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# $E$-BOCHNER CURVATURE TENSOR ON $(\kappa,\ \mu)$-CONTACT METRIC MANIFOLDS

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*Dedicated to memory of Proffessor Franki Dillen*

Abstract. We study $E$-Bochner curvature tensor $\mathrm{B}^e$ satisfying $\mathrm{R}\cdot\mathrm{B}^e = 0$, $\mathrm{B}^e\cdot\mathrm{R} = 0$, $\mathrm{B}^e\cdot\mathrm{B}^e = 0$ and $\mathrm{B}^e\cdot\mathrm{S} = 0$ in n-dimensional $(\kappa,\ \mu)$-contact metric manifolds.

## 1. Introduction

In [3], Blair, Koufogiorgos and Papantoniou introduced $(\kappa,\ \mu)$-contact metric manifolds. A class of contact metric manifold $M$ with contact metric structure $(\varphi,\ \xi,\ \eta,\ g)$ in which the curvature tensor $R$ satisfies the condition

$$R(X,Y)\xi = (KI + \mu h)(\eta(Y)X - \eta(X)Y), \tag{1.1}$$

for all $X, Y \in TM$ is called $(\kappa,\ \mu)$-manifolds. On the other hand, Bochner [5] introduced a Kahler analogue of Weyl conformal curvature tensor by purely formal consideration which is known as Bochner Curvature Tensor. A geometric meaning of Bochner Curvature Tensor was given by Blair [2]. By using Boothby-Wang's fibration [7], Matsumoto and Chuman [15] constructed C-Bochner curvature tensor. In [9], Endo defined E-Bochner curvature tensor as an extended C-Bochner curvature tensor. E-Bochner curvature tensor is defined as
$\mathrm{B}^e(X,Y)Z = \mathrm{B}(X,Y)Z - \eta(X)\mathrm{B}(\xi,Y)Z - \eta(Y)\mathrm{B}(X,\xi)Z - \eta(Z)\mathrm{B}(X,Y)\xi$, for all

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$X, Y, Z$ belongs to TM, B is the C-Bochner curvature tensor defined by

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z + \frac{1}{n+3}\big[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &\quad -g(Y,Z)QX + S(\varphi X,Z)\varphi Y - S(\varphi Y,Z)\varphi X + g(\varphi X,Z) \\ &\quad Q\varphi Y - g(\varphi Y,Z)Q\varphi X + 2S(\varphi X,Y)\varphi Z + 2g(\varphi X,Y)Q\varphi Z \\ &\quad -S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y) \\ &\quad \eta(Z)QX\big] - \frac{p+n-1}{n+3}\big[g(\varphi X,Y)\varphi Y - g(\varphi Y,Z)\varphi X + 2 \\ &\quad g(\varphi X,Y)\varphi Z\big] - \frac{p-4}{n+3}\big[g(X,Z)Y - g(Y,Z)X\big] + \frac{p}{n+3} \\ &\quad \big[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\big], \end{aligned} \tag{1.2}$$

where $S$ is Ricci tensor of type of type $(0,2)$, $Q$ is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and $p = \frac{n+r-1}{n+1}$, $r$ is the scalar curvature of the manifold. E-Bochner curvature tensor is denoted by $B^e$.

A Riemannian manifold $(M^{2m+1}, g)$ is said to be semisymmetric if its curvature tensor $R$ satisfies the condition $R(X,Y) \cdot R = 0$, for all $X, Y$ belongs to TM where $R(X,Y)$ acts on $R$ as a derivation ([14], [18]).

