}$. No redefinition was needed for $D_{\underline{m}}{}^{\boldsymbol{\alpha}}$, but we have not checked if one is needed for $D_{\underline{m}}$. (The latter constrained 4-form is absent in $G^{\underline{m}}{}_{\boldsymbol{\alpha}}$.)

5 Component results from superspace

Here we verify that the use of the proposed generalized superdiffeomorphisms and constraints on torsion and curvatures produce the component results.

5.1 Component fields and supersymmetry transformations

First, we must identify the component fields in terms of the various superfields. For the component one-forms, the correct procedure is to identify them as the $\theta = \mathrm{d}\theta = 0$ part of the superspace one-form. Formally, this corresponds to the pullback of the inclusion map embedding spacetime into superspace. For the vierbein, this amounts to

$$e^a := E^a|_{\theta=\mathrm{d}\theta=0} \quad\Longrightarrow\quad e^a = \mathrm{d}x^m e_m{}^a\,, \quad e_m{}^a(x,y) := E_m{}^a(Z)|_{\theta=0}\,. \tag{5.1}$$

For the gravitino, it is conventional to include an additional factor of 2,

$$\psi_\alpha{}^i := 2\,E_\alpha{}^i|_{\theta=\mathrm{d}\theta=0} \quad\Longrightarrow\quad \psi_{m\alpha}{}^i(x,y) = 2\,E_{m\alpha}{}^i(Z)|_{\theta=0}\,. \tag{5.2}$$

For all other one-forms, we make the analogous choices, i.e.

$$\begin{aligned} A_m{}^{\underline{m}}(x,y) &:= A_m{}^{\underline{m}}(Z)|_{\theta=0}\,, \qquad & B_{mn\,\boldsymbol{\alpha}}(x,y) &:= B_{mn\,\boldsymbol{\alpha}}(Z)|_{\theta=0}\,, \\ B_{mn\,\underline{m}}(x,y) &:= B_{mn\,\underline{m}}(Z)|_{\theta=0}\,, \end{aligned} \tag{5.3}$$

for the fundamental one-forms, and similarly for the composite one-forms,

$$\omega_m{}^{ab}(x,y) := \Omega_m{}^{ab}(Z)|_{\theta=0}\,, \qquad \mathcal{Q}_{mi}{}^j(x,y) := \mathcal{Q}_{mi}{}^j(Z)|_{\theta=0}\,. \tag{5.4}$$

All other component fields correspond to $\theta = 0$ parts of identically named superfields. For example, $\chi_\alpha{}^{ijk}(x,y) := \chi_\alpha{}^{ijk}(Z)|_{\theta=0}$, and so forth.

To derive their symmetry transformations, we must compute their transformations under *covariant* external diffeomorphisms where the diffeomorphism parameter, written in tangent space, takes the form $\xi^A := \xi^M E_M{}^A = (0, \epsilon^{\alpha i}, \epsilon_{\dot\alpha i})$. Let's discuss first how this works with the vierbein and gravitino. From (4.24) and (4.50), we find for the vierbein

$$\delta E_M{}^a = 2\, E_M{}^{\beta j} \epsilon_{\dot\gamma k} (\gamma^a)_\beta{}^{\dot\gamma} \delta_j{}^k + 2\, E_{M\,\dot\beta j} \epsilon^{\gamma k} (\gamma^a)_\gamma{}^{\dot\beta} \delta_k{}^j\,, \tag{5.5}$$

which reduces to the component result, rewritten in four-component notation,

$$\delta e_m{}^a = \bar\epsilon_j \gamma^a \psi_m{}^j + \bar\epsilon^j \gamma^a \psi_{mj}\;. \tag{5.6}$$

From (4.27) and (4.59), we find

$$\begin{aligned}\delta E_M{}^{\alpha i} &= \mathcal{D}_M \epsilon^{\alpha i} + 2i \mathcal{V}^{\underline{m}ij} \nabla_{\underline{m}} \Big(\epsilon_{\dot\beta j} E_M{}^c (\gamma_c)^{\dot\beta\alpha} \Big) \\ &\quad + E_{M\,\dot\beta j} \epsilon_{\dot\gamma k} \sqrt{2}\, \epsilon^{\dot\gamma\dot\beta} \chi^{\alpha kji} + \frac{1}{8} E_M{}^c \epsilon^{\beta j} (\bar\chi^{ikl} \gamma^a \chi_{jkl}) (\gamma_c \gamma_a)_\beta{}^\alpha \\ &\quad + \frac{1}{8} E_M{}^c \epsilon_{\dot\beta j} (\gamma_c \gamma^{ab})^{\dot\beta\alpha}\, \widehat{\mathcal{F}}_{ab}{}^{ij} - \frac{1}{8} E_M{}^c \epsilon_{\dot\beta j} (\gamma^{ab} \gamma_c)^{\dot\beta\alpha}\, \widehat{\mathcal{F}}_{ab}{}^{ij} \\ &\quad + i E_M{}^c \epsilon_{\dot\beta j} (\gamma_c)^{\dot\beta\alpha} \mathcal{V}^{\underline{m}}{}_{kl} \mathcal{P}_{\underline{m}}{}^{ijkl} - E_M{}^c \epsilon_{\dot\beta j} (\gamma_c)^{\dot\beta\alpha} A_1{}^{ij}\;. \end{aligned} \tag{5.7}$$

Lowering and suppressing the spinor index, and then reducing to the $\theta = \mathrm{d}\theta = 0$ part gives the gravitino supersymmetry transformation

$$\begin{aligned}\delta\psi_m{}^i &= 2\mathcal{D}_m \epsilon^i - 4i\, \mathcal{V}^{\underline{m}ij} \nabla_{\underline{m}} (\gamma_m \epsilon_j) - \sqrt{2}\, \bar\epsilon_j \psi_{mk}\, \chi^{ijk} + \frac{1}{4} (\bar\chi^{ikl} \gamma^a \chi_{jkl})\, \gamma_a \gamma_m \epsilon^j \\ &\quad + \frac{1}{4} \gamma^{ab} \gamma_m \epsilon_j\, \widehat{\mathcal{F}}_{ab}{}^{ij} - \frac{1}{4} \gamma_m \gamma^{ab} \epsilon_j\, \widehat{\mathcal{F}}_{ab}{}^{ij} + 2\gamma_m \epsilon_j \Big(A_1{}^{ij} + i \mathcal{V}^{\underline{m}}{}_{kl} \mathcal{P}_{\underline{m}}{}^{ijkl} \Big)\;. \end{aligned} \tag{5.8}$$

For the 56-bein, (4.30) and (4.49) lead to

$$\delta \mathcal{V}_{\underline{m}}{}^{ij} = 2\sqrt{2}\, \bar\epsilon^i \chi^{jkl} \mathcal{V}_{\underline{m}kl} + \frac{\sqrt{2}}{12}\, \varepsilon^{ijklpqrs}\, \bar\epsilon_p \chi_{qrs} \mathcal{V}_{\underline{m}kl} \tag{5.9}$$

whereas for χ (4.32) and (4.62) imply

$$\begin{aligned}\delta\chi^{ijk} &= -2\sqrt{2}\, \mathcal{P}_a{}^{ijkl}\, \gamma^a \epsilon_l - 12i\sqrt{2}\, \mathcal{V}^{\underline{n}[ij} \nabla_{\underline{n}} \epsilon^{k]} + \frac{3\sqrt{2}}{4}\, \widehat{\mathcal{F}}_{ab}{}^{[ij}\, \gamma_{ab} \epsilon^{k]} - \frac{\sqrt{2}}{24} \varepsilon^{ijkpqrst} \bar\chi_{pqr} \chi_{stl}\, \epsilon^l \\ &\quad - 2\sqrt{2}\, A_{2l}{}^{ijk}\, \epsilon^l - 6i\sqrt{2}\, \mathcal{V}^{\underline{n}}{}_{pq} \mathcal{P}_{\underline{n}}{}^{pq[ij}\, \epsilon^{k]} - 8i\sqrt{2}\, \mathcal{V}^{\underline{n}}{}_{pq} \mathcal{P}_{\underline{n}}{}^{ijkp}\, \epsilon^q\;. \end{aligned} \tag{5.10}$$

For $A_m{}^{\underline{m}}$, we combine (4.27) with (4.52) to recover

$$\delta A_m{}^{\underline{m}} = 4i\, \mathcal{V}^{\underline{M}}{}_{ij}\, \bar\epsilon^i \psi_m{}^j - i\sqrt{2}\, \mathcal{V}^{\underline{M}ij} \bar\epsilon^k \gamma_m \chi_{ijk} + \text{c.c.} \tag{5.11}$$

For the adjoint-valued two-forms, (4.33) and (4.54) lead to

$$\Delta B_{mn\,\boldsymbol{\alpha}} = -\frac{2}{3}\sqrt{2}\, (t_{\boldsymbol{\alpha}})^{\underline{mn}} \mathcal{V}_{\underline{m}ij} \mathcal{V}_{\underline{n}kl}\, \bar\epsilon^i \gamma_{mn} \chi^{jkl} - \frac{8}{3}\, (t_{\boldsymbol{\alpha}})^{\underline{mn}} \mathcal{V}_{\underline{m}jl} \mathcal{V}_{\underline{n}}{}^{kl}\, \bar\epsilon_k\, \gamma_{[m} \psi_{n]}{}^j + \text{c.c.} \tag{5.12}$$

