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η-Ricci solitons in Kenmotsu manifolds

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Abstract. The object of the present paper is to study generalized weakly symmetric and generalized weakly Ricci symmetric Kenmotsu manifolds whose metric tensor is η-Ricci soliton. The paper also aims to bring out curvature conditions for which η-Ricci solitons in Kenmotsu manifolds are sometimes shrinking or expanding and some other time remain steady. The existence of each generalized weakly symmetric and generalized weakly Ricci symmetric Kenmotsu manifold is ensured by an example.

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1 Introduction

The notion of weakly symmetric Riemannian manifold has been introduced by Tamássy and Binh [32]. Thereafter, it has become of interest for many geometers. For details, we refer to [17], [21], [23], [27], [28], [29], [30], [31] and the references therein.

Keeping the spirit of Dubey [20], recently the present author has introduced a new type of space called generalized weakly symmetric manifold [9] which is abbreviated hereafter as $(GWS)_n$-manifold. An n-dimensional Riemannian manifold is said to be generalized weakly symmetric if it satisfies the equation

$$
\begin{aligned}
(\nabla_X \bar{R})(Y,U,V,W) = {} & A_1(X)\bar{R}(Y,U,V,W) + B_1(Y)\bar{R}(X,U,V,W) \\
& + B_1(U)\bar{R}(Y,X,V,W) + D_1(V)\bar{R}(Y,U,X,W) \\
& + D_1(W)\bar{R}(Y,U,V,X) + A_2(X)\bar{G}(Y,U,V,W) \\
& + B_2(Y)\bar{G}(X,U,V,W) + B_2(U)\ \bar{G}(Y,X,V,W) \\
& + D_2(V)\ \bar{G}(Y,U,X,W) + D_2(W)\ \bar{G}(Y,U,V,X)
\end{aligned} \tag{1.1}
$$

where

$$\bar{G}(Y,U,V,W) = [g(U,V)g(Y,\ W) - g(Y,\ V)g(U,\ W)] \tag{1.2}$$

and A_i, B_i and D_i are non-zero 1-forms defined by $A_i(X) = g(X,\sigma_i)$, $B_i(X) = g(X,\varrho_i)$, and $D_i(X) = g(X,\pi_i)$, for $i = 1,2$. The beauty of such $(GWS)_n$-manifold is that it has the flavour of

(i) locally symmetric space [14] (for $A_i = B_i = D_i = 0$),

(ii) recurrent space [34] (for $A_1 \neq 0$, $A_2 = B_i = D_i = 0$),

(iii) generalized recurrent space [20] (for $A_i \neq 0$, $B_i = D_i = 0$),

(iv) pseudo symmetric space [15] (for $\frac{A_1}{2} = B_1 = D_1 = H_1 \neq 0$, $A_2 = B_2 = D_2 = 0$),

(v) generalized pseudo symmetric space [7] (for $\frac{A_i}{2} = B_i = D_i = H_i \neq 0$),

(vi) semi-pseudo symmetric space [33] (for $A_i = B_2 = D_2 = 0, B_1 = D_1 \neq 0$),

(vii) generalized semi-pseudo symmetric space [10] (for $A_i = 0, B_i = D_i \neq 0$),

(viii) almost pseudo symmetric space [15] (for $A_1 = H_1 + K_1, B_1 = D_1 = H_1 \neq 0$ and $A_2 = B_2 = D_2 = 0$),

(ix) almost generalized pseudo symmetric space ([8], [6]) (for $A_i = H_i + K_i, B_i = D_i = H_i \neq 0$),

(x) weakly symmetric space [32] (for A_1, B_1, $D_i \neq 0$, $A_2 = B_2 = D_2 = 0$).

Analogously, we have defined generalized weakly Ricci symmetric Kenmotsu manifold as follows.

An n-dimensional Riemannian manifold is said to be generalized weakly Ricci symmetric if it satisfies the equation

$$
\begin{aligned}
(\nabla_X S)(Y,Z) = {} & A_1^*(X)S(Y,Z) + B_1^*(Y)S(X,Z) + D_1^*(Z)S(Y,X) \\
& + A_2^*(X)g(Y,Z) + B_2^*(Y)g(X,Z) + D_2^*(Z)g(Y,X)
\end{aligned} \tag{1.3}
$$

where A_i^*, B_i^* and D_i^* are non-zero 1-forms which are defined as $A_i^*(X) = g(X,\theta_i), B_i^*(X) = g(X,\phi_i), D_i^*(X) = g(X,\pi_i)$ for $i = 1,2$. The beauty of generalized weakly Ricci symmetric manifold is that it has the flavour of Ricci symmetric, Ricci recurrent, generalized Ricci recurrent, pseudo Ricci symmetric, generalized pseudo Ricci symmetric, semi-pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric [16], almost generalized pseudo Ricci symmetric and weakly Ricci symmetric space as special cases.

