E_8 orbits of IR dualities

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ABSTRACT: We discuss $\mathrm{USp}(2n)$ supersymmetric models with eight fundamental fields and a field in the antisymmetric representation. Turning on the most generic superpotentials, coupling pairs of fundamental fields to powers of the antisymmetric field while preserving an R symmetry, we give evidence for the statement that the models are connected by a large network of dualities which can be organized into orbits of the Weyl group of E_8. We make also several curious observations about such models. In particular, we argue that a $\mathrm{USp}(2m)$ model with the addition of singlet fields and even rank m flows in the IR to a CFT with $E_7 \times \mathrm{U}(1)$ symmetry. We also discuss an infinite number of duals for the $\mathrm{USp}(2)$ theory with eight fundamentals and no superpotential.

Contents

1 Introduction

Different QFTs can flow to the same conformal field theory in the IR. This phenomenon is usually referred to as IR duality in high energy physics or universality in statistical physics. Considering supersymmetric theories in four dimensions following [1], one can discover various examples of seemingly very different models, for example possessing different gauge groups, which nevertheless reside in the same universality class. An understanding why two different looking theories in the UV flow to the same fixed point is rather lacking at the moment.[1] An interesting question one can ask to aid such an understanding is whether there is any structure relating different theories in a certain universality class.

In this short note we will discuss such a structure in a very particular setup. We consider USp($2m$) gauge theories with eight fundamental chiral fields Q_i, a field in the antisymmetric representation X, and possibly a superpotential with gauge singlet fields. It so happens that these models are interrelated by a large network of dualities and this network has intriguing group theoretic structure. In particular there are dualities relating models with fixed rank but different superpotentials and a non trivial map of operators/symmetries between various sides of the duality. This duality web forms [9, 10] orbits of the Weyl group of E_7. Here we discuss yet another duality transformation which relates theories with different rank. In particular a USp($2m$) model with superpotential $Q_2 Q_1 X^n$ is in the same universality class as a USp($2n$) model with superpotential $Q_2 Q_1 X^m$ when certain singlet fields and superpotentials involving them are added. This duality transformation was considered implicitly in [9] as a property of integrals which imply equality of the supersymmetric index of the two dual models. We will argue that turning on most general superpotentials of the form above, breaking all the flavor symmetry of the gauge models but preserving R symmetry, this duality transformation together with permutations

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[1]There is though growing evidence that such dualities can be understood by constructing dual four dimensional theories as geometrically equivalent but different looking compactifications of a six dimensional model (see for examples [2–8]).

of the eight quarks generates orbits of the Weyl group of SO(16). Then together with the dualities generating E_7 orbits the full duality web is that of orbits of the Weyl group of E_8.

The note is organized as follows. We start the discussion with a review of dualities transforming USp(2m) models on an E_7 orbit. We also make a curious observation about a special property of the USp(2m) with even m. We argue that this model with a particular superpotential flows to a SCFT with $E_7 \times$ U(1) symmetry. In the particular case of USp(4) this model sits on the same conformal manifold as the E_7 surprise of Dimofte and Gaiotto [11]. In section three we discuss a duality which relates USp(2m) and USp(2n) models with superpotentials on both sides. In section four we finish by combining the two types of dualities and explain how they build orbits of the Weyl group of E_8.

2 The E_7 orbits of dualities

The basic theory we consider is a USp(2n) gauge theory with four flavours, Q_a with $a \in \{1, \cdots, 8\}$ in the fundamental representation, and one field in the antisymmetric representation, X. This model was studied by various authors, in particular see [12, 13] and references below. The symmetry is SU(8) $\times$ U(1). We call this model $\mathfrak{T}_0^{(n)}$. The charges are summarized in table 2.1.

	USp(2n)	SU(8)	U(1)	U(1)$_r$
Q_j	**2n**	**8**	$-\frac{n-1}{4}$	$\frac{1}{2}$
X	**n(2n − 1) − 1**	**1**	1	0

(2.1)

When $n = 1$ the field X does not exist and the model is SU(2) gauge theory with four fundamental flavors.

