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# $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M theory

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Abstract: We show that generalised geometry gives a unified description of bosonic eleven-dimensional supergravity restricted to a $d$-dimensional manifold for all $d \leq 7$. The theory is based on an extended tangent space which admits a natural $E_{d(d)} \times \mathbb{R}^+$ action. The bosonic degrees of freedom are unified as a "generalised metric", as are the diffeomorphism and gauge symmetries, while the local $O(d)$ symmetry is promoted to $H_d$, the maximally compact subgroup of $E_{d(d)}$. We introduce the analogue of the Levi-Civita connection and the Ricci tensor and show that the bosonic action and equations of motion are simply given by the generalised Ricci scalar and the vanishing of the generalised Ricci tensor respectively. The formalism also gives a unified description of the bosonic NSNS and RR sectors of type II supergravity in $d-1$ dimensions. Locally the formulation also describes M-theory variants of double field theory and we derive the corresponding section condition in general dimension. We comment on the relation to other approaches to M theory with $E_{d(d)}$ symmetry, as well as the connections to flux compactifications and the embedding tensor formalism.

Keywords: Flux compactifications, Differential and Algebraic Geometry, Supergravity Models, String Duality

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## Contents

## 1 Introduction

The idea that eleven-dimensional supergravity, or for that matter M theory, might have a more unified description incorporating a larger symmetry group is a long-standing one. Following the original observation that the dimensionally reduced supergravity has a hidden $E_{d(d)}$ global symmetry [1–3], formulations using exceptional groups, as well as their infinite-dimensional extensions, have appeared in various guises [4–18].

In this paper we show that generalised geometry [19, 20] gives a unified geometrical description of bosonic eleven-dimensional supergravity restricted to a $d$-dimensional manifold for $d \leq 7$. One starts with an extended tangent space [21, 22] which admits a natural $E_{d(d)} \times \mathbb{R}^+$ action. The bosonic degrees of freedom are unified as a "generalised metric" $G$, while the diffeomorphism and gauge symmetries are encoded as a "generalised Lie derivative." The local $O(d)$ symmetry is promoted to $H_d$, the maximally compact subgroup of $E_{d(d)}$. Remarkably, the dynamics are simply the generalised geometrical analogue of Einstein gravity. The bosonic action is given by

$$S_{\mathrm{B}} = \int \mathrm{vol}_G \, R, \tag{1.1}$$

where $\mathrm{vol}_G$ is the volume form associated to the generalised metric and $R$ is the analogue of the Ricci scalar. The corresponding equations of motion are simply

$$R_{MN} = 0, \tag{1.2}$$

where $R_{MN}$ is the analogue of the Ricci tensor. This work extends the corresponding description of type II theories in terms of $O(10,10) \times \mathbb{R}^+$ generalised geometry given in [23].

The formalism also describes type II theories restricted to $d-1$ dimensions, geometrising not only the NSNS sector but also the RR fields. Even though here we focus our attention on the bosonic sector, we will find that, in fact, the supersymmetry variations of the fermions are already encoded by the geometry. In a forthcoming paper [24] we extend the construction to include the fermion fields to leading order, thus completing the reformulation of restricted eleven-dimensional supergravity.

That eleven-dimensional supergravity could be reformulated with a manifest local $H_7 = \mathrm{SU}(8)/\mathbb{Z}_2$ symmetry, and fields transforming in $E_{7(7)}$ representations was first shown by de Wit and Nicolai [4, 5], who also conjectured that formulations using other $E_{d(d)}$ groups should exist. This was elaborated on in [7, 8] for the case of $H_8 = \mathrm{SO}(16)$ local symmetry and $E_{8(8)}$ representations. Julia [3] had earlier noted that for dimensional reductions to three-dimensions the global $E_{8(8)}$ symmetry includes part of the three-dimensional helicity group and wondered if $E_{8(8)}$ could be a symmetry of the theory in all dimensions, while Duff [6] also independently conjectured that $E_{8(8)}$ was a global symmetry of the eleven-dimensional equations of motion. The construction of $E_{d(d)} \times \mathbb{R}^+$ generalised geometry can be viewed as providing a geometrical basis for these results for $d \leq 7$. (Note that the relevant $E_{d(d)}$ action is by the continuous group rather than the discrete U-duality group that appears in toroidal reductions of M-theory [26].)



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The idea that dimensional reductions to less than three dimensions should realise infinite dimensional Kac-Moody algebras was first proposed in [3, 25]. The case of $E_9$ was analysed in [27, 28], while $E_{10}$ was discussed in [29–31]. That such algebras might appear as symmetries of or in classifying the degrees of freedom of the uncompactified theory is mentioned in [3–6]. It is West [10] who was the first to conjecture that $E_{11}$ is a symmetry of the full eleven-dimensional theory, and to give a proposal for how it is realised. At around the same time, Damour, Henneaux and Nicolai [11] introduced an $E_{10}$ description of the full theory, showing that there is a coset formulation of the small-tension expansion near a spacelike singularity.

In West's $E_{11}$ proposal [10], the symmetry is realised non-linearly over an extended spacetime with an infinite number of coordinates [32]. The corresponding $E_{7(7)}$ non-linear realisation, following the construction of West and using the finite extended spacetime originally conjectured in [9], has been discussed in considerable detail by Hillmann [12, 13]. Truncating to conventional spacetime, he was able to show an equivalence with [4, 5], and, again, the current paper can be viewed as the corresponding geometrical formulation, analogous to the relation between gravity as Riemannian geometry and as a non-linear realisation of $\mathrm{GL}(4) \ltimes \mathbb{R}^4$ introduced by Borisov and Ogievetsky [33].

A related approach to realising $E_{d(d)}$ symmetries is based on the double field theory of Hull and Zwiebach [34], which describes string backgrounds in terms of fields on a doubled spacetime that admits an action of $O(d,d)$, and also connects to earlier work by Duff [35], Tseytlin [36, 37] and Siegel [38, 39]. The dynamics [40–43] are ultimately encoded in a version of a curvature tensor (first constructed by Siegel [38, 39] and introduced from a different perspective in [44, 45]) provided the fields are required to satisfy the "strong constraint", or "section condition". This implies that they depend on only half the coordinates, so locally the theory is equivalent to the $O(d,d)\times\mathbb{R}^+$ generalised geometry described in [23]. (Interestingly, in the double field theory realisations of the mass-deformed type IIA theory [46] and of generic Scherk-Schwarz reductions [47, 48] the strong constraint can be slightly weakened, and so the relation to generalised geometry becomes less clear.) The corresponding formulation of M theory with $E_{d(d)}$ groups was introduced by Berman and Perry [14] for the case of $d = 4$, following earlier work by Duff and Lu [49] (see also [50]). This was extended to $d = 5$ in [15] and subsequently to $d = 6, 7$ in [18], using the $E_{11}$ non-linear formalism of [10] (while the relation to $O(d,d)$ double field theory was discussed in [16]). In these papers a bosonic action is constructed in terms of first-order derivatives of the generalised metric in a generic $E_{d(d)}$ form by brute force. Arbitrary coefficients are fixed by requiring diffeomorphism invariance upon restriction to dependence on $d$ coordinates, and the resulting expression matches the supergravity action up to integration by parts. This coordinate restriction means that locally the generalised geometrical theory constructed here is equally applicable to the double field theory approach to M theory. In this work we are able to derive the $E_{d(d)}$ form of the action directly in terms of the scalar curvature of the generalised connection, which is therefore automatically invariant. Furthermore, we find a generic $E_{d(d)}$ covariant form of the "section condition" [17] that encodes the restriction of the M theory version of double field theory to $d$ coordinates.

At their core, generalised geometries[1] [19, 20] rely on the idea of extending the tangent space of a manifold $M$, such that it can accommodate a larger symmetry group that includes not only diffeomorphisms but also the gauge transformations of supergravity. In its original form, one studies structures on a generalised tangent space $E \simeq TM \oplus T^*M$, with a symmetry group combining diffeomorphisms with the gauge transformations of a two-form potential $B$. There is a natural $O(d,d)$ structure on $E$, where $d$ is the dimension of $M$, and a natural bracket between generalised vectors giving $E$ the structure of a Courant algebroid [51]. Slightly extending the structure group to $O(d,d) \times \mathbb{R}^+$, we showed in an earlier paper [23] that generalised geometry gives a natural rewriting of type II supergravity unifying the NSNS fields as a generalised metric preserved by an $O(9,1)\times O(1,9)$ subgroup, which then becomes a manifest local symmetry of the theory.

The original version of generalised geometry was extended by Hull [21] and Pacheco and Waldram [22] to include the symmetries appearing in M theory. This gives a generalised tangent space $E \simeq TM \oplus \Lambda^2T^*M \oplus \Lambda^5T^*M \oplus \big(T^*M \otimes \Lambda^7T^*M\big)$, relevant to eleven-dimensional supergravity restricted to $d \leq 7$ dimensions and admitting a natural $E_{d(d)}$ structure. One can construct the corresponding generalised metric and also the analogue of the Courant bracket. Applied to type II theories, it allowed the geometrisation of the RR fields and was then used to study the origin of general gaugings of supergravity [52] and to reformulate the effective theory of generic supergravity compactifications to four dimensions as well as the conditions for existence of a supersymmetric background [53, 54].

The generalised tangent space contains objects familiar from Riemannian geometry, namely a bracket structure, covariant derivatives, torsion, and by introducing the generalised metric, the analogue of the Levi-Civita connection, and curvature tensors. Still, there are important differences with respect to ordinary geometry, such as the failure of the generalised bracket to satisfy the Jacobi identity and the fact that, unlike the Levi-Civita connection, there is a family of torsion-free, metric-compatible generalised connections. We discuss all these concepts for $E_{d(d)} \times \mathbb{R}^+$ generalised geometry with a local compact subgroup $H_d \subset E_{d(d)}$ in a manner that treats all dimensions uniformly by decomposing under the appropriate $\mathrm{GL}(d,\mathbb{R})$ and $O(d)$ subgroups. By constructing the natural generalised geometrical equivalent of Einstein gravity, we then find that it contains the entire bosonic supergravity field content — metric, warp factor, three- and dual six-form gauge fields — and precisely describes eleven-dimensional supergravity reduced to $d$ dimensions in the simple forms (1.1) and (1.2).

The paper is arranged as follows. In section 2 we describe the key concepts of $E_{d(d)}\times\mathbb{R}^+$ generalised geometry, including the generalised tangent bundle, its differential structure and the notions of generalised connection and torsion. Next, in section 3 we introduce the local $H_d$ structure and show that one can always construct a torsion-free, $H_d$-compatible generalised connection $D$, the analogue of the Levi-Civita connection. Finally, in section 4 we review the bosonic sector of restrictions of eleven-dimensional supergravity and show

[1] Note that the term "generalised geometry" is sometimes used to refer to formulations where spacetime is extended to include more coordinates. Although the two notions are closely related, here we will limit it to the narrow sense of structures on a Courant algebroid as first introduced by Hitchin and Gualtieri [19, 20], and the related extensions relevant to M theory.



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that it can be reformulated in terms of the generalised geometry. We also comment on the relation to type II theories, generic flux compactifications and the embedding tensor formalism of gauged supergravity [55–57]. We conclude with some summary and discussion in section 5.

## 2 $E_{d(d)} \times \mathbb{R}^+$ generalised geometry

Following closely the construction given in section 3 of [23], we introduce the generalised geometry versions of the tangent space, frame bundle, Lie derivative, connections and torsion, now in the more subtle context of an $E_{d(d)} \times \mathbb{R}^+$ structure. The $E_{d(d)}$ generalised tangent space was first developed in [21] and independently in [22], where the exceptional Courant bracket was also given for the first time. We slightly generalise those notions by introducing an $\mathbb{R}^+$ factor, known as the "trombone symmetry" [58], as it allows one to specify the isomorphism between the generalised tangent space and a sum of vectors and forms. Physically, it is known to be related to the "warp factor" of warped supergravity reductions. The need for this extra factor in the context of $E_{7(7)}$ geometries has already been identified in [12, 13, 18, 59].

### 2.1 Generalised bundles and frames

#### 2.1.1 Generalised tangent space

We start by recalling the definition of the generalised tangent space for $E_{d(d)} \times \mathbb{R}^+$ generalised geometry [21, 22] and defining what is meant by the "generalised structure".

Let $M$ be a $d$-dimensional spin manifold with $d \leq 7$. The generalised tangent space is isomorphic to a sum of tensor bundles

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus \left(T^*M \otimes \Lambda^7 T^*M\right), \tag{2.1}$$

where for $d < 7$ some of these terms will of course be absent. The isomorphism is not unique. The bundle is actually described using a specific patching. If we write

$$\begin{aligned} V_{(i)} &= v_{(i)} + \omega_{(i)} + \sigma_{(i)} + \tau_{(i)} \\ &\in \Gamma\big(TU_i \oplus \Lambda^2 T^*U_i \oplus \Lambda^5 T^*U_i \oplus \left(T^*U_i \otimes \Lambda^7 T^*U_i\right)\big), \end{aligned} \tag{2.2}$$

for a section of $E$ over the patch $U_i$, then

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} V_{(j)}, \tag{2.3}$$

on the overlap $U_i \cap U_j$ where $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally two- and five-forms respectively. The exponentiated action is given by

$$\begin{aligned} v_{(i)} &= v_{(j)}, \\ \omega_{(i)} &= \omega_{(j)} + i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}, \\ \sigma_{(i)} &= \sigma_{(j)} + \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)} + i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)}, \\ \tau_{(i)} &= \tau_{(j)} + j\mathrm{d}\Lambda_{(ij)} \wedge \sigma_{(j)} - j\mathrm{d}\tilde{\Lambda}_{(ij)} \wedge \omega_{(j)} + j\mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)} \\ &\quad + \frac{1}{2} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \frac{1}{6} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}, \end{aligned} \tag{2.4}$$

where we are using the notation of (B.4). Technically this defines $E$ as a result of a series of extensions

$$\begin{aligned}
&0 \longrightarrow \Lambda^2 T^*M \longrightarrow E'' \longrightarrow TM \longrightarrow 0, \\
&0 \longrightarrow \Lambda^5 T^*M \longrightarrow E' \longrightarrow E'' \longrightarrow 0, \\
&0 \longrightarrow T^*M \otimes \Lambda^7 T^*M \longrightarrow E \longrightarrow E' \longrightarrow 0.
\end{aligned} \tag{2.5}$$

Note that while the $v_{(i)}$ globally are equivalent to a choice of vector, the $\omega_{(i)}$, $\sigma_{(i)}$ and $\tau_{(i)}$ are not globally tensors.

Note that the collection $\Lambda_{(ij)}$ formally define a "connective structures on gerbe" (for a review see, for example, [60]). This essentially means there is a hierarchy of successive gauge transformations on the multiple intersections

$$\begin{aligned}
\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} &= \mathrm{d}\Lambda_{(ijk)} \qquad \text{on } U_i \cap U_j \cap U_k, \\
\Lambda_{(jkl)} - \Lambda_{(ikl)} + \Lambda_{(ijl)} - \Lambda_{(ijk)} &= \mathrm{d}\Lambda_{(ijkl)} \qquad \text{on } U_i \cap U_j \cap U_k \cap U_l.
\end{aligned} \tag{2.6}$$

If the supergravity flux is quantised, we will have $g_{(ijkl)} = \mathrm{e}^{\mathrm{i}\Lambda_{(ijkl)}} \in \mathrm{U}(1)$ with the cocycle condition

$$g_{(jklm)} g^{-1}_{(iklm)} g_{(ijlm)} g^{-1}_{(ijkm)} g_{(ijkl)} = 1, \tag{2.7}$$

on $U_i \cap \cdots \cap U_m$. For $\tilde{\Lambda}_{(ij)}$ there is a similar set of structures,

$$\begin{aligned}
&\tilde{\Lambda}_{(ij)} - \tilde{\Lambda}_{(ik)} + \tilde{\Lambda}_{(jk)} \\
&\qquad = \mathrm{d}\tilde{\Lambda}_{(ijk)} + \frac{1}{2}\frac{1}{3!}\Big(\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(jk)} + \text{antisymmetrisation in } [ijk]\Big) \\
&\qquad\quad \text{on } U_i \cap U_j \cap U_k, \\
&\tilde{\Lambda}_{(ijk)} - \tilde{\Lambda}_{(ijl)} + \tilde{\Lambda}_{(ikl)} - \tilde{\Lambda}_{(jkl)} \\
&\qquad = \mathrm{d}\tilde{\Lambda}_{(ijkl)} + \frac{1}{2}\frac{1}{4!}\Big(\Lambda_{(ijk)} \wedge \mathrm{d}\Lambda_{(kl)} + \text{antisymmetrisation in } [ijkl]\Big) \\
&\qquad\quad \text{on } U_i \cap U_j \cap U_k \cap U_l, \\
&\text{etc.}
\end{aligned} \tag{2.8}$$

with the final cocycle condition defined on a octuple intersection $U_{i_1} \cap \cdots \cap U_{i_8}$. Note that this gives a generalisation of the conventional gerbe structure, where the $\tilde{\Lambda}_{(ij)}$ connective structure depends on the $\Lambda_{(ij)}$ gerbe, ultimately reflecting the Chern-Simons coupling in eleven-dimensional supergravity [61].