For the constrained two-forms, (4.38) and (4.56) lead to

$$\begin{aligned}\Delta B_{mn\,\underline{m}} + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}} \Delta B_{mn\,\boldsymbol{\alpha}} &= -8i\, \bar\epsilon^i\, \gamma_{[m} \nabla_{\underline{m}} \psi_{n]i} + 8i\, \nabla_{\underline{m}} \bar\epsilon^i\, \gamma_{[m} \psi_{n]i} \\ &\quad + 8i\, \bar\epsilon_i\, \gamma_{[m} \nabla_{\underline{m}} \psi_{n]}{}^i - 8i\, \nabla_{\underline{m}} \bar\epsilon_i\, \gamma_{[m} \psi_{n]}{}^i \\ &\quad - \frac{4i}{3}\sqrt{2}\, \bar\epsilon^i \gamma_{mn} \chi^{jkl} \mathcal{P}_{\underline{n}ijkl} + \frac{4i}{3}\sqrt{2}\, \bar\epsilon_i \gamma_{mn} \chi_{jkl} \mathcal{P}_{\underline{n}}{}^{ijkl}\;. \end{aligned} \tag{5.13}$$

Above, we have recovered the SUSY transformations of $e_m{}^a$, $\mathcal{V}_{\underline{m}}{}^{ij}$, and $B_{mn\,\boldsymbol{\alpha}}$ in the form of (2.46). For the gravitino, χ^{ijk}, and $B_{mn\,\underline{m}}$, one must keep in mind that the above rules involve the redefined field $B'_{mn\,\underline{m}}$ (which we have denoted in section 4 and onward without a prime) and so one should compare instead with (3.20). Aside from higher fermionic corrections, some deviations arise in these transformations having to do with allowing the internal GVP to take a more general form. For example, in comparing with the gravitino transformation (5.8), one finds in addition to (3.20), two explicit higher-order fermion terms, one implicit higher-order fermion term (the second $\widehat{\mathcal{F}}_{ab}{}^{ij}$ term, which is on-shell related to a fermion bilinear via (4.53)), and the last two terms involving the A_1 component of the $\mathrm{E}_{7(7)}$ torsion tensor and the non-metricity $\mathcal{P}_{\underline{m}}{}^{ijkl}$, which vanish under the internal GVP assumptions made in sections 2 and 3.

We emphasize that as in [22] one can confirm that all of the undetermined components of the internal connections drop out from the above transformations. This is most easily seen by using (3.33) to rewrite Γ in terms of the Weitzenböck connection, isolating the undetermined pieces in the fields $\mathcal{Q}_{\underline{m}i}{}^j$ and $\mathcal{P}_{\underline{m}}{}^{ijkl}$. The latter two fields then cancel out of all equations.

5.2 Composite connections and supercovariant curvatures

The supersymmetry transformations discussed above involve several composite quantities — the spin connection, the SU(8) connection, the covariant field strength $\widehat{\mathcal{F}}_{ab}{}^{ij}$ — and their component definitions need to be given for the component SUSY transformations to be fully realized.

From the constraints on the torsion two-form, one determines the component external spin connection by projecting (4.51) to spacetime,

$$2\,\mathcal{D}_{[m}e_{n]}{}^a = 2\,D_{[m}e_{n]}{}^a + 2\,\omega_{mn}{}^a = \bar{\psi}_{[m}{}^i\gamma^a\psi_{n]i} + \frac{1}{12}\varepsilon_{mn}{}^{ab}\,\bar{\chi}^{ijk}\gamma_b\chi_{ijk}\,, \tag{5.14}$$

and solving for $\omega_m{}^{ab}$ in the usual way. One similarly obtains the component SU(8) from (4.30),

$$\begin{aligned}\mathcal{D}_m\mathcal{V}_{\underline{n}}{}^{ij} &= D_m\mathcal{V}_{\underline{n}}{}^{ij} + \mathcal{Q}_{mk}{}^{[i}\mathcal{V}_{\underline{n}}{}^{j]k}\\ &= \left(e_m{}^a\mathcal{P}_a{}^{ijkl} + \sqrt{2}\,\bar{\psi}_m{}^{[i}\chi^{jkl]} + \frac{\sqrt{2}}{24}\varepsilon^{ijklpqrs}\,\bar{\psi}_{mp}\chi_{qrs}\right)\mathcal{V}_{\underline{n}\,kl}\,,\end{aligned} \tag{5.15}$$

and inverting the relation to solve for $\mathcal{Q}_{mi}{}^j$. Both expressions for $\omega_m{}^{ab}$ and $\mathcal{Q}_{\underline{m}i}{}^j$ match those of ungauged $N=8$ supergravity upon replacing $\partial_m \to D_m$. Note that this expression defines $\mathcal{P}_a{}^{ijkl}$ to coincide with the supercovariant one-form of ungauged $N=8$ supergravity, where it is usually denoted $\hat{\mathcal{P}}_a{}^{ijkl}$.

The supercovariant field strength for the vector fields arises by projecting (4.21) to components, using the constraints (4.52), and solving for $\widehat{\mathcal{F}}_{ab}{}^{\underline{m}}$ as

$$\begin{aligned}\widehat{\mathcal{F}}_{ab}{}^{\underline{m}} &= e_a{}^m e_b{}^n\mathcal{F}_{mn}{}^{\underline{m}} + i\,\mathcal{V}^{\underline{m}\,ij}\left(4\,\bar{\psi}_{ai}\psi_{bj} + \sqrt{2}\bar{\psi}_{[a}{}^k\gamma_{b]}\chi_{ijk}\right)\\ &\quad - i\,\mathcal{V}^{\underline{m}}{}_{ij}\left(4\,\bar{\psi}_a{}^i\psi_b{}^j + \sqrt{2}\bar{\psi}_{[ak}\gamma_{b]}\chi^{ijk}\right) - \mathcal{M}^{\underline{mn}}e_{[a}{}^m\nabla_{\underline{n}}e_{mb]}\,.\end{aligned} \tag{5.16}$$

It is this quantity that obeys the twisted self-duality condition (4.53), equivalently written

$$\widehat{\mathcal{F}}_{ab}{}^{\underline{m}} = \frac{i}{2}\,\varepsilon_{abcd}\,\Omega^{\underline{mn}}\,\mathcal{M}_{\underline{nk}}\,\widehat{\mathcal{F}}^{cd\,\underline{k}}\ . \tag{5.17}$$

Although we don't need them to realize the SUSY transformations, it is worth giving the explicit formulae for the supercovariant 3-form field strengths of the 2-form fields. The supercovariant form of $H_{mnp\,\boldsymbol{\alpha}}$ corresponds to the lowest component of the superspace tensor $H_{abc\,\boldsymbol{\alpha}}$, which is

$$\begin{aligned} H_{abc\,\boldsymbol{\alpha}} &= e_a{}^m e_b{}^n e_c{}^p H_{mnp\,\boldsymbol{\alpha}} + \sqrt{2}\,(t_{\boldsymbol{\alpha}})^{\underline{mn}}\Big(\mathcal{V}_{\underline{m}ij}\mathcal{V}_{\underline{n}kl}\,\bar\psi_{[a}{}^i\gamma_{bc]}\chi^{jkl} + \text{c.c.}\Big) \\ &\quad + 4\,(t_{\boldsymbol{\alpha}})^{\underline{mn}}\mathcal{V}_{\underline{m}jk}\mathcal{V}_{\underline{n}}{}^{ik}\,\bar\psi_{[a}{}^i\gamma_b\psi_{c]j}\ . \end{aligned} \tag{5.18}$$

The constraint (4.54d) corresponds to the on-shell duality equation

$$H_{abc\,\boldsymbol{\alpha}} = -\frac{1}{6}\varepsilon_{abcd}\,(t_{\boldsymbol{\alpha}})^{\underline{mn}}\Big(\mathcal{V}_{\underline{m}}{}^{ij}\mathcal{V}_{\underline{n}}{}^{kl}\mathcal{P}^d{}_{ijkl} - \mathcal{V}_{\underline{m}ij}\mathcal{V}_{\underline{n}kl}\mathcal{P}^{dijkl} - 2\,\mathcal{V}_{\underline{m}}{}^{ik}\mathcal{V}_{\underline{n}jk}\,\bar\chi^{jpq}\gamma^d\chi_{ipq}\Big) \tag{5.19}$$

which is a natural generalization of the bosonic result (2.33).