The model above has multiple known dual descriptions. The dual description preserving manifestly the most symmetry is given by the same gauge theory but with a collection of singlet fields coupled to gauge invariant operators through the superpotential [14],

$$W = \sum_{y=1}^{n} \sum_{i,j} M_{i,j/y}^n q_i q_j \widetilde{X}^{y-1} \,. \tag{2.2}$$

The charges of the fields are in table 2.3.

	USp(2n)	SU(8)	U(1)	U(1)$_r$
q_j	**2n**	$\overline{\mathbf{8}}$	$-\frac{n-1}{4}$	$\frac{1}{2}$
$\widetilde{X}$	**n(2n − 1) − 1**	**1**	1	0
$M_{i,j/y}$	**1**	**28**	$\frac{n+1}{2} - y$	1

(2.3)

The map between the operators is,

$$Q_i Q_j X^l \to M_{ij/n-l} \,, \qquad X^l \to \widetilde{X}^l \,. \tag{2.4}$$

This is a generalization of Intriligator-Pouliot duality [15] for n equals one case. We can build many other duality frames which will have less symmetry manifest in a similar manner

to the $n = 1$ case. The number of duals is $72 = W(E_7)/W(A_7)$. To construct these dualities we split the eight fundamental fields into two groups of four. The matter content is then written in table 2.5.

	USp(2n)	SU(4)$_1$	SU(4)$_2$	U(1)$_b$	U(1)	U(1)$_r$
$Q_{1,\dots,4}$	**2n**	**4**	**1**	1	$-\frac{n-1}{4}$	$\frac{1}{2}$
$Q_{5,\dots,8}$	**2n**	**1**	$\mathbf{\bar{4}}$	-1	$-\frac{n-1}{4}$	$\frac{1}{2}$
X	**n(2n − 1) − 1**	**1**	**1**	0	1	0

(2.5)

We have thirty five choices to perform such splitting. One set of thirty five duals is then given by the following matter content,

	USp(2n)	SU(4)$_1$	SU(4)$_2$	U(1)$_b$	U(1)	U(1)$_r$
$q_{1,\dots,4}$	**2n**	$\mathbf{\bar{4}}$	**1**	1	$-\frac{n-1}{4}$	$\frac{1}{2}$
$q_{5,\dots,8}$	**2n**	**1**	**4**	-1	$-\frac{n-1}{4}$	$\frac{1}{2}$
X	**n(2n − 1) − 1**	**1**	**1**	0	1	0
$M_{/l}$	**1**	**4**	$\mathbf{\bar{4}}$	0	$\frac{n+1}{2} - l$	1

(2.6)

with superpotential given here,

$$W = \sum_{l=1}^{n} \sum_{i,j=1}^{4} M_{i,j/l} \tilde{X}^{l-1} q_i q_{j+4} \,.$$

(2.7)

The mesons map to singlets and "baryons" to "baryons" as antisymmetric square of $\mathbf{\bar{4}}$ (**4**) is the real representation **6**. This is the analogue of [1] Seiberg duality. Another thirty five duals have the superpotential given by the following,

$$W = \sum_{l=1}^{n} \sum_{i,j=1,i\neq j}^{4} \left(\hat{M}_{i,j/l} q_i q_j \widetilde{X}^{l-1} + \hat{M}'_{i,j/l} q_{j+4} q_{i+4} \widetilde{X}^{l-1} \right),$$

(2.8)

with the matter content given in table 2.9.

	USp(2n)	SU(4)$_1$	SU(4)$_2$	U(1)$_b$	U(1)	U(1)$_r$
$q_{1,\dots,4}$	**2n**	**4**	**1**	-1	$-\frac{n-1}{4}$	$\frac{1}{2}$
$q_{5,\dots,8}$	**2n**	**1**	$\mathbf{\bar{4}}$	1	$-\frac{n-1}{4}$	$\frac{1}{2}$
X	**n(2n − 1) − 1**	**1**	**1**	0	1	0
$\hat{M}'_{/l}$	**1**	**1**	**6**	-2	$\frac{n+1}{2} - l$	1
$\hat{M}_{/l}$	**1**	**6**	**1**	2	$\frac{n+1}{2} - l$	1

(2.9)

The "baryons" map to the singlets and mesons to mesons. This is an analogue of the duality [16] discussed by Csaki, Schmaltz, Skiba, and Terning. The last two sets of dual descriptions were considered in [10].

