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E-Strings and $N = 4$ Topological Yang-Mills Theories

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We study certain properties of six-dimensional tensionless E-strings (arising from zero size E_8 instantons). In particular we show that n E-strings form a bound string which carries an E_8 level n current algebra as well as a left-over conformal system with $c = 12n - 4 - \frac{248n}{n+30}$, whose characters can be computed. Moreover we show that the characters of the n-string bound state are captured by $N = 4$ $U(n)$ topological Yang-Mills theory on $\frac{1}{2}K3$. This relation not only illuminates certain aspects of E-strings but can also be used to shed light on the properties of $N = 4$ topological Yang-Mills theories on manifolds with $b_2^+ = 1$. In particular the E-string partition functions, which can be computed using local mirror symmetry on a Calabi-Yau three-fold, give the Euler characteristics of the Yang-Mills instanton moduli space on $\frac{1}{2}K3$. Moreover, the partition functions are determined by a gap condition combined with a simple recurrence relation which has its origins in a holomorphic anomaly that has been conjectured to exist for $N = 4$ topological Yang-Mills on manifolds with $b_2^+ = 1$ and is also related to the holomorphic anomaly for higher genus topological strings on Calabi-Yau threefolds.

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1. Introduction

One of the interesting new results in string theory has been the discovery of various kinds of strings in addition to the fundamental string. Among these, there are strings where the tension is large, and these have applications in the counting of microstates of blackholes [1]. At the other extreme there are constructions in which the string can become tensionless, and one typically ends up with a critical conformal theory. There are several natural questions one could ask about the physical properties of such strings: How are they different from fundamental strings? What are their excitations? Do tensionless strings bind together?

One of the commonalities between fundamental and non-critical strings is that, for a sufficient number of compactified dimensions, all of these strings can be related to M5 branes wrapped around a four cycle $\mathcal{N}$ (for applications to black hole counting in this context see [2,3]). For example, the type II string can be viewed as an M5 brane wrapped around T^4, and the heterotic string can be viewed as an M5 brane wrapped around $K3$. There are two well-known examples of non-critical strings in six dimensions, one with $(0,2)$ (space-time) supersymmetry [4] and the other with $(0,1)$ (space-time) supersymmetry. The second is dual to the heterotic string with an E_8 instanton of zero size [5,6], which in F-theory corresponds to a particular $\mathbf{P}^1$ shrinking to zero size [7,8]. The $(0,2)$ non-critical string can be viewed as an M5 brane wrapped around $\mathbf{P}^1 \times T^2$, while the $(0,1)$ non-critical string can be viewed as an M5 brane wrapped around $\frac{1}{2}K3$. The purpose of this paper is to continue the study of the BPS states of the $(0,1)$ non-critical string, initiated in [9] and further studied in detail in [10,11]. We will call such strings, E-strings, for their association with the shrinking E_8 instanton. These strings have less world-sheet supersymmetry, $(0,4)$, in comparison to the other string, which has $(4,4)$ world-sheet supersymmetry. The generalized elliptic genus, which counts BPS states weighted by $(-1)^F$, vanishes for the $(0,2)$ string, but it is much more informative for the $(0,1)$ string, and will be discussed extensively in this paper[1].

[1] Note that $\mathbf{P}^1 \times T^2$ can be viewed as $\frac{1}{2}T^4$ in the sense that $\mathbf{P}^1 = T^2/Z_2$. It turns out that the low energy degrees of freedom on the $(4,4)$ string is exactly half of the degrees of freedom of the type II string (in the lightcone) which in turn corresponds to an M5 brane wrapped around T^4. Similarly for the $(0,4)$ case, which corresponds to an M5 brane wrapped around $\frac{1}{2}K3$, the degrees of freedom of the single E-string is half that of the $E_8 \times E_8$ heterotic string, which in turn corresponds to the M5 brane wrapped around $K3$. So perhaps another set of suitable names for these strings are $\frac{1}{2}$ type II strings and $\frac{1}{2}$ heterotic strings!