In [20], Yildiz and De studied $h$-projectively semisymmetric on $(\kappa, \mu)$-contact metric manifolds. Besides this, in [13] Kim, Tripathi and Choi proved that a $(\kappa, \mu)$-contact metric manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold. Motivated by the above studies, we characterize a $(\kappa, \mu)$-contact metric manifold satisfying certain curvature conditions on E-Bochner curvature tensor. The present paper is organized as follows:

After preliminaries in section 3 and 4, we characterize $(\kappa, \mu)$-contact metric manifolds satisfying $\mathrm{R} \cdot \mathrm{B}^e = 0$ and $\mathrm{B}^e \cdot \mathrm{R} = 0$ respectively. Besides these we prove that a $(\kappa, \mu)$-contact metric manifold is Sasakian if and only if it satisfies $\mathrm{B}^e \cdot \mathrm{B}^e = 0$. Finally, we prove that a $(\kappa, \mu)$-contact metric manifold satisfying $\mathrm{B}^e \cdot \mathrm{S} = 0$ is an $\eta$-Einstein manifold. Also we obtain some important corollaries.

## 2. Preliminaries

An $n(= 2m+1)$ dimensional differentiable manifold $M$ is called an almost contact manifold if there is an almost contact structure $(\varphi, \xi, \eta)$ consisting of a $(1,1)$ tensor field $\varphi$, a vector field $\xi$, a 1-form $\eta$ satisfying

$$\varphi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta \circ \varphi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\mathrm{M}^n \times \mathbb{R}$ defined by
$\mathrm{J}(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable where X is tangent to $M$, t is the coordinate of R and f is a smooth function on $\mathrm{M}^n \times \mathbb{R}$.

The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$ where $[\varphi, \varphi]$ is the Nijenhuis tensor of $\varphi$.

Let g be a compatible Reimannian metric with $(\varphi,\xi,\eta)$, that is,

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{2.2}$$

or equivalently,

$$g(X,\xi)=\eta(X), g(\varphi X,Y)=-g(X,\varphi Y), \tag{2.3}$$

for all $X, Y$ belongs to TM.

An almost contact metric structure becomes a contact metric structure if

$$g(X,\varphi Y)=d\eta(X,Y), \tag{2.4}$$

for all $X, Y$ belongs to TM. Given a contact metric manifold $M^n(\varphi,\xi,\eta,g)$ we define a $(1,1)$ tensor field $h$ by $h=\frac{1}{2}\mathrm{L}_\xi\varphi$ where L denotes the Lie differentiation. Then $h$ is symmetric and satisfies

$$h\xi=0,\ h\varphi+\varphi h=0, \tag{2.5}$$

$$\nabla\xi=-\varphi-\varphi h,\ trace(h)=trace(\varphi h)=0, \tag{2.6}$$

where $\nabla$ is the Levi-Civita connection.
A contact metric manifold is said to be an $\eta$-Einstein if

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{2.7}$$

where $a, b$ are smooth functions and $S$ is the Ricci tensor.
A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\varphi)Y=g(X,Y)\xi-\eta(Y)X. \tag{2.8}$$

On a Sasakian manifold the following relation holds

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y, \tag{2.9}$$

for all $X, Y$ belongs to TM.Blair, Koufogiorgos and Papantoniou [3] considered the ($\kappa$, $\mu$)-nullity condition and gave several reasons for studying it. The ($\kappa$, $\mu$)-nullity distribution N($\kappa$, $\mu$) ([3], [17]) of a contact metric manifold $M$ is defined by

$$N(\kappa,\mu):p\mapsto N_p(\kappa,\mu)=[U\in T_pM\mid R(X,Y)U=(\kappa I+\mu h)(g(Y,U)X-g(X,U)Y)]$$

for all $X, Y$ belongs to TM, where $(\kappa,\mu)\in\mathbb{R}^2$.
A contact metric manifold $M^n$ with $\xi\in N(\kappa,\mu)$ is called a ($\kappa$, $\mu$)- contact metric manifold. Then we have

$$\begin{aligned} R(X,Y)\xi &= \kappa[\eta(Y)X-\eta(X)Y]+ \\ &\quad \mu[\eta(Y)hX-\eta(X)hY], \end{aligned} \tag{2.10}$$