The bundle $E$ encodes all the topological information of the supergravity background: the twisting of the tangent space $TM$ as well as that of the gerbes, which encode the topology of the supergravity form-field potentials.

#### 2.1.2 Generalised $E_{d(d)} \times \mathbb{R}^+$ structure bundle and split frames

In all dimensions[2] $d \leq 7$ the fibre $E_x$ of the generalised vector bundle at $x \in M$ forms a representation space of $E_{d(d)} \times \mathbb{R}^+$ [21, 22]. These are listed in table 1. They correspond to the set of U-dual momentum and brane central charges in the corresponding dimensionally reduced theories [26], and also appear in the dimensional reduction of West's $E_{11}$ theory [62–64].

[2]In fact the $d \leq 2$ cases essentially reduce to normal Riemannian geometry, so in what follows we will always take $d \geq 3$.

As we discuss below, the explicit action is defined using the $\mathrm{GL}(d,\mathbb{R})$ subgroup that acts on the component spaces $T_xM$, $\Lambda^2T^*_xM$, $\Lambda^5T^*_xM$ and $T^*_xM \otimes \Lambda^7T^*_xM$. Note that without the additional $\mathbb{R}^+$ action, sections of $E$ would transform as tensors weighted by a power of $\det T^*M$. Thus it is key to extend the action to $E_{d(d)} \times \mathbb{R}^+$ in order to define $E$ directly as the extension (2.5).

Crucially, the patching defined in (2.3) is compatible with this $E_{d(d)} \times \mathbb{R}^+$ action. This means that one can define a generalised structure bundle as a sub-bundle of the frame bundle $F$ for $E$. Let $\{\hat{E}_A\}$ be a basis for $E_x$, where the label $A$ runs over the dimension $n$ of the generalised tangent space as listed in table 1. The frame bundle $F$ formed from all such bases is, by construction, a $\mathrm{GL}(n,\mathbb{R})$ principal bundle. We can then define the generalised structure bundle as the natural $E_{d(d)} \times \mathbb{R}^+$ principal sub-bundle of $F$ compatible with the patching (2.3) as follows.

Let $\hat{e}_a$ be a basis for $T_xM$ and $e^a$ the dual basis for $T^*_xM$. We can use these to construct an explicit basis of $E_x$ as

$$\{\hat{E}_A\} = \{\hat{e}_a\} \cup \{e^{ab}\} \cup \{e^{a_1\dots a_5}\} \cup \{e^{a,a_1\dots a_7}\}, \tag{2.9}$$

where $e^{a_1\dots a_p} = e^{a_1} \wedge \dots \wedge e^{a_p}$ and $e^{a,a_1\dots a_7} = e^a \otimes e^{a_1} \wedge \dots \wedge e^{a_7}$. A generic section of $E$ at $x \in U_i$ takes the form

$$V = V^A\hat{E}_A = v^a\hat{e}_a + \frac{1}{2}\omega_{ab}e^{ab} + \frac{1}{5!}\sigma_{a_1\dots a_5}e^{a_1\dots a_5} + \frac{1}{7!}\tau_{a,a_1\dots a_7}e^{a,a_1\dots a_7}. \tag{2.10}$$

As usual, a choice of coordinates on $U_i$ defines a particular such basis where $\{\hat{E}_A\} = \{\partial/\partial x^m\} \cup \{\mathrm{d}x^m \wedge \mathrm{d}x^n\} \cup \dots$. We will denote the components of $V$ in such a coordinate frame by an index $M$, namely $V^M = (v^m, \omega_{mn}, \sigma_{m_1\dots m_5}, \tau_{m,m_1\dots m_7})$.

We then define a $E_{d(d)} \times \mathbb{R}^+$ *basis* as one related to (2.9) by an $E_{d(d)} \times \mathbb{R}^+$ transformation

$$V^A \mapsto V'^A = M^A{}_BV^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B\left(M^{-1}\right)^B{}_A, \tag{2.11}$$

where the explicit action of $M$ is defined in appendix C. The action has a $\mathrm{GL}(d,\mathbb{R})$ subgroup that acts in a conventional way on the bases $\hat{e}_a$, $e^{ab}$ etc, and includes the patching transformation (2.3).[3]

The fact that the definition of the $E_{d(d)} \times \mathbb{R}^+$ action is compatible with the patching means that we can then define the *generalised* $E_{d(d)} \times \mathbb{R}^+$ *structure bundle* $\tilde{F}$ as a sub-bundle of the frame bundle for $E$ given by

$$\tilde{F} = \left\{\left(x, \{\hat{E}_A\}\right) : x \in M, \text{ and } \{\hat{E}_A\} \text{ is an } E_{d(d)} \times \mathbb{R}^+ \text{ basis of } E_x\right\}. \tag{2.12}$$

By construction, this is a principal bundle with fibre $E_{d(d)} \times \mathbb{R}^+$. The bundle $\tilde{F}$ is the direct analogue of the frame bundle of conventional differential geometry, with $E_{d(d)} \times \mathbb{R}^+$ playing the role of $GL(d,\mathbb{R})$.

[3]In analogy to the definitions for $O(d,d) \times \mathbb{R}^+$ generalised geometry [23], we could equivalently define an $E_{d(d)} \times \mathbb{R}^+$ basis using invariants constructed from sections of $E$. For example, in $d = 7$ there is a natural symplectic pairing and symmetric quartic invariant that can be used to define $E_{7(7)}$ (in the context of generalised geometry see [22]). However, these invariants differ in different dimension $d$ so it is more useful here to define $E_{d(d)}$ by an explicit action.



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| $E_{d(d)}$ group | $E_{d(d)} \times \mathbb{R}^+$ rep. |
|---|---|
| $E_{7(7)}$ | $\mathbf{56_1}$ |
| $E_{6(6)}$ | $\mathbf{27'_1}$ |
| $E_{5(5)} \simeq \mathrm{Spin}(5,5)$ | $\mathbf{16^c_1}$ |
| $E_{4(4)} \simeq \mathrm{SL}(5,\mathbb{R})$ | $\mathbf{10'_1}$ |
| $E_{3(3)} \simeq \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ | $(\mathbf{3'},\mathbf{2})_\mathbf{1}$ |

**Table 1**. Generalised tangent space representations where the subscript denotes the $\mathbb{R}^+$ weight, where $\mathbf{1_1} \simeq (\det T^*M)^{1/(9-d)}$.



A special class of $E_{d(d)} \times \mathbb{R}^+$ frames are those defined by a splitting of the generalised tangent space $E$, that is, an isomorphism of the form (2.1). Let $A$ and $\tilde{A}$ be three- and six-form (gerbe) connections patched on $U_i \cap U_j$ by

$$\begin{aligned} A_{(i)} &= A_{(j)} + \mathrm{d}\Lambda_{(ij)}, \\ \tilde{A}_{(i)} &= \tilde{A}_{(j)} + \mathrm{d}\tilde{\Lambda}_{(ij)} - \frac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(j)}. \end{aligned} \tag{2.13}$$

Note that from these one can construct the globally defined field strengths

$$\begin{aligned} F &= \mathrm{d}A_{(i)}, \\ \tilde{F} &= \mathrm{d}\tilde{A}_{(i)} - \frac{1}{2}A_{(i)} \wedge F. \end{aligned} \tag{2.14}$$

Given a generic basis $\{\hat{e}_a\}$ for $TM$ with $\{e^a\}$ the dual basis on $T^*M$ and a scalar function $\Delta$, we define a *conformal split frame* $\{\hat{E}_A\}$ for $E$ by

$$\begin{aligned} \hat{E}_a &= \mathrm{e}^\Delta \Big( \hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \frac{1}{2}A \wedge i_{\hat{e}_a}A \\ &\qquad + jA \wedge i_{\hat{e}_a}\tilde{A} + \frac{1}{6}jA \wedge A \wedge i_{\hat{e}_a}A \Big), \\ \hat{E}^{ab} &= \mathrm{e}^\Delta \Big( e^{ab} + A \wedge e^{ab} - j\tilde{A} \wedge e^{ab} + \frac{1}{2}jA \wedge A \wedge e^{ab} \Big), \\ \hat{E}^{a_1\dots a_5} &= \mathrm{e}^\Delta \left( e^{a_1\dots a_5} + jA \wedge e^{a_1\dots a_5} \right), \\ \hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^\Delta e^{a,a_1\dots a_7}, \end{aligned} \tag{2.15}$$

while a *split frame* has the same form but with $\Delta = 0$. To see that $A$ and $\tilde{A}$ define an isomorphism (2.1) note that, in the conformal split frame,

$$\begin{aligned} V^{(A,\tilde{A},\Delta)} &= \mathrm{e}^{-\Delta}\mathrm{e}^{-A_{(i)} - \tilde{A}_{(i)}} V_{(i)} \\ &= v^a \hat{e}_a + \frac{1}{2}\omega_{ab}e^{ab} + \frac{1}{5!}\sigma_{a_1\dots a_5}e^{a_1\dots a_5} + \frac{1}{7!}\tau_{a,a_1\dots a_7}e^{a,a_1\dots a_7} \\ &\in \Gamma\big(TM \oplus \Lambda^2T^*M \oplus \Lambda^5T^*M \oplus \big(T^*M \otimes \Lambda^7T^*M\big)\big), \end{aligned} \tag{2.16}$$

since the patching implies $\mathrm{e}^{-A_{(i)} - \tilde{A}_{(i)}} V_{(i)} = \mathrm{e}^{-A_{(j)} - \tilde{A}_{(j)}} V_{(j)}$ on $U_i \cap U_j$.

| dimension | $E^*$ | $\mathrm{ad}\,\tilde{F} \subset E \otimes E^*$ | $N \subset S^2 E$ | $K \subset E^* \otimes \mathrm{ad}\,\tilde{F}$ |
|---|---|---|---|---|
| 7 | $\mathbf{56}_{-\mathbf{1}}$ | $\mathbf{133}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$ | $\mathbf{133}_{+\mathbf{2}}$ | $\mathbf{912}_{-\mathbf{1}}$ |
| 6 | $\mathbf{27}_{-\mathbf{1}}$ | $\mathbf{78}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$ | $\mathbf{27}_{+\mathbf{2}}$ | $\mathbf{351}'_{-\mathbf{1}}$ |
| 5 | $\mathbf{16}^s_{-\mathbf{1}}$ | $\mathbf{45}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$ | $\mathbf{10}_{+\mathbf{2}}$ | $\mathbf{144}^c_{-\mathbf{1}}$ |
| 4 | $\mathbf{10}_{-\mathbf{1}}$ | $\mathbf{24}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$ | $\mathbf{5}_{+\mathbf{2}}$ | $\mathbf{40}_{-\mathbf{1}} + \mathbf{15}'_{-\mathbf{1}}$ |
| 3 | $(\mathbf{3},\mathbf{2})_{-\mathbf{1}}$ | $(\mathbf{8},\mathbf{1})_{\mathbf{0}} + (\mathbf{1},\mathbf{3})_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$ | $(\mathbf{3},\mathbf{1})_{+\mathbf{2}}$ | $(\mathbf{3},\mathbf{2})_{-\mathbf{1}} + (\mathbf{6}',\mathbf{2})_{-\mathbf{1}}$ |

**Table 2**. Some generalised tensor bundles.

The class of split frames defines a sub-bundle of $\tilde{F}$

$$P_{\text{split}} = \left\{ \left( x, \{\hat{E}_A\} \right) : x \in M, \text{ and } \{\hat{E}_A\} \text{ is split frame} \right\} \subset \tilde{F}. \tag{2.17}$$

Split frames are related by transformations (2.11) where $M$ takes the form $M = \mathrm{e}^{a+\tilde{a}} m$ with $m \in \mathrm{GL}(d,\mathbb{R})$. The action of $a + \tilde{a}$ shifts $A \mapsto A + a$ and $\tilde{A} \mapsto \tilde{A} + \tilde{a}$. This forms a parabolic subgroup $G_{\text{split}} = \mathrm{GL}(d,\mathbb{R}) \ltimes (a+\tilde{a})\text{-shifts} \subset E_{d(d)} \times \mathbb{R}^+$ where $(a+\tilde{a})$-shifts is the nilpotent group of order two formed of elements $M = \mathrm{e}^{a+\tilde{a}}$. Hence $P_{\text{split}}$ is a $G_{\text{split}}$ principal sub-bundle of $\tilde{F}$, that is a $G_{\text{split}}$-structure. This reflects the fact that the patching elements in the definition of $E$ lie only in this subgroup of $E_{d(d)} \times \mathbb{R}^+$.

### 2.1.3 Generalised tensors

Generalised tensors are simply sections of vector bundles constructed from the generalised structure bundle using different representations of $E_{d(d)} \times \mathbb{R}^+$. We have already discussed the generalised tangent space $E$. There are four other vector bundles which will be of particular importance in the following. The relevant representations are summarised in table 2.[4]

The first is the dual generalised tangent space

$$E^* \simeq T^*M \oplus \Lambda^2 TM \oplus \Lambda^5 TM \oplus \left( TM \otimes \Lambda^7 TM \right). \tag{2.18}$$

Given a basis $\{\hat{E}_A\}$ for $E$ we have a dual basis $\{E^A\}$ on $E^*$ and sections of $E^*$ can be written as $Z = Z_A E^A$.

Next we then have the adjoint bundle $\mathrm{ad}\,\tilde{F}$ associated with the $E_{d(d)} \times \mathbb{R}^+$ principal bundle $\tilde{F}$

$$\mathrm{ad}\,\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM. \tag{2.19}$$

By construction $\mathrm{ad}\,\tilde{F} \subset E \otimes E^*$ and hence we can write sections as $R = R^A{}_B \hat{E}_A \otimes E^B$. We write the projection on the adjoint representation as

$$\times_{\text{ad}} : E^* \otimes E \to \mathrm{ad}\,\tilde{F}. \tag{2.20}$$

It is given explicitly in (C.13).

[4] Note that these representations have already appeared in both the dimensional reduction of the $E_{11}$ theory [62–64] and the tensor hierarchy formulation of gauged supergravity [65, 66].

We also consider the sub-bundle of the symmetric product of two generalised tangent bundles $N \subset S^2E$,

$$\begin{aligned} N \simeq T^*M \oplus \Lambda^4T^*M \oplus \left(T^*M \otimes \Lambda^6T^*M\right) \\ \oplus \left(\Lambda^3T^*M \otimes \Lambda^7T^*M\right) \oplus \left(\Lambda^6T^*M \otimes \Lambda^7T^*M\right). \end{aligned} \tag{2.21}$$

We can write sections as $Y = Y^{AB}\hat{E}_A \otimes \hat{E}_B$ with the projection

$$\times_N : E \otimes E \to N. \tag{2.22}$$

It is given explicitly in (C.15).