It is convenient to encode the dualities as transformations on an ordered set of fugacities for the different symmetries. We will parametrize the symmetries as follows using

supersymmetric index nomenclature. Remember that a chiral field of R charge r contributes to the index as $\Gamma_e((qp)^{\frac{r}{2}}h)$ [17, 18] with h being the fugacity of the U(1) symmetry under which the field transforms. We will then denote by $u_i = (qp)^{\frac{1}{4}}h_i t^{-\frac{n-1}{4}}$ the weight for the ith quark. Here the fugacity t is for the U(1) symmetry, and fugacities h_j for SU(8) (then $\prod_{k=1}^{8} h_k = 1$). We can take the parameters t and u_k to be general with one constraint coming from anomaly cancelation, $t^{2n-2}\prod_j u_j = (qp)^2$. We define an ordered set of fugacities parametrizing the gauge sector of the USp($2n$) theory to be $(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, n, t)$. Denote $u_+^4 = \prod_{j=1}^{4} u_j$ and $u_-^4 = \prod_{j=5}^{8} u_j$. Then the three dualities we discussed imply the following transformations of this set,

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, n, t) \rightarrow$$
$$\rightarrow \left(\frac{u_+ u_-}{u_1}, \frac{u_+ u_-}{u_2}, \frac{u_+ u_-}{u_3}, \frac{u_+ u_-}{u_4}, \frac{u_+ u_-}{u_5}, \frac{u_+ u_-}{u_6}, \frac{u_+ u_-}{u_7}, \frac{u_+ u_-}{u_8}, n, t \right)$$
$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, n, t) \rightarrow \left(\frac{u_+^2}{u_1}, \frac{u_+^2}{u_2}, \frac{u_+^2}{u_3}, \frac{u_+^2}{u_4}, \frac{u_-^2}{u_5}, \frac{u_-^2}{u_6}, \frac{u_-^2}{u_7}, \frac{u_-^2}{u_8}, n, t \right)$$
$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, n, t) \rightarrow$$
$$\rightarrow \left(\frac{u_-}{u_+}u_1, \frac{u_-}{u_+}u_2, \frac{u_-}{u_+}u_3, \frac{u_-}{u_+}u_4, \frac{u_+}{u_-}u_5, \frac{u_+}{u_-}u_6, \frac{u_+}{u_-}u_7, \frac{u_+}{u_-}u_8, n, t \right) \qquad (2.10)$$

These transformations together with permutations of the first eight terms generate the Weyl group of E_7.

It has been observed by Dimofte and Gaiotto [11] that combining two copies of USp(2) theories by coupling the gauge invariant operators as $Q_j Q_i q_j q_i$, the theory has a point on the conformal manifold in the IR with, at least, E_7 symmetry. This fact was related in [8] to a statement that the two copies of USp(2) with that superpotential can be obtained by compactification of the E string theory on a torus. We will now discuss here a generalizations of the former fact to higher rank.

2.1 $E_7 \times$ U(1) surprise

Consider $\mathfrak{T}_0^{(m)}$ and assume m even. Turn on superpotential,

$$\sum_{j=1}^{m/2} Q_l Q_i X^{j-1} M_{il/j} + \sum_{i=2}^{m} X^i x_i \,.$$

We claim this model is self dual under the dualities we have considered. Note that under all the dualities it is either that $Q_i Q_j X^{l-1}$ maps to itself or to $M_{ij/n-l}$. The powers of X map to the same powers on the dual side. This implies that the superpotential maps to itself under the three dualities. The only effect of the duality is the non trivial identification of symmetries. These imply that for example the protected spectrum is invariant under th Weyl group of E_7 and thus forms representations of E_7. It is then plausible that on some point on the conformal manifold of this model the group enhances to U(1) $\times E_7$. Let us analyze one example in detail.