for all $X, Y$ belongs to TM.For ($\kappa$, $\mu$)-metric manifolds, it follows that $h^2=(\kappa-1)\varphi^2$. This class contains Sasakian manifolds for $\kappa=1$ and $h=0$. In fact, for a ($\kappa$, $\mu$)-metric manifold, the condition of being Sasakian manifold, $\kappa$-contact manifold, $\kappa=1$ and $h=0$ are equivalent. If $\mu=0$, then the ($\kappa$, $\mu$)-nullity distribution N($\kappa$, $\mu$) is reduced to $\kappa$-nullity distribution N($\kappa$) [19]. If $\xi\in N(\kappa)$ , then we call contact metric manifold M an N($\kappa$)- contact metric manifold.
($\kappa$, $\mu$)-contact metric manifolds have been studied by several authors ([16], [1], [8], [10], [11], [12]) and many others.

In a ($\kappa$, $\mu$)-contact metric manifold the following relations hold [3]:

$$h^2=(\kappa-1)\varphi^2, \tag{2.11}$$

$$(\nabla_X\varphi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{2.12}$$

$$R(\xi, X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.13}$$

$$S(X,\xi) = (n-1)\kappa\eta(X), \tag{2.14}$$

$$\begin{aligned} S(X,Y) &= [(n-3) - \frac{n-1}{2}\mu]g(X,Y) + \\ &\quad [(n-3)+\mu]g(hX,Y) + \big[(3-n) + \frac{n-1}{2} \\ &\quad (2\kappa+\mu)\big]\eta(X)\eta(Y), \end{aligned} \tag{2.15}$$

$$r = (n-1)\big(n-3+\kappa - \frac{n-1}{2}\mu\big), \tag{2.16}$$

$$\begin{aligned} S(X,hY) &= \big[(n-3) - \frac{(n-1)}{2}\mu\big] \\ &\quad g(X,hY) - (\kappa-1)[(n-3)+\mu]g(X,Y) \\ &\quad +(\kappa-1)[(n-3)+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.17}$$

$$Q\varphi - \varphi Q = 2[(n-3)+\mu]h\varphi,$$

where $Q$ is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. Let us recall the following result:

**Lemma 2.1.** [4] *A contact metric manifold $M^{2m+1}$ satisfying $R(X,Y)\xi = 0$ is locally isometric to the Riemannian product $E^{m+1} \times S^m(4)$ for $m > 1$.*

Besides these, it can be easily verified that in a ($\kappa$, $\mu$)-contact metric manifold $M^n$, $n \geq 5$, the E-Bochner curvature tensor satisfies the following conditions

$$B^e(X,Y)\xi = \frac{4(\kappa-1)}{n+3}(\eta(X)Y - \eta(Y)X) + \mu(\eta(X)hY - \eta(Y)hX), \tag{2.18}$$

$$B^e(\xi,X)Z = \eta(Z)\frac{4(\kappa-1)}{n+3}(X - \eta(X)\xi) + \eta(Z)\mu hX, \tag{2.19}$$

$$B^e(X,\xi)Z = \eta(Z)\frac{4(\kappa-1)}{n+3}(\eta(X)\xi - X) - \eta(Z)\mu hX, \tag{2.20}$$

$$B^e(X,\xi)\xi = \frac{4(\kappa-1)}{n+3}[\eta(X)\xi - X] - \mu hX, \tag{2.21}$$

$$B^e(\xi,X)\xi = \frac{4(\kappa-1)}{n+3}[X - \eta(X)\xi] + \mu hX, \tag{2.22}$$

$$B^e(\xi,\xi)\xi = 0, \tag{2.23}$$

$$\begin{aligned} \sum_{i=1}^{n} \widetilde{B^e}(e_i,Y,Z,e_i) &= \frac{6(n-3)+\mu)}{n+3}g(hY,Z) \\ &\quad -\frac{4(\kappa-1)}{(n+3)}g(Y,Z) - \\ &\quad [\frac{(n-1)\kappa + (2-n)p + r}{n+3} \\ &\quad -\frac{4(\kappa-1)}{n+3}]\eta(Y)\eta(Z), \end{aligned} \tag{2.24}$$

where $p = \frac{n+r-1}{n+1}$, r is the scalar curvature of $M$, $\{e_1, e_2, ..., e_m, e_{m+1} = \varphi e_1, ..e_{2m} = \varphi e_m, e_{2m+1} = \xi\}$ is a $\varphi$ basis of $M$ and $\widetilde{B^e}(X,Y,Z,W) = g((B^e(X,Y)Z,W)$.