The study of the Ricci solitons in contact geometry has begun with the work of Ramesh Sharma [22], Cornelia Livia Bejan and Mircea Crasmareanu [11] and the references therein. Ricci solitons are defined as triples (g, V, λ), where (M, g) is a Riemannian manifold and V is a vector field (the potential vector field) so that the following equation is satisfied

$$\frac{1}{2}\pounds_V g + S + \lambda g = 0 \tag{1.4}$$

where $\pounds$ denotes the Lie derivative, S is the Ricci tensor and λ is a constant. A Ricci soliton is said to be shrinking, steady or expanding according to λ negative, zero and positive respectively. A Ricci soliton with V equal to zero reduces to Einstein equation.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians ([1], [2], [3]). η -Ricci solitons (M, g, λ, μ) is the generalization of Ricci solitons (M, g, λ) which is defined as ([12], [13], [4])

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g , λ and μ are real constants.

Our paper is structured as follows. Section 2 is concerned with Kenmotsu manifolds and some known results. Generalized weakly symmetric Kenmotsu manifolds whose metric tensor is η-Ricci soliton have been studied in Section 3. It is observed that (i) Ricci soliton in each of locally symmetric, locally recurrent, generalized recurrent, pseudo symmetric, semi-pseudo symmetric, almost pseudo symmetric and weakly symmetric Kenmotsu manifold is always expanding,(ii) Ricci soliton in each of generalized pseudo symmetric, generalized semi-pseudo symmetric and almost generalized pseudo symmetric Kenmotsu manifold is expanding, steady or shrinking according as $\left[\frac{1+D_2(\xi)}{1+D_1(\xi)}\right] \gtreqless 0$. In Section 4, we study generalized weakly Ricci-symmetric Kenmotsu manifolds whose metric tensor is η-Ricci soliton and obtain some interesting results. Finally, the existence of generalized weakly symmetric Kenmotsu manifolds and generalized weakly Ricci-symmetric Kenmotsu manifolds are ensured by non-trivial examples.

2 Kenmotsu manifolds and some known results

Let M be a n-dimensional connected differentiable manifold of class C^∞-covered by a system of coordinate neighborhoods (U, x^h) in which there are given a tensor field ϕ of type $(1,1)$, a contravariant vector field ξ and a $1-$form η such that

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, && (2.1)\\ \eta(\xi) &= 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, && (2.2) \end{aligned}$$

for any vector field X on M. Then the structure (ϕ, ξ, η) is called contact structure and the manifold M^n equipped with such structure is said to be an almost contact manifold. If there is given a Riemannian compatible metric g such that

$$\begin{aligned} g(\phi X, Y) &= -g(X, \phi Y), \quad g(X, \xi) = \eta(X), && (2.3)\\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), && (2.4) \end{aligned}$$

for all vector fields X and Y, then we say that M is an almost contact metric manifold. An almost contact metric manifold M is called a Kenmotsu manifold if it satisfies [26]

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi \ - \ \eta(Y)\phi(X), \tag{2.5}$$

for all vector fields X and Y, where ∇ is a Levi-Civita connection of the Riemannian metric. From the above it follows that

$$\begin{aligned} \nabla_X \xi &= X - \eta(X)\xi, && (2.6)\\ (\nabla_X \eta)Y &= g(X, Y) - \eta(X)\ \eta(Y). && (2.7) \end{aligned}$$

In a Kenmotsu manifold the following relations hold ([5], [6], [19], [18], [24], [25])

$$\begin{aligned} R(X, Y)\xi &= \eta(X)\ Y - \eta(Y)X, && (2.8)\\ S(X, \xi) &= -(n-1)\eta(X), && (2.9)\\ R(X, \xi)Y &= g(X, Y)\xi - \eta(Y)X, && (2.10)\\ R(\xi, X)Y &= \eta(Y)X - \ g(X, Y)\xi && (2.11) \end{aligned}$$

for any vector fields X, Y, Z, where R is the Riemannian curvature tensor of the manifold.

Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold satisfying (1.4). Writing $L_\xi g$ in terms of the Levi-Civita connection ∇ , we obtain from (1.4) that

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \quad (2.12)$$

for any $X, Y \in \chi(M)$. As a consequence of (2.5) and (2.6), the above equation becomes

$$S(X,Y) = -(\lambda+1)g(X,Y) - (\mu+1)\eta(X)\eta(Y). \quad (2.13)$$

In the sequel, we shall use the following Lemma.

Lemma 2.1. *Let (M^n, g) be a Kenmotsu manifold. Then for any X, Y, Z, the following relation holds:*

$$(\nabla_X S)(U, \xi) = -(n-1)g(X,U) - S(U,X). \quad (2.14)$$

Proof. By virtue of (2.6) and (2.9), we can easily get (2.14). □

3 Generalized weakly symmetric Kenmotsu manifolds whose metric is η-Ricci soliton

In this section, we consider a generalized weakly symmetric Kenmotsu manifold $(M^n, g)(n > 2)$. Now, contracting Y over W in both sides of (1.3), we get

$$\begin{aligned}(\nabla_X S)(U,V) = {} & A_1(X)S(U,V) + B_1(U)S(X,V) + B_1(R(X,U)V) \\ & + D_1(R(X,V)U) + D_1(V)S(U,X) \\ & + (n-1)[A_2(X)g(U,V) + B_2(U)g(X,V) \\ & + D_2(V)g(U,X)] + B_2(G(X,U)V) + D_2(G(X,V)U).\end{aligned} \quad (3.1)$$