Consider the USp(4) theory with the mesons and X^2 flipped.[2] That is the superpotential is $Q_l Q_m M_{ml} + X^2 x$ with M and x gauge singlet fields. The superconformal R symmetry derived by a maximization [21] assigns R charge zero to X and R charge half to the quarks. The superconformal (a, c) anomalies coincide with two copies of SU(2) theories glued together with a superpotential, the E_7 surprise model. This suggests that the USp(4) model sits on the same conformal manifold. In particular as the superconformal R charge of X is zero,[3] giving it a vacuum expectation value takes us on the conformal manifold of that model.[4] Giving such an expectation value Higgses the USp(4) gauge group to $SU(2)^2$. This generates the E_7 surprise model if the mesons are flipped. We thus expect the USp(4) model to have E_7 symmetry somewhere on its conformal manifold. The model has $U(1) \times SU(8)$ symmetry visible in the Lagrangian and if we study the index we see that the symmetry is enhanced to $E_7 \times U(1)$. The index is given by,

$$1 + \mathbf{56} t^{\frac{1}{2}} (pq)^{\frac{1}{2}} + \mathbf{56} t^{\frac{1}{2}} (qp)^{\frac{1}{2}} (q + p) + \left(\mathbf{1463} t + \frac{1}{t^2} - \mathbf{133} - 1 \right) qp + \dots \,. \qquad (2.11)$$

In particular we see explicitly the conserved current of E_7 appearing at order qp in the expansion of the index (see [22]). The conserved currents multiplet contributes at order qp as a fermionic operator, $-\mathbf{133}$. It is worth verifying what are the operators giving such a contribution. Let us list all the operators which contribute at order qp and have vanishing charge under the U(1) symmetry,

$$\bar{\psi}_i Q_j, \qquad \bar{\psi}_{ji}^M Q_l Q_m X, \qquad \bar{\psi}_{ij}^M M_{lm}, \qquad Q_j Q_i M_{lm}, \qquad Q_i Q_l Q_j Q_k X,$$
$$\lambda \lambda, \qquad \bar{\psi}^X X, \qquad x X^2, \qquad \bar{\psi}^x x. \qquad (2.12)$$

Here λ is the gaugino. The fields ψ_i are fermionic partners of Q_i and the fields ψ^L are fermionic partners of fields L with L being one of (M, X, x). The operator $\bar{\psi}_i Q_j$ forms the $\mathbf{63} + \mathbf{1}$ of $SU(8) \times U(1)$ and to construct the adjoint of E_7 we also need the $\mathbf{70}$, fourth completely antisymmetric power of the $\mathbf{8}$, in addition to the $\mathbf{63}$. The operators $Q_i Q_l M_{kl}$ and $\bar{\psi}_{km}^M M_{ij}$ cancel each other in the index computation as a consequence of the chiral ring relation. The operator $Q_i Q_j Q_k Q_n X$ is in the representation $\mathbf{378} + \mathbf{336}$ of SU(8). In particular this lacks the rank 4 antisymmetric representation of SU(8). This arises as to get it one has to contract the Q's antisymmetrically in the flavor index, but, since these are bosonic fields, they must then be contracted antisymmetrically also in the gauge indices. However the fourth totally antisymmetric product of the $\mathbf{4}$ of USp(4) is a singlet, and so the product with X then cannot be made gauge invariant. The operator $\bar{\psi}_{lk}^M Q_i Q_j X$ is in the $\mathbf{28} \times \mathbf{28} = \mathbf{336} + \mathbf{378} + \mathbf{70}$. The operator which gives us the required $-\mathbf{70}$ is then obtained

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[2]Flipping, that is introducing chiral fields $\phi_{\mathfrak{O}}$ and coupling them to a theory as $\phi_{\mathfrak{O}} \mathfrak{O}$ with $\mathfrak{O}$ being an operator to be removed in the IR, is a standard technique in CFT. For recent discussion of some aspects of this procedure see [19, 20].

[3]This will cease to be the case for higher rank, and in particular there will be no conformal manifold.