## 3. E-Bochner Semisymmetric ($\kappa$, $\mu$)-Contact Metric Manifolds

**Definition** 3.1: An n-dimensional ($\kappa$, $\mu$)-contact metric manifold is said to be E-Bochner semisymmetric if it satisfies the following equation

$$R(X,Y) \cdot B^e = 0, \tag{3.1}$$

for all $X, Y$ belongs to TM and $B^e$ is E-Bochner curvature tensor.

Let us consider that M be an $n(= 2m+1)$ dimensional ($\kappa$, $\mu$)-contact metric manifold and M is E-Bochner semisymmetric. From (3.1) we have $(R(X,Y) \cdot B^e)(U,V)W = 0$ , which implies that

$$\begin{aligned} &R(X,Y)B^e(U,V)W - B^e(R(X,Y)U,V)W - B^e(U,R(X,Y)V)W - \\ &B^e(U,V)R(X,Y)W = 0. \end{aligned} \tag{3.2}$$

Putting $Y = \xi$ in (3.2) and using (2.13), we obtain

$$\begin{aligned} &\kappa[-g(X,B^e(U,V)W)\xi + \eta(B^e(U,V)W)X + g(X,U)B^e(\xi,V)W \\ &-\eta(U)B^e(X,V)W + g(X,V)B^e(U,\xi)W - \eta(V)B^e(U,X)W \\ &+g(X,W)B^e(U,V)\xi - \eta(W)B^e(U,V)X] + \\ &\mu[-g(hX,B^e(U,V)W)\xi + \eta(B^e(U,V)W)hX + g(hX,U)B^e(\xi,V)W \\ &-\eta(U)B^e(hX,V)W + g(hX,V)B^e(\xi,V)W - \eta(V)B^e(hX,V)W + \\ &g(hX,W)B^e(U,V)\xi - \eta(W)B^e(U,V)hX] = 0. \end{aligned} \tag{3.3}$$

Again putting $W = \xi$ in (3.3) and using (2.18), (2.20), (2.21) and (2.22) we have

$$\begin{aligned} &\frac{4\kappa(\kappa-1)}{n+3}[g(X,U)V - g(X,V)U] + \frac{4(\kappa-1)\mu}{n+3}[g(hX,U) - g(hX,V)U] \\ &+\mu\kappa[\eta(V)g(X,hU)\xi - \eta(U)g(X,hV)\xi + g(X,U)hV - g(X,V)hU] \\ &+\mu^2[\eta(V)g(hX,hU)\xi - \eta(U)g(hX,hV)\xi + g(hX,U)hV - \\ &g(hX,V)hU] - \kappa B^e(U,V)X - \mu B^e(U,V)hX = 0. \end{aligned} \tag{3.4}$$

Taking inner product of (3.4) with Z we obtain

$$\begin{aligned} &\frac{4\kappa(\kappa-1)}{n+3}[g(X,U)g(V,Z) - g(X,V)g(U,Z)] + \frac{4(\kappa-1)\mu}{n+3}[g(hX,U)g(V,Z) \\ &-g(hX,V)g(U,Z)] + \mu\kappa[\eta(V)\eta(Z)g(X,hU) - \eta(U)\eta(Z)g(X,hV) \\ &+g(X,U)g(hV,Z) - g(X,V)g(hU,Z)] + \mu^2[\eta(V)g(hX,hU)\eta(Z) - \eta(U)\eta(Z) \\ &g(hX,hV) + g(hX,U)g(hV,Z) + g(hX,U)g(hV,Z)] - \kappa\widetilde{B^e}(U,V,X,Z) - \\ &\mu\widetilde{B^e}(U,V,hX,Z) = 0. \end{aligned}$$