As a consequence of (2.8), (2.9) and (2.10) the above equation yields

$$\begin{aligned}(\nabla_X S)(U,\xi) = {} & -(n-1)A_1(X)\eta(U) - (n-2)B_1(U)\eta(X) \\ & + D_1(\xi)S(U,X) - \eta(U)B_1(X) - \eta(U)D_1(X) \\ & + g(X,U)D_1(\xi) + (n-1)[A_2(X)\eta(U) \\ & + B_2(U)\eta(X) + D_2(\xi)g(U,X)] + \eta(U)B_2(X) \\ & - \eta(X)B_2(U) + \eta(U)D_2(X) - g(U,X)D_2(\xi)\end{aligned} \quad (3.2)$$

for $V = \xi$. Now, using (2.14) in (3.2), we have

$$\begin{aligned}-(n-1)g(X,U)-S(U,X) = &-(n-1)A_1(X)\eta(U)-(n-2)B_1(U)\eta(X)\\ &+D_1(\xi)S(U,X)-\eta(U)B_1(X)+g(X,U)D_1(\xi)-\eta(U)D_1(X)\\ &+(n-1)[A_2(X)\eta(U)+B_2(U)\eta(X)+D_2(\xi)g(U,X)]\\ &+\eta(U)B_2(X)-\eta(X)B_2(U)+\eta(U)D_2(X)-g(U,X)D_2(\xi)\end{aligned} \tag{3.3}$$

which gives

$$[A_1(\xi)+B_1(\xi)+D_1(\xi)] = [A_2(\xi)+B_2(\xi)+D_2(\xi)] \tag{3.4}$$

for $X = U = \xi$. In particular, if $A_2(\xi) = B_2(\xi) = B_2(\xi) = D_2(\xi) = 0$, (3.4) turns into

$$A_1(\xi)+B_1(\xi)+D_1(\xi) = 0. \tag{3.5}$$

In a similar manner, we have

$$\begin{aligned}-(n-1)g(X,V)-S(V,X) = &-(n-1)A_1(X)\eta(V)-(n-2)D_1(V)\eta(X)]\\ &+B_1(\xi)S(X,V)+g(X,V)B_1(\xi)-\eta(V)B_1(X)-\eta(V)D_1(X)\\ &+(n-1)[\{A_2(X)\}\eta(V)+B_2(\xi)g(X,V)\\ &+D_2(V)\eta(X)]+\eta(V)B_2(X)-g(X,V)B_2(\xi)\\ &+\eta(V)D_2(X)-\eta(X)D_2(V).\end{aligned} \tag{3.6}$$

Now, putting $V = \xi$ in (3.6) and using (2.1), (2.9), we obtain

$$\begin{aligned}(n-1)A_1(X)+B_1(X)+D_1(X)+(n-2)[B_1(\xi)+D_1(\xi)]\eta(X) &=\\ = [(n-1)A_2(X)+(n-2)\{B_2(\xi)+D_2(\xi)\}\eta(X)]&\\ +B_2(X)+D_2(X).&\end{aligned} \tag{3.7}$$

Putting $X = \xi$ in (3.6) and using (2.1), (2.2), (2.9), we obtain

$$\begin{aligned}(n-1)[A_1(\xi)+B_1(\xi)]\eta(V)+(n-2)D_1(V)+\eta(V)D_1(\xi) &=\\ = (n-1)[\{A_2(\xi)+B_2(\xi)\}\eta(V)+D_2(V)]&\\ +\eta(V)D_2(\xi)-D_2(V).&\end{aligned} \tag{3.8}$$

Replacing V by X in the above equation and using (3.4), we get

$$D_1(X)-D_1(\xi)\eta(X) = D_2(X)-D_2(\xi)\eta(X). \tag{3.9}$$

Moreover, in view of (3.4), (3.7) and (3.9), we get

$$B_1(X)-B_1(\xi)\eta(X) = B_2(X)-B_2(\xi)\eta(X). \tag{3.10}$$

Subtracting (3.9), (3.10) from (3.7), we get

$$A_1(X)+[B_1(\xi)+D_1(\xi)]\eta(X) = A_2(X) + [B_2(\xi) + D_2(\xi)]\eta(X)]. \tag{3.11}$$

Again, adding (3.9), (3.10) and (3.11), we get

$$A_1(X) + B_1(X) + D_1(X) = A_2(X) + B_2(X) + D_2(X). \tag{3.12}$$

Next, for the choice of $A_2 = B_2 = C_2 = D_2 = 0$, the relation (3.12) yields

$$A_1(X) + B_1(X) + D_1(X) = 0. \tag{3.13}$$

Again from (3.3), putting $X = \xi$, we have

$$\begin{aligned}(n-1)[-\{A_1(\xi) - A_2(\xi)\} - \{D_1(\xi) - D_2(\xi)\}]\eta(U) = \\ = \{B_1(\xi) - B_2(\xi)\}\eta(U) + (n-2)\{B_1(U) - B_2(U)\}.\end{aligned} \tag{3.14}$$

Using (3.4), above equation becomes

$$\{B_1(\xi) - B_2(\xi)\}\eta(U) = B_1(U) - B_2(U). \tag{3.15}$$

Also, setting $U = \xi$, we have

$$\begin{aligned}-(n-1)\{A_1(X) - A_2(X)\}-\{B_1(X) - B_2(X)\}-\{D_1(X) - D_2(X)\} = \\ = -(n-2)\{A_1(\xi) - A_2(\xi)\}.\end{aligned} \tag{3.16}$$