[4]More specifically, an expectation value for X is forbidden by both the F-term and D-term conditions. To turn it on we must deform the superpotential by a term linear in x, in which case the F-term conditions forces an expectation value. This operator has conformal R charge 2 and so this is a marginal deformation in the SCFT.

from $Q_l Q_m X \bar\psi^M_{ik}$. The relevant operators in the **56** are constructed from M_{ij} which form the $\overline{\mathbf{28}}$ of SU(8) and from $X Q_i Q_j$ which forms the **28**. Both operators have charge $\frac{1}{2}$ under the U(1). We note that the fact that at order qp we see $-\mathbf{133}-1$, assuming that the theory flows to interacting SCFT in the IR, is a proof, following from the superconformal representation theory [22], that the symmetry of the fixed point enhances to at least $\mathrm{U}(1) \times E_7$.

3 Rank changing duality

We consider a deformation of $\mathfrak{T}_0^{(n)}$ by a superpotential term,

$$W_m^n = Q_1 Q_2 X^m. \tag{3.1}$$

The theory then will be denoted by $\mathfrak{T}_m^{(n)}$.[5] We claim that if m is bigger than n $\mathfrak{T}_m^{(n)}$ is dual to $\mathfrak{T}_n^{(m)}$ with the additional superpotential and singlet fields,

$$\Delta W_n^m = \sum_{k=n+1}^{m} x_k \widetilde{X}^k + \sum_{k=1}^{m-n} \sum_{2<i<j<9} q_i q_j M_{ij/k} \widetilde{X}^{k-1}. \tag{3.2}$$

The symmetries of the theories are $\mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and the R-symmetry. When we consider n to be one, the symmetry should enhance to SU(8) though it is not apparent on the USp($2m$) side of the duality. The representations under non-abelian symmetries are the same across the duality. The charges on the USp($2n$) side are detailed in table 3.3.

	USp($2n$)	SU(6)	SU(2)	U(1)	U(1)$_r$
(Q_1, Q_2)	$\mathbf{2n}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}m$	1
$Q_{3,\dots,8}$	$\mathbf{2n}$	$\mathbf{6}$	$\mathbf{1}$	$-\frac{1}{6}(2n-m-2)$	$\frac{1}{3}$
X	$\mathbf{n(2n-1)-1}$	$\mathbf{1}$	$\mathbf{1}$	1	0

$$(3.3)$$

On the USp($2m$) side we obtain the charges in table 3.4.

	USp($2m$)	SU(6)	SU(2)	U(1)	U(1)$_r$
(q_1, q_2)	$\mathbf{2m}$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}n$	1
$q_{3,\dots,8}$	$\mathbf{2m}$	$\mathbf{6}$	$\mathbf{1}$	$-\frac{1}{6}(2m-n-2)$	$\frac{1}{3}$
$\widetilde{X}$	$\mathbf{m(2m-1)-1}$	$\mathbf{1}$	$\mathbf{1}$	1	0
$M_{/y}$	$\mathbf{1}$	$\mathbf{15}$	$\mathbf{1}$	$-\frac{1}{3}(n-2m+3y-1)$	$\frac{4}{3}$
x_y	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-y$	2

$$(3.4)$$

The map of the operators is as follows,

$$X^j \to \widetilde{X}^j, \qquad i,l \neq 1,2 \quad Q_i Q_l X^j \to q_i q_l \widetilde{X}^{m-n+j},$$
$$Q_{1,2} Q_{l>2} X^{j-1} \to q_{1,2} q_{l>2} \widetilde{X}^{j-1},$$
$$j \leq m-n \quad Q_1 Q_2 X^{j-1} \to x_{m-j+1}, \qquad j > m-n \quad Q_1 Q_2 X^{j-1} \to q_1 q_2 \widetilde{X}^{j-1-m+n},$$
$$j \leq n-1 \quad (Q^4)_{ck} X^{j-1} \to M_{ck/n-j}. \tag{3.5}$$

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[5]Note that because of the anomaly condition, the superpotential $Q_3 Q_4 Q_5 Q_6 Q_7 Q_8 X^{2n-2-m}$ has R charge 4 minus the R charge of W_m^n and opposite charges under other symmetries. This implies that at least one of these operators is relevant if $m < 2n-2$ and that the two terms are marginal in IR.