For the constrained two-form, the supercovariant form of its field strength corresponds to the lowest component of $\widehat{\mathscr{H}}_{abc\,\underline{m}}$, which is a component of the superspace tensor $\widehat{\mathscr{H}}_{\underline{m}}$ defined in (4.37). Using the constraints (4.56), we find

$$\begin{aligned} \widehat{\mathscr{H}}_{abc\,\underline{m}} = e_{[a}{}^m e_b{}^n e_{c]}{}^p \Big[& H_{mnp\,\underline{m}} + 2\,\Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}} H_{mnp\,\boldsymbol{\alpha}} + 6i\,\varepsilon_{mn\,ef}\nabla_{\underline{m}}\omega_p{}^{ef} \\ & + 12i\,\bar\psi_m{}^i\gamma_n\nabla_{\underline{m}}\psi_{pi} - 12i\,\bar\psi_{mi}\gamma_n\nabla_{\underline{m}}\psi_p{}^i \\ & + 2i\sqrt{2}\,\bar\psi_m{}^i\gamma_{np}\chi^{jkl}\,\mathcal{P}_{\underline{m}\,ijkl} - 2i\sqrt{2}\,\bar\psi_{mi}\gamma_{np}\chi_{jkl}\,\mathcal{P}_{\underline{m}}{}^{ijkl}\Big]\ , \end{aligned} \tag{5.20}$$

where the on-shell duality equation is given by

$$\widehat{\mathscr{H}}_{abc\,\underline{m}} = \frac{2i}{3}\varepsilon_{abcd}\left(\mathcal{P}_{\underline{m}}{}^{ijkl}\mathcal{P}^d{}_{ijkl} + \frac{1}{2}\bar\chi^{ijk}\gamma^d\overset{\leftrightarrow}{\nabla}_{\underline{m}}\chi_{ijk}\right)\ . \tag{5.21}$$

This generalizes the bosonic result (2.34), where one must take care to note that the terms involving the internal and external spin connections, $\partial_{\underline{m}}\omega_n{}^{ab} - D_n\omega_{\underline{m}}{}^{ab}$, corresponding to a mixed internal/external Riemann tensor $R_{\underline{m}n}{}^{ab}$, have been eliminated in different ways: the former by absorption into the definition of $\mathscr{H}_{abc\,\underline{m}}$, and the latter by redefining B to B'.

6 Consistent Scherk-Schwarz reductions in superspace

It has already been shown in [19] that the (bosonic) $\mathrm{E}_{7(7)}$ ExFT admits a consistent Scherk-Schwarz reduction to gauged supergravity with an embedding tensor related to the twist matrices associated with the reduction, provided the twist matrices themselves obey the section condition. It is no surprise that a similar statement can be made connecting $\mathrm{E}_{7(7)}$ ExFT superspace with $N = 8$ superspace with an arbitrary embedding tensor. We sketch the construction here for two reasons. First, with the more generic internal GVP we have advocated, the connection between ExFT and gauged supergravity becomes completely transparent. Second, to our knowledge, the corresponding $N = 8$ superspace with generic embedding tensor has not actually appeared explicitly in the literature, although it is by no means difficult to construct it directly from the component results [34].

In complete analogy to [19], a generalized Scherk-Schwarz reduction in superspace arises by assuming that the y-dependence of any superfield is sequestered into two special fields, a so-called twist matrix $U_{\underline{m}}{}^{\mathtt{M}}(y)$ and a scale factor $\rho(y)$; hereafter we refer to these collectively as twist matrices. We employ $\mathtt{M}, \mathtt{N}, \mathtt{P}, \cdots$ to denote the "flat" $\mathrm{E}_{7(7)}$ 56-plet indices of gauged supergravity.[18] For a superfield $\Phi_{\underline{m}}$ of weight λ, carrying a single fundamental $\mathrm{E}_{7(7)}$ index, we call a *covariant twist* one for which

$$\Phi_{\underline{m}}(x, \theta, y) = \rho^{-2\lambda}\, U_{\underline{m}}{}^{\mathtt{M}}\, \Phi_{\mathtt{M}}(x, \theta)\,, \tag{6.1}$$

with a straightforward generalization to different $\mathrm{E}_{7(7)}$ representations. Nearly every superfield is covariantly twisted, e.g.

$$E_M{}^a(x, \theta, y) = \rho^{-1}\, E_M{}^a(x, \theta)\,, \tag{6.2a}$$
$$E_M{}^{\alpha i}(x, \theta, y) = \rho^{-1/2}\, E_M{}^{\alpha i}(x, \theta)\,, \tag{6.2b}$$
$$\mathcal{V}_{\underline{m}}{}^{ij}(x, \theta, y) = U_{\underline{m}}{}^{\mathtt{M}}\, \mathcal{V}_{\mathtt{M}}{}^{ij}(x, \theta)\,, \tag{6.2c}$$
$$\chi_\alpha{}^{ijk}(x, \theta, y) = \rho^{1/4}\, \chi_\alpha{}^{ijk}(x, \theta)\,, \tag{6.2d}$$
$$A_M{}^{\underline{m}}(x, \theta, y) = \rho^{-1}\, (U^{-1})_{\mathtt{M}}{}^{\underline{m}}\, A_M{}^{\mathtt{M}}(x, \theta)\ , \tag{6.2e}$$
$$B_{NM\,\boldsymbol{\alpha}}(x, \theta, y) = \rho^{-2}\, U_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\, B_{NM\,\boldsymbol{\beta}}(x, \theta)\ . \tag{6.2f}$$

The exception is for the constrained p-form fields, e.g. $B_{NM\,\underline{m}}$, which we will describe shortly.

The twist matrices cannot be chosen arbitrarily. Rather, they must obey the following two conditions

$$7\left[(U^{-1})_{\mathtt{M}}{}^{\underline{m}}(U^{-1})_{\mathtt{N}}{}^{\underline{n}}\partial_{\underline{m}} U_{\underline{n}}{}^{\mathtt{P}}\right]_{\mathbf{912}} = \rho\, X_{\mathtt{MN}}{}^{\mathtt{P}} \equiv \rho\, \Theta_{\mathtt{M}}{}^{\boldsymbol{\alpha}}(t_{\boldsymbol{\alpha}})_{\mathtt{N}}{}^{\mathtt{P}}\,, \tag{6.3}$$
$$\partial_{\underline{n}}(U^{-1})_{\mathtt{M}}{}^{\underline{n}} - 3\rho^{-1}\partial_{\underline{n}}\rho\,(U^{-1})_{\mathtt{M}}{}^{\underline{n}} = 2\,\rho\,\vartheta_{\mathtt{M}}\,, \tag{6.4}$$

where $X_{\mathtt{MN}}{}^{\mathtt{P}}$ and $\vartheta_{\mathtt{M}}$ are constant matrices. These correspond to the two components of the embedding tensor of gauged supergravity, with $X_{\mathtt{MN}}{}^{\mathtt{P}} = \Theta_{\mathtt{M}}{}^{\boldsymbol{\alpha}}(t_{\boldsymbol{\alpha}})_{\mathtt{N}}{}^{\mathtt{P}}$ corresponding to the **912** component [34] and $\vartheta_{\mathtt{M}}$ corresponding to the **56** component associated only to trombone gaugings [43]. Provided one can choose twist matrices in this way, one can show that the two pieces of the embedding tensor, $X_{\mathtt{MN}}{}^{\mathtt{P}}$ and $\vartheta_{\mathtt{M}}$, obey the quadratic constraints [19].

In order to convert the various ExFT formulae, it is useful to eliminate the ambiguity inherent in the internal GVP. The easiest way to do this is to choose the Weitzenböck connection where $\mathring{\Gamma}_{\underline{m}}{}^{\boldsymbol{\alpha}}$ is determined entirely in terms of the derivative of the 56-bein with $\mathring{\mathcal{Q}}_{\underline{m}i}{}^j$ and $\mathring{\mathcal{P}}_{\underline{m}}{}^{ijkl}$ both vanishing. Because of the ansatz made for the coset fields, it is easy to see that the $\mathrm{E}_{7(7)}$ connection is given purely by the twist matrices

$$\mathring{\Gamma}_{\underline{mn}}{}^{\underline{p}} = \partial_{\underline{m}} U_{\underline{n}}{}^{\mathtt{P}}(U^{-1})_{\mathtt{P}}{}^{\underline{p}}\ . \tag{6.5}$$

[18] These "flat" indices should not be confused with the SU(8) tangent space indices ^{ij} and ${}_{ij}$ which arise when one contracts with $\mathcal{V}_{\underline{m}}{}^{ij}$ or $\mathcal{V}_{\underline{m}ij}$.

In particular, it follows that the torsion tensor is

$$\mathring{\mathcal{T}}_{\underline{mn}}{}^{\underline{p}} = \rho\, U_{\underline{m}}{}^{\mathtt{M}} U_{\underline{n}}{}^{\mathtt{N}} X_{\mathtt{MN}}{}^{\mathtt{P}} \, (U^{-1})_{\mathtt{P}}{}^{\underline{p}} \,. \tag{6.6}$$

Considering the torsion tensor as an internal tensor of weight $\lambda = -1/2$, we identify its "flattened" version as the embedding tensor. It is also straightforward to show that the covariant derivative of any superfield obeying the covariant twist ansatz (6.1) is

$$\mathring{\nabla}_{\underline{n}} \Phi_{\underline{m}} = \rho^{-2(\lambda-1)}\, U_{\underline{n}}{}^{\mathtt{N}}\, U_{\underline{m}}{}^{\mathtt{M}} \left(\frac{4}{3} \lambda\, \vartheta_{\mathtt{N}} \Phi_{\mathtt{M}} \right) \,. \tag{6.7}$$

This generalizes easily to any other $\mathrm{E}_{7(7)}$ representation carried by Φ. Thus, covariant derivatives of covariantly twisted objects just map to the trombone part of the embedding tensor, multiplied by a factor of $\frac{4}{3}\lambda$.