Using (3.4) in (3.16), we obtain

$$\{A_1(\xi) - A_2(\xi)\}\eta(X) = A_1(X) - A_2(X). \tag{3.17}$$

As a consequence of the above equations, (3.3) yields

$$\begin{aligned}S(U,X) = &-\frac{[(n-1)\{1 + D_2(\xi)\} + D_1(\xi) - D_2(\xi)]}{[1+D_1(\xi)]}g(X,U) \\ &+ \frac{(n-2)[\{A_1(\xi) - A_2(\xi)\}-\{B_1(\xi) - B_2(\xi)\}]}{[1+D_1(\xi)]}\eta(U)\eta(X)\end{aligned} \tag{3.18}$$

for any $X,\ U \in \chi(M)$. Comparing (2.13) and (3.18) we obtain

$$\lambda = \left[\frac{[(n-1)\{1 + D_2(\xi)\} + D_1(\xi) - D_2(\xi)]}{[1+D_1(\xi)]} - 1\right]. \tag{3.19}$$

This leads to the following

Theorem 3.1. *The Ricci soliton in each of the locally symmetric, locally recurrent, generalized recurrent, pseudo symmetric, semi-pseudo symmetric, almost pseudo symmetric and weakly symmetric Kenmotsu manifold is always expanding.*

Theorem 3.2. *The Ricci soliton for each of the generalized pseudo symmetric, generalized semi-pseudo symmetric, almost generalized pseudo symmetric Kenmotsu manifold is expanding, steady or shrinking according to* $\left[\frac{1+D_2(\xi)}{1+D_1(\xi)}\right] \gtreqless 0$.

4 Generalized weakly Ricci-symmetric Kenmotsu manifolds whose metric is η-Ricci soliton

A Kenmotsu manifold $(M^n, g)(n > 2)$ is said to be generalized weakly Ricci-symmetric if there exist 1-forms $\bar{A}_i, \bar{B}_i$ and $\bar{D}_i$ which satisfy the condition

$$\begin{aligned}(\nabla_X S)(U,V) = \bar{A}_1(X)S(U,V) + \bar{B}_1(U)S(X,V) + \bar{D}_1(V)S(U,X) \\ + \bar{A}_2(X)g(U,V) + \bar{B}_2(U)g(X,V) + \bar{D}_2(V)g(U,X).\end{aligned} \tag{4.1}$$

Putting $V = \xi$ in (4.1), we obtain

$$\begin{aligned}(\nabla_X S)(U,\xi) = (n-2)[\bar{A}_1(X)\eta(U) + \bar{B}_1(U)\eta(X)] + \bar{D}_1(\xi)S(U,X) \\ + \bar{A}_2(X)\eta(U) + \bar{B}_2(U)\eta(X) + \bar{D}_2(\xi)g(U,X).\end{aligned} \tag{4.2}$$

In view of (2.14), the relation (4.2) becomes

$$\begin{aligned}&-(n-1)g(X,U) - S(U,X) = \\ &\quad = -(n-1)[\{\bar{A}_1(X) + \bar{B}_1(X)\}\eta(U) + \bar{B}_1(U)\eta(X)] + \bar{D}_1(\xi)S(U,X) \\ &\quad + \bar{A}_2(X)\eta(U) + \bar{B}_2(U)\eta(X) + \bar{D}_2(\xi)g(U,X).\end{aligned} \tag{4.3}$$

Setting $X = U = \xi$ in (4.3) and using (2.1), (2.2) and (2.9), we get

$$(n-1)[\bar{A}_1(\xi) + 2\bar{B}_1(\xi) + \bar{D}_1(\xi)] = [\bar{A}_2(\xi) + 2\bar{B}_2(\xi) + \bar{D}_2(\xi)]. \tag{4.4}$$

Again, putting $X = \xi$ in (4.3), we get

$$\begin{aligned}(n-1)[\{\bar{A}_1(\xi) + \bar{B}_1(\xi) + \bar{D}_1(\xi)\}\eta(U) + \bar{B}_1(U)] = \\ = [\bar{A}_2(\xi) + \bar{B}_2(\xi) + \bar{D}_2(\xi)]\eta(U) + \bar{B}_2(U).\end{aligned} \tag{4.5}$$

Setting $U = \xi$ in (4.3) and then using (2.1), (2.2) and (2.9), we obtain

$$\begin{aligned}(n-1)[\{\bar{A}_1(X)+\bar{B}_1(X)\}+\{\bar{B}_1(\xi)+\bar{D}_1(\xi)\}\eta(X)] = \\ = \bar{A}_2(X)+\bar{B}_2(\xi)\eta(X)+\bar{D}_2(\xi)\eta(X).\end{aligned} \tag{4.6}$$

Replacing U by X in (4.5) and then adding the resultant with (4.6), we have

$$\begin{aligned}(n-1)[\bar{A}_1(X)+\bar{B}_1(X)]-[\bar{A}_2(X)+\bar{B}_2(X)] = \\ = -(n-1)[\bar{A}_1(\xi)+\bar{B}_1(\xi)+\bar{D}_1(\xi)]\eta(X)+[\bar{A}_2(\xi)+\bar{B}_2(\xi) \\ +\bar{D}_2(\xi)]\eta(X)-(n-1)\bar{D}_1(\xi)\eta(X)+\bar{D}_2(\xi)\eta(X).\end{aligned} \tag{4.7}$$