Let $\{e_i\}_{i=1}^n$, be an orthonormal basis of the tangent space. Putting $U = Z = e_i$ and summing up over 1 to $n$, we have

$$g(hX,V) = a_1 g(X,V) + b_1\eta(X)\eta(V), \tag{3.5}$$

where

$$a_1 = \Big[\frac{-4\kappa(\kappa-1)(2-n) - 6\mu(n-3+\mu)(\kappa-1) + \mu^2(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big],$$

and

$$b_1 = \Big[\frac{\kappa^2(n-1) + \kappa p(2-n) + r\kappa + 4\kappa(\kappa-1) + 6\mu(n-3+\mu)(\kappa-1) + \mu^2(\kappa-1)(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big],$$

where $p = \frac{n+r-1}{n+1}$, r being the scalar curvature of $M$. From (2.15) and (3.5) we obtain

$$S(X,V) = ag(X,V) + b\eta(X)\eta(V), \tag{3.6}$$

where

$$\begin{aligned} a &= \big[(n-3) - \frac{n-1}{2}\mu\big] + [(n-3)+\mu] \\ &\quad \Big[\frac{-4\kappa(\kappa-1)(2-n) - 6\mu(n-3+\mu)(\kappa-1) + \mu^2(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big], \end{aligned}$$

and

$$\begin{aligned} b &= \big[(3-n) + \frac{n-1}{2}(2\kappa+\mu)\big] + [(n-3)+\mu] \\ &\quad \Big[\frac{\kappa^2(n-1) + \kappa p(2-n) + r\kappa + 4\kappa(\kappa-1) + 6\mu(n-3+\mu)(\kappa-1) + \mu^2(\kappa-1)(n+3)}{4\mu(\kappa-1)(1-n) - (n-1)\mu^2 - 6\kappa(n-3+\mu) + \mu\kappa(n+3) + 4\mu(\kappa-1)}\Big]. \end{aligned}$$

Thus, from (3.6) we can state the following:

**Theorem 3.1.** *Let $M$ be an n-dimensional $(n \geq 5)$ E-Bochner semisymmetric $(\kappa, \mu)$-contact metric manifold. Then the manifold is an $\eta$-Einstein manifold.*

Putting the value $\kappa = 1$ and $h = 0$ in the equation (3.5) we have the following:

**Corollary 3.1.** *An E-Bochner semisymmetric Sasakian manifold $M^n(n \geq 5)$ is E-Bochner flat.*

In general, in a $(\kappa, \mu)$-contact metric manifold the Ricci operator Q does not commute with $\varphi$. However, Yildiz and De [20] proved the following:

**Lemma 3.1.** *In a non-Sasakian $(\kappa, \mu)$-contact metric manifold the following conditions are equivalent:*
*(a) $\eta$-Einstein manifold,*
*(b) $Q\varphi = \varphi Q$.*

From Lemma 3.1 we can state that

**Corollary 3.2.** *Let $M^n$ be an n-dimensional $(n \geq 5)$ E-Bochner semisymmetric non-Sasakian $(\kappa, \mu)$-contact metric manifold. Then the Ricci operator $Q$ commutes with $\varphi$.*

## 4. $(\kappa, \mu)$-Contact Metric Manifolds Satisfying $B^e(\xi, U) \cdot R = 0$

In this section,we consider an n-dimensional $(\kappa, \mu)$-contact metric manifold satisfying $\big(B^e(\xi,U)\cdot R\big)(X,Y)Z = 0$. Therefore, we have

$$\begin{aligned} &B^e(\xi,U)R(X,Y)Z - R(B^e(\xi,U)X,Y)Z - R(X,B^e(\xi,U)Y)Z - \\ &R(X,Y)B^e(\xi,U)Z = 0. \end{aligned} \tag{4.1}$$