The notable exception to the covariant twist ansatz is the constrained two-form $B_{NM\,\underline{m}}$ (and the higher constrained p-form fields). The appropriate ansatz, given in component form in [19], can be motivated by considering a covariantized version of $B_{NM\,\underline{m}}$,

$$\mathscr{B}_{NM\underline{m}} = B_{NM\,\underline{m}} + 2\, \Gamma_{\underline{m}}{}^{\boldsymbol{\alpha}} B_{NM\,\boldsymbol{\alpha}} \,. \tag{6.8}$$

This redefined 2-form is the natural potential associated with the 3-form field strength $\mathscr{H}_{PNM\,\underline{m}}$ that we have been employing. For example, the field strength superform $\mathcal{F}^{\underline{m}}$ can be rewritten

$$\begin{aligned} \mathcal{F}^{\underline{m}} = \mathrm{d}A^{\underline{m}} + A^{\underline{n}} \nabla_{\underline{n}} A^{\underline{m}} + \frac{1}{4}(24\, t^{\boldsymbol{\alpha}\,\underline{mn}} t_{\boldsymbol{\alpha}\,\underline{rs}} - \Omega^{\underline{mn}} \Omega_{\underline{rs}}) A^{\underline{r}} \nabla_{\underline{n}} A^{\underline{s}} - \frac{1}{2} A^{\underline{n}} A^{\underline{p}} \mathcal{T}_{\underline{np}}{}^{\underline{m}} \\ - 12\, t^{\boldsymbol{\alpha}\underline{mn}} \nabla_{\underline{n}} B_{\boldsymbol{\alpha}} + \Omega^{\underline{mn}}\, T_{\underline{n}}{}^{\boldsymbol{\alpha}}\, B_{\boldsymbol{\alpha}} - \frac{1}{2} \Omega^{\underline{mn}} \mathscr{B}_{\underline{n}} \,, \end{aligned} \tag{6.9}$$

in terms of $\mathscr{B}$. Above, we have converted all internal derivatives to covariant ones. Here we are using a generic $\mathrm{E}_{7(7)}$ connection, but now we will specialize to the Weitzenböck connection. The reduction ansatz for $\mathscr{B}$, when the Weitzenböck connection is chosen, can be simply written as

$$\mathring{\mathscr{B}}_{NM\underline{m}}(x,\theta,y) = 0 \,. \tag{6.10}$$

It is straightforward now to apply the reduction ansatz to all of the various curvature superforms. For example, the field strength superfield $\mathcal{F}^{\mathtt{M}}$ becomes

$$\begin{aligned} \mathcal{F}^{\mathtt{M}} = \mathrm{d}A^{\mathtt{M}} + \frac{2}{3} \vartheta_{\mathtt{N}} A^{\mathtt{N}} A^{\mathtt{M}} - \frac{1}{6} \Omega^{\mathtt{MN}} \vartheta_{\mathtt{N}}\, \Omega_{\mathtt{RS}}\, A^{\mathtt{R}} A^{\mathtt{S}} - \frac{1}{2} A^{\mathtt{N}} A^{\mathtt{P}} X_{\mathtt{NP}}{}^{\mathtt{M}} \\ - 16\, t^{\boldsymbol{\alpha}\mathtt{MN}} \vartheta_{\mathtt{N}} B_{\boldsymbol{\alpha}} + \Omega^{\mathtt{MN}} \Theta_{\mathtt{N}}{}^{\boldsymbol{\alpha}} B_{\boldsymbol{\alpha}} \,, \end{aligned} \tag{6.11}$$

as expected for gauged supergravity [43]. Note that there is no longer any difference between $\mathcal{F}^{\mathtt{M}}$ and $\widehat{\mathcal{F}}^{\mathtt{M}}$. Now the superspace constraints on $\mathcal{F}^{\mathtt{M}}$ are just given by (4.52) and (4.53), with the index $\underline{m}$ replaced by $\mathtt{M}$. Similar considerations apply to the higher p-form field strengths in the tensor hierarchy (with the exception of the constrained field strengths discussed below). For example, the field strength $H_{\boldsymbol{\alpha}}$ of gauged supergravity will

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obey the same constraints (4.54), although its explicit form in terms of the potentials will now involve the embedding tensor as in (6.11).

For quantities that are covariant under internal diffeomorphisms, it is useful to first fully covariantize any internal derivatives. In particular, the external covariant derivative $\mathcal{D}$ of any superfield that transforms covariantly under internal diffeomorphisms is altered as follows. For the prototypical superfield $\Phi_{\underline{m}}$ of weight λ discussed above,

$$\mathcal{D}_N \Phi_{\underline{m}} := \partial_N \Phi_{\underline{m}} - \mathbb{L}_{A_N} \Phi_{\underline{m}} = \partial_N \Phi_{\underline{m}} - \mathbb{L}^{\nabla}_{A_N} \Phi_{\underline{m}} - A_N{}^{\underline{n}} \mathcal{T}_{\underline{nm}}{}^{\underline{p}} \Phi_{\underline{p}} \,. \tag{6.12}$$

Now covariantly twisting quantities and specializing to the Weitzenböck connection, this becomes

$$\mathcal{D}_N \Phi_{\mathtt{M}} = \partial_N \Phi_{\underline{m}} - 2\lambda\, \vartheta_{\mathtt{K}} A_N{}^{\mathtt{K}} \Phi_{\mathtt{M}} - 8\, \vartheta_{\mathtt{K}} A_N{}^{\mathtt{L}} \mathbb{P}^{\mathtt{K}}{}_{\mathtt{L}}{}^{\mathtt{N}}{}_{\mathtt{M}} \Phi_{\mathtt{N}} - A_N{}^{\mathtt{N}} X_{\mathtt{NM}}{}^{\mathtt{P}} \Phi_{\mathtt{P}} \,. \tag{6.13}$$

The last term is the usual embedding tensor contribution, whereas the middle two terms correspond to trombone contributions.

For the vierbein $E_M{}^a$, the new torsion tensor $T^a = \mathcal{D}E^a$ is unchanged. Similarly, the constraints on $\mathcal{D}\mathcal{V}_{\mathtt{M}}{}^{ij} = \mathcal{P}^{ijkl}\mathcal{V}_{\mathtt{M}kl}$ exactly match the superspace ExFT results. For the gravitino $E_M{}^{\alpha i}$, we define $T^{\alpha i} = \mathcal{D}E^{\alpha i}$, and using the definition (4.28) of $\widehat{T}^{\alpha i}$ with its ExFT constraints (4.59), leads to the gauged supergravity constraints

$$T_{\gamma k\, \beta j}{}^{\alpha i} = 0\,, \tag{6.14a}$$

$$T^{\dot\gamma k}{}_{\beta j}{}^{\alpha i} = 0\,, \tag{6.14b}$$

$$T^{\dot\gamma k\, \dot\beta j\, \alpha i} = \sqrt{2}\, \epsilon^{\dot\gamma\dot\beta} \chi^{\alpha k j i}\,, \tag{6.14c}$$

$$T_{\beta j c}{}^{\alpha i} = \frac{1}{8} (\bar\chi^{ikl} \gamma^a \chi_{jkl}) (\gamma_c \gamma_a)_\beta{}^\alpha\,, \tag{6.14d}$$

$$\begin{aligned} T^{\dot\beta j}{}_c{}^{\alpha i} &= \frac{1}{8} (\gamma_c \gamma^{ab})^{\dot\beta\alpha} \mathcal{F}_{ab}{}^{ij} - \frac{1}{1152} \varepsilon^{ijklpqrs} (\bar\chi_{klp} \gamma^{ab} \chi_{qrs}) (\gamma_{ab}\gamma_c)^{\dot\beta\alpha} \\ &\quad + (\gamma_c)^{\dot\beta\alpha} (2B^{ij} - A_1{}^{ij}) \end{aligned} \tag{6.14e}$$

It helps to recall here that $A_1{}^{ij}$ in (4.59) corresponded to a specific component of the $E_{7(7)}$ torsion tensor. Adopting the Weitzenböck connection and making the reduction ansatz converts this to the corresponding component of the embedding tensor. The trombone contribution $B^{ij} := i\mathcal{V}^{\mathtt{M}ij}\vartheta_{\mathtt{M}}$ arises from the second term in the definition of $\widehat{T}^{\alpha i}$. We also emphasize that taking the Weitzenböck connection has eliminated all factors of $\mathcal{P}$ and $\mathcal{Q}$. For the one-form χ curvature, $\tau^{\alpha ijk} := \mathcal{D}\chi^{\alpha ijk}$, we recover the constraints

$$\begin{aligned} \tau_{\beta l}{}^{\alpha ijk} &= -\frac{3}{4}\sqrt{2}\, \delta_l^{[i} (\gamma_{ab})_\beta{}^\alpha \mathcal{F}_{ab}{}^{jk]} - \frac{\sqrt{2}}{24} \delta_\beta{}^\alpha \varepsilon^{ijkpqrst} \bar\chi_{pqr} \chi_{stl} \\ &\qquad - 2\sqrt{2}\, \delta_\beta{}^\alpha A_{2l}{}^{ijk} - 4\sqrt{2}\, \delta_\beta{}^\alpha \delta_l{}^{[i} B^{jk]}\,, \\ \tau^{\dot\beta l\, \alpha ijk} &= 2\sqrt{2} (\gamma_a)^{\dot\beta\alpha} \mathcal{P}_a{}^{ijkl}\,. \end{aligned} \tag{6.15}$$

Using these constraints, one can recover the expected SUSY transformations of the component gravitino and χ field.

