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η-Ricci Solitons on Quasi-Sasakian Manifolds

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Abstract. The object of the present paper is to study η-Ricci solitons in a 3-dimensional non-cosymplectic quasi-Sasakian manifolds. We study a particular type of second order parallel tensor in this manifold. Beside this we consider this manifold satisfying some curvature properties of Ricci tensor.

AMS Subject Classification (2000). 53C15; 53C25
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1 Introduction

Ricci soliton is a natural generalization of Einstein metrics on a Riemannian manifold, being generalized fixed points of Hamilton's Ricci flow $\frac{\partial}{\partial t}g = -2S$ [14]. The evaluation equation defining the Ricci flow is a kind of nonlinear diffusion equation, an analogue of the heat equation of the metrics. Under the Ricci flow a metric can be improved to evolve into a more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold: it will expand in the directions of the negative Ricci curvature and

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shrink in the positive case.

A more general notion of Ricci soliton named η-Ricci soliton was introduced by J. T. Cho and M. Kimura [8], which was treated by C. Calin and M. Crasmareanu on Hopf hypersufaces in complex space forms [7]. An η-Ricci soliton is a quadruple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1}$$

where S is the Ricci tensor associated to g and η is an one form. In this connection we mention the works of Blaga ([2], [3]), Majhi et al. [15] and Prakasha and Hadimani [17]. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ).

An important geometrical object in studying Ricci solitons is well-known to be a symmetric $(0, 2)$-tensor field which is parallel with respect to the Levi-Civita connection, some of its geometric properties being described in ([1], [10]) etc. In [3], the author studied the second order parallel tensor field in the context of para-Kenmotsu manifolds admitting η-Ricci solitons. In [17] authors studied non-existence of certain kinds of para-Sasakian manifolds admitting η-Ricci soliton. Beside these Ricci solitons on a three dimensional quasi-Sasakian manifolds have been studied by U. C. De and A. K. Mondal in [11]. η-Ricci solitons on Sasakian 3-manifolds have been studied in [15].

Quasi-Sasakian manifold is a natural generalization of Sasakian manifold. Motivated by the above studies, in this paper we consider the η-Ricci solitons in a 3-dimensional quasi-Sasakian manifold.

The present paper is organized as follows:
After preliminaries in section 2, section 3 is devoted to study η-Ricci solitons in a 3-dimensional non-cosymplectic quasi-Sasakian manifold and we prove that the manifold is an η-Einstein β-Sasakian manifold with $\lambda + \mu = -2\beta^2$. Here we construct an example to verify the result. In the next part of this section we prove that if the symmetric second order tensor field $\alpha = \pounds_\xi g + 2S + 2\mu\eta\otimes\eta$ is parallel with respect to the Levi-Civita connection of the metric tensor, then the manifold admits an η-Ricci soliton. Section 4 deals with the study η-Ricci solitons satisfying some curvature properties of Ricci tensor in 3-dimensional non-cosymplectic quasi-Sasakian manifolds. In the first part of this section we prove that an η-Ricci soliton in non-cosymplectic quasi-

Sasakian manifolds with Ricci tensor of Codazzi-type becomes a Ricci soliton. We also prove that in an η-Ricci soliton in non-cosymplectic 3-dimensional quasi-Sasakian manifolds the Ricci tensor is cyclic parallel. Beside these, we prove a corollary in this section. Finally, in section 5, we prove that ϕ-Ricci symmetric η-Ricci soliton on 3-dimensional non-cosymplectic quasi-Sasakian manifolds becomes a Ricci soliton.

2 Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([4], [5])

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0 \quad \text{and} \quad \eta \circ \phi = 0, \tag{2}$$

where $X \in TM$.