Using (2.19) in (4.1), we get

$$\begin{aligned}\frac{4(\kappa-1)}{n+3}\quad\Big[\ & -\eta(U)\eta(R(X,Y)Z)\xi+\eta(R(X,Y)Z)U+\\ & \eta(X)\eta(U)R(\xi,Y)Z-\eta(X)R(U,Y)Z+\eta(Y)\eta(U)R(X,\xi)Z\\ & -\eta(Y)R(X,U)Z+\eta(Z)\eta(U)R(X,Y)\xi-\eta(Z)R(X,Y)U]\\ & +\mu[\eta(R(X,Y)Z)hU-\eta(X)R(hU,Y)Z-\\ & \eta(Y)R(X,hU)Z-\eta(Z)R(X,Y)hU\Big]=0,\end{aligned}\tag{4.2}$$

Taking inner product with $\xi$ of (4.2) and using $h\xi=0$, $g(R(X,Y)\xi,\xi)=0$ we get

$$\begin{aligned}\frac{4(\kappa-1)}{n+3}\quad\Big[\ & \eta(X)\eta(U)g(R(\xi,Y)Z,\xi)-\eta(X)g((U,Y)Z,\xi)+\\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi)-\eta(Y)g(R(X,U)Z,\xi)-\\ & \eta(Z)g(R(X,Y)U,\xi]+\\ & \mu[-\eta(X)g(R(hU,Y)Z,\xi)-\eta(Y)g(R(X,hU)Z,\xi)-\\ & \eta(Z)g(R(X,Y)hU,\xi)\Big]=0,\end{aligned}\tag{4.3}$$

Let us consider the following cases:
**CASE 1.** $\kappa=0=\mu$.
**CASE 2.** $\kappa=0$, $\mu\neq 0$.
**CASE 3.** $\kappa\neq 0$, $\mu=0$.
**CASE 4.** $\kappa\neq 0$, $\mu\neq 0$
For Case 1, we observe that $R(X,Y)\xi=0$, for all X,Y. Hence, by Lemma 2.1, $M$ is locally the Remannian product $E^{m+1}\times S^m(4)$.
For Case 2, from (4.3) we get

$$\begin{aligned}\frac{4(-1)}{n+3}\quad\Big[\ & \eta(X)\eta(U)g(R(\xi,Y)Z,\xi)-\eta(X)g((U,Y)Z,\xi)+\\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi)-\eta(Y)g(R(X,U)Z,\xi)-\\ & \eta(Z)g(R(X,Y)U,\xi]+\\ & \mu[-\eta(X)g(R(hU,Y)Z,\xi)-\eta(Y)g(R(X,hU)Z,\xi)\\ & -\eta(Z)g(R(X,Y)hU,\xi)\Big]=0.\end{aligned}\tag{4.4}$$

Let $\{e_i\}_{i=1}^n$, be an orthonormal basis of the tangent space. Putting $Y=Z=e_i$ in (4.3) and summing over 1 to $n$, we get

$$g(X,hU)=a_1 g(X,U)+b_1\eta(X)\eta(U),\tag{4.5}$$

where

$$a_1=[\frac{\mu(n+3)}{4}],$$

and

$$b_1=-[\frac{\mu(n+3)}{4}].$$

From (2.15) and (4.5) we get

$$S(X,U)=ag(X,U)+b\eta(X)\eta(U),\tag{4.6}$$

where

$$a=\big[(n-3)-\frac{n-1}{2}\mu\big]+[(n-3)+\mu][\frac{\mu(n+3)}{4}],$$

$$b = \big[(3-n) + \frac{n-1}{2}\mu\big] - [(n-3)+\mu]\big[\frac{\mu(n+3)}{4}\big].$$