The study of the BPS states suggest many new things about the E-strings. It shows that n of them form bound states at threshold and that they carry an E_8 level n Kac-Moody algebra. Moreover, there are additional left-over degrees of freedom on the left-movers for which we compute the corresponding characters.

One of the key observations in this paper is to use the connection of such BPS counts with the computation of $N = 4$ topological Yang-Mills [12]. This relation shows, in particular that the number of BPS states consisting of n wrapped E-strings with total momentum k around the circle, which is given by counting holomorphic curves on $\frac{1}{2}K3$, is related (by an application of mirror symmetry) to the partition function of $N = 4$ topological $U(n)$ Yang-Mills on $\frac{1}{2}K3$ with instanton number k. Using this observation we relate the holomorphic anomaly discovered for $N = 4$ $SU(2)$ Yang-Mills on manifolds with $b_2^+ = 1$ [12], to the holomorphic anomaly in the BPS state partition function derived in [11]. In [11] the holomorphic anomaly was derived for arbitrary n, and it now gives us the holomorphic anomaly of $SU(n)$ $N = 4$ gauge theories. We will also give an interpretation of the holomorphic anomaly from the viewpoint of counting holomorphic curves, which thus connects to the holomorphic anomaly of Kodaira-Spencer theory [13]. Furthermore we interpret the existence of this holomorphic anomaly as coming from the fact that n E-strings form bound states at threshold.

The organization of this paper is as follows: In section 2 we review various interconnected ways of looking at the E-strings. We also relate the BPS count with the $N = 4$ topological partition function. In section 3 we discuss general aspects of $N = 4$ topological Yang-Mills theory on four manifolds, and we develop the $N = 4$ $U(n)$ Yang-Mills theory on $K3$ in detail (generalizing the results in [12] from $SU(2)$ to $U(n)$). We then discuss aspects of $N = 4$ Yang-Mills on $\frac{1}{2}K3$ which follow from general principles and a few conjectures. In section 4 we discuss the general results concerning the BPS state partition function for E-strings. In section 5 we review, and re-derive in a new way, the mirror of $\frac{1}{2}K3$ which is needed for the BPS count. In section 6 we write very explicitly the partition function for low winding numbers (up to 4). The results are compared with the predictions based on $N = 4$ Yang-Mills. In section 7 we try to interpret some aspects of the BPS partition function as well as the meaning of the anomaly in terms of the E-string. We conclude, in section 8, with some suggestions for further study.

2. F-theory, M-theory, Yang-Mills theory and E-strings

In this section we will review the various descriptions of solitonic strings in F-theory and M-theory. We start with the $N=1$ supersymmetric compactification of F-theory to six dimensions, and then we discuss how the same string looks from the M-theory perspective. That is, upon a further compactification to five dimensions on an S^1, such a string theory has two dual descriptions in M-theory that depend upon whether the string wraps S^1 or not. Both descriptions are relevant for us and will be reviewed here. However it is the M5-brane description of the string that enables us to connect the partition function of the string with that of topologically twisted $N=4$ supersymmetric Yang-Mills theory.

2.1. General overview

Consider compactification of F-theory down to six dimensions on an elliptic Calabi-Yau threefold, K, with a section, *i.e.* with a two complex dimensional base sub-manifold B [14,7]. Let $\Sigma \subset B$ denote a Riemann surface in the base. Let $\hat{\Sigma}$ denote the complex two dimensional elliptic manifold consisting of Σ and the elliptic fiber of K over each point. Consider the 3-brane of type IIB wrapped around Σ, which gives rise to a string in six dimensions. One is interested in the low energy degrees of freedom on this string. One aspect of these degrees of freedom is reflected in the BPS states that result upon further wrapping of this string n times around a circle, carrying some momenta p. This gives rise to BPS particles in five dimensions. From the M-theory perspective these are described as follows: The compactification of F-theory on $K \times S^1$, where S^1 has radius R, is equivalent to an M-theory compactification on K in which the Kähler class of the elliptic fiber is $1/R$. In M-Theory, the BPS states described above correspond to M2 branes wrapped around K whose H_2 class is given by $n[\Sigma]+p[T^2]$, where $[T^2]$ denotes the class of the elliptic fiber. Thus counting the BPS states of the six-dimensional string wrapped around the circle amounts to counting holomorphic curves on K, which in turn can be done using mirror symmetry techniques. This was the method exploited in [9] and will be discussed further in sections 4 and 5.