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Because of the structure of the generalized Scherk-Schwarz reduction, where group-valued twist matrices govern the entirety of the y-dependence, consistency of the reduced theory is straightforward. The only meaningful check is to ensure that the trivial ansatz (6.10) for $\mathscr{B}_{NM\,\underline{m}}$ is consistent with the other ansätze, where a general (x, θ) dependent piece remained. This amounts to checking that the curvature associated with $\mathscr{B}_{NM\,\underline{m}}$ actually vanishes. Although we have not discussed this explicitly, it is relatively straightforward to show that $\mathscr{H}_{\underline{m}}$ can be defined directly in terms of $\mathscr{B}_{\underline{m}}$ and corresponds to the covariantization of its exterior derivative. Now upon specializing to the Weitzenböck connection and making the ansätze discussed above, one can see that $\mathring{\mathscr{H}}_{\underline{m}}$ does indeed vanish.[19] (The same is true for $\mathring{\mathscr{G}}_{\underline{m}}{}^{\underline{n}}$, ensuring that the constrained 3-forms drop out as well.)

In summary, the constraints discussed above characterize the structure of gauged supergravity in superspace.

7 Conclusions

In this paper we have provided the superspace formulation for $\mathrm{E}_{7(7)}$ exceptional field theory. We have shown how the external diffeomorphisms and local supersymmetry transformations can be understood in a unified fashion as superdiffeomorphisms. In doing so, we have found that a redefinition of constrained 2-form potential is necessary, and it provides a geometrical framework in which the internal Lorentz connection is removed everywhere. Interestingly, a similar field redefinition in $\mathrm{E}_{8(8)}$ exceptional field theory, this time involving a constrained 1-form, allowed a reinterpretation of the theory as a Chern-Simons theory [38]. As an application of our superspace $\mathrm{E}_{7(7)}$ ExFT, we have performed a generalized Scherk-Schwarz reduction to obtain the superspace formulation of maximal gauged supergravities parametrized by an embedding tensor.

The ideas of this paper are expected to be applicable to all other exceptional field theories. Two challenging future directions are as follows. The first is an application of our results to the construction of actions for particle, string and brane actions as suitable sigma models in which the target space manifold is the superspace we have constructed here. The second is to aim for a further unification. Although we have combined supersymmetry and external diffeomorphisms, they remain distinct from internal diffeomorphisms. A master formulation should exist where these emerge as different parts of a single set of generalized (super)diffeomorphisms. In such a formulation, including fermions and their local supersymmetry transformations, it would be interesting to understand better the reason for the redefinitions we have encountered and whether it is indeed essential for some of the constrained p-form fields to adopt anomalous R-symmetry transformations.

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[19] This is slightly more subtle than we have described, because one must ensure that the full definition of $\mathscr{H}_{\underline{m}}$ in terms of $\mathscr{B}_{\underline{m}}$ consistently vanishes. This expression involves $R_{[P\underline{m}}{}^{\boldsymbol{\alpha}} B_{NM]\,\boldsymbol{\alpha}}$, the first factor being the mixed external/internal $\mathrm{E}_{7(7)}$ curvature, and indeed one can show this vanishes in the Weitzenböck connection under the ansätze.

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A Conventions

The Lorentz metric is $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ and the antisymmetric tensor ε_{abcd} is imaginary, with $\varepsilon_{0123} = -i$. We employ the pseudotensor $\varepsilon_{mnpq} := e_m{}^a e_n{}^b e_p{}^c e_q{}^d \varepsilon_{abcd}$, which introduces some factors of $e = \det e_m{}^a$ versus corresponding formulae in [22].

A.1 Spinor conventions

We employ both four-component and two-component conventions. Our two-component conventions follow mainly Wess and Bagger [39]. Left-handed spinors are denoted with two-component Greek indices $\alpha, \beta, \gamma, \cdots$, while right-handed spinors are denoted with dotted indices $\dot\alpha, \dot\beta, \dot\gamma, \cdots$. Spinor indices are raised and lowered using the antisymmetric tensor $\epsilon_{\alpha\beta}$,

$$\psi^\beta = \epsilon^{\beta\alpha}\psi_\alpha\,, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta\,, \qquad \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma\,, \qquad \epsilon^{12} = \epsilon_{21} = 1\,, \tag{A.1}$$

and similarly with dotted indices. A complex four-component Dirac spinor Ψ decomposes into left-handed and right-handed spinors ψ_α and $\chi^{\dot\alpha}$. Its charge conjugate Ψ^c decomposes into $\chi_\alpha = (\chi_{\dot\alpha})^*$ and $\psi^{\dot\alpha} = (\psi^\alpha)^*$ so that Ψ and its Dirac conjugate $\bar\Psi$ are given by

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot\alpha} \end{pmatrix}\,, \qquad \bar\Psi = \begin{pmatrix} \chi^\alpha\,,\ \psi_{\dot\alpha} \end{pmatrix}\,. \tag{A.2}$$

For a Majorana spinor, $\chi = \psi$ above. Our 4D gamma matrices obey

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\,, \qquad (\gamma^a)^\dagger = \gamma_a\,, \qquad \gamma_5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{A.3}$$

They decompose as

$$\gamma^a = \begin{pmatrix} 0 & (\gamma^a)_{\alpha\dot\beta} \\ (\gamma^a)^{\dot\alpha\beta} & 0 \end{pmatrix}\,, \qquad \gamma_5 = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & -\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}\,. \tag{A.4}$$

The two-component matrices $(\gamma^a)_{\alpha\dot\alpha}$ are formally identical to $i(\sigma^a)_{\alpha\dot\alpha}$ where σ^a obey the same relations as in [39], i.e.

$$(\gamma^a)^{\dot\alpha\alpha} := \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\gamma^a)_{\beta\dot\beta}\,. \tag{A.5}$$

Antisymmetric combinations of γ matrices are

$$\gamma^{ab} = \gamma^{[a}\gamma^{b]} = \begin{pmatrix} (\gamma^{ab})_\alpha{}^\beta & 0 \\ 0 & (\gamma^{ab})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}\,, \tag{A.6}$$

where $(\gamma^{ab})_{\alpha\beta} = \epsilon_{\beta\gamma}(\gamma^{ab})_\alpha{}^\gamma$ is *symmetric* in its spinor indices and similarly for $(\gamma^{ab})_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\gamma}(\gamma^{ab})^{\dot\gamma}{}_{\dot\beta}$. These obey the duality properties

$$\frac{1}{2}\varepsilon_{abcd}\gamma^{cd} = -\gamma_5\gamma_{ab}\,, \qquad \frac{1}{2}\varepsilon_{abcd}(\gamma^{cd})_\alpha{}^\beta = -(\gamma_{ab})_\alpha{}^\beta\,, \qquad \frac{1}{2}\varepsilon_{abcd}(\gamma^{cd})^{\dot\alpha}{}_{\dot\beta} = +(\gamma_{ab})^{\dot\alpha}{}_{\dot\beta}\,. \tag{A.7}$$

A.2 SO(1,3) and SU(8) transformations and connections

Lorentz transformations act as

$$\begin{aligned}
\delta e_m{}^a &= \lambda^a{}_b\, e_m{}^b = -e_m{}^b\, \lambda_b{}^a\,, \\
\delta \psi_{m\,\alpha}{}^i &= \lambda_\alpha{}^\beta \psi_{m\,\beta i} = \frac{1}{4}\lambda^{cd}(\gamma_{cd})_\alpha{}^\beta \psi_{m\,\beta i}\,, \\
\delta \psi_m{}^{\alpha i} &= -\psi_m{}^{\beta i}\lambda_\beta{}^\alpha = -\frac{1}{4}\lambda^{cd}\,\psi_m{}^{\beta i}(\gamma_{cd})_\beta{}^\alpha\,,
\end{aligned} \tag{A.8}$$

Our conventions for SU(8) indices follow [34]. In particular, the SU(8) transformations of the $\mathbf{8}$ and $\overline{\mathbf{8}}$ involve factors of $\frac{1}{2}$ as

$$\delta V_i = -\frac{1}{2}\lambda_i{}^j V_j\,, \qquad \delta V^i = +\frac{1}{2}\lambda_j{}^i V^j \qquad \Longrightarrow \qquad \delta \mathcal{V}_{\underline{m}}{}^{ij} = \lambda_{kl}{}^{ij}\mathcal{V}_{\underline{m}}{}^{kl} = \delta_k{}^{[i}\lambda_l{}^{j]}\,\mathcal{V}_{\underline{m}}{}^{kl}\,. \tag{A.9}$$

The corresponding connections appear in the covariant derivative with a minus sign so that

$$\mathcal{D}_m \psi_n{}^i = \partial_m \psi_n{}^i - \frac{1}{4}\omega_m{}^{cd}\gamma_{cd}\psi_n{}^i - \frac{1}{2}\mathcal{Q}_{mj}{}^i\psi_n{}^j\,. \tag{A.10}$$