By virtue of (4.4), the above equation becomes

$$\begin{aligned}(n-1)[\bar{A}_1(X)+\bar{B}_1(X)]+(n-1)\bar{D}_1(\xi)\eta(X) = \\ = [\bar{A}_2(X)+\bar{B}_2(X)]+\bar{D}_2(\xi)\eta(X).\end{aligned} \tag{4.8}$$

Next, putting $X = U = \xi$ in (4.1), we get

$$\begin{aligned}(n-1)[\bar{A}_1(\xi)+\bar{B}_1(\xi)]\eta(V)+(n-1)\bar{D}_1(V) = \\ = [\bar{A}_2(\xi)+\bar{B}_2(\xi)]\eta(V)+\bar{D}_2(V).\end{aligned} \tag{4.9}$$

Replacing V by X in (4.9) and adding with (4.8), we obtain

$$\begin{aligned}(n-1)[\bar{A}_1(X)+\bar{B}_1(X)+\bar{D}_1(X)]+(n-1)[\bar{A}_1(\xi)+\bar{B}_1(\xi)+\bar{D}_1(\xi)]\eta(V) = \\ = [\bar{A}_2(X)+\bar{B}_2(X)+\bar{D}_1(X)]+[\bar{A}_2(\xi)+\bar{B}_2(\xi)+\bar{D}_2(\xi)]\eta(V).\end{aligned} \tag{4.10}$$

By virtue of (4.4), the above equation becomes

$$(n-1)[\bar{A}_1(X)+\bar{B}_1(X)+\bar{D}_1(X)] = [\bar{A}_2(X)+\bar{B}_2(X)+\bar{D}_1(X)]. \tag{4.11}$$

Again from (4.3), we have

$$\begin{aligned}S(U,X) = \frac{1}{-[1+\bar{D}_1(\xi)]}[\bar{D}_2(\xi)+(n-1)]g(X,U) \\ +\frac{1}{[1+\bar{D}_1(\xi)]}[(n-1)\{\bar{A}_1(X)+\bar{B}_1(X)\}-\bar{A}_2(X)]\eta(U) \\ +\frac{1}{[1+\bar{D}_1(\xi)]}[(n-1)\bar{B}_1(U)-\bar{B}_2(U)]\eta(X).\end{aligned} \tag{4.12}$$

From (4.6), we have

$$\begin{aligned}(n-1)[\{\bar{A}_1(X)+\bar{B}_1(X)\}-\bar{A}_2(X)] = \\ = [-(n-1)\{\bar{B}_1(\xi)+\bar{D}_1(\xi)\}+\bar{B}_2(\xi)+\bar{D}_2(\xi)]\eta(X).\end{aligned} \tag{4.13}$$

Using (4.4) in (4.5), we have

$$(n-1)\bar{B}_1(U)-\bar{B}_2(U)=-(n-1)\bar{B}_1(\xi)+\bar{B}_2(\xi). \tag{4.14}$$

In view of (4.12), (4.13) and (4.14), we have

$$\begin{aligned} S(U,X)=&-\left[\frac{\bar{D}_2(\xi)+(n-1)}{1+\bar{D}_1(\xi)}\right]g(X,U)\\ &+\left[\frac{-2\{(n-1)\bar{B}_1(\xi)-\bar{B}_2(\xi)\}-\{(n-1)\bar{D}_1(\xi)-\bar{D}_2(\xi)\}}{[1+\bar{D}_1(\xi)]}\right]\eta(U)\eta(X) \end{aligned} \tag{4.15}$$

Now, comparing (2.13) and (4.15) we have

$$\lambda=\left[\frac{\bar{D}_2(\xi)+(n-1)}{1+\bar{D}_1(\xi)}-1\right].$$

This leads to the following

Theorem 4.1. *The Ricci soliton in each of Ricci symmetric, Ricci recurrent and generalized Ricci recurrent Kenmotsu manifold is always expanding.*

Theorem 4.2. *The Ricci soliton in each of pseudo Ricci symmetric, semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric and weakly Ricci symmetric Kenmotsu manifold is expanding, steady or shrinking according to* $[n-1]\gtreqless\left[1+\bar{D}_1(\xi)\right]$.

Theorem 4.3. *The Ricci soliton in each of generalized pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost generalized pseudo Ricci symmetric Kenmotsu manifold is expanding, steady or shrinking according to* $\left[\bar{D}_2(\xi)+n\right]\gtreqless\left[2+\bar{D}_1(\xi)\right]$.