For Case 2, $M$ becomes an $\eta$-Einstein manifold.
For Case 3, from (4.3) we get

$$\begin{aligned}\frac{4(\kappa-1)}{n+3}\quad [\quad & \eta(X)\eta(U)g(R(\xi,Y)Z,\xi) - \eta(X)g((U,Y)Z,\xi) + \\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi) - \eta(Y)g(R(X,U)Z,\xi) - \\ & \eta(Z)g(R(X,Y)U,\xi] = 0. \end{aligned} \tag{4.7}$$

Therefore, either $\kappa = 1$ or

$$\begin{aligned} & [\eta(X)\eta(U)g(R(\xi,Y)Z,\xi) - \eta(X)g((U,Y)Z,\xi) + \\ & \eta(Y)\eta(U)g(R(X,\xi)Z,\xi) - \eta(Y)g(R(X,U)Z,\xi) - \\ & \eta(Z)g(R(X,Y)U,\xi] = 0. \end{aligned} \tag{4.8}$$

Let $\{e_i\}_{i=1}^n$, be an orthonormal basis of the tangent space. Putting $Y = Z = e_i$ in (4.8) and summing over 1 to $n$, we get

$$g(X,U) = \eta(X)\eta(U), \tag{4.9}$$

which is not possible.
Thus, for Case 3, $M$ is a Sasakian manifold.
For Case 4, putting $Y = Z = e_i$ in (4.3) we get

$$g(X,hU) = a_1 g(X,U) + b_1\eta(X)\eta(U), \tag{4.10}$$

where

$$a_1 = \big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big],$$

$$b_{1=} = -\big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big].$$

Form (2.15) and (4.10) we get

$$S(X,U) = ag(X,U) + b\eta(X)\eta(U), \tag{4.11}$$

where

$$a = \big[(n-3) - \frac{n-1}{2}\mu\big] + [(n-3)+\mu]\big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big],$$

and

$$b = \big[(3-n) + \frac{n-1}{2}(2\kappa+\mu)\big] - [(n-3)+\mu]\big[\frac{\mu^2(\kappa-1)(n+3) - 4\kappa(\kappa-1)}{4\mu(\kappa-1)+\kappa\mu(n+3)}\big].$$

Summing up we can state the following:

**Theorem 4.1.** *Let $M^n$ be an $(n = 2m+1)$-dimensional $(\kappa,\ \mu)$-contact metric manifold satisfying $B^e(\xi,U)\cdot R = 0$. Then we have one of the following:*
*(a) $M^n$ is locally the Remannian product $E^{m+1}\times S^m(4)$.*
*(b) $M^n$ is an $\eta$-Einstein manifold.*
*(c) $M^n$ is a Sasakian manifold.*

## 5. ($\kappa$, $\mu$)-Contact Metric Manifolds Satisfying $B^e(\xi, U) \cdot B^e = 0$

Let $M^n$ be an n-dimensional ($\kappa$, $\mu$)-contact metric manifold satisfying $B^e(\xi, U) \cdot B^e = 0$
Then we have

$$B^e(\xi,U)B^e(X,Y)Z - B^e(B^e(\xi,U)X,Y)Z - B^e(X,B^e(\xi,U)Y)Z - B^e(X,Y)B^e(\xi,U)Z = 0, \tag{5.1}$$

Using (2.19), we get

$$\frac{4(\kappa-1)}{n+3}[\eta(B^e(X,Y)Z)U - \eta(B^e(X,Y)Z)\eta(U)\xi - \eta(X)B^e(U,Y)Z + \eta(U)\eta(X)B^e(\xi,Y)Z - \eta(Y)B^e(X,U)Z + \eta(Y)\eta(U)B^e(X,\xi)Z - \eta(Z)B^e(X,Y)U + \eta(Z)\eta(U)B^e(X,Y)\xi] + \mu[\eta(B^e(X,Y)Z)hU - \eta(X)B^e(hU,Y)Z - \eta(Y)B^e(X,hU)Z - \eta(Z)B^e(X,Y)hU] = 0. \tag{5.2}$$