Finally, we also need the higher dimensional representation $K \subset E^* \otimes \operatorname{ad}\tilde{F}$ listed in the last column of table 2. Decomposing under $\mathrm{GL}(d,\mathbb{R})$ one has

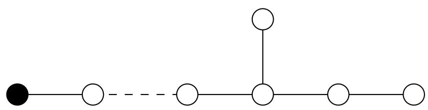
$$\begin{aligned} K \simeq T^*M \oplus S^2TM \oplus \Lambda^2TM \oplus \left(\Lambda^2T^*M \otimes TM\right)_0 \oplus \left(\Lambda^3TM \otimes T^*M\right)_0 \\ \oplus \Lambda^4T^*M \oplus \left(\Lambda^4TM \otimes TM\right)_0 \oplus \Lambda^5TM \oplus \left(\Lambda^2TM \otimes \Lambda^6TM\right)_0 \\ \oplus \Lambda^7T^*M \oplus \left(TM \otimes \Lambda^7TM\right) \oplus \left(\Lambda^7TM \otimes \Lambda^7TM\right) \\ \oplus \left(S^2T^*M \otimes \Lambda^7TM\right) \oplus \left(\Lambda^4TM \otimes \Lambda^7TM\right), \end{aligned} \tag{2.23}$$

where, in fact, the $\Lambda^5TM$ term is absent when $d = 5$. Note also that the zero subscripts are defined such that

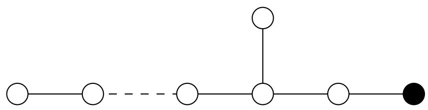
$$\begin{aligned} a_{mn}{}^n &= 0, && \text{if } a \in \Gamma\left(\left(\Lambda^2T^*M \otimes TM\right)_0\right), \\ a^{mnp}{}_p &= 0, && \text{if } a \in \Gamma\left(\left(\Lambda^3TM \otimes T^*M\right)_0\right), \\ a^{[m_1m_2m_3m_4,m_5]} &= 0, && \text{if } a \in \Gamma\left(\left(\Lambda^4TM \otimes TM\right)_0\right), \\ a^{m[n_1,n_2\dots,n_7]} &= 0, && \text{if } a \in \Gamma\left(\left(\Lambda^2TM \otimes \Lambda^6TM\right)_0\right). \end{aligned} \tag{2.24}$$

Since $K \subset E^* \otimes \operatorname{ad}\tilde{F}$ we can write sections as $T = T_A{}^B{}_C E^A \otimes \hat{E}_B \otimes E^C$.

It is interesting to note that, up to symmetries of the $E_d$ Dynkin diagram, the Dynkin labels of the representations $E$ and $N$ follow patterns as $d$ varies. For each value of $d$, the Dynkin label for $E$ can be represented on the Dynkin diagram as



while $N$ has the label



## 2.2 The Dorfman derivative and Courant bracket

An important property of the generalised tangent space is that it admits a generalisation of the Lie derivative which encodes the bosonic symmetries of the supergravity. Given $V = v + \omega + \sigma + \tau \in \Gamma(E)$, one can define an operator $L_V$ acting on any generalised tensor, which combines the action of an infinitesimal diffeomorphism generated by $v$ and $A$- and $\tilde{A}$-field gauge transformations generated by $\omega$ and $\sigma$. Formally this gives $E$ the structure of a "Leibniz algebroid" [59].

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Acting on $V' = v' + \omega' + \sigma' + \tau' \in \Gamma(E)$, one defines the *Dorfman derivative*[5] or "generalised Lie derivative"

$$\begin{aligned} L_V V' = &\mathcal{L}_v v' + \left(\mathcal{L}_v \omega' - i_{v'} \mathrm{d}\omega\right) + \left(\mathcal{L}_v \sigma' - i_{v'} \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega\right) \\ &+ \left(\mathcal{L}_v \tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma\right). \end{aligned} \tag{2.25}$$

Defining the action on a function $f$ as simply $L_V f = \mathcal{L}_v f$, one can then extend the notion of Dorfman derivative to a derivative on the space of $E_{d(d)} \times \mathbb{R}^+$ tensors using the Leibniz property.

To see this, first note that we can rewrite (2.25) in a more $E_{d(d)} \times \mathbb{R}^+$ covariant way, in analogy with the corresponding expressions for the conventional Lie derivative and the Dorfman derivative in $O(d,d) \times \mathbb{R}^+$ generalised geometry [23]. One can embed the action of the partial derivative operator via the map $T^*M \to E^*$ defined by the dual of the exact sequences (2.5). In coordinate indices $M$, as viewed as mapping to a section of $E^*$, one defines

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m \\ 0 & \text{otherwise} \end{cases}. \tag{2.26}$$

Such an embedding has the property that under the projection onto $N^*$ we have

$$\partial f \times_{N^*} \partial g = 0, \tag{2.27}$$

for arbitrary functions $f, g$. We will comment on this observation in section 2.4.

One can then rewrite (2.25) in terms of generalised objects as

$$L_V V'^M = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N, \tag{2.28}$$

where $\times_{\mathrm{ad}}$ denotes the projection onto $\mathrm{ad}\,\tilde{F}$ given in (2.20). Concretely, from (C.13) we have

$$\partial \times_{\mathrm{ad}} V = r + a + \tilde{a}, \tag{2.29}$$

where $r^m{}_n = \partial_n v^m$, $a = \mathrm{d}\omega$ and $\tilde{a} = \mathrm{d}\sigma$. We see that the action actually lies in the adjoint of the $G_{\mathrm{split}} \subset E_{d(d)} \times \mathbb{R}^+$ group. This form of the Dorfman derivative can then be naturally extended to an arbitrary $E_{d(d)} \times \mathbb{R}^+$ tensor by taking that appropriate adjoint action on the $E_{d(d)} \times \mathbb{R}^+$ representation.

Note that we can also define a bracket by taking the antisymmetrisation of the Dorfman derivative. This was originally given in [22] where it was called the "exceptional Courant bracket", and re-derived in [59]. It is given by

$$\begin{aligned} [\![V, V']\!] &= \frac{1}{2}\left(L_V V' - L_{V'} V\right) \\ &= [v, v'] + \mathcal{L}_v \omega' - \mathcal{L}_{v'} \omega - \frac{1}{2}\mathrm{d}\left(i_v \omega' - i_{v'} \omega\right) \\ &\quad + \mathcal{L}_v \sigma' - \mathcal{L}_{v'} \sigma - \frac{1}{2}\mathrm{d}\left(i_v \sigma' - i_{v'} \sigma\right) + \frac{1}{2}\omega \wedge \mathrm{d}\omega' - \frac{1}{2}\omega' \wedge \mathrm{d}\omega \\ &\quad + \frac{1}{2}\mathcal{L}_v \tau' - \frac{1}{2}\mathcal{L}_{v'} \tau + \frac{1}{2}\left(j\omega \wedge \mathrm{d}\sigma' - j\sigma' \wedge \mathrm{d}\omega\right) - \frac{1}{2}\left(j\omega' \wedge \mathrm{d}\sigma - j\sigma \wedge \mathrm{d}\omega'\right). \end{aligned} \tag{2.30}$$

[5] The corresponding object on a Courant algebroid, where the generalised structure is $O(d,d)$ is known as the Dorfman bracket and, following [59], we use the same nomenclature in this case too.



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Note that the group generated by closed $A$ and $\tilde{A}$ shifts is a semi-direct product $\Omega^3_{\rm cl}(M) \ltimes \Omega^6_{\rm cl}(M)$ and corresponds to the symmetry group of gauge transformations in the supergravity. The full automorphism group of the exceptional Courant bracket is then the local symmetry group of the supergravity $G_{\rm sugra} = {\rm Diff}(M) \ltimes \left(\Omega^3_{\rm cl}(M) \ltimes \Omega^6_{\rm cl}(M)\right)$.

For $U, V, W \in \Gamma(E)$, the Dorfman derivative also satisfies the Leibniz identity

$$L_U(L_V W) - L_V(L_U W) = L_{L_U V} W, \tag{2.31}$$

and hence $E$ is a "Leibniz algebroid". On first inspection, one might expect that the bracket of $[\![U, V]\!]$ should appear on the r.h.s. . However, the statement is correct since one can show that

$$L_{[\![U,V]\!]} W = L_{L_U V} W, \tag{2.32}$$

so that the r.h.s. is automatically antisymmetric in $U$ and $V$.

### 2.3 Generalised $E_{d(d)} \times \mathbb{R}^+$ connections and torsion

We now turn to the definitions of generalised connections and torsion. Generalised connections on algebroids were first introduced by Alekseev and Xu [67, 68]. To study the dynamics of $E_{7(7)}$ geometries with an eleven-dimensional supergravity origin and supersymmetric backgrounds, related notions were also developed by [12, 13, 54]. Here, for the $E_{d(d)} \times \mathbb{R}^+$ case, we follow much the same procedure and conventions as in [23], where we gave the precise definitions relevant for type II supergravity, taking care to include an $\mathbb{R}^+$ factor in the generalised structure bundle.

#### 2.3.1 Generalised connections

We first define generalised connections that are compatible with the $E_{d(d)} \times \mathbb{R}^+$ structure. These are first-order linear differential operators $D$, such that, given $W \in \Gamma(E)$, in frame indices,

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B, \tag{2.33}$$

where $\Omega$ is a section of $E^*$ (denoted by the $M$ index) taking values in $E_{d(d)} \times \mathbb{R}^+$ (denoted by the $A$ and $B$ frame indices), and as such, the action of $D$ then extends naturally to any generalised $E_{d(d)} \times \mathbb{R}^+$ tensor.

A simple example of a generalised connection can be constructed as follows. One starts with a conventional connection $\nabla$ and a conformal split frame of the form (2.15). Given the isomorphism (2.16), by construction $v^a \hat{e}_a \in \Gamma(TM)$, $\frac{1}{2}\omega_{ab} e^{ab} \in \Gamma\left(\Lambda^2 T^* M\right)$ etc and hence $\nabla_m v^a$ and $\nabla_m \omega_{ab}$ are well-defined. The generalised connection defined by $\nabla$ lifted to an action on $E$ by the conformal split frame then defines a generalised connection $D^\nabla$ as

$$D^\nabla_M V = \begin{cases} (\nabla_m v^a) \hat{E}_a + \frac{1}{2}(\nabla_m \omega_{ab}) \hat{E}^{ab} & \\ \quad + \frac{1}{5!}(\nabla_m \sigma_{a_1 \dots a_5}) \hat{E}^{a_1 \dots a_5} + \frac{1}{7!}(\nabla_m \tau_{a, a_1 \dots a_7}) \hat{E}^{a, a_1 \dots a_7} & \text{for } M = m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.34}$$

### 2.3.2 Generalised torsion

We define the *generalised torsion* $T$ of a generalised connection $D$ in direct analogy to the conventional definition and to the one we defined in the $O(d,d)\times\mathbb{R}^+$ description of type II theories [23].

Let $\alpha$ be any generalised $E_{d(d)}\times\mathbb{R}^+$ tensor and let $L^D_V\alpha$ be the Dorfman derivative (2.28) with $\partial$ replaced by $D$. The generalised torsion is a linear map $T : E \to \mathrm{ad}\,(\tilde{F})$ defined by

$$T(V)\cdot\alpha = L^D_V\alpha - L_V\alpha, \tag{2.35}$$

for any $V\in\Gamma\,(E)$ and where $T(V)$ acts via the adjoint representation on $\alpha$. Let $\{\hat{E}_A\}$ be an $E_{d(d)}\times\mathbb{R}^+$ frame for $E$ and $\{E^A\}$ be the dual frame for $E^*$ satisfying $E^A(\hat{E}_B)=\delta^A{}_B$. We then have the explicit expression

$$T(V) = V^C\left[\Omega_C{}^A{}_B - \Omega_B{}^A{}_C - E^A\left(L_{\hat{E}_C}\hat{E}_B\right)\right]\hat{E}_A\times_{\mathrm{ad}}E^B. \tag{2.36}$$

Note that we are projecting onto the adjoint representation on the $A$ and $B$ indices. Note also that in a coordinate frame the last term vanishes.

Viewed as a generalised $E_{d(d)}\times\mathbb{R}^+$ tensor we have $T\in\Gamma\left(E^*\otimes\mathrm{ad}\,\tilde{F}\right)$. However, the form of the Dorfman derivative means that fewer components actually survive and we find

$$T\in\Gamma\,(K\oplus E^*)\,, \tag{2.37}$$

where $K$ was defined in table 2. Note that these representations are exactly the same ones that appear in the embedding tensor formulation of gauged supergravities [55, 56], including gaugings [57] of the so-called "trombone" symmetry [58]. We will comment on this further in section 4.4.

As an example, we can calculate the torsion of the generalised connection $D^\nabla$ defined by a conventional connection $\nabla$ and a conformal split frame as given in (2.34). We find

$$\begin{aligned} T(V) = \mathrm{e}^\Delta\big(&-i_v\mathrm{d}\Delta + v\otimes\mathrm{d}\Delta + i_vT - i_vF + \mathrm{d}\Delta\wedge\omega + T\cdot\omega \\ &+\omega\wedge F + \mathrm{d}\Delta\wedge\sigma - i_v\tilde{F} + T\cdot\sigma\big), \end{aligned} \tag{2.38}$$

where we are using the isomorphism (2.19), $F$ and $\tilde{F}$ are the field strengths (2.14), $T$ is the conventional torsion of $\nabla$ and we use the notation $(i_vT)^\mu{}_\nu = v^\lambda T^\mu{}_{\lambda\nu}$ and $(T\cdot\lambda)_{\mu_1\dots\mu_p} = \frac{p!}{(p-2)!2!}T^\nu{}_{[\mu_1\mu_2}\lambda_{|\nu|\mu_3\dots\mu_p]}$.

## 2.4 The "section condition", Jacobi identity and the absence of generalised curvature

Restricting our analysis to $d\leq 6$, we find that the bundle $N$ given in (2.21) measures the failure of the generalised tangent bundle to satisfy the properties of a Lie algebroid. This follows from the observation that the difference between the Dorfman derivative and the exceptional Courant bracket (that is, the symmetric part of the Dorfman derivative), for $V,V'\in\Gamma\,(E)$, is precisely given by[6]

$$L_VV' - [\![V,V']\!] = \frac{1}{2}\mathrm{d}\left(i_v\omega' + i_{v'}\omega - i_v\sigma' - i_{v'}\sigma + \omega\wedge\omega'\right) = \partial\times_E(V\times_N V'), \tag{2.39}$$

[6]For $d\geq 7$ the r.h.s. can no longer be written covariantly as a derivative of an $E_{d(d)}\times\mathbb{R}^+$ tensor built from $U$ and $V$. Similar complications occur in the discussion of the curvature below. This is the reason for the restriction to $d\leq 6$ in this section.

where the last equality stresses the $E_{d(d)} \times \mathbb{R}^+$ covariant form of the exact term. Therefore, while the Dorfman derivative satisfies a sort of Jacobi identity via the Leibniz identity (2.31), the Jacobiator of the exceptional Courant bracket, like that of the $O(d,d)$ Courant bracket, does not vanish in general. In fact, it can be shown that

$$\mathrm{Jac}(U,V,W) = [\![ [\![U,V]\!] , W ]\!] + \text{c.p.} = \frac{1}{3}\partial \times_E ([\![U,V]\!] \times_N W + \text{c.p.}) \,, \tag{2.40}$$

where $U,V,W \in \Gamma(E)$ and c.p. denotes cyclic permutations in $U,V$ and $W$. We see that both the failure of the exceptional Courant bracket to be Jacobi and the Dorfman derivative to be antisymmetric is measured by an exact term given by the $\times_N$ projection. The proof is essentially the same as the one for the $O(d,d)$ case, see for example [20], section 3.2.[7]

Similarly, and as was the case with $O(d,d)\times\mathbb{R}^+$ generalised connections, for notions of generalised curvature one finds the naive definition $[D_U, D_V]\,W - D_{[\![U,V]\!]}W$ is not a tensor and its failure to be covariant is measured by the projection of the first two arguments to $N$. Explicitly, taking $U \to fU$, $V \to gV$ and $W \to hW$ for some scalar functions $f,g,h$, we obtain

$$\begin{aligned} &[D_{fU}, D_{gV}]\,hW - D_{[\![fU,gV]\!]}hW \\ &\qquad = fgh\left([D_U,D_V]\,W - D_{[\![U,V]\!]}W\right) - \frac{1}{2}hD_{(f\partial g - g\partial f)\times_E(U\times_N V)}W. \end{aligned} \tag{2.41}$$

Note, however, that it is still possible to define analogues of the Ricci tensor and scalar when there is additional structure on the generalised tangent space, as we see in the following section.