5 Example of a $(\mathrm{GWS})_3$ Kenmotsu manifold

(See [19], Ex. 3.1, pp. 21-22) Let $M^3(\phi,\xi,\eta,g)$ be a Kenmotsu manifold (M^3,g) with a ϕ-basis

$$e_1=e^{-z}\frac{\partial}{\partial x},\ e_2=e^{-z}\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}.$$

Then from Koszul's formula for Riemannian metric g, we obtain the Levi-Civita connection as follows

$$\begin{aligned}
\nabla_{e_1}e_3 &= e_1, & \nabla_{e_1}e_2 &= 0, & \nabla_{e_1}e_1 &= -e_3,\\
\nabla_{e_2}e_3 &= e_2, & \nabla_{e_2}e_2 &= -e_3, & \nabla_{e_2}e_1 &= 0,\\
\nabla_{e_3}e_3 &= 0, & \nabla_{e_3}e_2 &= 0, & \nabla_{e_3}e_1 &= 0.
\end{aligned}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

$$\bar{R}(e_1,e_3,e_1,e_3) = \bar{R}(e_2,e_3,e_2,e_3) = \bar{R}(e_1,e_2,e_1,e_2) = 1.$$

Since $\{e_1,e_2,e_3\}$ is a basis, any vector field X, Y, U, $V \in \chi(M)$ can be written as

$$\begin{aligned}
X &= \sum_1^3 a_ie_i,\ Y = \sum_1^3 b_ie_i,\ U = \sum_1^3 c_ie_i,\ \ V = \sum_1^3 d_ie_i,\\
\bar{R}(X,Y,U,V) &= (a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1) - (a_1b_3 - a_3b_1)(c_1d_3\\
&\quad - c_3d_1) - (a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2)\\
&= H_1 \text{ (say)}\\
\bar{R}(e_1,Y,U,V) &= b_2\ (c_1d_2 - c_2d_1) + b_3(c_1d_3 - c_3d_1) = \Gamma_1 \text{ (say)}\\
\bar{R}(e_2,Y,U,V) &= b_3(c_2d_3 - c_3d_2) - b_1\ (c_1d_2 - c_2d_1) = \Gamma_2 \text{ (say)}\\
\bar{R}(e_3,Y,U,V) &= -b_1(c_1d_3 - c_3d_{1-}) - b_2\ (c_2d_3 - c_3d_2) = \Gamma_3 \text{ (say)}\\
\bar{R}(X,e_1,U,V) &= -a_3(c_1d_3 - c_3d_1) - a_2\ (c_1d_2 - c_2d_1) = \Gamma_4 \text{ (say)}\\
\bar{R}(X,e_2,U,V) &= -a_3(c_2d_3 - c_3d_2) + a_1\ (c_1d_2 - c_2d_1) = \Gamma_5 \text{ (say)}\\
\bar{R}(X,e_3,U,V) &= a_1(c_1d_3 - c_3d_1\ ) + a_2\ (c_2d_3 - c_3d_2) = \Gamma_6 \text{ (say)}\\
\bar{R}(X,Y,e_1,V) &= d_2\ (a_1b_2 - a_2b_1\ ) + d_3(a_1b_3 - a_3b_1) = \Gamma_7 \text{ (say)}\\
\bar{R}(X,Y,e_2,V) &= d_3(a_2b_3 - a_3b_2) - d_1\ (a_1b_2 - a_2b_1) = \Gamma_8 \text{ (say)}\\
\bar{R}(X,Y,e_3,V) &= -d_1(a_1b_3 - a_3b_1) - d_2\ (a_2b_3 - a_3b_2) = \Gamma_9 \text{ (say)}\\
\bar{R}(X,Y,U,e_1) &= -c_3(a_1b_3 - a_3b_1) - c_2\ (a_1b_2 - a_2b_1\ ) = \Gamma_{10} \text{ (say)}\\
\bar{R}(X,Y,U,e_2) &= -c_3(a_2b_3 - a_3b_2\ ) + c_1\ (a_1b_2 - a_2b_1) = \Gamma_{11} \text{ (say)}\\
\bar{R}(X,Y,U,e_3) &= c_1(a_1b_3 - a_3b_1) + c_2\ (a_2b_3 - a_3b_2\ ) = \Gamma_{12} \text{ (say)}\\
\bar{G}(X,Y,U,V) &= (b_1c_1 + b_2c_2 + b_3c_3)(a_1d_1 + a_2d_2 + a_3d_3)\\
&\quad -(a_1c_1 + a_2c_2 + a_3c_3)(b_1d_1 + b_2d_2 + b_3d_3) = H_2 \text{ (say)}\\
\bar{G}(e_1,Y,U,V) &= (b_2c_2 + b_3c_3)d_1 - (b_2d_2 + b_3d_3)c_1 = \Omega_1 \text{ (say)}\\
\bar{G}(e_2,Y,U,V) &= (b_1c_1 + b_3c_3)d_2 - (b_1d_1 + b_3d_3)c_2 = \Omega_2 \text{ (say)}\\
\bar{G}(e_3,Y,U,V) &= (b_1c_1 + b_2c_2)d_3 - (b_1d_1 + b_2d_2)c_3 = \Omega_3 \text{ (say)}
\end{aligned}$$