The supersymmetric index of the two sides of the duality agrees as was shown by Rains [9]. The fact that the index agrees guarantees in particular that the anomalies agree and that the protected operators map to each other. Nevertheles, let us detail the anomalies here. We can encode anomalies involving abelian symmetries in the trial c and a anomalies. Defining $R = R' + sq$ with R' and q the R symmetry and the U(1) charge in the tables above, with s a parameter, the conformal anomalies are,

$$
\begin{aligned}
a(s) = {} & \frac{1}{32}\big(s^3\big(-3\big(m^2+4m-2\big)n^2+6(m+2)n^3-(m-1)^2(4m+5)n-4n^4-9\big) \\
& -3\,s^2\big(\big(2m^2+8m-1\big)n+(2-8m)n^2+8n^3-9\big)+12s\big(2mn+n^2+n-2\big)+4n+6\big)
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
c(s) = {} & \frac{1}{32}\big(s^3\big(-3\big(m^2+4m-2\big)n^2+6(m+2)n^3-(m-1)^2(4m+5)n-4n^4-9\big) \\
& -3\,s^2\big(\big(2m^2+8m-1\big)n+(2-8m)n^2+8n^3-9\big)+2s\big(3(4m+1)n+8n^2-11\big)+16n+4\big).
\end{aligned}
$$

A more symmetric way to think about the duality is to define the model $\mathfrak{T}_0^{(n)}$ with the superpotential given by the following.

$$
W_0^{(n)} = \sum_{k=2}^{n} x_k \widetilde{X}^k + \sum_{k=1}^{n} \sum_{2<i<j<9} q_i q_j M_{ij/k} \widetilde{X}^{k-1}.
\tag{3.7}
$$

We denote this theory as $\bar{\mathfrak{I}}_0^{(n)}$ and the model with $Q_1 Q_2 X^m$ as $\bar{\mathfrak{I}}_m^{(n)}$. We then claim that $\bar{\mathfrak{I}}_m^{(n)}$ is dual to $\bar{\mathfrak{I}}_n^{(m)}$.

Let us define the index of a theory $\bar{\mathfrak{I}}_m^{(n)}$ to be $I_n^m(u_1, u_2, \cdots, u_8, t)$ with the condition coming from anomalies $t^{2n-2}\prod_{i=1}^{8} u_i = p^2 q^2$. Here again u_i are the weights of the quarks as defined in the previous section and t is the fugacity for the U(1) under which the antisymmetric chiral field is charged. Turning on the superpotential $Q_1 Q_2 X^m$ we identify $u_1 u_2 t^m = pq$. If we define $u^2 = pq\frac{t^{-n}}{u_2 u_1}$, the duality implies that the index satisfies the following identity,

$$
I_n^m(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, t) = I_m^n\left(u_1 u, u_2 u, \frac{u_3}{u}, \frac{u_4}{u}, \frac{u_5}{u}, \frac{u_6}{u}, \frac{u_7}{u}, \frac{u_8}{u}, t\right).
\tag{3.8}
$$

This can be derived from the map of symmetries between the two dualities.

3.1 Duals of SU(2) SQCD with four flavors

Consider an example of the rank changing duality with $n = 1$ and general m. The theory on one side is always USp(2) = SU(2) SQCD with eight fundamental chiral fields and no superpotential. On the dual side we have a USp($2m$) model with superpotential $W = q_2 q_1 \widetilde{X}$ and other terms involving singlet fields we have discussed. We then have an infinite number of duals for the SU(2) theory with eight fundamental chiral fields. The fact that this model has an infinite number of duals is not surprising [4], but the surprising point is that the duals are rather simple.