Finally, we note that from the point of view of "double field theory"-like geometries [14–18, 34, 38–43], the equation

$$\partial f \times_{N^*} \partial g = 0, \tag{2.42}$$

for any functions $f$ and $g$ acquires a special interpretation. In these theories, one starts by enlarging the spacetime manifold so that its dimension matches that of the generalised tangent space. The partial derivative $\partial_M f$ is then generically non-zero for all $M$. However, the corresponding Dorfman derivative does not then satisfy the Leibniz property, nor is the action for the generalised metric invariant. One must instead impose a "section condition" or "strong constraint". In the original $O(d,d)$ double field theory the condition takes the form $(\partial^M f)(\partial_M g) = 0$. It implies that, in fact, the fields only depend on half the coordinates. For exceptional geometries, the $d = 4$ case was thoroughly analysed in [17], and is given by (2.42). Again it implies that the fields depend on only $d$ of the coordinates.

It is in fact easy to show that satisfying (2.42) always implies the Leibniz property. Thus it gives the section condition in general dimension. In generalised geometry it is satisfied identically by taking $\partial_M$ of the form (2.26). However given the $E_{d(d)}\times\mathbb{R}^+$ covariant form of the Dorfman derivative (2.28), any subspace of $E^*$ in the same orbit under $E_{d(d)} \times \mathbb{R}^+$ will also satisfy the Leibniz condition. Note further that any such subspace, like $T^*$, is invariant under an action of the parabolic subgroup $G_{\text{split}}$.

[7] Note that in the $O(d,d)$ case the fibre of $N$ is the $\mathbf{1}_{+2}$ representation, so $N$ is a trivial bundle.

| $E_{d(d)}$ group | $\tilde{H}_d$ group | $\operatorname{ad} P^\perp = \operatorname{ad} \tilde{F} / \operatorname{ad} P$ |
|---|---|---|
| $E_{7(7)}$ | $\mathrm{SU}(8)$ | $\mathbf{35} + \bar{\mathbf{35}} + \mathbf{1}$ |
| $E_{6(6)}$ | $\mathrm{USp}(8)$ | $\mathbf{42} + \mathbf{1}$ |
| $E_{5(5)} \simeq \mathrm{Spin}(5,5)$ | $\mathrm{Spin}(5) \times \mathrm{Spin}(5)$ | $(\mathbf{5}, \mathbf{5}) + (\mathbf{1}, \mathbf{1})$ |
| $E_{4(4)} \simeq \mathrm{SL}(5,\mathbb{R})$ | $\mathrm{Spin}(5)$ | $\mathbf{14} + \mathbf{1}$ |
| $E_{3(3)} \simeq \mathrm{SL}(3,\mathbb{R})) \times \mathrm{SL}(2,\mathbb{R})$ | $\mathrm{Spin}(3) \times \mathrm{Spin}(2)$ | $(\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{1})$ |

**Table 3**. Double covers of the maximal compact subgroups of $E_{d(d)}$ and $H_d$ representations of the coset bundle.

# 3 $H_d$ structures and torsion-free connections

We now turn to the construction of the analogue of the Levi-Civita connection by considering additional structure on the generalised tangent space. Again, this closely follows the constructions in $O(d,d) \times \mathbb{R}^+$ generalised geometry [23].

We consider $H_d$ structures on $E$ where $H_d$ is the maximally compact[8] subgroup of $E_{d(d)}$. These, or rather their double covers[9] $\tilde{H}_d$ are listed in table 3. We will then be interested in generalised connections $D$ that preserve the $H_d$ structure. We find it is always possible to construct torsion-free connections of this type but they are not unique. Nonetheless we show that, using the $H_d$ structure, one can construct unique projections of $D$, and that these can be used to define analogues of the Ricci tensor and scalar curvatures with a local $H_d$ symmetry.

## 3.1 $H_d$ structures and the generalised metric

An $H_d$ structure on the tangent space is a set of frames related by $H_d$ transformations. This is the direct analogy of metric structure, where one considers the set of orthonormal frames related by $O(d)$ transformations. Formally it defines an $H_d$ principal sub-bundle of the generalised structure bundle $\tilde{F}$, that is

$$P \subset \tilde{F} \text{ with fibre } H_d. \tag{3.1}$$

The choice of such a structure is parametrised, at each point on the manifold, by an element of the coset $(E_{d(d)} \times \mathbb{R}^+)/H_d$. The corresponding representations are listed in table 3. Note that there is always a singlet corresponding to the $\mathbb{R}^+$ factor.

[8]Note that one could equally consider the non-compact versions of $H_d$ by switching the signature of the metric in appendix C.3 so that it defines an $\mathrm{SO}(p,q)$ subgroup of $\mathrm{GL}(d,\mathbb{R})$, and the corresponding results then follow identically. For instance, if in $d = 7$ one chooses the $\mathrm{SO}(6,1)$ signature, one would obtain the non-compact $\mathrm{SU}^*(8)$ subgroup of $E_{7(7)} \times \mathbb{R}^+$, which would be relevant for discussing timelike reductions of 11-dimensional supergravity [69].

[9]We give the double covers of the maximally compact group, since we will be interested in the analogues of spinor representations. A necessary and sufficient condition for the existence of the double cover is the vanishing of the 2nd Stiefel-Whitney class of the generalised tangent bundle [21]. As the underlying manifold is spin by assumption, this is automatically satisfied.



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One can construct elements of $P$ concretely, that is, identify the analogues of "orthonormal" frames, in the following way. Given an $H_d$ structure, it is always possible to put the $H_d$ frame in a conformal split form, namely,

$$
\begin{aligned}
\hat{E}_a &= \mathrm{e}^{\Delta}\Big(\hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \frac{1}{2}A\wedge i_{\hat{e}_a}A \\
&\qquad + jA\wedge i_{\hat{e}_a}\tilde{A} + \frac{1}{6}jA\wedge A\wedge i_{\hat{e}_a}A\Big), \\
\hat{E}^{ab} &= \mathrm{e}^{\Delta}\left(e^{ab} + A\wedge e^{ab} - j\tilde{A}\wedge e^{ab} + \frac{1}{2}jA\wedge A\wedge e^{ab}\right), \\
\hat{E}^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(e^{a_1\dots a_5} + jA\wedge e^{a_1\dots a_5}\right), \\
\hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}e^{a,a_1\dots a_7}.
\end{aligned}
\tag{3.2}
$$

Any other frame is then related by an $H_d$ transformation of the form given in appendix C.3. Concretely given $V{=}V^A\hat{E}_A{\in}\Gamma(E)$ expanded in such a frame, different frames are related by

$$
V^A \mapsto V'^A = H^A{}_B V^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B\left(H^{-1}\right)^B{}_A, \tag{3.3}
$$

where $H$ is defined in (C.19). Note that the $O(d)\subset H_d$ action simply rotates the $\hat{e}_a$ basis, defining a set of orthonormal frames for a conventional metric $g$. It also keeps the frame in the conformal split form. Thus the set of conformal split $H_d$ frames actually forms an $O(d)$ structure on $E$, that is

$$
(P\cap P_{\text{split}})\subset\tilde{F}\text{ with fibre }O(d). \tag{3.4}
$$

One can also define the generalised metric acting on $V\in\Gamma(E)$ as

$$
G(V,V) = v^2 + \frac{1}{2!}\omega^2 + \frac{1}{5!}\sigma^2 + \frac{1}{7!}\tau^2, \tag{3.5}
$$

$v^2 = v_a v^a$, $\omega^2 = \omega_{ab}\omega^{ab}$, etc as in (B.5), are evaluated in an $H_d$ frame and indices are contracted using the flat frame metric $\delta_{ab}$ (as used to define the $H_d$ subgroup in appendix C.3). Since, by definition, this is independent of the choice of $H_d$ frame, it can be evaluated in the conformal split representative (3.2). Hence one sees explicitly that the metric is defined by the fields $g$, $A$, $\tilde{A}$ and $\Delta$ that determine the coset element.

Note that the $H_d$ structure embeds as $H_d\subset E_{d(d)}\subset E_{d(d)}\times\mathbb{R}^+$. This mirrors the chain of embeddings in Riemannian geometry $\mathrm{SO}(d)\subset\mathrm{SL}(d,\mathbb{R})\subset GL(d,\mathbb{R})$ which allows one to define a $\det T^*M$ density that is $\mathrm{SO}(d)$ invariant, $\sqrt{g}$. Likewise, here we can define a density that is $H_d$ (and $E_{d(d)}$) invariant, corresponding to the choice of $\mathbb{R}^+$ factor which, in terms of the conformal split frame, is given by

$$
\mathrm{vol}_G = \sqrt{g}\,\mathrm{e}^{(9-d)\Delta}, \tag{3.6}
$$

as can be seen from appendix C.1. This can also be defined as the determinant of $G$ to a suitable power.



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### 3.2 Torsion-free, compatible connections

A generalised connection $D$ is compatible with the $H_d$ structure $P \subset \tilde{F}$ if

$$DG = 0, \tag{3.7}$$

or, equivalently, if the derivative acts only in the $H_d$ sub-bundle. In this subsection we will show, in analogy to the construction of the Levi-Civita connection, that

> *Given an $H_d$ structure $P \subset \tilde{F}$ there always exists a torsion-free, compatible generalised connection $D$. However, it is not unique.*

We construct the compatible connection explicitly by working in the conformal split $H_d$ frame (3.2). However the connection is $H_d$ covariant, so the form in any another frame simply follows from an $H_d$ transformation.

Let $\nabla$ be the Levi-Civita connection for the metric $g$. We can lift the connection to an action on $V \in \Gamma(E)$ by defining, as in (2.34),

$$D^{\nabla}_M V = \begin{cases} (\nabla_m v^a)\hat{E}_a + \frac{1}{2}(\nabla_m \omega_{ab})\hat{E}^{ab} & \\ \quad + \frac{1}{5!}(\nabla_m \sigma_{a_1\dots a_5})\hat{E}^{a_1\dots a_5} + \frac{1}{7!}(\nabla_m \tau_{a,a_1\dots a_7})\hat{E}^{a,a_1\dots a_7} & \text{for } M = m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Since $\nabla$ is compatible with the $O(d) \subset H_d$ subgroup, it is necessarily an $H_d$-compatible connection. However, $D^{\nabla}$ is not torsion-free. From (2.38), since $\nabla$ is torsion-free (in the conventional sense), we have

$$T(V) = \mathrm{e}^{\Delta}\left(-i_v \mathrm{d}\Delta + v \otimes \mathrm{d}\Delta - i_v F + \mathrm{d}\Delta \wedge \omega - i_v \tilde{F} + \omega \wedge F + \mathrm{d}\Delta \wedge \sigma\right). \tag{3.9}$$

To construct a torsion-free compatible connection we simply modify $D^{\nabla}$. A generic generalised connection $D$ can always be written as

$$D_M W^A = D^{\nabla}_M W^A + \Sigma_M{}^A{}_B W^B. \tag{3.10}$$

If $D$ is compatible with the $H_d$ structure then

$$\Sigma \in \Gamma\left(E^* \otimes \operatorname{ad} P\right), \tag{3.11}$$

that is, it is a generalised covector taking values in the adjoint of $H_d$. The problem is then to find a suitable $\Sigma$ such that the torsion of $D$ vanishes. Fortunately, decomposing under $H_d$ one finds that all the representations that appear in the torsion are already contained in $\Sigma$. Thus a solution always exists, but is not unique.[10] The relevant representations are listed in table 4. As $H_d$ tensor bundles one has

$$E^* \otimes \operatorname{ad} P \simeq (K \oplus E^*) \oplus U, \tag{3.12}$$

so that the torsion $T \in \Gamma(K \oplus E^*)$ and the unconstrained part of $\Sigma$ is a section of $U$.

[10]In $d = 3$ all the components of $\Sigma$ are contained in the torsion representations, $E^* \otimes \operatorname{ad} P \simeq K \oplus E^*$, and so, in that particular case, the generalised connection is in fact completely determined.

| dimension | $K \oplus E^*$ | $U \simeq (E^* \otimes \mathrm{ad}\, P)/(K \oplus E^*)$ |
|---|---|---|
| 7 | $\mathbf{28} + \bar{\mathbf{28}} + \mathbf{36} + \bar{\mathbf{36}} + \mathbf{420} + \bar{\mathbf{420}}$ | $\mathbf{1280} + \bar{\mathbf{1280}}$ |
| 6 | $\mathbf{27} + \mathbf{36} + \mathbf{315}$ | $\mathbf{594}$ |
| 5 | $(\mathbf{4},\mathbf{4}) + (\mathbf{4},\mathbf{4}) + (\mathbf{16},\mathbf{4}) + (\mathbf{4},\mathbf{16})$ | $(\mathbf{20},\mathbf{4}) + (\mathbf{4},\mathbf{20})$ |
| 4 | $\mathbf{1} + \mathbf{5} + \mathbf{10} + \mathbf{14} + \mathbf{35}'$ | $\mathbf{35}$ |
| 3 | $(\mathbf{1},\mathbf{2}) + (\mathbf{3},\mathbf{2}) + (\mathbf{3},\mathbf{2}) + (\mathbf{5},\mathbf{2})$ | - |

**Table 4**. Components of the connection $\Sigma$ that are constrained by the torsion, $T$, and the unconstrained ones, $U$, as $H_d$ representations.

The solution for $\Sigma$ can be written very explicitly as follows. Contracting with $V \in \Gamma(E)$ so $\Sigma(V) \in \mathrm{ad}\, P$ and using the basis for the adjoint of $H_d$ given in (C.17) and (C.18) we have

$$
\begin{aligned}
\Sigma(V)_{ab} &= \mathrm{e}^{\Delta} \left( 2 \left( \frac{7-d}{d-1} \right) v_{[a} \partial_{b]} \Delta + \frac{1}{4!} \omega_{cd} F^{cd}{}_{ab} + \frac{1}{7!} \sigma_{c_1 \dots c_5} \tilde{F}^{c_1 \dots c_5}{}_{ab} + Q(V)_{ab} \right) , \\
\Sigma(V)_{abc} &= \mathrm{e}^{\Delta} \left( \frac{6}{(d-1)(d-2)} (\mathrm{d}\Delta \wedge \omega)_{abc} + \frac{1}{4} v^d F_{dabc} + Q(V)_{abc} \right) , \\
\Sigma(V)_{a_1 \dots a_6} &= \mathrm{e}^{\Delta} \left( \frac{1}{7} v^b \tilde{F}_{b a_1 \dots a_6} + Q(V)_{a_1 \dots a_6} \right) ,
\end{aligned}
\tag{3.13}
$$

where the ambiguous part of the connection $Q \in \Gamma(E^* \otimes \mathrm{ad}\, P)$ projects to zero under the map to the torsion representation $K \oplus E^*$, that is

$$
Q \in \Gamma(U) . \tag{3.14}
$$

Using the embedding of $\tilde{H}_d$ in $\mathrm{Cliff}(d;\mathbb{R})$ given in (C.20) we can thus write the full connection as

$$
\begin{aligned}
D_a &= \mathrm{e}^{\Delta} \left( \nabla_a + \frac{1}{2} \left( \frac{7-d}{d-1} \right) (\partial_b \Delta) \gamma_a{}^b - \frac{1}{2} \frac{1}{4!} F_{a b_1 b_2 b_3} \gamma^{b_1 b_2 b_3} - \frac{1}{2} \frac{1}{7!} \tilde{F}_{a b_1 \dots b_6} \gamma^{b_1 \dots b_6} + \not{Q}_a \right) , \\
D^{a_1 a_2} &= \mathrm{e}^{\Delta} \left( \frac{1}{4} \frac{2!}{4!} F^{a_1 a_2}{}_{b_1 b_2} \gamma^{b_1 b_2} - \frac{3}{(d-1)(d-2)} (\partial_b \Delta) \gamma^{a_1 a_2 b} + \not{Q}^{a_1 a_2} \right) , \\
D^{a_1 \dots a_5} &= \mathrm{e}^{\Delta} \left( \frac{1}{4} \frac{5!}{7!} \tilde{F}^{a_1 \dots a_5}{}_{b_1 b_2} \gamma^{b_1 b_2} + \not{Q}^{a_1 \dots a_5} \right) , \\
D^{a, a_1 \dots a_7} &= \mathrm{e}^{\Delta} \left( \not{Q}^{a, a_1 \dots a_7} \right) ,
\end{aligned}
\tag{3.15}
$$

where

$$
\begin{aligned}
\not{Q}_m &= \frac{1}{2} \left( \frac{1}{2!} Q_{m,ab} \gamma^{ab} - \frac{1}{3!} Q_{m, a_1 a_2 a_3} \gamma^{a_1 a_2 a_3} - \frac{1}{6!} Q_{m, a_1 \dots a_6} \gamma^{a_1 \dots a_6} \right) , \\
\not{Q}^{m_1 m_2} &= \frac{1}{2} \left( \frac{1}{2!} Q^{m_1 m_2}{}_{ab} \gamma^{ab} - \frac{1}{3!} Q^{m_1 m_2}{}_{a_1 a_2 a_3} \gamma^{a_1 a_2 a_3} - \frac{1}{6!} Q^{m_1 m_2}{}_{a_1 \dots a_6} \gamma^{a_1 \dots a_6} \right) , \\
&\text{etc.}
\end{aligned}
\tag{3.16}
$$

is the embedding of the ambiguous part of the connection.[11]