Let g be the compatible Riemannian metric with an almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for $X, Y \in TM$. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2) and (3) it can be easily seen that for $X, Y \in TM$

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X). \tag{4}$$

An almost complex structure J can be defined on the product $M \times \mathbb{R}$ of M and the real line $\mathbb{R}$ by $J(X, \nu\frac{d}{dt}) = (\phi X - \nu\xi, \eta(X)\frac{d}{dt})$, where ν is a scalar field on $M \times \mathbb{R}$. If the structure J is complex analytic, the almost contact metric structure (ϕ, ξ, η, g) is said to be normal. A necessary and sufficient condition of an almost contact metric manifold to be normal is that the Nijehaus tensor field N vanishes on the manifold ([4], [18]). A normal almost contact metric manifold is quasi-Sasakian if the fundamental 2-form Φ, defined by $\Phi(X, Y) = g(\phi X, Y)$ is closed ([6], [19]). Beside this, Olszak [16] proved that a 3-dimensional almost contact metric manifold M is a quasi-Sasakian manifold if and only if

$$\nabla_X \xi = -\beta\phi X, \tag{5}$$

where $X \in TM$ and β is some function on M, such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of M. Hence a 3-dimensional quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. For $\beta =$ constant, the manifold reduces to a β-Sasakian manifold and $\beta = 1$ gives the Sasakian structure.

In a non-cosymplectic 3-dimensional quasi-Sasakian manifold the following relations hold ([12], [16]):

$$(\nabla_X\phi)Y = \beta[g(X,Y)\xi - \eta(Y)X], \tag{6}$$

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2[\eta(Y)X - \eta(X)Y], \tag{7}$$

$$\begin{aligned} S(X,Y) &= (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) \\ &\quad -\eta(X)d\beta(\phi Y) - \eta(Y)d\beta(\phi X), \end{aligned} \tag{8}$$

$$(\nabla_X\eta)(Y) = -\beta g(\phi X, Y), \tag{9}$$

where $X, Y \in TM$, R, S and r being the curvature tensor of type $(1,3)$, Ricci tensor of type (0, 2) and the scalar curvature of the manifold respectively.

A manifold M is said to be an η-Einstein manifold if the Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{10}$$

and Einstein if

$$S(X,Y) = \kappa g(X,Y) \tag{11}$$

where a and b are smooth functions on the manifold and κ is a constant.

A. Gray [13] introduced the notion of cyclic parallel Ricci tensor and Codazzi-type of Ricci tensor. A Riemannian manifold is said to have cyclic parallel Ricci tensor if the Ricci tensor of the manifold satisfies the following:

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{12}$$

Again a Riemannian manifold is said to have Codazzi-type of Ricci tensor if the Ricci tensor S is non-zero and satisfies the following:

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{13}$$

3 η-Ricci soliton on 3-dimensional quasi-Sasakian manifolds

Let M be a 3-dimensional quasi-Sasakian manifold. We write (1) as

$$2S(X,Y) = -(\pounds_\xi g)(X,Y) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{14}$$

for $X, Y \in TM$. In a quasi-Sasakian manifold, ξ is a Killing vector field [6]. Therefore $(\pounds_\xi g)(X,Y) = 0$. Hence (14) takes the form

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{15}$$

In view of (15) we state the following:

Proposition 3.1. *A 3-dimensional quasi-Sasakian manifold admitting η-Ricci soliton is an η-Einstein manifold.*

Comparing the equations (8) and (15), we have

$$\begin{aligned}(\lambda + \frac{r}{2} - \beta^2)g(X,Y) \quad + \quad & (\mu + 3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) \\ & = \eta(X)d\beta(\phi Y) + \eta(Y)d\beta(\phi X).\end{aligned} \tag{16}$$

Replacing X by ξ in (16) and using (2), we get

$$(\lambda + \mu + 2\beta^2)\eta(Y) = d\beta(\phi Y). \tag{17}$$

Replacing Y by ϕY in (17) and using (2) yields

$$d\beta(\phi^2 Y) = 0. \tag{18}$$

Since in a 3-dimensional quasi-Sasakian manifold $\phi^2 Y \neq 0$, in general, therefore (18) gives $d\beta = 0$ i.e., $\beta =$ constant. Again using $d\beta = 0$ and in virtue of the fact $\eta(Y) \neq 0$, in general, (17) gives $\lambda + \mu = -2\beta^2$. Therefore we state the following:

Theorem 3.1. *A 3-dimensional non-cosymplectic quasi-Sasakian manifold admitting an η-Ricci soliton is an η-Einstein β-Sasakian manifold with $\lambda + \mu = -2\beta^2$.*

Example:

Here we construct an example [11] of a 3-dimensional non-cosymplectic quasi-Sasakian manifold which verifies the above theorem. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates of $\mathbb{R}^3$. We consider the linearly independent vector fields

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

at each point of M. Let g be a Riemannian metric given by

$$\begin{aligned} g(e_i, e_i) &= 1 \\ g(e_i, e_j) &= 0, \qquad \text{for} \qquad i \neq j. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in TM$.

Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \phi^2 Z = -Z + \eta(Z)e_3,$$
$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for $Z, W \in TM$. Thus for $e_3 = \xi$, (M, ϕ, ξ, η, g) becomes an almost contact metric manifold.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]), \end{aligned} \tag{19}$$

which is known as Koszul's formula.

Using (19) we calculate the following:

$$\begin{aligned} &\nabla_{e_1} e_3 = -\frac{1}{2}e_2, \nabla_{e_1} e_2 = \frac{1}{2}e_3, \nabla_{e_1} e_1 = 0, \\ &\nabla_{e_2} e_3 = \frac{1}{2}e_1, \nabla_{e_2} e_2 = 0, \nabla_{e_2} e_1 = -\frac{1}{2}e_3, \\ &\nabla_{e_3} e_3 = 0, \nabla_{e_3} e_2 = \frac{1}{2}e_1, \nabla_{e_3} e_1 = -\frac{1}{2}e_2. \end{aligned} \tag{20}$$

Therefore for $X = a_1e_1 + a_2e_2 + a_3e_3$, a_1, a_2, a_3 being functions, we have

$$\nabla_X e_3 = \frac{1}{2}\phi X. \tag{21}$$

Therefore the manifold under consideration is a 3-dimensional non-cosymplectic quasi-Sasakian manifold with $\beta = -\frac{1}{2}$.

It is known that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{22}$$

Using the above results, (22) gives

$$\begin{aligned} &R(e_1,e_2)e_3 = 0, R(e_2,e_3)e_3 = \frac{1}{4}e_2, R(e_1,e_3)e_3 = \frac{1}{4}e_1, \\ &R(e_1,e_2)e_2 = -\frac{3}{4}e_1, R(e_2,e_3)e_2 = \frac{1}{4}e_3, R(e_1,e_3)e_2 = 0, \\ &R(e_1,e_2)e_1 = \frac{3}{4}e_2, R(e_2,e_3)e_1 = 0, R(e_1,e_3)e_1 = \frac{1}{4}e_3. \end{aligned}$$

From the above relations we obtain the following non-vanishing Ricci tensors:

$$S(e_1,e_1) = -\frac{1}{2}, S(e_2,e_2) = -\frac{1}{2}, S(e_3,e_3) = \frac{1}{2}.$$

Therefore for $Y = b_1e_1 + b_2e_2 + b_3e_3$ we have

$$S(X,Y) = -\frac{1}{2}a_1b_1 - \frac{1}{2}a_2 + \frac{1}{2}a_3b_3. \tag{23}$$

Hence

$$\begin{aligned} & 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) \\ = \; & (2\lambda - 1)a_1b_1 + (2\lambda - 1)a_2b_2 + (2\lambda + 2\mu + 1)a_3b_3. \end{aligned} \tag{24}$$

From (24) it is clear that for $\lambda = \frac{1}{2}$ and $\mu = -1$

$$2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

i.e.,

$$\pounds_{e_3} g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{25}$$

since in 3-dimensional quasi-Sasakian manifold $\pounds_{e_3} g = 0$.