$$
\begin{aligned}
\bar{G}(X, e_1, U, V) &= (a_2d_2 + a_3d_3)c_1 - (a_2c_2 + a_3c_3)d_1 = \Omega_4 \text{ (say)}\\
\bar{G}(X, e_2, U, V) &= (a_1d_1 + a_3d_3)c_2 - (a_1c_1 + a_3c_3)d_2 = \Omega_5 \text{ (say)}\\
\bar{G}(X, e_3, U, V) &= (a_1d_1 + a_2d_2)c_3 - (a_1c_1 + a_2c_2)d_3 = \Omega_6 \text{ (say)}\\
\bar{G}(X, Y, e_1, V) &= (a_2d_2 + a_3d_3)b_1 - (b_2d_2 + b_3d_3)a_1 = \Omega_7 \text{ (say)}\\
\bar{G}(X, Y, e_2, V) &= (a_1d_1 + a_3d_3)b_2 - (b_1d_1 + b_3d_3)a_2 = \Omega_8 \text{ (say)}\\
\bar{G}(X, Y, e_3, V) &= (b_1d_1 + b_2d_2)a_3 - (a_1d_1 + a_2d_2)b_3 = \Omega_9 \text{ (say)}\\
\bar{G}(X, Y, U, e_1) &= (b_2c_2 + b_3c_3)a_1 - (a_2c_2 + a_3c_3)b_1, = \Omega_{10} \text{ (say)}\\
\bar{G}(X, Y, U, e_2) &= (b_1c_1 + b_3c_3)a_2 - (a_1c_1 + a_3c_3)b_2 = \Omega_{11} \text{ (say)}\\
\bar{G}(X, Y, U, e_3) &= (b_1c_1 + b_2c_2)a_3 - (a_1c_1 + a_2c_2)b_3 = \Omega_{12} \text{ (say)}
\end{aligned}
$$

and the components which can be obtained from these by the symmetry properties. The covariant derivatives of the curvature tensor are as follows

$$
\begin{aligned}
(\nabla_{e_1}\bar{R})(X, Y, U, V) = a_1\Gamma_3 - a_3\Gamma_1 + b_1\Gamma_6 - b_3\Gamma_4 \\
+ c_1\Gamma_9 - c_3\Gamma_7 + d_1\Gamma_{12} - d_3\Gamma_{10} = H_3 \text{ (say)},
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_{e_2}\bar{R})(X, Y, U, V) = a_2\Gamma_3 - a_3\Gamma_2 + b_2\Gamma_6 - b_3\Gamma_5 \\
+ c_2\Gamma_9 - c_3\Gamma_8 + d_2\Gamma_{12} - d_3\Gamma_{11} = H_4 \text{ (say)},\\
(\nabla_{e_3}\bar{R})(X, Y, U, V) = 0.
\end{aligned}
$$

For the following choice of the 1-forms

$$
\begin{aligned}
A_1(e_1) &= \frac{H_3}{H_1},\\
A_2(e_1) &= \frac{1}{H_2}\left[\left(\frac{c_3\Omega_7 + d_3\Omega_{10}}{c_3\Omega_8 + d_3\Omega_{11}}\right) - \left(\frac{a_3\Omega_1 + b_3\Omega_4}{a_3\Omega_2 + b_3\Omega_5}\right)\right],\\
A_1(e_2) &= \frac{1}{H_1}\left[\left(\frac{c_3\Gamma_8 + d_3\Gamma_{11}}{c_3\Gamma_7 + d_3\Gamma_{10}}\right) - \left(\frac{a_3\Gamma_2 + b_3\Gamma_5}{a_3\Gamma_1 + b_3\Gamma_4}\right)\right],\\
A_2(e_2) &= \frac{H_4}{H_2},\\
A_1(e_3) &= \frac{1}{H_1}\left[\left(\frac{c_3\Gamma_9 + d_3\Gamma_{12}}{c_3\Gamma_7 + d_3\Gamma_{10}}\right) - \left(\frac{a_3\Gamma_3 + b_3\Gamma_6}{a_3\Gamma_1 + b_3\Gamma_4}\right)\right],\\
A_2(e_3) &= \frac{1}{H_2}\left[\left(\frac{c_3\Omega_9 + d_3\Omega_{12}}{c_3\Omega_8 + d_3\Omega_{11}}\right) - \left(\frac{a_3\Omega_3 + b_3\Omega_6}{a_3\Omega_2 + b_3\Omega_5}\right)\right],\\
B_1(e_3) &= \frac{1}{(a_3\Gamma_1 + b_3\Gamma_4)},\ B_2(e_3) = \frac{1}{(a_3\Omega_2 + b_3\Omega_5)},\\
D_1(e_3) &= -\frac{1}{(c_3\Gamma_7 + d_3\Gamma_{10})},\ D_2(e_3) = -\frac{1}{(c_3\Omega_8 + d_3\Omega_{11})},
\end{aligned}
$$

one can easily verify the relations

$$\begin{aligned}(\nabla_{e_i}\bar{R})(X,Y,U,V) &= A_1(e_i)\bar{R}(X,Y,U,V)\\ &+ B_1(X)\bar{R}(e_i,Y,U,V) + B_1(Y)\bar{R}(X,e_i,U,V)\\ &+ D_1(U)\bar{R}(X,Y,e_i,V) + D_1(V)\bar{R}(X,Y,U,e_i)\\ &+ A_2(e_i)\ \bar{G}(X,Y,U,V)\\ &+ B_2(X)\bar{G}(e_i,Y,U,V) + B_2(Y)\bar{G}(X,e_i,U,V)\\ &+ D_2(U)\bar{G}(X,Y,e_i,V) + D_2(V)\bar{G}(X,Y,U,e_i)\end{aligned}$$

for $i = 1, 2, 3$. From the above, we can state that

Theorem 5.1. *There exists a Kenmotsu manifold (M^3, g) which is generalized weakly symmetric.*