[11]It is interesting to compare this connection to the one defined in [54]. There $\Sigma$ is chosen to lie solely in the **912** representation. This leads to a unique torsion-free connection, which is, however, not compatible with the generalised metric.

| $\tilde{H}_d$ | $S$ | $J$ |
|---|---|---|
| $\mathrm{SU}(8)$ | $\mathbf{8}+\bar{\mathbf{8}}$ | $\mathbf{56}+\bar{\mathbf{56}}$ |
| $\mathrm{USp}(8)$ | $\mathbf{8}$ | $\mathbf{48}$ |
| $\mathrm{USp}(4)\times\mathrm{USp}(4)$ | $(\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{4})$ | $(\mathbf{4},\mathbf{5})+(\mathbf{5},\mathbf{4})$ |
| $\mathrm{USp}(4)$ | $\mathbf{4}$ | $\mathbf{16}$ |
| $\mathrm{SU}(2)\times\mathrm{U}(1)$ | $\mathbf{2_1}+\mathbf{2_{-1}}$ | $\mathbf{4_1}+\mathbf{4_{-1}}+\mathbf{2_3}+\mathbf{2_{-3}}$ |

**Table 5**. Spinor and gravitino representations in each dimension.

### 3.3 Unique operators and generalised $H_d$ curvatures

We now turn to the construction of unique operators and curvatures from the torsion-free and $\tilde{H}_d$-compatible connection $D$ constructed in the previous section. To keep the $\tilde{H}_d$ covariance manifest in all dimensions, we will necessarily have to maintain the discussion in this section fairly abstract. We should note, however, that the entire construction can be made very concrete. In [24] we will present the details for particular dimensions, such as the $d=7$ case where the unique operators and the curvatures can be explicitly written out in $\tilde{H}_7 = \mathrm{SU}(8)$ indices.

Given a bundle $X$ transforming as some representation of $\tilde{H}_d$, we define the map

$$\mathcal{Q}_X : U \otimes X \longrightarrow E^* \otimes X, \tag{3.17}$$

via the embedding $U \subset E^* \otimes \operatorname{ad} P$ and the adjoint action of $\operatorname{ad} P$ on $X$. We then have the projection

$$\mathcal{P}_X : E^* \otimes X \longrightarrow \frac{E^* \otimes X}{\operatorname{Im} \mathcal{Q}_X}. \tag{3.18}$$

Recall that the ambiguous part $Q$ of the connection $D$ is a section of $U$, which acts on $X$ via the map $\mathcal{Q}_X$. If $\alpha \in \Gamma(X)$, then, by construction, $\mathcal{P}_X(D \otimes \alpha)$ is uniquely defined, independent of $Q$.

We can construct explicit examples of such operators as follows. Consider two real $\tilde{H}_d$ bundles $S$ and $J$, which we refer to as the "spinor" bundle and the "gravitino" bundle respectively, since the supersymmetry parameter and the gravitino field in supergravity are sections of them. The relevant $\tilde{H}_d$ representations are listed in table 5. Note that the spinor representation is simply the $\mathrm{Cliff}(d;\mathbb{R})$ spinor representation using the embedding (C.20).

One finds that under the projection $\mathcal{P}_X$ we have[12]

$$\begin{aligned} \mathcal{P}_S(E^* \otimes S) &\simeq S \oplus J, \\ \mathcal{P}_J(E^* \otimes J) &\simeq S \oplus J. \end{aligned} \tag{3.19}$$

[12]Note that there is an exception for $d=3$ since, as was previously mentioned, in that case the entire metric compatible, torsion-free connection is uniquely determined, and so $\mathcal{P}_X$ is just the identity map and $\mathcal{P}_X(E^* \otimes X) = E^* \otimes X$ for any bundle $X$.

Therefore, for any $\varepsilon \in \Gamma(S)$ and $\psi \in \Gamma(J)$, one has that the following are unique for any torsion-free connection

$$\begin{aligned} &D \times_J \varepsilon, && D \times_S \varepsilon, \\ &D \times_J \psi, && D \times_S \psi, \end{aligned} \tag{3.20}$$

where $\times_X$ denotes the projection onto the $X$ bundle.

One can show that the first two expressions encode the supersymmetry variation of the internal and external gravitino respectively, while the latter two are related to the gravitino equation of motion. This will be described in more detail in [24].

We would now like to define measures of generalised curvature. As was mentioned in section 2.4, the natural definition of a Riemann curvature does not result in a tensor. Nonetheless, for a torsion-free, $\tilde{H}_d$-compatible connection $D$ there does exist a generalised Ricci tensor $R_{AB}$, and it is a section of the bundle

$$\operatorname{ad} P^{\perp} = \operatorname{ad} \tilde{F} / \operatorname{ad} P \subset E^* \otimes E^*, \tag{3.21}$$

where the last relation follows because, as representations of $H_d$, $E \simeq E^*$. It is not immediately apparent that we can make such a definition, but $R_{AB}$ can in fact be constructed from compositions of the unique operators (3.20) as

$$\begin{aligned} D \times_J (D \times_J \varepsilon) + D \times_J (D \times_S \varepsilon) &= R^0 \cdot \varepsilon, \\ D \times_S (D \times_J \varepsilon) + D \times_S (D \times_S \varepsilon) &= R\, \varepsilon, \end{aligned} \tag{3.22}$$

where $R$ and $R^0_{AB}$ provide the scalar and non-scalar parts of $R_{AB}$ respectively.[13] The existence of expressions of this type is a non-trivial statement. By computing in the split frame, it can be shown that the l.h.s. is linear in $\varepsilon$, and since $\varepsilon$ and the l.h.s. are manifestly covariant, these expressions define a tensor. We will write the components explicitly in section 4.2, equation (4.11). This calculation further provides the non-trivial result that $R_{AB}$ is restricted to be a section of $\operatorname{ad} P^{\perp}$, rather than a more general section of $(S \otimes J) \oplus \mathbb{R}$. In the context of supergravity, this calculation exactly corresponds to the closure of the supersymmetry algebra on the fermionic equations of motion, as will be discussed further in [24]. Finally, since it is built from unique operators, the generalised curvature is automatically unique for a torsion-free compatible connection.

The expressions (3.22) can be written with a different sequence of projections. This helps elucidate the nature of the curvature in terms of certain second-order differential operators. In conventional differential geometry the commutator of two connections $[\nabla_m, \nabla_n]$ has no second-derivative term simply because the partial derivatives commute. This is a necessary condition for the curvature to be tensorial. In $E_{d(d)}$ indices one can similarly write the commutator of two generalised derivatives formally as $(D \wedge D)_{AB} = [D_A, D_B]$. More precisely, acting on an $E_{d(d)} \times \mathbb{R}^+$ vector bundle $X$ we have

$$(D \wedge D) : X \to \Lambda^2 E^* \otimes X. \tag{3.23}$$

[13]Note that $\operatorname{ad} P^{\perp} \subset (S \otimes J) \oplus \mathbb{R}$ and the $\tilde{H}_d$ structure gives an isomorphism $S \simeq S^*$ and $J \simeq J^*$. Thus, as in the first line of (3.22), we can also view $R^0$ as a map from $S$ to $J$.

Since again the partial derivatives commute this operator contains no second-order derivative term, and so can potentially be used to construct a curvature tensor. However, in $E_{d(d)} \times \mathbb{R}^+$ generalised geometry we also have $\partial f \times_{N^*} \partial g = 0$ for any $f$ and $g$, and so we can take the projection to the bundle $N^*$ defined earlier, giving a similar operator

$$(D \times_{N^*} D) : X \to N^* \otimes X, \tag{3.24}$$

which will again contain no second-order derivatives. One thus expects that these two operators, which can be defined for an arbitrary $E_{d(d)} \times \mathbb{R}^+$ connection, should appear in any definition of generalised curvature. Given an $\tilde{H}_d$ structure and a torsion-free compatible connection $D$, they indeed enter the definition of $R_{AB}$. Using $\tilde{H}_d$ covariant projections one finds

$$\begin{aligned} (D \wedge D) \times_J \varepsilon + (D \times_{N^*} D) \times_J \varepsilon &= R^0 \cdot \varepsilon, \\ (D \wedge D) \times_S \varepsilon + (D \times_{N^*} D) \times_S \varepsilon &= R\, \varepsilon. \end{aligned} \tag{3.25}$$

This structure suggests there will be similar definitions of curvature in terms of the operators $(D \wedge D)$ and $(D \times_{N^*} D)$ independent of the representation on which they act, and potentially without the need for additional structure.

# 4 Supergravity as $H_d$ generalised geometry

We now show how the generalised geometrical structures we have described in the previous sections allow us to rewrite the bosonic sector of eleven-dimensional supergravity with the local $H_d$-covariance manifest. We also cover the relation to type II theories and the embedding tensor formalism.

## 4.1 Eleven-dimensional supergravity in $d$-dimensions

We will be interested in "restrictions" of eleven-dimensional supergravity where the space-time is assumed to be a product $\mathbb{R}^{10-d,1} \times M$ of Minkowski space with a $d$-dimensional spin manifold $M$, with $d \leq 7$. The metric is taken to be

$$\mathrm{d}s^2_{11} = \mathrm{e}^{2\Delta} \mathrm{d}s^2(\mathbb{R}^{10-d,1}) + \mathrm{d}s^2_d(M), \tag{4.1}$$

where $\mathrm{d}s^2\left(\mathbb{R}^{10-d,1}\right)$ is the flat metric on $\mathbb{R}^{10-d,1}$ and $\mathrm{d}s^2_d(M)$ is a general metric on $M$. The warp factor $\Delta$ and all the other fields are assumed to be independent of the flat $\mathbb{R}^{10-d,1}$ space. In this sense we restrict the full eleven-dimensional theory to $M$. We will split the eleven-dimensional indices as external indices $\mu = 0, 1, \ldots, \hat{c} - 1$ and internal indices $m = 1, \ldots, d$ where $\hat{c} + d = 11$. The full eleven-dimensional theory and the conventions we are using are summarised in appendix A.

In the restricted theory, the surviving fields include the obvious internal components of the eleven-dimensional fields (namely the metric $g$ and three-form $A$) as well as the warp factor $\Delta$. If $d = 7$, the eleven-dimensional Hodge dual of the 4-form $F$ can have a purely internal 7-form component. This leads one to introduce in addition a dual six-form potential $\tilde{A}$ on $M$ which is related to the seven-form field strength $\tilde{F}$ by

$$\tilde{F} = \mathrm{d}\tilde{A} - \frac{1}{2} A \wedge F. \tag{4.2}$$



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The Bianchi identities satisfied by $F = \mathrm{d}A$ and $\tilde{F}$ are then

$$\begin{aligned} \mathrm{d}F &= 0, \\ \mathrm{d}\tilde{F} + \frac{1}{2}F \wedge F &= 0. \end{aligned} \tag{4.3}$$

With these definitions we see that $F$ and $\tilde{F}$ are related to the eleven dimensional 4-form field strength $\mathcal{F}$ by

$$F_{m_1\dots m_4} = \mathcal{F}_{m_1\dots m_4}, \qquad \tilde{F}_{m_1\dots m_7} = (*_{11}\mathcal{F})_{m_1\dots m_7}. \tag{4.4}$$

One obtains the internal bosonic action

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \sqrt{g}\, \mathrm{e}^{\hat{c}\Delta}\left(\mathcal{R} + \hat{c}(\hat{c}-1)(\partial\Delta)^2 - \frac{1}{2}\frac{1}{4!}F^2 - \frac{1}{2}\frac{1}{7!}\tilde{F}^2\right), \tag{4.5}$$

by requiring that its associated equations of motion

$$\begin{aligned} \mathcal{R}_{mn} - \hat{c}\nabla_m\nabla_n\Delta - \hat{c}(\partial_m\Delta)(\partial_n\Delta) - \frac{1}{2}\frac{1}{4!}\left(4F_{mp_1p_2p_3}F_n{}^{p_1p_2p_3} - \frac{1}{3}g_{mn}F^2\right)& \\ -\frac{1}{2}\frac{1}{7!}\left(7\tilde{F}_{mp_1\dots p_6}\tilde{F}_n{}^{p_1\dots p_6} - \frac{2}{3}g_{mn}\tilde{F}^2\right) &= 0, \\ \mathcal{R} - 2(\hat{c}-1)\nabla^2\Delta - \hat{c}(\hat{c}-1)(\partial\Delta)^2 - \frac{1}{2}\frac{1}{4!}F^2 - \frac{1}{2}\frac{1}{7!}\tilde{F}^2 &= 0, \\ \mathrm{d}*\left(\mathrm{e}^{\hat{c}\Delta}F\right) - \mathrm{e}^{\hat{c}\Delta}\left(*\tilde{F}\right)\wedge F &= 0, \\ \mathrm{d}*\left(\mathrm{e}^{\hat{c}\Delta}\tilde{F}\right) &= 0, \end{aligned} \tag{4.6}$$

are those obtained by substituting the field ansatz into (A.3).

Although here we are interested in the bosonic sector of supergravity, note that the supersymmetry variations of the gravitino can also be written as

$$\begin{aligned} \delta\rho &= \gamma^m\nabla_m\epsilon - \frac{1}{4}\frac{1}{4!}\gamma^{m_1\dots m_4}F_{m_1\dots m_4}\epsilon - \frac{1}{4}\frac{1}{7!}\tilde{F}_{m_1\dots m_7}\gamma^{m_1\dots m_7}\epsilon + \frac{\hat{c}-2}{2}(\gamma^m\partial_m\Delta)\epsilon, \\ \delta\psi_m &= \nabla_m\epsilon + \frac{1}{288}\left(\gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\gamma^{n_2n_3n_4}\right)F_{n_1\dots n_4}\epsilon - \frac{1}{12}\frac{1}{6!}\tilde{F}_{mn_1\dots n_6}\gamma^{n_1\dots n_6}\epsilon, \end{aligned} \tag{4.7}$$

where $\rho$ is related to the trace of the gravitino in the external space and $\gamma^m$ are $\mathrm{Cliff}(d;\mathbb{R})$ gamma matrices. These expressions will be discussed in more detail in [24].

### 4.2 Reformulation as $H_d$ generalised geometry

It is well known [2, 3] that the bosonic fields of the reduced supergravity parametrise a $(E_{d(d)} \times \mathbb{R}^+)/H_d$ coset, that is, at each point $x \in M$,

$$\left\{g, A, \tilde{A}, \Delta\right\} \in \frac{E_{d(d)}}{H_d} \times \mathbb{R}^+. \tag{4.8}$$

Thus giving the bosonic fields is equivalent to specifying a generalised metric $G$. Furthermore, the infinitesimal bosonic symmetry transformation (diffeomorphisms and gauge transformations of $A$ and $\tilde{A}$) are encoded by the Dorfman derivative [70]

$$\delta_V G = L_V G, \tag{4.9}$$

and the algebra of these transformations is given by the Courant bracket.