Let us work out a simple example. We consider m to be two. At the fixed point of the gauge theory with no superpotential the operator $q_2 q_1 \widetilde{X}$ has R charge 1.15749 and thus is relevant. The operator $\widetilde{X}^2$ violates the unitarity bound and needs to be decoupled by introducing a flip x appearing in the superpotential as $x\widetilde{X}^2$. After decoupling the $\widetilde{X}^2$

operator we get that the R charge of $\widetilde{X}q_1q_2$ is 1.15331. We turn it on and flow to a new fixed point. At that point the operators $(i,j \neq 1,2)$ q_iq_j violate the unitarity bound and need to be decoupled by introducing flippers. After this there are no operators violating unitarity bounds and we get precisely the superpotential we obtain from the duality. Thus USp(4) theory with the superpotenial flipping $\widetilde{X}^2$ and q_iq_j $(i,j \neq 1,2)$, and superpotential term $q_2q_1\widetilde{X}$ flows to USp(2) with no superpotential. For low values of m we can repeat such an analysis though for higher values it becomes rather intricate.

4 The duality orbit

We consider the more general superpotential for a USp($2\mathfrak{a}_9$) theory,

$$W_{\mathfrak{a}_9;\mathfrak{a}_1,\cdots,\mathfrak{a}_8} = \sum_{i \neq j} Q_i Q_j X^{\mathfrak{a}_i + \mathfrak{a}_j} \,. \tag{4.1}$$

This superpotential breaks all the flavor symmetry but preserves the R symmetry.[6] The parameters $\mathfrak{a}_j$ are either all integer or all half integer. The R charges are,

$$r_{Q_i} = 1 - \mathfrak{a}_i r_X \,, \qquad r_X = \frac{4}{2 - 2\mathfrak{a}_9 + \sum_{j=1}^{8} \mathfrak{a}_j} \,. \tag{4.2}$$

Some operators violate the unitarity bounds for general choices of $\mathfrak{a}_i$ and need to be decoupled. We define the parameters u_i and t to be as in previous sections,

$$u_i = (pq)^{\frac{1}{2}} t^{-\mathfrak{a}_i} \,. \tag{4.3}$$

Note also that now $t = (qp)^{\frac{1}{2}r_X}$. The index is given by

$$I_{\mathfrak{a}_9}^{\mathfrak{a}_1 + \mathfrak{a}_2}((pq)^{\frac{1}{2}} t^{-\mathfrak{a}_1}, \cdots, (pq)^{\frac{1}{2}} t^{-\mathfrak{a}_8}, t) \,. \tag{4.4}$$

The duality of the previous section then implies that this index is equal to,

$$I_{\mathfrak{a}_1 + \mathfrak{a}_2}^{\mathfrak{a}_9}\left((pq)^{\frac{1}{2}} t^{-\frac{1}{2}(\mathfrak{a}_1 - \mathfrak{a}_2 + \mathfrak{a}_9)}, (pq)^{\frac{1}{2}} t^{-\frac{1}{2}(\mathfrak{a}_2 - \mathfrak{a}_1 + \mathfrak{a}_9)}, (pq)^{\frac{1}{2}} t^{-\frac{1}{2}(2\mathfrak{a}_3 + \mathfrak{a}_2 + \mathfrak{a}_1 - \mathfrak{a}_9)}, \cdots, t\right) \tag{4.5}$$

We parametrize again the theory by the nine numbers, which here are integers (or half integers), $(\mathfrak{a}_9, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8)$. The duality transforms

$$(\mathfrak{a}_9, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8) \rightarrow \tag{4.6}$$
$$\rightarrow \left(\mathfrak{a}_1 + \mathfrak{a}_2, \frac{1}{2}(\mathfrak{a}_1 - \mathfrak{a}_2 + \mathfrak{a}_9), \frac{1}{2}(\mathfrak{a}_2 - \mathfrak{a}_1 + \mathfrak{a}_9), \frac{1}{2}(2\mathfrak{a}_3 + \mathfrak{a}_1 + \mathfrak{a}_2 - \mathfrak{a}_9), \cdots\right) \,.$$

The transformations with permutations of the last eight numbers generate the Weyl group of SO(16).

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[6]Here we assume that $\mathfrak{a}_9 \neq 1 + \sum_{j=1}^{8} \frac{\mathfrak{a}_j}{2}$. If this is not true then there is no R symmetry, but instead there is an anomaly free U(1) global symmetry. We shall not discuss this case in any detail.