A.3 Differential forms

Our conventions for differential forms follow the usual superspace conventions. For a p-form Ω, we write

$$\Omega = \frac{1}{p!}\mathrm{d}Z^{M_1}\cdots \mathrm{d}Z^{M_p}\Omega_{M_p\cdots M_1} = \frac{1}{p!}E^{A_1}\cdots E^{A_p}\Omega_{A_p\cdots A_1}\,. \tag{A.11}$$

Differential forms and interior products act from the right, so that

$$\mathrm{d}\Omega = \frac{1}{p!}\mathrm{d}Z^{M_1}\cdots \mathrm{d}Z^{M_p}\mathrm{d}Z^N\partial_N\Omega_{M_p\cdots M_1}\,, \tag{A.12}$$

$$\imath_V\Omega = \frac{1}{(p-1)!}\mathrm{d}Z^{M_1}\cdots V^{M_p}\Omega_{M_p\cdots M_1}\,. \tag{A.13}$$

Whenever superindices M and N are antisymmetrized, this carries a usual grading so that

$$2\,V_{[M}W_{N]} = V_M W_N - (-1)^{\epsilon(M)\epsilon(N)}V_N W_M \tag{A.14}$$

where $\epsilon(M) = 0$ or 1 depending on if M is a bosonic or fermionic index. The grading can be understood as arising because the indices M and N have been interchanged from their ordering on the left-hand side. In a similar way, gradings appear in expressions like (4.22), which should actually be read as

$$\widehat{\mathcal{F}}_{NM}{}^{\underline{m}} = E_N{}^B E_M{}^A \widehat{\mathcal{F}}_{BA}{}^{\underline{m}}\,(-1)^{\epsilon(B)(\epsilon(M)+\epsilon(A))}\,. \tag{A.15}$$

Gradings also arise from pushing super-indices past other fermionic indices. For example, (4.26) should be read as

$$\begin{aligned}
\delta_{\mathrm{cov}}(\xi)E_M{}^{\alpha i} &= \mathcal{D}_M\xi^{\alpha i} + \xi^N T_{NM}{}^{\alpha i} \\
&\quad + 2i\,\mathcal{V}^{\underline{m}ij}\nabla_{\underline{m}}\xi^N\Big((-1)^{\epsilon(M)}E_{N\,\dot\beta j}E_M{}^c - E_N{}^c E_{M\,\dot\beta j}\Big)(\gamma_c)^{\dot\beta\alpha} \\
&\quad - \frac{i}{2\sqrt{2}}\mathcal{V}^{\underline{m}}{}_{jk}\nabla_{\underline{m}}\xi^N E_N{}^d E_M{}^c\chi^{\beta jki}(\gamma_c\gamma_d)_\beta{}^\alpha\,,
\end{aligned} \tag{A.16}$$

as the M index must be pushed all the way to the left and picks up a sign when passing $\dot\beta j$.

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B Algebra of external and internal derivatives

In analyzing the superspace Bianchi identities, it is useful to employ covariant external and internal derivatives to maintain manifest internal diffeomorphism covariance. In this appendix, we summarize the commutation relations of these covariant derivatives. A number of these formulae have appeared elsewhere (see e.g. [37]), but we present them here in a unified way in our conventions.

External derivative algebra. Defining the exterior (external) covariant differential $D := \mathrm{d} - \mathbb{L}_A$, we have as usual $D^2 = -\mathbb{L}_{\mathcal{F}}$. Because $\mathcal{F}^{\underline{m}}$ has weight $\frac{1}{2}$, it follows that for arbitrary tensor $V^{\underline{m}}$,

$$D^2 V^{\underline{m}} = -\mathbb{L}_{\mathcal{F}} V^{\underline{m}} = -\mathbb{L}^{\nabla}_{\mathcal{F}} V^{\underline{m}} + \mathcal{F}^{\underline{n}} \mathcal{T}_{\underline{np}}{}^{\underline{m}} V^{\underline{p}} \,. \tag{B.1}$$

The last equation is useful because it allows us to maintain manifest internal diffeomorphism covariance.

Mixed external/internal derivative algebra. On an internal vector $V^{\underline{m}}$ of weight λ, one can show

$$[D_M, \nabla_{\underline{m}}] V^{\underline{n}} = R_{M\underline{mp}}{}^{\underline{n}} V^{\underline{p}} - \frac{2}{3} \lambda \, R_{M\underline{k}\,\underline{m}}{}^{\underline{k}} V^{\underline{n}} \,, \tag{B.2}$$

where the mixed $\mathrm{E}_{7(7)}$ curvature

$$R_{M\underline{mp}}{}^{\underline{n}} := D_M \Gamma_{\underline{mp}}{}^{\underline{n}} - 12\, \mathbb{P}^{\underline{n}}{}_{\underline{p}}{}^{\underline{k}}{}_{\underline{l}}\, \partial_{\underline{m}} \partial_{\underline{k}} A_M{}^{\underline{l}} \,. \tag{B.3}$$

Provided the $\mathrm{E}_{7(7)}$ connection Γ transforms as a proper affine connection, the mixed $\mathrm{E}_{7(7)}$ curvature transforms covariantly, i.e. as a proper curvature.

Internal derivative algebra. The commutator of internal derivatives $V^{\underline{m}}$ can be written

$$[\nabla_{\underline{m}}, \nabla_{\underline{n}}] V^{\underline{p}} = -\mathbf{T}_{\underline{mn}}{}^{\underline{r}} \nabla_{\underline{r}} V^{\underline{p}} + R_{\underline{mnq}}{}^{\underline{p}} V^{\underline{q}} - \frac{2}{3} \lambda R_{\underline{mn}} V^{\underline{p}} \,. \tag{B.4}$$

While the full right-hand side is a covariant expression by construction, the individual terms are not. The "torsion tensor" defined by

$$\mathbf{T}_{\underline{mn}}{}^{\underline{p}} := 2\, \Gamma_{[\underline{mn}]}{}^{\underline{p}} - \frac{2}{3} \Gamma_{[\underline{m}} \delta_{\underline{n}]}{}^{\underline{p}} \,, \qquad \Gamma_{\underline{m}} := \Gamma_{\underline{km}}{}^{\underline{k}} \,, \tag{B.5}$$

is only a tensor if it is contracted with a constrained vector on the $\underline{p}$ index, as one finds a non-covariant part to its transformation,

$$\begin{aligned} \delta_{\rm nc} \mathbf{T}_{\underline{mn}}{}^{\underline{p}} &= Y_{\underline{nl}}{}^{\underline{pk}}\, \partial_{\underline{m}} \partial_{\underline{k}} \Lambda^{\underline{l}} - Y_{\underline{ml}}{}^{\underline{pk}}\, \partial_{\underline{n}} \partial_{\underline{k}} \Lambda^{\underline{l}} \,, \\ Y_{\underline{mn}}{}^{\underline{kl}} &:= 12 (t_{\boldsymbol{\alpha}})_{\underline{mn}} (t^{\boldsymbol{\alpha}})^{\underline{kl}} - \frac{1}{2} \Omega_{\underline{mn}} \Omega^{\underline{kl}} \,. \end{aligned} \tag{B.6}$$

It cannot generically be chosen to vanish for this reason, although one can have a situation where it vanishes always upon contraction with a constrained vector, as in [37]. Note that the internal covariant derivative $\nabla_{\underline{m}}$ is not necessarily a constrained object.

Some observations are in order. The object $\mathbf{T}$ defined above and the actual $\mathrm{E}_{7(7)}$ torsion tensor $\mathcal{T}$ are related by

$$\mathcal{T}_{\underline{mn}}{}^{\underline{p}} = \mathbf{T}_{\underline{mn}}{}^{\underline{p}} - Y_{\underline{nr}}{}^{\underline{ps}}\Gamma_{\underline{sm}}{}^{\underline{r}} + \frac{1}{3}Y_{\underline{nm}}{}^{\underline{pq}}\Gamma_{\underline{rp}}{}^{\underline{r}} \,. \tag{B.7}$$

These happen to coincide when using the Weitzenböck connection and contracting with a constrained vector, i.e. $\mathring{\mathcal{T}}_{\underline{mn}}{}^{\underline{p}}\partial_{\underline{p}} = \mathring{\mathbf{T}}_{[\underline{mn}]}{}^{\underline{p}}\partial_{\underline{p}}$.