We now show that the dynamics of the reduced theory are encoded by the torsion-free $H_d$ connection $D$. By doing so we show that the theory can be reformulated geometrically with a local $H_d$ invariance. Such local symmetries were first considered, for $d = 7$, by de Wit and Nicolai [4, 5]. The generalised geometry here can be viewed as a geometrical explanation of their original rewriting. In order to match the dynamics we work in a particular frame, namely the conformal split $H_d$ frame, which is equivalent to an $O(d)$ structure on $E$. It is worth stressing that the generalised geometrical theory is $H_d$ covariant, it is simply that supergravity is conventionally written with only an $O(d) \subset H_d$ manifest.

We have already seen that in the conformal split frame $D$ takes the form (3.15). Viewing sections of $S$ and $J$ in Spin($d$) representations one can then write the unique operators (3.20) in this basis. For example, taking $\epsilon = \mathrm{e}^{\Delta/2}\varepsilon$ to be the supersymmetry parameter, one finds

$$\begin{aligned}
\mathrm{e}^{-\Delta/2}(D \times_J \varepsilon)_a &= \nabla_a \epsilon + \frac{1}{288}\left(\gamma_a{}^{b_1\dots b_4} - 8\delta_a{}^{b_1}\gamma^{b_2b_3b_4}\right) F_{b_1\dots b_4}\epsilon \\
&\quad - \frac{1}{12}\frac{1}{6!}\tilde{F}_{ab_1\dots b_6}\gamma^{b_1\dots b_6}\epsilon, \\
\mathrm{e}^{-\Delta/2}(D \times_S \varepsilon) &= \gamma^m \nabla_m \epsilon - \frac{1}{4}\frac{1}{4!}\gamma^{m_1\dots m_4}F_{m_1\dots m_4}\epsilon \\
&\quad - \frac{1}{4}\frac{1}{7!}\tilde{F}_{m_1\dots m_7}\gamma^{m_1\dots m_7}\epsilon + \frac{\hat{c}-2}{2}(\gamma^m\partial_m\Delta)\epsilon.
\end{aligned} \tag{4.10}$$

These exactly match the operators that appear in the supersymmetry variations (4.7). Given such expressions one can then calculate the Ricci tensor (3.22) in this frame, finding,

$$\begin{aligned}
\mathrm{e}^{-2\Delta}R_{ab} &= \mathcal{R}_{ab} - \hat{c}\nabla_a\nabla_b\Delta - \hat{c}(\partial_a\Delta)(\partial_b\Delta) \\
&\quad - \frac{1}{2}\frac{1}{4!}\left(4F_{ac_1c_2c_3}F_b{}^{c_1c_2c_3} - \frac{1}{3}g_{ab}F^2\right) \\
&\quad - \frac{1}{2}\frac{1}{7!}\left(7\tilde{F}_{ac_1\dots c_6}\tilde{F}_b{}^{c_1\dots c_6} - \frac{2}{3}g_{ab}\tilde{F}^2\right), \\
\mathrm{e}^{-2\Delta}R_{abc} &= \frac{1}{2}\left[\mathrm{e}^{-\hat{c}\Delta} * \mathrm{d} * \mathrm{e}^{\hat{c}\Delta}F - *\big(*\tilde{F}\wedge F\big)\right]_{abc}, \\
\mathrm{e}^{-2\Delta}R_{a_1\dots a_6} &= \frac{1}{2}\left[\mathrm{e}^{-\hat{c}\Delta} * \mathrm{d} * \mathrm{e}^{\hat{c}\Delta}\tilde{F}\right]_{a_1\dots a_6}, \\
\mathrm{e}^{-2\Delta}R &= \mathcal{R} - 2(\hat{c}-1)\nabla^2\Delta - \hat{c}(\hat{c}-1)(\partial\Delta)^2 - \frac{1}{2}\frac{1}{4!}F^2 - \frac{1}{2}\frac{1}{7!}\tilde{F}^2.
\end{aligned} \tag{4.11}$$

Comparing with (4.5) and (4.6) we see that the bosonic action is given by

$$S_{\mathrm{B}} = \int \mathrm{vol}_G\, R, \tag{4.12}$$

where $\mathrm{vol}_G$ is the $E_{d(d)}$-invariant scalar given in (3.6), and that the bosonic equations of motion are equivalent to

$$R_{MN} = 0. \tag{4.13}$$

As advertised, we have rewritten the bosonic dynamics in terms of generalised curvatures with a manifest $H_d$ local symmetry.

### 4.3 Relation to type II supergravity

The $(E_{d(d)} \times \mathbb{R}^+)/H_d$ coset structure can equally well describe the fields of type II theories in $d-1$ dimensions. Specifically

$$\left\{g, B, \tilde{B}, \phi, A^{\pm}, \Delta\right\} \in \frac{E_{d(d)}}{H_d} \times \mathbb{R}^+, \tag{4.14}$$

where $B$ is the NSNS two-form field, $\tilde{B}$ is the six-form potential dual to $B$, $\phi$ is the dilaton and $A^{\pm}$ are the RR potentials (in a democratic formalism) where $A^-$ is a sum of odd-degree forms in type IIA and $A^+$ is a sum of even-degree forms in type IIB. All the fields now depend on a $d-1$ dimensional manifold $M'$.

The construction of torsion-free compatible connections $D$ follows exactly as above. The only difference is that the bundles, partial derivative and split frames are now naturally written in terms of a $\mathrm{GL}(d-1,\mathbb{R})$ subgroup of $E_{d(d)}$ as opposed to $\mathrm{GL}(d,\mathbb{R})$ (the appropriate subgroups are defined in appendix C.4.) In particular, the generalised tangent space takes the form [21, 52–54]

$$E \simeq TM' \oplus T^*M' \oplus \Lambda^5 T^*M' \oplus \left(T^*M' \otimes \Lambda^6 T^*M'\right) \oplus \Lambda^{\text{even/odd}} T^*M', \tag{4.15}$$

where "even" refers to type IIA and "odd" to IIB. The partial derivative $\partial$ now acts via the embedding $T^*M' \to E^*$ so that $\partial f = \mathrm{d}f \in \Gamma(E^*)$, for any function $f$. One still has the "section condition" $\partial f \times_{N^*} \partial g = 0$ but now the space spanned by $\partial f$ is not the maximal such subspace in $E^*$ (since $\partial_M$ spans one less dimension).

We will not give the expressions for the type II decompositions here, though given the $\mathrm{Spin}(d-1)$ spinor decomposition in appendix C.4, it is relatively straightforward to calculate them directly from $\mathrm{Spin}(d)$ expressions given in the previous section. The central point is that the bosonic equations of motion and action given by (4.12) and (4.13) are left unchanged. What changes is the decomposition of these expressions in the bosonic fields, and the partial derivative action $\partial$.

### 4.4 Identity structures, fluxes and relation to the embedding tensor

The embedding tensor is the object that determines general gaugings of (typically maximally) supersymmetric theories in $11-d$ dimensions [55–57]. It is striking that it transforms in the same $E_{d(d)}$ representations as the generalised torsion (2.35), namely $K \oplus E^*$. That such representations appear in gauged supersymmetric theories has been discussed in detail in [71–76] in the context of $E_{11}$ theory (as well as [77] in the case of $E_{10}$). Here we simply show why, in the current context, the generalised torsion and the embedding tensor are related when the gauged supergravity arises from a dimensional reduction of eleven-dimensional supergravity on a $d$-dimensional manifold $M$. This also relates to the observation that the generic set of fluxes, both geometrical and non-geometrical, are sections of the same bundle $K$ [52].

To make the connection we first need to identify what structures on the internal space $M$ lead to maximally supersymmetric theories in $11-d$ dimensions. Metrically the eleven-dimensional space is a fibration

$$\mathrm{d}s^2 = \hat{g}_{\mu\nu}(y)\mathrm{d}y^\mu \mathrm{d}y^\nu + g_{mn}(x,y)(\mathrm{d}x^m + A^m(x,y))(\mathrm{d}x^n + A^n(x,y)), \tag{4.16}$$

where $y$ and $x$ are coordinates on the external and internal spaces respectively, and $A^m$ are one-forms on the external space. It is well known that tori give suitable supersymmetric compactifications, as do generic twisted tori (or local group manifolds) [78–82], including their non-geometrical extensions [47, 48, 70, 83–87]. The corresponding relation to the embedding tensor is also well established (for a review see [88]). The characteristic feature of these backgrounds is that the associated generalised tangent space $E$ admits an "identity structure", that is, a $G$-structure $P \subset \tilde{F}$ where $G$ is just the trivial group, with a single element, the identity. This means that the space admits a single *globally defined* frame $\{\hat{E}^A\}$ and is a necessary condition for a reduction to a maximally supersymmetric effective theory. As discussed in the context of $N = 2$ supersymmetry in [53, 89–91], such reductions require globally defined spinors on $M$. For a maximally symmetric theory, there is a maximal number of such spinors and hence the $\tilde{H}_d$-bundle is trivial, implying we have an identity structure

$$\begin{Bmatrix}\text{maximal supersymmetric}\\ \text{effective theory}\end{Bmatrix} \Longleftrightarrow \begin{Bmatrix}\text{exists globally defined}\\ \text{frame on } M\end{Bmatrix}. \tag{4.17}$$

Such structures are also sometimes referred to as "generalised parallelizable" spaces [70]. They are the generalised analogues of parallelizable spaces, where there is a globally defined frame $\{\hat{e}^a\}$. Note that twisted tori give examples of such generalised parallelizable spaces, but in principle one could also have a generalised parallelization of $E$ without a parallelization of $TM$.

In making the connection to the embedding tensor let us focus on the scalar moduli fields, which parametrise the coset $E_{d(d)}/H_d$. Recall that given a conventional global frame $e^a(x)$ (for example left-invariant one-forms on a local group manifold) one can write a family of frames $e'^a(x, y) = m^a{}_b(y)e^b(x)$ and make a Scherk-Schwarz reduction [92, 93]. The corresponding metrics are given by

$$g' = \delta_{ab}\, e'^a(x, y) \otimes e'^b(x, y) = h_{ab}(y)\, e^a(x) \otimes e^b(x), \tag{4.18}$$

where $h_{ab}(y) = \delta_{cd}m^c{}_a(y)m^b{}_d(y)$ are moduli parametrising $\mathrm{GL}(d, \mathbb{R})/O(d)$. Now suppose we have a generalised parallelization $\hat{E}_A(x)$ and a dual basis $E^A(x)$. The scalar fields in the effective theory similarly can be regarded as parametrising generic $E_{d(d)}$ transformations $E'^A(x, y) = M^A{}_B(y)E^B(x)$, defining the generalised metric,

$$G' = \delta_{AB}\, E'^A(x, y) \otimes E'^B(x, y) = H_{AB}(y)\, E^A(x) \otimes E^B(x), \tag{4.19}$$

where $H_{AB}(y) = \delta_{CD}M^C{}_A(y)M^B{}_D(y)$ are moduli parametrising an $E_{d(d)}/H_d$ coset. Note that we ignore the $\mathbb{R}^+$ degree of freedom that rescales $G$. Since this factor was associated with warping of the external space, which in the dimensionally reduced theory is encoded in the conformal rescaling of the external metric $\hat{g}$, this does not lose any degrees of freedom.

The potential for the scalar moduli arises from the dimensional reduction of the action on the internal space, which, as we have seen, can be written in terms of the generalised Ricci scalar as in (4.12). This in turn is completely determined by the torsion-free connection $G'$-compatible connection $D'$. One can construct $D'$ as follows. Given the transformed



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frame $\{\hat{E}'_A\}$ we can always define a connection $D''$ that is compatible with the corresponding identity structure, that is, for all $A$,

$$D''\hat{E}'_A = 0, \tag{4.20}$$

but in general it will be torsionful. By definition the torsion is simply given by the algebra of the basis $\{\hat{E}'_A\}$ under the Dorfman derivative, namely,

$$L_{\hat{E}'_A}\hat{E}'_C = -T'_A{}^B{}_C\hat{E}'_B, \tag{4.21}$$

where $T' \in \Gamma(K \oplus E^*)$ and is moduli dependent. By construction $D''$ is compatible with the generalised metric (4.19). The torsion-free metric compatible connection can then be constructed as

$$D' = D'' + \Sigma', \tag{4.22}$$

where, in the notation of section 3.2, $\Sigma' \in \Gamma(E^* \otimes \operatorname{ad} P)$ and for $D'$ to be torsion-free we require

$$\Sigma' = -T' + Q', \tag{4.23}$$

where the ambiguous part $Q' \in \Gamma(U)$. Since the supergravity does not depend on the ambiguous part $Q'$ we see that effective theory, and in particularly the scalar potential, is determined completely by the moduli-dependent $T'$ defined in (4.21).

We could also consider the corresponding tensor $T$ for the fixed frame $\{\hat{E}_A\}$. This is independent of the moduli, and is related to $T'$, the corresponding "dressed" version, simply by transforming indices with $M$ or $M^{-1}$ as appropriate. The undressed $T$ can be directly identified with the embedding tensor if we make the further assumption that its components $T_A{}^B{}_C$ are constant. First we note that it is in the same representations of $E_{d(d)}$ as the embedding tensor. Second it satisfies the same quadratic relation [55, 56], giving the embedding of the gauged symmetry group in $E_{d(d)}$. Here this condition arises from the Jacobi-like relation, following from the fact that $L_U$ satisfies the Leibniz identity,

$$L_{\mathrm{U}}(L_V W) - L_V(L_U W) = L_{L_U V}W. \tag{4.24}$$

Taking $U = \hat{E}^A$, $V = \hat{E}^B$ and $W = \hat{E}^C$ this gives

$$[T_A, T_B] = T_A{}^C{}_B T_C, \tag{4.25}$$

where we view $(T_A)^B{}_C = T_A{}^B{}_C$ as a set of elements in the adjoint representation of $E_{d(d)}$ labelled by $A$.

We can then make the connection to [52], where it was shown that generic fluxes in string compactifications correlate with the embedding tensor. The definition (4.21) gives $T$ a direct geometrical interpretation which matches the fluxes identified in [52] in the context of type IIB backgrounds. Suppose, for example we have a twisted torus (that is a local group manifold) with a global frame $\hat{e}_a$. We can then define a generalised parallelization by taking $\{\hat{E}^A\}$ in the split form (2.15) (that is, with $\Delta = 0$). Let $\nabla$ be the conventional connection that satisfies $\nabla\hat{e}_a = 0$ and has torsion $T^a{}_{bc} = -f_{bc}{}^a$ where $f$ are the structure constants given by $[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c$. By definition we then have that $D^\nabla \hat{E}_A = 0$. Using (2.38), we can calculate the components of $T$ of the torsion of $D^\nabla$ and find

$$T(V) = -i_v f - f \cdot \omega - i_v F - f \cdot \sigma - i_v \tilde{F} + \omega \wedge F. \tag{4.26}$$

Thus only certain components of $T$, the so-called geometrical fluxes, are non-zero. The corresponding split frame for type IIB generates the geometrical fluxes identified in [52].

## 5 Conclusions and discussion

We have seen that the action, equations of motion and symmetries for the bosonic fields in reductions of M theory to $d$ dimensions actually have a remarkably simple and unified form. The fields unify as a generalised metric. The symmetries are simply the generalised geometry extensions of diffeomorphisms, and the action is simply the analogue of the Ricci scalar. The formalism works for all dimensions $d \leq 7$ and the theory has an extended local $H_d \supset O(d)$ symmetry. It is a direct extension of our earlier work [23] on reformulating type II supergravities using $O(10,10) \times \mathbb{R}^+$ generalised geometry.

It is natural to ask how this structure might extend to higher dimensions. The basic obstruction, even for $d = 8$, is that although the generalised tangent space exists, together with an $E_{8(8)} \times \mathbb{R}^+$ structure bundle, and a Dorfman derivative, one cannot write the derivative in the form (2.28), since this gives a non-covariant expression. Consequently, one does not have a natural way to define the generalised torsion. The problem with writing the derivative in this form is the presence of the $\tau \in \Gamma\left(T^*M \otimes \Lambda^7 T^*M\right)$ tensors in $E$. Physically these correspond to Kaluza-Klein monopole charges in the U-duality algebra and should be associated to the symmetries of "dual gravitons". Note that these terms already meant, even in $d = 7$, that we could not write the difference of between the Dorfman derivative and the bracket (2.39) an the $E_{d(d)} \times \mathbb{R}^+$ covariant form. One possible way out is to include dependence on the "non-compact" $(11-d)$-dimensional space $M^{10-d,1}_{\text{ext}}$. Allowing for diffeomorphisms in $M^{10-d,1}_{\text{ext}}$ may then correct the non-covariance of the naive structure.