Therefore $(M, \xi, g, \lambda, \mu)$ is an η-Ricci soliton for $\lambda = \frac{1}{2}$ and $\mu = -1$. Also we see that $S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y)$ and $\lambda + \mu = -\frac{1}{2} = -2\beta^2$. Therefore the manifold is an η-Einstein β-Sasakian manifold with $\lambda + \mu = -2\beta^2$, which verifies the Theorem 3.1.

Next part of this section is devoted to the study of second order parallel tensor in a 3-dimensional quasi-Sasakian manifold. Let M be a 3-dimensional quasi-Sasakian manifold. Suppose that M is endowed with a second order covariant tensor field α satisfying $\nabla\alpha = 0$, where ∇ is the Levi-Civita connection on M.

Therefore we have

$$\alpha(R(X,Y)Z, W) + \alpha(Z, R(X,Y)W) = 0. \tag{26}$$

In particular for $W = Z = \xi$, (26) yields

$$\alpha(R(X,Y)\xi, \xi) = 0. \tag{27}$$

In a non-cosymplectic 3-dimensional quasi-Sasakian manifold admitting an η-Ricci soliton, from (7) we have

$$R(X,Y)\xi = \beta^2[\eta(Y)X - \eta(X)Y], \tag{28}$$

since $\beta =$ constant.

Therefore using (28) in (27) yields

$$\beta^2[\eta(Y)\alpha(X,\xi) - \eta(X)\alpha(Y,\xi)] = 0. \tag{29}$$

Since the manifold under consideration is non-cosymplectic, hence for $Y = \xi$ (29) gives

$$\alpha(X,\xi) = g(X,\xi)\alpha(\xi,\xi). \tag{30}$$

Differentiating (30) covariantly along Y, we get

$$\alpha(\nabla_Y X,\xi)+\alpha(X,\nabla_Y\xi) = \alpha(\xi,\xi)[g(\nabla_Y X,\xi)+g(X,\nabla_Y\xi)+2g(X,\xi)\alpha(\nabla_Y\xi,\xi). \tag{31}$$

Using (5) and (30) in (31), we obtain

$$\alpha(X,\phi Y) = [\alpha(\xi,\xi)g(X,\phi Y) + 2\eta(X)\alpha(\phi Y,\xi)], \tag{32}$$

since $\beta \neq 0$.

Now, from (30) we see that $\alpha(\phi X,\xi) = 0$. Therefore from (32), we have

$$\alpha(X,\phi Y) = \alpha(\xi,\xi)g(X,\phi Y). \tag{33}$$

Replacing Y by ϕY in (33) and using (2) and (30), we obtain

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y). \tag{34}$$

As α is a parallel tensor field, $\alpha(\xi,\xi)$ is constant. Therefore in view of (34) we state the following:

Proposition 3.2. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold, any parallel symmetric* $(0,2)$*-tensor field is a constant multiple of the metric tensor.*

Now, we consider the symmetric parallel tensor field $\alpha = \pounds_\xi g + 2S + 2\mu\eta\otimes\eta$. Therefore $\alpha(\xi,\xi) = 2S(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi)$.

In view of (15) we have $S(\xi,\xi) = -\lambda - \mu$. Therefore $\alpha(\xi,\xi) = -2\lambda$ and we have $\pounds_\xi g + 2S + 2\mu = -2\lambda g$. This relation defines an η-Ricci soliton on M. Therefore we state the following:

Theorem 3.2. *Let M be a 3-dimensional non-cosymplectic quasi-Sasakian manifold. If the symmetric second order tensor field $\alpha = \pounds_\xi g + 2S + 2\mu\eta\otimes\eta$ is parallel with respect to the Levi-Civita connection of the metric tensor, then M admits an η-Ricci soliton.*

4 η-Ricci solitons in quasi-Sasakian manifolds with some curvature properties of Ricci tensor

First part of this section deals with the η-Ricci soliton in 3-dimensional non-cosymplectic quasi-Sasakian manifolds with Ricci tensor of Codazzi-type.