Putting $X = Z = \xi$ in (5.2) and using (2.22) and (2.23), we obtain

$$\frac{4(\kappa-1)}{n+3}\big[2(\eta(U)\frac{4(\kappa-1)}{n+3}(Y - \eta(Y)\xi) + \mu\eta(U)hY)\big] = 0. \tag{5.3}$$

From(5.3) and (2.19) it follows that either $\kappa = 1$, or $B^e(\xi, Y)U = 0$.
For $\kappa = 1$, the manifold is Sasakian.
If $B^e(\xi, Y)U = 0$, then $B^e(\xi, Y) \cdot B^e = 0$ and if the manifold is Sasakian, then from (2.19) we obtain $B^e(\xi, Y)U = 0$ and hence $B^e(\xi, Y) \cdot B^e = 0$

**Lemma 5.1.** [13] *A ($\kappa$, $\mu$)-contact metric manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold.*

From the above discussion and Lemma 5.1 we conclude that

**Theorem 5.1.** *A ($\kappa$, $\mu$)-contact metric manifold $M^n(n \geq 5)$ satisfies $B^e(\xi, U) \cdot B^e = 0$ if and only if the manifold is Sasakian.*

## 6. ($\kappa$, $\mu$)-Contact Metric Manifolds Satisfyng $B^e(\xi, X) \cdot S = 0$

Let M be an n-dimensional ($\kappa$, $\mu$)-contact metric manifold satisfying $B^e(\xi, X) \cdot S = 0$.
Therefore, $\big(B^e(\xi, X) \cdot S\big)(U, V) = 0$ implies

$$S(B^e(\xi,X)U,V) + S(U,B^e(\xi,X)V) = 0. \tag{6.1}$$

Using (2.19), we get

$$\frac{4(\kappa-1)}{n+3}[\eta(U)S(X,V) + \eta(V)S(U,X)] + \mu[\eta(U)S(hX,V) + \eta(V)S(U,hX)] = 0. \tag{6.2}$$

Putting $V = \xi$ in (6.2), we get

$$\frac{4(\kappa-1)}{n+3}[2\kappa(n-1)\eta(U)\eta(X) + S(U,X)] + \mu S(U,hX) = 0. \tag{6.3}$$

Using (2.15) and (2.17) in (6.3) we have

$$S(U,X) = ag(U,X) + b\eta(U)\eta(X), \tag{6.4}$$

where

$$a = \big[(n-3) - \frac{(n-1)}{2}\mu\big] + [(n-3)+\mu]$$
$$\frac{[-4(\kappa-1)((n-3) - \frac{(n-1)}{2}\mu) + \mu((\kappa-1)(n^2-9) - \frac{(n+3)(n-1)}{2}\mu)]}{[4(\kappa-1)((n-3)+\mu) + \mu((n^2-9) - \frac{n+3)(n-1)}{2}\mu)]},$$

and

$$b = \big[(3-n) + \frac{(n-1)}{2}(2\kappa+\mu)\big] - [(n-3)+\mu]$$
$$\frac{[4(\kappa-1)(3\kappa(n-1) + (3-n) + \frac{(n-1)}{2}\mu) + \mu((\kappa-1)(n^2-9) + \mu(\kappa-1)(n+3))]}{[4(\kappa-1)((n-3)+\mu) + \mu((n^2-9) - \frac{n+3)(n-1)}{2}\mu)]}.$$

From (6.4) we conclude that

**Theorem 6.1.** *Let $M^n(n \geq 5)$ be a $(\kappa,\ \mu)$-contact metric manifold satisfying $B^e(\xi, X)\cdot S = 0$. Then the manifold is an $\eta$-Einstein Manifold.*

From Lemma 3.1 we can state that

**Corollary 6.1.** *Let$M^n$ be an n-dimensional $(n \geq 5)$ non-Sasakian $(\kappa,\ \mu)$-contact metric manifold satisfying $B^e(\xi, X)\cdot S = 0$. Then the Ricci operator $Q$ commutes with $\varphi$.*

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