The possibility of extending the formalism to the Kac-Moody algebras $E_{10}$ or $E_{11}$ is particularly intriguing. Since we assume an underlying manifold, the connection to the $E_{10}$ formalism of [11] is less direct, since there the spacetime is emergent, the $E_{10}$ fields encoding a spatial gradient expansion around a spacetime point. The $E_{11}$ formalism on the other hand starts with (an infinite number) of coordinates [32] transforming in a particular representation $l_1$ (which corresponds to the generalised tangent space representation upon reduction to $E_{d(d)}$). One important question is how the dependence on these coordinates might be truncated to define eleven-dimensional supergravity. The results here would suggest one imposes a quadratic section condition (2.42) projecting onto the corresponding $N^*$ representation defined by the appropriate node in the $e_{11}$ Dynkin diagram as described in section 2.1.3. Another very interesting possibility is that, if a generalised geometrical formulation can be found for $d > 8$ it may be that the existence of the torsion-free compatible connection $D$ actually constrains the $H_d$-structure. This is what happens for instance with conventional connections compatible with an almost complex structure, where the existence of a torsion-free compatible connections requires the structure to be integrable. Such a situation would impose differential conditions on the fields defining the coset $(E_{d(d)} \times \mathbb{R}^+)/H_d$, perhaps truncating the infinite set to a finite number of independent components corresponding to the degrees of freedom of supergravity. This may be connected to the recent result [94] that, given fairly weak assumptions, only a finite number of the fields in the Kac-Moody algebra are propagating.

As we have already stressed, the formalism used here and in [23] is locally equivalent to the standard formulation of double field theory and its M theory variants. The derivations relied only on the partial derivative satisfying the section condition (2.42) (or the corresponding condition for $O(d,d)$). The maximal subspace of $E^*$ satisfying this condition is $d$-dimensional and is stabilised by the parabolic subgroup $G_{\text{split}}$. In the context of double field theory it defines the set of coordinates on which the fields depend, and is a necessary condition for the formulation of an action and a closed symmetry algebra. This defines a foliation and reducing along the isometries, the theory is locally defined on a $d$-dimensional manifold as in generalised geometry. In either formulation there is a global $O(d,d)$ or $E_{d(d)}$ symmetry acting on the frame bundle. However, while the strong constraint is covariant, the particular choice of a maximal subspace is not invariant under the global symmetry group.

There are number of other natural extensions to the geometry described here. It would be interesting to understand if similar constructions can be used for other supergravity theories. One might also wonder if the formalism can be used to describe higher-derivative corrections and their $E_{d(d)}$ transformation properties. A more direct, key application is the description of supersymmetric backgrounds. Formulations of $N = 1$ and $N = 2$ backgrounds in $E_{7(7)}$ language have already been given in [54]. In the current context one expects that generic supersymmetric backgrounds in $d \leq 7$ should correspond to special holonomy $G \subset \tilde{H}_d$ for the operator $D$. Note that it is the holonomy of $D$ and not its projections (4.10) that are relevant, and hence $G$ is a subgroup of $\tilde{H}_d$. This is in contrast to [95, 96] where the holonomy of the operators appearing directly in the supersymmetry variations was considered, and larger groups can appear. The most obvious extension, though, is the completion of the description of the supergravity by including the fermionic degrees of freedom and supersymmetry transformations, at least to leading order. This will be the main result of [24].

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## A Eleven-dimensional supergravity

Let us start by summarising the action, equations of motion and supersymmetry variations of eleven-dimensional supergravity, to leading order in the fermions, following the conventions of [97].

The fields are simply

$$\{g_{\mu\nu}, \mathcal{A}_{\mu\nu\rho}, \psi_\mu\}, \tag{A.1}$$

where $g_{\mu\nu}$ is the metric, $\mathcal{A}_{\mu\nu\rho}$ the three-form potential and $\psi_\mu$ is the gravitino. The bosonic action is given by

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int\left(\mathrm{vol}_g\,\mathcal{R} - \frac{1}{2}\mathcal{F}\wedge *\mathcal{F} - \frac{1}{6}\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}\right), \tag{A.2}$$

where $\mathcal{R}$ is the Ricci scalar and $\mathcal{F} = \mathrm{d}\mathcal{A}$. This leads to the equations of motion

$$\begin{aligned}\mathcal{R}_{\mu\nu} - \frac{1}{12}\left(\mathcal{F}_{\mu\rho_1\rho_2\rho_3}\mathcal{F}_\nu{}^{\rho_1\rho_2\rho_3} - \frac{1}{12}g_{\mu\nu}\mathcal{F}^2\right) &= 0,\\ \mathrm{d}*\mathcal{F} + \frac{1}{2}\mathcal{F}\wedge\mathcal{F} &= 0,\end{aligned} \tag{A.3}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor. Note that we are using the notation $K^2 = K_{\mu_1\dots\mu_k}K^{\mu_1\dots\mu_k}$ for a rank $k$ tensor $K$.

The supersymmetry variation of the gravitino is

$$\delta\psi_\mu = \nabla_\mu\epsilon + \frac{1}{288}\left(\Gamma_\mu{}^{\nu_1\dots\nu_4} - 8\delta_\mu{}^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4}\right)\mathcal{F}_{\nu_1\dots\nu_4}\epsilon, \tag{A.4}$$

where $\Gamma^\mu$ are the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma matrices and $\epsilon$ is the supersymmetry parameter.

## B Conventions in Euclidean signature

We use the indices $m, n, p, \dots$ as the coordinate indices and $a, b, c\dots$ for frame indices. We take symmetrisation of indices with weight one. Given a polyvector $w \in \Lambda^p TM$ and a form $\lambda \in \Lambda^q T^*M$, we write in components

$$\begin{aligned}w &= \frac{1}{p!}w^{m_1\dots m_p}\frac{\partial}{\partial x^{m_1}}\wedge\dots\wedge\frac{\partial}{\partial x^{m_p}},\\ \lambda &= \frac{1}{q!}\lambda_{m_1\dots m_q}\mathrm{d}x^{m_1}\wedge\dots\wedge\mathrm{d}x^{m_q},\end{aligned} \tag{B.1}$$

so that wedge products and contractions are given by

$$\begin{aligned}\left(w\wedge w'\right)^{m_1\dots m_{p+p'}} &= \frac{(p+p')!}{p!p'!}w^{[m_1\dots m_p}w'^{m_{p+1}\dots m_{p+p'}]},\\ \left(\lambda\wedge\lambda'\right)_{m_1\dots m_{q+q'}} &= \frac{(q+q')!}{q!q'!}\lambda_{[m_1\dots m_q}\lambda'_{m_{q+1}\dots m_{q+q'}]},\\ \left(w\lrcorner\lambda\right)_{m_1\dots m_{q-p}} &:= \frac{1}{p!}w^{n_1\dots n_p}\lambda_{n_1\dots n_p m_1\dots m_{q-p}} && \text{if } p\le q,\\ \left(w\lrcorner\lambda\right)^{m_1\dots m_{p-q}} &:= \frac{1}{q!}w^{m_1\dots m_{p-q}n_1\dots n_q}\lambda_{n_1\dots n_q} && \text{if } p\ge q.\end{aligned} \tag{B.2}$$

Given the tensors $t \in TM \otimes \Lambda^7 TM$, $\tau \in T^*M \otimes \Lambda^7 T^*M$ and $a \in TM \otimes T^*M$ with components

$$
\begin{aligned}
t &= \frac{1}{7!} w^{m,m_1\dots m_7} \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^{m_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{m_7}}, \\
\tau &= \frac{1}{7!} \tau_{m,m_1\dots m_7} \mathrm{d}x^m \otimes \mathrm{d}x^{m_1} \wedge \dots \wedge \mathrm{d}x^{m_q}, \\
a &= a^m{}_n \frac{\partial}{\partial x^m} \otimes \mathrm{d}x^n,
\end{aligned}
\tag{B.3}
$$

and also a form $\sigma \in \Lambda^5 T^*M$ and a vector $v \in TM$, we use the "$j$-notation" from [22], defining

$$
\begin{aligned}
(w \lrcorner \tau)_{m_1\dots m_{8-p}} &:= \frac{1}{(p-1)!} w^{n_1\dots n_p} \tau_{n_1,n_2\dots n_p m_1\dots m_{8-p}}, \\
(t \lrcorner \lambda)^{m_1\dots m_{8-q}} &:= \frac{1}{(q-1)!} t^{n_1,n_2\dots n_q m_1\dots m_{8-q}} \lambda_{n_1\dots n_q}, \\
(t \lrcorner \tau) &:= \frac{1}{7!} t^{m,n_1\dots n_7} \tau_{m,n_1\dots n_7}, \\
\left(jw \wedge w'\right)^{m,m_1\dots m_7} &:= \frac{7!}{(p-1)!(8-p)!} w^{m[m_1\dots m_{p-1}} w'^{m_p\dots m_7]}, \\
\left(j\lambda \wedge \lambda'\right)_{m,m_1\dots m_7} &:= \frac{7!}{(q-1)!(8-q)!} \lambda_{m[m_1\dots m_{q-1}} \lambda'_{m_q\dots m_7]}, \\
(jw \lrcorner j\lambda)^m{}_n &:= \frac{1}{(p-1)!} w^{mn_1\dots n_{p-1}} \lambda_{nn_1\dots n_{p-1}}, \\
(jt \lrcorner j\tau)^m{}_n &:= \frac{1}{7!} t^{m,n_1\dots n_7} \tau_{n,n_1\dots n_7}, \\
\left(j^{p+1}\lambda \wedge \tau\right)_{m_1\dots m_{p+1},n_1\dots n_7} &:= (p+1) \lambda_{[m_1\dots} \tau_{m_{p+1}],n_1\dots n_7}, \\
\left(j^3\sigma \wedge \sigma'\right)_{m_1\dots m_3,n_1\dots n_7} &:= \frac{7!}{5!\cdot 2!} \sigma_{m_1\dots m_3[n_1n_2} \sigma'_{\dots n_7]}, \\
(v \lrcorner j\tau)_{mn_1\dots n_6} &:= v^n \tau_{m,nn_1\dots n_6}.
\end{aligned}
\tag{B.4}
$$

The $d$-dimensional metric $g$ is always positive definite. We define the orientation, $\epsilon_{1\dots d} = \epsilon^{1\dots d} = +1$, and use the conventions

$$
\begin{aligned}
*\lambda_{m_1\dots m_{d-q}} &= \frac{1}{q!} \sqrt{|g|} \epsilon_{m_1\dots m_{d-q} n_1\dots n_q} \lambda^{n_1\dots n_q}, \\
\lambda^2 &= \lambda_{m_1\dots m_q} \lambda^{m_1\dots m_q}.
\end{aligned}
\tag{B.5}
$$

# C $\quad E_{d(d)} \times \mathbb{R}^+$ and $H_d$

## C.1 Construction of $E_{d(d)} \times \mathbb{R}^+$ from $\mathrm{GL}(d, \mathbb{R})$

In this section we give an explicit construction of $E_{d(d)} \times \mathbb{R}^+$ for $d \le 7$ based on the $\mathrm{GL}(d, \mathbb{R})$ subgroup. If $\mathrm{GL}(d, \mathbb{R})$ acts linearly on the $d$-dimensional vector space $F$, we define

$$
\begin{aligned}
W_1 &= F \oplus \Lambda^2 F^* \oplus \Lambda^5 F^* \oplus \left(F^* \otimes \Lambda^7 F^*\right), \\
W_1^* &= F^* \oplus \Lambda^2 F \oplus \Lambda^5 F \oplus \left(F \otimes \Lambda^7 F\right), \\
W_{\mathrm{ad}} &= \mathbb{R} \oplus (F \otimes F^*) \oplus \Lambda^3 F^* \oplus \Lambda^6 F^* \oplus \Lambda^3 F \oplus \Lambda^6 F.
\end{aligned}
\tag{C.1}
$$



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The corresponding $E_{d(d)} \times \mathbb{R}^+$ representations are listed in table 1. We write elements as

$$\begin{aligned} V &= v + \omega + \sigma + \tau && \in W_1, \\ Z &= \zeta + u + s + t && \in W_1^*, \\ R &= c + r + a + \tilde{a} + \alpha + \tilde{\alpha} && \in W_{\text{ad}}, \end{aligned} \tag{C.2}$$

so that $v \in F$, $\omega \in \Lambda^2 F^*$, $\zeta \in F^*$, $c \in \mathbb{R}$ etc. If $\{\hat{e}_a\}$ is a basis for $F$ with a dual basis $\{e^a\}$ on $F^*$ then there is a natural $\text{gl}(d,\mathbb{R})$ action on each tensor component. For instance

$$(r \cdot v)^a = r^a{}_b v^b, \qquad (r \cdot \omega)_{ab} = -r^c{}_a \omega_{cb} - r^c{}_b \omega_{ac}, \qquad \text{etc.} \tag{C.3}$$

Writing $V' = R \cdot V$ for the adjoint $E_{d(d)} \times \mathbb{R}^+$ action of $R \in W_{\text{ad}}$ on $V \in F$, the components of $V'$, using the notation of appendix B, are given by

$$\begin{aligned} v' &= cv + r \cdot v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma, \\ \omega' &= c\omega + r \cdot \omega + v \lrcorner a + \alpha \lrcorner \sigma + \tilde{\alpha} \lrcorner \tau, \\ \sigma' &= c\sigma + r \cdot \sigma + v \lrcorner \tilde{a} + a \wedge \omega + \alpha \lrcorner \tau, \\ \tau' &= c\tau + r \cdot \tau - j\tilde{a} \wedge \omega + ja \wedge \sigma. \end{aligned} \tag{C.4}$$

Note that, the $E_{d(d)}$ sub-algebra is generated by setting $c = \frac{1}{(9-d)} r^a{}_a$. Similarly, given $Z \in W_1^*$ we have

$$\begin{aligned} \zeta' &= -c\zeta + r \cdot \zeta - u \lrcorner a + s \lrcorner \tilde{a}, \\ u' &= -cu + r \cdot u - \alpha \lrcorner \zeta - s \lrcorner a + t \lrcorner \tilde{a}, \\ s' &= -cs + r \cdot s - \tilde{\alpha} \lrcorner \zeta - \alpha \wedge u - t \lrcorner a, \\ t' &= -ct + r \cdot t - j\alpha \wedge s - j\tilde{\alpha} \wedge u. \end{aligned} \tag{C.5}$$

Finally the adjoint commutator

$$R'' = [R, R'] \tag{C.6}$$

has components

$$\begin{aligned} c'' &= \frac{1}{3}\left(\alpha \lrcorner a' - \alpha' \lrcorner a\right) + \frac{2}{3}\left(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'\right), \\ r'' &= [r, r'] + j\alpha \lrcorner ja' - j\alpha' \lrcorner ja - \frac{1}{3}\left(\alpha \lrcorner a' - \alpha' \lrcorner a\right) \mathbb{1} \\ &\qquad + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \frac{2}{3}\left(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'\right) \mathbb{1}, \\ a'' &= r \cdot a' - r' \cdot a + \alpha' \lrcorner \tilde{a} - \alpha \lrcorner \tilde{a}', \\ \tilde{a}'' &= r \cdot \tilde{a}' - r' \cdot \tilde{a} - a \wedge a', \\ \alpha'' &= r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a', \\ \tilde{\alpha}'' &= r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha'. \end{aligned} \tag{C.7}$$

Here we have $c'' = \frac{1}{9-d} r''^a{}_a$, as $R''$ lies in the $E_{d(d)}$ sub-algebra.