Differentiating (15) covariantly along Z and using (9), we have

$$(\nabla_Z S)(X,Y) = \mu\beta[g(\phi Z,X)\eta(Y) + g(\phi Z,Y)\eta(X)]. \tag{35}$$

Using (35) in (13), we get

$$\mu\beta[g(\phi Z,X)\eta(Y) + g(\phi Z,Y)\eta(X)] = \mu\beta[g(\phi Y,Z)\eta(X) + g(\phi Y,X)\eta(Z)]. \tag{36}$$

Replacing Z by ξ in (36) and using (2), we obtain

$$\mu\beta g(\phi Y,X) = 0. \tag{37}$$

Since the manifold under consideration is non-cosymplectic and $g(\phi Y,X)\neq 0$, in general, therefore (36) yields $\mu = 0$. Therefore the η-Ricci soliton becomes Ricci soliton. Hence we state the following:

Theorem 4.1. *An η-Ricci soliton in a non-cosymplectic 3-dimensional quasi-Sasakian manifold whose Ricci tensor is of Codazzi-type becomes a Ricci soliton.*

Again for $\mu = 0$, (15) becomes

$$S(X,Y) = -\lambda g(X,Y). \tag{38}$$

Therefore the manifold becomes an Einstein manifold. Again it is known that [20] a 3-dimensional Einstein manifold is a manifold of constant curvature. Thus we have:

Corollary 4.1. *An η-Ricci soliton in a non-cosymplectic 3-dimensional quasi-Sasakian manifold whose Ricci tensor is of Codazzi-type is a manifold of constant curvature.*

In the next part of this section we prove the following:

Theorem 4.2. *In a non-cosymplectic 3-dimensional quasi-Sasakian manifold admitting an η-Ricci soliton, the Ricci tensor is cyclic parallel.*

Proof. In view of (35) and (4), we have

$$\begin{aligned} & (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = \; & \mu\beta[\eta(Z)\{g(\phi X,Y) + g(X,\phi Y)\} + \eta(Y)\{g(\phi X,Z) \\ & + g(\phi Z,X)\} + \eta(X)\{g(\phi Y,Z) + g(\phi Z,X)\}] \\ = \; & 0. \end{aligned}$$

Hence the theorem.

5 ϕ-Ricci symmetric η-Ricci soliton in 3-dimensional quasi-Sasakian manifold

In this section we study ϕ-Ricci symmetric η-Ricci soliton on 3-dimensional quasi-Sasakian manifolds. A quasi-Sasakian manifold is said to be ϕ-Ricci symmetric if

$$\phi^2(\nabla_X Q)Y = 0, \tag{39}$$

for all vector fields X, Y.

In view of (15), the Ricci operator Q can be written as

$$QY = -\lambda Y - \mu\eta(Y)\xi, \tag{40}$$

for $Y \in TM$.

Differentiating (40) covariantly along the vector field X and using (5) and (9), we obtain

$$(\nabla_X Q)Y = \mu\beta[g(\phi X, Y)\xi + \eta(Y)\phi X]. \tag{41}$$

Therefore in view of (2), we get

$$\phi^2(\nabla_X Q)Y = -\beta\mu\eta(Y)\phi X. \tag{42}$$

Here we consider the manifold to be non-cosymplectic. Therefore (42) yields $\mu = 0$. Hence we state the following:

Theorem 5.1. *A ϕ-Ricci symmetric η-Ricci soliton on 3-dimensional non-cosymplectic quasi-Sasakian manifolds becomes a Ricci soliton.*

Again $\mu = 0$ implies that the manifold is an Einstein one and hence the manifold is of constant curvature. Thus we have:

Corollary 5.1. *A ϕ-Ricci symmetric η-Ricci soliton on a 3-dimensional non-cosymplectic quasi-Sasakian manifold is a manifold of constant curvature.*

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