6 Example of a (GWRS)$_3$ Kenmotsu manifold

(See [19], Ex. 3.2, p. 21) Let $M^3(\phi,\xi,\eta,g)$ be a Kenmotsu manifold (M^3, g) with a ϕ-basis

$$e_1= e^{-z}\frac{\partial}{\partial x},\ e_2= e^{-z+x}\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}.$$

Then from Koszul's formula forRiemannian metric g, we obtain the Levi-Civita connection as follows

$$\begin{array}{lll}\nabla_{e_1}e_3 = e_1, & \nabla_{e_2}e_3 = e_2, & \nabla_{e_3}e_3 = 0,\\ \nabla_{e_1}e_2 = 0, & \nabla_{e_2}e_2 = -e_3, & \nabla_{e_3}e_2 = 0,\\ \nabla_{e_1}e_1 = -e_3, & \nabla_{e_2}e_1 = e^{-z}e_2, & \nabla_{e_3}e_1 = 0.\end{array}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

$$\bar{R}(e_1,e_3,e_1,e_3) =\bar{R}(e_2,e_3,e_2,e_3) = 1,\ \bar{R}(e_1,e_2,e_1,e_2) = e^{-2z},$$

and the components which can be obtained from these by the symmetry properties. From the above, we can easily calculate the non-vanishing components of the Ricci tensor S as follows

$$S(e_1,e_1) = S(e_2,e_2) = -(e^{-2z}+1),\quad S(e_3,e_3) = -2.$$

Since $\{e_1,\ e_2, e_3\}$ is a basis, any vector field $X,\ Y \in \chi(M)$ can be written as

$$X = \sum_1^3 a_ie_i,\ Y = \sum_1^3 b_ie_i,$$

$$\begin{aligned}
S(X,Y) &= -(e^{-2z}+1)(a_1b_1+a_2b_2)-2a_3b_3,\\
g(X,Y) &= (a_1b_1+a_2b_2+a_3b_3),\\
-\frac{1}{(e^{-2z}+1)}S(X,e_1) &= a_1 = g(X,e_1),\\
-\frac{1}{(e^{-2z}+1)}S(X,e_2) &= a_2 = g(X,e_2),\\
-\frac{1}{2}S(X,e_3) &= a_3 = g(X,e_3),\\
-\frac{1}{(e^{-2z}+1)}S(e_1,Y) &= b_1 = g(e_1,Y),\\
-\frac{1}{(e^{-2z}+1)}S(e_2,Y) &= b_2 = g(e_2,Y),\\
-\frac{1}{2}S(e_3,Y) &= b_3 = g(e_3,Y).
\end{aligned}$$

The covariant derivatives of the Ricci tensor are as follows

$$\begin{aligned}
(\nabla_{e_1}S)(X,Y) &= = (e^{-2z}-1)(a_1b_3+a_3b_1),\\
(\nabla_{e2}S)(X,Y) &= = (e^{-2z}-1)(a_2b_3+a_3b_2),\\
(\nabla_{e_3}S)(X,Y) &= 2e^{-2z}(a_1b_1+a_2b_2).
\end{aligned}$$

For the following choice of the 1-forms

$$\begin{aligned}
\bar{A}_1(e_1) &= -\frac{e^{-2z}(a_1b_3+a_3b_1)}{(e^{-2z}+1)(a_1b_1+a_2b_2)+2a_3b_3},\\
\bar{A}_2(e_1) &= \frac{(a_1b_3+a_3b_1)}{(e^{-2z}+1)(a_1b_1+a_2b_2)+2a_3b_3},\\
\bar{A}_1(e_2) &= \frac{(a_2b_3+a_3b_2)}{(e^{-2z}+1)(a_1b_1+a_2b_2)+2a_3b_3},\\
\bar{A}_2(e_2) &= -\frac{e^{-2z}(a_2b_3+a_3b_2)}{(e^{-2z}+1)(a_1b_1+a_2b_2)+2a_3b_3},\\
\bar{B}_1(e_3) &= -\left(\frac{a_1b_1-2}{2a_3b_3}\right)e^{-2z},\ \ \bar{B}_2(e_3) = \left(\frac{a_1b_1+2}{a_3b_3}\right)e^{-2z},\\
\bar{D}_1(e_3) &= -\left(\frac{a_2b_2+1}{2a_3b_3}\right)e^{-2z}, \bar{D}_2(e_3) = \left(\frac{a_2b_2-1}{2a_3b_3}\right)e^{-2z},
\end{aligned}$$

one can easily verify the relations

$$\begin{aligned}
(\nabla_{e_i}S)(X,Y) &= \bar{A}_1(e_i)S(X,Y)+\bar{B}_1(X)S(e_i,Y)+\bar{D}_1(Y)S(X,e_i)\\
&\quad +\bar{A}_2(e_i)g(X,Y)+\bar{B}_2(X)g(e_i,Y)+\bar{D}_2(Y)g(X,e_i)
\end{aligned}$$

for $i = 1, 2, 3$. From the above, we can state that

Theorem 6.1. *There exists a Kenmotsu manifold (M^3, g) which is generalized weakly Ricci symmetric.*

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