The $E_{d(d)} \times \mathbb{R}^+$ Lie group can then be constructed starting with $\text{GL}(d,\mathbb{R})$ and using the exponentiated action of $a$, $\tilde{a}$, $\alpha$ and $\tilde{\alpha}$. The $\text{GL}(d,\mathbb{R})$ action by an element $m$ is standard so

$$(m \cdot v)^a = m^a{}_b v^b, \qquad (m \cdot \omega)_{ab} = \left(m^{-1}\right)^c{}_a \left(m^{-1}\right)^d{}_b \omega_{cd}, \qquad \text{etc.} \tag{C.8}$$

The action of $a$ and $\tilde{a}$ form a nilpotent subgroup of nilpotency class two. One has

$$\begin{aligned}
e^{a+\tilde{a}}V = v &+ (\omega + i_v a) \\
&+ \left(\sigma + a\wedge\omega + \frac{1}{2}a\wedge i_v a + i_v\tilde{a}\right) \\
&+ \left(\tau + ja\wedge\sigma - j\tilde{a}\wedge\omega + \frac{1}{2}ja\wedge a\wedge\omega\right. \\
&\qquad \left. + \frac{1}{2}ja\wedge i_v\tilde{a} - \frac{1}{2}j\tilde{a}\wedge i_v a + \frac{1}{6}ja\wedge a\wedge i_v a\right),
\end{aligned} \tag{C.9}$$

with no terms higher than cubic in the expansion. The action of $\alpha$ and $\tilde{\alpha}$ form a similar nilpotent subgroup of nilpotency class two with

$$\begin{aligned}
e^{\alpha+\tilde{\alpha}}V = &\left(v + \alpha\lrcorner\omega - \tilde{\alpha}\lrcorner\sigma + \frac{1}{2}\alpha\lrcorner\alpha\lrcorner\sigma\right. \\
&\qquad \left. + \frac{1}{2}\alpha\lrcorner\tilde{\alpha}\lrcorner\tau + \frac{1}{2}\tilde{\alpha}\lrcorner\alpha\lrcorner\tau + \frac{1}{6}\alpha\lrcorner\alpha\lrcorner\alpha\lrcorner\tau\right) \\
&+ (\omega + \alpha\lrcorner\sigma + \tilde{\alpha}\lrcorner\tau + \alpha\lrcorner\alpha\lrcorner\sigma) \\
&+ (\sigma + \alpha\lrcorner\tau) + \tau.
\end{aligned} \tag{C.10}$$

A general element of $E_{d(d)} \times \mathbb{R}^+$ then has the form

$$M\cdot V = e^{\lambda}\, e^{\alpha+\tilde{\alpha}}\, e^{a+\tilde{a}}\, m\cdot V, \tag{C.11}$$

where $e^{\lambda}$ with $\lambda\in\mathbb{R}$ is included to give a general $\mathbb{R}^+$ scaling.

### C.2 Some tensor products

We also define two tensor products. We have the map into the adjoint

$$\times_{\mathrm{ad}} : W_1^* \otimes W_1 \to W_{\mathrm{ad}}. \tag{C.12}$$

Writing $R = Z\times_{\mathrm{ad}} V$ we have

$$\begin{aligned}
c &= -\frac{1}{3}u\lrcorner\omega - \frac{2}{3}s\lrcorner\sigma - t\lrcorner\tau, \\
r &= v\otimes\zeta - ju\lrcorner j\omega + \frac{1}{3}(u\lrcorner\omega)\mathbb{1} - js\lrcorner j\sigma + \frac{2}{3}(s\lrcorner\sigma)\mathbb{1} - jt\lrcorner j\tau, \\
\alpha &= v\wedge u + s\lrcorner\omega + t\lrcorner\sigma, \\
\tilde{\alpha} &= -v\wedge s - t\lrcorner\omega, \\
a &= \zeta\wedge\omega + u\lrcorner\sigma + s\lrcorner\tau, \\
\tilde{a} &= \zeta\wedge\sigma + u\lrcorner\tau.
\end{aligned} \tag{C.13}$$

We can also consider the space $W_2$ as given in table 2. Taking

$$\begin{aligned}
W_2 &= F^*\oplus\Lambda^4F^*\oplus\left(F^*\otimes\Lambda^6F^*\right)\oplus\left(\Lambda^3F^*\otimes\Lambda^7F^*\right)\oplus\left(\Lambda^6F^*\otimes\Lambda^7F^*\right), \\
Y &= \lambda + \kappa + \mu + \nu + \pi,
\end{aligned} \tag{C.14}$$

we have that the symmetric map $W_1 \otimes W_1 \to W_2$ is

$$
\begin{aligned}
\lambda &= v \lrcorner \omega' + v' \lrcorner \omega, \\
\kappa &= v \lrcorner \sigma' + v' \lrcorner \sigma - \omega \wedge \omega', \\
\mu &= \left(j\omega \wedge \sigma' + j\omega' \wedge \sigma\right) - \frac{1}{4}\left(\sigma \wedge \omega' + \sigma' \wedge \omega\right) \\
&\quad + \left(v \lrcorner j\tau\right) + \left(v \lrcorner j\tau'\right) - \frac{1}{4}\left(v \lrcorner \tau' + v' \lrcorner \tau\right), \\
\nu &= j^3\omega \wedge \tau' + j^3\omega' \wedge \tau - j^3\sigma \wedge \sigma', \\
\pi &= j^6\sigma \wedge \tau' + j^6\sigma' \wedge \tau.
\end{aligned}
\tag{C.15}
$$

### C.3 $H_d$ and $O(d)$

Given a positive definite metric $g$ on $F$, which for convenience we take to be in standard form $\delta_{ab}$ in frame indices, we can define a metric on $W_1$ by

$$G(V,V) = v^2 + \frac{1}{2!}\omega^2 + \frac{1}{5!}\sigma^2 + \frac{1}{7!}\tau^2, \tag{C.16}$$

where $v^2 = v_a v^a$, $\omega^2 = \omega_{ab}\omega^{ab}$, etc as in (B.5). Note that this metric allows us to identify $W_1 \simeq W_1^*$.

The subgroup of $E_{d(d)} \times \mathbb{R}^+$ that leaves the metric is invariant is $H_d$, the maximal compact subgroup of $E_{d(d)}$ (see table 3). The corresponding Lie algebra is parametrised by

$$N = n + b + \tilde{b} \in \Lambda^2 F^* \oplus \Lambda^3 F^* \oplus \Lambda^6 F^*, \tag{C.17}$$

and embeds in $W_{\mathrm{ad}}$ as

$$
\begin{aligned}
c &= 0, \\
r_{ab} &= n_{ab}, \\
a_{abc} &= -\alpha_{abc} = b_{abc}, \\
\tilde{a}_{a_1\dots a_6} &= \tilde{\alpha}_{a_1\dots a_6} = \tilde{b}_{a_1\dots a_6},
\end{aligned}
\tag{C.18}
$$

where indices are lowered with the metric $g$. Note that $n_{ab}$ generates the $O(d) \subset \mathrm{GL}(d,\mathbb{R})$ subgroup that preserves $g$. Concretely a general group element can be written as

$$H \cdot V = \mathrm{e}^{\alpha+\tilde{\alpha}}\, \mathrm{e}^{a+\tilde{a}}\, h \cdot V, \tag{C.19}$$

where $h \in O(d)$ and $a$ and $\alpha$ and $\tilde{a}$ and $\tilde{\alpha}$ are related as in (C.18).

Finally we note that the double cover $\tilde{H}_d$ of $H_d$ has a realisation in terms of the Clifford algebra $\mathrm{Cliff}(d;\mathbb{R})$. Consider the gamma matrices $\gamma^a$ satisfying $\{\gamma^a, \gamma^b\} = 2g^{ab}$. The $H_d$ Lie algebra can be realised on $\mathrm{Cliff}(d;\mathbb{R})$ spinors by taking

$$N = \frac{1}{2}\left(\frac{1}{2!}n_{ab}\gamma^{ab} - \frac{1}{3!}b_{abc}\gamma^{abc} - \frac{1}{6!}\tilde{b}_{a_1\dots a_6}\gamma^{a_1\dots a_6}\right). \tag{C.20}$$

Again $n_{ab}$ generates the $\mathrm{Spin}(d)$ subgroup of $\tilde{H}_d$.

## C.4 Type II GL$(d-1,\mathbb{R})$ and $O(d-1)$ subgroups of $E_{d(d)} \times \mathbb{R}^+$

We can identify two distinct GL$(d-1,\mathbb{R})$ subgroups of $E_{d(d)}$ appropriate to type IIA and type IIB.

For type IIA, GL$(d-1,\mathbb{R})$ is a subgroup of the GL$(d,\mathbb{R})$ group used to define the $E_{d(d)} \times \mathbb{R}^+$ group in section C.1. We simply decompose the $d$-dimensional space as

$$F \simeq L \oplus \mathbb{R}, \tag{C.21}$$

with a GL$(d-1,\mathbb{R})$ action on $L$. Concretely, if we write the GL$(d,\mathbb{R})$ index $a=(1,i)$ then the GL$(d-1,\mathbb{R})$ Lie algebra $p \in L \otimes L^*$ embeds as

$$r^i{}_j = p^i{}_j. \tag{C.22}$$

Under this decomposition one has

$$\begin{aligned} W_1 &= L \oplus L^* \oplus \Lambda^5 L^* \oplus \left(L^* \otimes \Lambda^6 L^*\right) \oplus \Lambda^{\text{even}} L^*, \\ W_{\text{ad}} &= \mathbb{R} \oplus \mathbb{R} \oplus (L \otimes L^*) \oplus \Lambda^2 L^* \oplus \Lambda^2 L^* \\ &\quad \oplus \Lambda^6 L^* \oplus \Lambda^6 L^* \oplus \Lambda^{\text{odd}} L^* \oplus \Lambda^{\text{odd}} L^*. \end{aligned} \tag{C.23}$$

For type IIB the embedding is slightly more complicated. We decompose GL$(d,\mathbb{R})$ under a GL$(d-2,\mathbb{R}) \times$ SL$(2,\mathbb{R})$ subgroup, i.e. we decompose $F$ as a $d-2$-dimensional space $A$ and 2-dimensional space $B$. We then identify

$$F \simeq A \oplus B, \qquad \hat{L} = A \oplus \Lambda^2 B^*, \tag{C.24}$$

where the GL$(d-1,\mathbb{R})$ action acts on $\hat{L}$ and under SL$(2,\mathbb{R})$ we have $\Lambda^2 B^* \simeq \mathbb{R}$ (this is needed for $\hat{L}$ to form a representation of GL$(d-1,\mathbb{R})$). Writing indices $a=(1,2,\hat{\imath})$, the GL$(d-1,\mathbb{R})$ Lie algebra element $\hat{p} \in \hat{L} \otimes \hat{L}^*$ embeds as

$$r^{\hat{\imath}}{}_{\hat{\jmath}} = \hat{p}^{\hat{\imath}}{}_{\hat{\jmath}}, \qquad \alpha^{\hat{\imath}12} = \hat{p}^{\hat{\imath}}{}_1, \qquad a_{\hat{\imath}12} = \hat{p}^1{}_{\hat{\imath}}, \qquad r^1{}_1 = r^2{}_2 = -\frac{1}{2}\hat{p}^1{}_1. \tag{C.25}$$

Decomposing under the GL$(d-2,\mathbb{R}) \times$ SL$(2,\mathbb{R})$ subgroup and then recombining the terms into GL$(d-1,\mathbb{R}) \times$ SL$(2,\mathbb{R})$ representations we find

$$\begin{aligned} W_1 &= \hat{L} \oplus \Lambda^3 \hat{L}^* \oplus \left(\hat{L}^* \otimes \Lambda^6 \hat{L}^*\right) \oplus \left[B \otimes \left(\hat{L}^* \oplus \Lambda^5 \hat{L}^*\right)\right], \\ W_{\text{ad}} &= \mathbb{R} \oplus \left(B \otimes B^*\right)_0 \oplus \left(\hat{L} \otimes \hat{L}^*\right) \oplus \Lambda^4 \hat{L}^* \oplus \Lambda^4 \hat{L} \\ &\quad \oplus \left[B \otimes \left(\Lambda^2 \hat{L}^* \oplus \Lambda^2 \hat{L} \oplus \Lambda^6 \hat{L}^* \oplus \Lambda^6 \hat{L}\right)\right]. \end{aligned} \tag{C.26}$$

After breaking the SL$(2,\mathbb{R})$ action on $B$ this becomes

$$\begin{aligned} W_1 &= \hat{L} \oplus \hat{L}^* \oplus \Lambda^5 \hat{L}^* \oplus \left(\hat{L}^* \otimes \Lambda^6 \hat{L}^*\right) \oplus \Lambda^{\text{odd}} \hat{L}^*, \\ W_{\text{ad}} &= \mathbb{R} \oplus \mathbb{R} \oplus \left(\hat{L} \otimes \hat{L}^*\right) \oplus \Lambda^2 \hat{L}^* \oplus \Lambda^2 \hat{L} \\ &\quad \oplus \Lambda^6 \hat{L}^* \oplus \Lambda^6 \hat{L} \oplus \Lambda^{\text{even}} \hat{L}^* \oplus \Lambda^{\text{even}} \hat{L}. \end{aligned} \tag{C.27}$$



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The corresponding embeddings of $O(d-1)$ in $H_d$ follow from the intersection of the embedding (C.18) with (C.22) and (C.25). The $H_d$ algebra element decomposes as

$$\begin{aligned} N &= q + s + \tilde{s} + t^- \in \ \Lambda^2 L^* \oplus \Lambda^2 L^* \oplus \Lambda^6 L^* \oplus \Lambda^{\text{odd}} L^*, \\ &= \hat{q} + \hat{s} + \hat{\tilde{s}} + \hat{t}^+ \in \Lambda^2 \hat{L}^* \oplus \Lambda^2 \hat{L}^* \oplus \Lambda^6 \hat{L}^* \oplus \Lambda^{\text{even}} \hat{L}^*. \end{aligned} \tag{C.28}$$

Lifting to a $\mathrm{Spin}(d-1)$ action, it is important to note that $\mathrm{Cliff}(d-1;\mathbb{R})$ for the type IIB spinors does not embed in $\mathrm{Cliff}(d;\mathbb{R})$; only the spin group $\mathrm{Spin}(d-1)$ embeds. Concretely, in both cases, one can decompose the $\mathrm{Cliff}(d;\mathbb{R})$ spinors under $\gamma^1$ by

$$\gamma^1 \epsilon^\pm = \pm \epsilon^\pm. \tag{C.29}$$

Each spinor $\epsilon^\pm$ then transforms under the $\mathrm{Spin}(d)$ group generated by

$$\begin{aligned} &\hat{\gamma}^{ij} = \gamma^{ij} && \text{type IIA} \\ &\hat{\gamma}^{ij} = \begin{cases} \gamma^{\hat{\imath}\hat{\jmath}} & \text{if } i = \hat{\imath}, j = \hat{\jmath} \\ \gamma^{\hat{\imath}12} & \text{if } i = \hat{\imath}, j = 1 \\ -\gamma^{\hat{\jmath}12} & \text{if } i = 1, j = \hat{\jmath} \end{cases} && \text{type IIB}\,. \end{aligned} \tag{C.30}$$

One then has the Clifford action for the type IIA decomposition

$$\begin{aligned} N\epsilon^\pm = &\ \frac{1}{2}\left(\frac{1}{2!} q_{ab}\hat{\gamma}^{ab} \mp \frac{1}{2!} s_{ab}\hat{\gamma}^{ab} - \frac{1}{6!}\tilde{s}_{a_1\dots a_6}\hat{\gamma}^{a_1\dots a_6}\right)\epsilon^\pm \\ &- \frac{1}{2}\sum_n \frac{1}{n!}(\pm)^{[(n+1)/2]} t^-_{a_1\dots a_n}\hat{\gamma}^{a_1\dots a_n}\epsilon^\mp \end{aligned} \tag{C.31}$$

and

$$\begin{aligned} N\epsilon^\pm = &\ \frac{1}{2}\left(\frac{1}{2!} \hat{q}_{ab}\hat{\gamma}^{ab} \mp \frac{1}{2!} \hat{s}_{ab}\hat{\gamma}^{ab} - \frac{1}{6!}\hat{\tilde{s}}_{a_1\dots a_6}\hat{\gamma}^{a_1\dots a_6}\right)\epsilon^\pm \\ &- \frac{1}{2}\sum_n \frac{1}{n!}(\pm)^{[(n+1)/2]} \hat{t}^+_{a_1\dots a_n}\hat{\gamma}^{a_1\dots a_n}\epsilon^\mp, \end{aligned} \tag{C.32}$$

for type IIB.



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