The $\mathrm{E}_{7(7)}$ "curvature"

$$R_{\underline{mnq}}{}^{\underline{p}} := \partial_{\underline{m}}\Gamma_{\underline{nq}}{}^{\underline{p}} - \partial_{\underline{n}}\Gamma_{\underline{mq}}{}^{\underline{p}} - \Gamma_{\underline{mq}}{}^{\underline{r}}\Gamma_{\underline{nr}}{}^{\underline{p}} + \Gamma_{\underline{nq}}{}^{\underline{r}}\Gamma_{\underline{mr}}{}^{\underline{p}} \,, \tag{B.8}$$

is also non-covariant, transforming as

$$\delta_{\mathrm{nc}} R_{\underline{mnq}}{}^{\underline{r}} = \delta_{\mathrm{nc}}\mathbf{T}_{\underline{mn}}{}^{\underline{r}}\,\Gamma_{\underline{rq}}{}^{\underline{p}} \,. \tag{B.9}$$

The scale curvature is

$$R_{\underline{mn}} := \partial_{\underline{m}}\Gamma_{\underline{kn}}{}^{\underline{k}} - \partial_{\underline{n}}\Gamma_{\underline{km}}{}^{\underline{k}} \,, \tag{B.10}$$

which is indeed covariant. This is not given by a simple contraction of $R_{\underline{mnq}}{}^{\underline{r}}$, but instead by

$$R_{\underline{mpn}}{}^{\underline{p}} - R_{\underline{npm}}{}^{\underline{p}} = \frac{4}{3}R_{\underline{mn}} - \nabla_{\underline{p}}\mathbf{T}_{\underline{mn}}{}^{\underline{p}} \,. \tag{B.11}$$

Covariant derivatives, their connections, and curvatures. It is straightforward to modify the definitions of D_M and $\nabla_{\underline{m}}$ so that they carry spin and SU(8) connections. That is, we take

$$\mathcal{D}_M := D_M - \frac{1}{2}\Omega_M{}^{ab}M_{ab} - \frac{1}{2}\mathcal{Q}_{Mj}{}^i I_i{}^j \,, \tag{B.12}$$

$$\nabla'_{\underline{m}} := \nabla_{\underline{m}} - \frac{1}{2}\Omega_{\underline{m}}{}^{ab}M_{ab} - \frac{1}{2}\mathcal{Q}_{\underline{m}j}{}^i I_i{}^j \,, \tag{B.13}$$

where M_{ab} and $I_i{}^j$ are the Lorentz and SU(8) generators, which act on a spinor $X_{\underline{m}\,i}$ as $M_{ab}X_{\underline{m}\,i} = \frac{1}{2}\gamma_{ab}X_{\underline{m}\,i}$ and $I_k{}^l X_{\underline{m}\,i} = -\delta^l_i X_{\underline{m}\,k} + \frac{1}{8}\delta^l_k X_{\underline{m}\,i}$. Henceforth, we drop the prime on $\nabla_{\underline{m}}$. It is now easy to show that

$$[\mathcal{D}_M, \nabla_{\underline{m}}]X_{\underline{n}\,i} = -\frac{1}{4}R_{M\underline{m}}{}^{ab}\gamma_{ab}\,X_{\underline{n}\,i} + \frac{1}{2}R_{M\underline{m}i}{}^j X_{\underline{n}\,j} - R_{M\underline{mn}}{}^{\underline{p}}X_{\underline{p}\,i} - \frac{2}{3}\lambda\, R_{M\underline{k}\,\underline{m}}{}^{\underline{k}}X_{\underline{n}\,i} \,, \tag{B.14}$$

involving the $R_{M\underline{n}\,ab}$ and $R_{M\underline{n}i}{}^j$, with their obvious definitions. In the body of the paper, we have taken the internal part of the spin connection to vanish, so that $R_{M\underline{n}}{}^{ab} = -\nabla_{\underline{n}}\Omega_M{}^{ab}$.

Curvature relations. As a consequence of the external and internal GVPs,

$$\mathcal{D}_M\mathcal{V}_{\underline{p}}{}^{ij} = \mathcal{P}_M{}^{ijkl}\mathcal{V}_{\underline{p}\,kl} \,, \qquad \nabla_{\underline{m}}\mathcal{V}_{\underline{p}}{}^{ij} = \mathcal{P}_{\underline{m}}{}^{ijkl}\mathcal{V}_{\underline{p}\,kl} \,, \tag{B.15}$$

one can show that

$$\begin{aligned} R_{M\underline{n}\,\underline{pq}} = 2i\Big(R_{M\underline{n}\,kl}{}^{ij} + \mathcal{P}_{\underline{n}}{}^{ijrs}\mathcal{P}_{Mrskl} - \mathcal{P}_M{}^{ijrs}\mathcal{P}_{\underline{n}rskl}\Big)\mathcal{V}_{(\underline{p}}{}^{kl}\mathcal{V}_{\underline{q})ij} \\ + i(\mathcal{D}_M\mathcal{P}_{\underline{n}}{}^{ijkl} - \nabla_{\underline{n}}\mathcal{P}_M{}^{ijkl})\mathcal{V}_{\underline{p}ij}\mathcal{V}_{\underline{q}kl} - i(\mathcal{D}_M\mathcal{P}_{\underline{n}ijkl} - \nabla_{\underline{n}}\mathcal{P}_{Mijkl})\mathcal{V}_{\underline{p}}{}^{ij}\mathcal{V}_{\underline{q}}{}^{kl} \,. \end{aligned} \tag{B.16}$$

This condition allows one to determine the $E_{7(7)}$ curvature $R_{M\underline{np}}{}^{\underline{q}}$ from the SU(8) curvature $R_{M\underline{n}\,kl}{}^{ij} = \delta_{[k}{}^{[i}R_{M\underline{n}l]}{}^{j]}$ and the mixed curl of $\mathcal{P}^{ijkl}$, or vice-versa, reflecting the ambiguity in the internal GVP. Similarly, using

$$\mathcal{D}^2\mathcal{V}_{\underline{m}}{}^{ij} = R_k{}^{[i}\mathcal{V}_{\underline{m}}{}^{j]k} - \mathbb{L}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij}\,, \tag{B.17}$$

one can determine the two-form $\mathcal{D}\mathcal{P}^{ijkl}$ and the external SU(8) curvature $R_i{}^j$ as

$$\mathcal{D}\mathcal{P}^{ijkl} = -i\mathcal{V}^{\underline{m}kl}\mathbb{L}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij}\,, \qquad R_{kl}{}^{ij} = i\mathcal{V}^{\underline{m}}{}_{kl}\mathbb{L}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij} - \mathcal{P}^{ijrs}\mathcal{P}_{rskl}\;. \tag{B.18}$$

For these last relations, it is helpful to use

$$\mathbb{L}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij} = \mathbb{L}^{\nabla}_{\mathcal{F}}\mathcal{V}_{\underline{m}}{}^{ij} + \mathcal{F}^{\underline{n}}\mathcal{T}_{\underline{nm}}{}^{\underline{p}}\mathcal{V}_{\underline{p}}{}^{ij}\;. \tag{B.19}$$

Then one can show for example that

$$\begin{aligned} R_{kl}{}^{ij} &= -\mathcal{P}^{ijrs}\wedge\mathcal{P}_{rskl} + 12i\mathcal{V}^{\underline{m}}{}_{kl}\mathcal{V}_{\underline{n}}{}^{ij}\mathbb{P}^{\underline{n}}{}_{\underline{m}}{}^{\underline{k}}{}_{\underline{l}}\nabla_{\underline{k}}\mathcal{F}^{\underline{l}} + i\mathcal{F}^{\underline{n}}\mathcal{T}_{\underline{nk}}{}^{\underline{l}}\mathcal{V}_{\underline{l}kl}\mathcal{V}^{\underline{k}ij} \\ &= -\mathcal{P}^{ijrs}\wedge\mathcal{P}_{rskl} - \Big(12\nabla_{\underline{p}}\mathcal{F}^{\underline{q}}\,(t^{\boldsymbol{\alpha}})_{\underline{q}}{}^{\underline{p}} + \mathcal{F}^{\underline{n}}\mathcal{T}_{\underline{n}}{}^{\boldsymbol{\alpha}}\Big)\mathcal{Q}_{\boldsymbol{\alpha}kl}{}^{ij}\;. \end{aligned} \tag{B.20}$$

Finally, we mention for the purely internal curvatures that

$$\begin{aligned} R_{\underline{mnp}}{}^{\underline{q}} = &\Big(2i\,\nabla_{[\underline{m}}\mathcal{P}_{\underline{n}]}{}^{ijkl} + i\,\mathbf{T}_{\underline{mn}}{}^{\underline{r}}\mathcal{P}_{\underline{r}}{}^{ijkl}\Big)\mathcal{V}_{\underline{p}kl}\mathcal{V}^{\underline{q}}{}_{ij} \\ &+ \Big(i\,R_{\underline{mn}kl}{}^{ij} - 2i\,\mathcal{P}_{[\underline{m}}{}^{ijrs}\mathcal{P}_{\underline{n}]rskl}\Big)\mathcal{V}_{\underline{p}}{}^{kl}\mathcal{V}^{\underline{q}}{}_{ij} + \text{c.c.} \end{aligned} \tag{B.21}$$

External derivatives of the $\mathbf{E_{7(7)}}$ torsion. One final set of relations prove useful: the external derivative of the $E_{7(7)}$ torsion tensor. This can be written

$$\mathcal{D}_M\mathcal{T}_{\underline{nm}}{}^{\underline{p}} = R_{M\underline{nm}}{}^{\underline{p}} - 12\,\mathbb{P}^{\underline{p}}{}_{\underline{m}}{}^{\underline{r}}{}_{\underline{s}}\,R_{M\underline{rn}}{}^{\underline{s}} + 4\,\mathbb{P}^{\underline{p}}{}_{\underline{m}}{}^{\underline{r}}{}_{\underline{n}}\,R_{M\underline{sr}}{}^{\underline{s}}\;. \tag{B.22}$$