We can also act with the duality generating the E_7 orbit. Denote $\mathfrak{a}_+ = \frac{1}{2}\sum_{i=1}^{4}\mathfrak{a}_i$, $\mathfrak{a}_- = \frac{1}{2}\sum_{i=5}^{8}\mathfrak{a}_i$, $\mathfrak{a} = \frac{1}{2}(\mathfrak{a}_- + \mathfrak{a}_+)$, $\mathfrak{a}' = \frac{1}{2}(\mathfrak{a}_- - \mathfrak{a}_+)$. Then the three dualities generating the E_7 orbit imply the following transformations of this set,

$$(\mathfrak{a}_9, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8) \rightarrow$$
$$\rightarrow (\mathfrak{a}_9, \mathfrak{a}_+ - \mathfrak{a}_1, \mathfrak{a}_+ - \mathfrak{a}_2, \mathfrak{a}_+ - \mathfrak{a}_3, \mathfrak{a}_+ - \mathfrak{a}_4, \mathfrak{a}_- - \mathfrak{a}_5, \mathfrak{a}_- - \mathfrak{a}_6, \mathfrak{a}_- - \mathfrak{a}_7, \mathfrak{a}_- - \mathfrak{a}_8) \rightarrow$$
$$\rightarrow (\mathfrak{a}_9, \mathfrak{a} - \mathfrak{a}_1, \mathfrak{a} - \mathfrak{a}_2, \mathfrak{a} - \mathfrak{a}_3, \mathfrak{a} - \mathfrak{a}_4, \mathfrak{a} - \mathfrak{a}_5, \mathfrak{a} - \mathfrak{a}_6, \mathfrak{a} - \mathfrak{a}_7, \mathfrak{a} - \mathfrak{a}_8) \rightarrow$$
$$\rightarrow (\mathfrak{a}_9, \mathfrak{a}' + \mathfrak{a}_1, \mathfrak{a}' + \mathfrak{a}_2, \mathfrak{a}' + \mathfrak{a}_3, \mathfrak{a}' + \mathfrak{a}_4, -\mathfrak{a}' + \mathfrak{a}_5, -\mathfrak{a}' + \mathfrak{a}_6, -\mathfrak{a}' + \mathfrak{a}_7, -\mathfrak{a}' + \mathfrak{a}_8). \quad (4.7)$$

These transformations and permutations of last eight elements generate the action of the Weyl group of E_7 as was observed previously. The SO(16) Weyl group with the E_7 transformations gives the Weyl group of E_8. Note that the combination $2 - 2\mathfrak{a}_9 + \sum_{i=1}^{8}\mathfrak{a}_i$ is invariant under all the transformations.

The transformations can generate a full orbit of the Weyl group of E_8 for general values of the parameters. For special values they might not act faithfully and generate smaller orbits. For example take all $\mathfrak{a}_i$ for $i = 1\ldots 8$ to be $\frac{1}{2}\mathfrak{a}_9$. Then the SO(16) transformations become self dualities.

There is an issue of whether or not the theories defined via the superpotential (4.1) are indeed all unique and define a non-trivial SCFT. The basic problem is that the superpotentials seem irrelevant in the IR so one may fear that the RG flow they generate is trivial. A related issue is that for generic values of $\mathfrak{a}_i$ there will be operators with R charge below the unitarity bound. To deal with the latter problem we can add, to all sides of the duality, flipping fields that remove these operators. The exact number of operators required may depend on the value of $2 - 2\mathfrak{a}_9 + \sum_{i=1}^{8}\mathfrak{a}_i$, but one can still perform this action so as to result in an orbit where all operators are above the unitarity bound. Here it is important that the R symmetry is fixed so this does not lead to any flow that may cause more operators to go below the unitarity bound. Therefore this statement can be phrased as an identity between a collection of theories where all operators are above the unitarity bound, and so there is no contradiction with them flowing to an SCFT. Of course we cannot rule out the possibility that in some cases the flow may be trivial and the superpotential are truly irrelevant rather then dangerously irrelevant.

It will be interesting to understand whether the fact that there are theories residing on an orbit of the Weyl group of E_8 implies that there is a model with E_8 symmetry, and in which particular way this fact is related to compactifications of six dimensional $(1,0)$ models.

Acknowledgments

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