From this equation, one can determine the covariant exterior derivatives of $A_1{}^{ij}$ and $A_{2i}{}^{jkl}$,

$$\mathcal{D}_M A_1{}^{ij} = \frac{1}{3}A_2{}^{(i}{}_{klp}\mathcal{P}_M{}^{j)klp} + iR_{M\underline{m}k}{}^{(i}\mathcal{V}^{\underline{m}j)k} - \frac{4i}{3}\mathcal{P}_{\underline{m}klpq}\mathcal{V}^{\underline{m}q(i}\mathcal{P}_M{}^{j)klp}\,, \tag{B.23a}$$

$$\begin{aligned} \mathcal{D}_M A_{2i}{}^{jkl} = &\Big[2A_{1ip}\mathcal{P}_M{}^{pjkl} + 3A_2{}^j{}_{ipq}\mathcal{P}_M{}^{pqkl} + 3i\,R_{M\underline{m}i}{}^j\mathcal{V}^{\underline{m}kl} \\ &+ 4i\,\mathcal{V}^{\underline{m}}{}_{ip}\,(\mathcal{D}_M\mathcal{P}_{\underline{m}}{}^{jklp} - \mathcal{D}_{\underline{m}}\mathcal{P}_M{}^{jklp}) - 4i\,\mathcal{P}_{\underline{m}ipqr}\,\mathcal{P}_M{}^{jpqr}\,\mathcal{V}^{\underline{m}kl}\Big]_{\mathbf{420}}\,, \end{aligned} \tag{B.23b}$$

where a projection onto the **420** of SU(8) is implied in the last equality.

References

[1] E. Cremmer and B. Julia, *The* SO(8) *supergravity*, *Nucl. Phys.* **B 159** (1979) 141 [INSPIRE].

[2] B. Julia, *Group disintegrations*, in *Superspace and supergravity*, S.W. Hawking and M. Roček eds., Cambridge University Press, Cambridge U.K. (1981).

[3] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, Nucl. Phys. **B 438** (1995) 109 [hep-th/9410167] [INSPIRE].

[4] B. de Wit and H. Nicolai, *Hidden symmetry in $d = 11$ supergravity*, Phys. Lett. **155B** (1985) 47 [INSPIRE].

[5] B. de Wit and H. Nicolai, *$d = 11$ supergravity with local* SU(8) *invariance*, Nucl. Phys. **B 274** (1986) 363 [INSPIRE].

[6] M. Duff, *$E_8 \times SO(16)$ symmetry of $d = 11$ supergravity?*, in *Quantum field theory and quantum statistics. Volume 2*, A. Batalin et al. eds., Adam Hilger, Bristol U.K. (1987).

[7] O. Hohm and H. Samtleben, *Exceptional form of $D = 11$ supergravity*, Phys. Rev. Lett. **111** (2013) 231601 [arXiv:1308.1673] [INSPIRE].

[8] O. Hohm and H. Samtleben, *Exceptional field theory I: $E_{6(6)}$ covariant form of M-theory and Type IIB*, Phys. Rev. **D 89** (2014) 066016 [arXiv:1312.0614] [INSPIRE].

[9] O. Hohm and H. Samtleben, *Exceptional field theory. II. $E_{7(7)}$*, Phys. Rev. **D 89** (2014) 066017 [arXiv:1312.4542] [INSPIRE].

[10] O. Hohm and H. Samtleben, *Exceptional field theory. III. $E_{8(8)}$*, Phys. Rev. **D 90** (2014) 066002 [arXiv:1406.3348] [INSPIRE].

[11] D.S. Berman and M.J. Perry, *Generalized geometry and M-theory*, JHEP **06** (2011) 074 [arXiv:1008.1763] [INSPIRE].

[12] D.S. Berman, H. Godazgar, M. Godazgar and M.J. Perry, *The Local symmetries of M-theory and their formulation in generalised geometry*, JHEP **01** (2012) 012 [arXiv:1110.3930] [INSPIRE].

[13] A. Coimbra, C. Strickland-Constable and D. Waldram, *$E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M-theory*, JHEP **02** (2014) 054 [arXiv:1112.3989] [INSPIRE].

[14] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as generalised geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M-theory*, JHEP **03** (2014) 019 [arXiv:1212.1586] [INSPIRE].

[15] G. Aldazabal, M. Graña, D. Marqués and J.A. Rosabal, *Extended geometry and gauged maximal supergravity*, JHEP **06** (2013) 046 [arXiv:1302.5419] [INSPIRE].

[16] W. Siegel, *Superspace duality in low-energy superstrings*, Phys. Rev. **D 48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[17] C. Hull and B. Zwiebach, *Double field theory*, JHEP **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[18] O. Hohm, C. Hull and B. Zwiebach, *Background independent action for double field theory*, JHEP **07** (2010) 016 [arXiv:1003.5027] [INSPIRE].

[19] O. Hohm and H. Samtleben, *Consistent Kaluza-Klein truncations via exceptional field theory*, JHEP **01** (2015) 131 [arXiv:1410.8145] [INSPIRE].

[20] G. Bossard and A. Kleinschmidt, *Loops in exceptional field theory*, JHEP **01** (2016) 164 [arXiv:1510.07859] [INSPIRE].

[21] O. Hohm and B. Zwiebach, *T-duality constraints on higher derivatives revisited*, JHEP **04** (2016) 101 [arXiv:1510.00005] [INSPIRE].

[22] H. Godazgar et al., *Supersymmetric $E_{7(7)}$ exceptional field theory*, JHEP **09** (2014) 044 [arXiv:1406.3235] [INSPIRE].

JHEP01(2019)087

[23] A.S. Arvanitakis and C.D.A. Blair, *The exceptional σ-model*, *JHEP* **04** (2018) 064 [arXiv:1802.00442] [INSPIRE].

[24] L. Brink and P.S. Howe, *The $N = 8$ supergravity in superspace*, *Phys. Lett.* **88B** (1979) 268 [INSPIRE].

[25] P.S. Howe, *Supergravity in superspace*, *Nucl. Phys.* **B 199** (1982) 309 [INSPIRE].

[26] I. Bandos and T. Ortın, *Tensor gauge fields of $N = 8$ supergravity*, *Phys. Rev.* **D 91** (2015) 085031 [arXiv:1502.00649] [INSPIRE].

[27] P. Howe and J. Palmkvist, *Forms and algebras in (half-)maximal supergravity theories*, *JHEP* **05** (2015) 032 [arXiv:1503.00015] [INSPIRE].

[28] P.S. Howe and U. Lindström, *Higher order invariants in extended supergravity*, *Nucl. Phys.* **B 181** (1981) 487 [INSPIRE].

[29] I. Bandos, *On section conditions of $E_{7(+7)}$ exceptional field theory and superparticle in $\mathcal{N} = 8$ central charge superspace*, *JHEP* **01** (2016) 132 [arXiv:1512.02287] [INSPIRE].

[30] M. Hatsuda, K. Kamimura and W. Siegel, *Superspace with manifest T-duality from type-II superstring*, *JHEP* **06** (2014) 039 [arXiv:1403.3887] [INSPIRE].

[31] M. Poláček and W. Siegel, *T-duality off shell in 3D type II superspace*, *JHEP* **06** (2014) 107 [arXiv:1403.6904] [INSPIRE].

[32] I. Bandos, *Superstring in doubled superspace*, *Phys. Lett.* **B 751** (2015) 408 [arXiv:1507.07779] [INSPIRE].

[33] M. Cederwall, *Double supergeometry*, *JHEP* **06** (2016) 155 [arXiv:1603.04684] [INSPIRE].

[34] B. de Wit, H. Samtleben and M. Trigiante, *The maximal $D = 4$ supergravities*, *JHEP* **06** (2007) 049 [arXiv:0705.2101] [INSPIRE].

[35] B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and gaugings of maximal supergravities*, *Nucl. Phys.* **B 655** (2003) 93 [hep-th/0212239] [INSPIRE].

[36] H. Godazgar, M. Godazgar and H. Nicolai, *Generalised geometry from the ground up*, *JHEP* **02** (2014) 075 [arXiv:1307.8295] [INSPIRE].

[37] H. Godazgar, M. Godazgar and H. Nicolai, *Einstein-Cartan calculus for exceptional geometry*, *JHEP* **06** (2014) 021 [arXiv:1401.5984] [INSPIRE].

[38] O. Hohm and H. Samtleben, *Leibniz-Chern-Simons Theory and Phases of Exceptional Field Theory*, arXiv:1805.03220 [INSPIRE].

[39] J. Wess and J.A. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1992).

[40] K. Peeters, *Introducing Cadabra: a symbolic computer algebra system for field theory problems*, hep-th/0701238 [INSPIRE].

[41] K. Peeters, *A field-theory motivated approach to symbolic computer algebra*, *Comput. Phys. Commun.* **176** (2007) 550 [cs/0608005] [INSPIRE].

[42] B. de Wit, H. Nicolai and H. Samtleben, *Gauged supergravities, tensor hierarchies and M-theory*, *JHEP* **02** (2008) 044 [arXiv:0801.1294] [INSPIRE].

[43] A. Le Diffon and H. Samtleben, *Supergravities without an action: gauging the trombone*, *Nucl. Phys.* **B 811** (2009) 1 [arXiv:0809.5180] [INSPIRE].

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