The counting of these five-dimensional BPS states can be related to another computation. Consider a string that remains a string in five dimensions, that is a string that does not wrap the circle. In terms of M-theory this is described by wrapping an M5 brane around $\hat{\Sigma}$. Now compactify one more dimension on a circle of radius R' and wrap this string around the new circle so that the M5 brane is wrapped around $\hat{\Sigma} \times S^1$. The BPS states of this wrapped state can be computed by its partition function, *i.e.* by taking the world-volume of the M5 brane to be $\hat{\Sigma} \times T^2$. Moreover if we are interested in strings wound n times around the second S^1 we should consider n M5 branes whose world-volume is given by $\hat{\Sigma} \times T^2$. However as is well known n M5 branes wrapped around T^2 give rise to $N = 4$ $U(n)$ Yang-Mills in four dimensions [4,15,16], (where the momentum number p gets mapped to the instanton number). We are thus left with the partition function of an $N = 4$ Yang-Mills theory on $\hat{\Sigma}$. More precisely we end up with a topologically twisted version of $N = 4$, as is expected on general grounds [17]. In the next section we show that the relevant twist is the one already considered in [12].

Putting this all together, we are relating the number of holomorphic curves[2] wrapped around $n[\Sigma] + p[T^2]$ to the partition function of $N = 4$ $U(n)$ Yang-Mills theory on $\hat{\Sigma}$. This final result has a relatively simple alternative explanation in terms of the type IIA description of the same states coming from a further compactification on a circle: Consider n D4 branes wrapped around $\hat{\Sigma}$ and consider the number of cohomology elements in the instanton moduli space with instanton number p. This is the same as the number of bound states of n D4 branes with p D0 branes [18,19]. Now, recall that $\hat{\Sigma}$ is elliptic and we are considering the limit where the size of the elliptic fiber is small. In this limit we can go to the T-dual description where the D4 branes become n D2 branes wrapped around $[\Sigma]$ and the D0 branes become p D2 branes wrapped around $[T^2]$. For BPS states we are interested in their bound states, which are represented by a holomorphic curve in $\hat{\Sigma}$ in the class $n[\Sigma] + p[T^2]$. Which thus brings us to the same conclusion via a more direct argument. In fact this T-duality, relating rank n-bundles on elliptic manifolds to holomorphic curves is also well known mathematically and is known as the spectral curve and has been studied in the context of F-theory in [20,21].

We can take this result one step further by a second application of mirror symmetry. Namely, we can use (local version of) mirror symmetry [9,22,23] to count the number of holomorphic curves in $\hat{\Sigma}$. We thus conclude that *the partition function of topologically-twisted $N = 4, U(n)$ Yang-Mills on an elliptic four-dimensional manifold $\hat{\Sigma}$, can be computed by a double application of mirror symmetry. Moreover, this partition function counts the BPS states of n times wrapped strings obtained in M-theory by wrapping M5 branes around $\hat{\Sigma}$ or in F-theory of D3 branes wrapped around Σ.*

[2] More precisely, an appropriate Euler characteristic for their moduli spaces.

2.2. Tensionless E-strings from F-theory

In the foregoing description we have described how one gets a string in six dimensions for every holomorphic curve $\Sigma \subset B$ in an F-theory compactification on a 3-fold. However to get an interesting critical theory the resulting string will have to be tensionless. Since the tension of the string is proportional to the volume of Σ, a tensionless string will arise only if that Σ is shrinkable. This can only happen if Σ is a $\mathbf{P}^1$. Moreover the normal bundle of $\mathbf{P}^1$ in the base B should be negative [7]. Denote its normal bundle in B by $O(-k)$. The fact that c_1 of the 3-fold is zero implies that the tangent direction on the elliptic fiber over $\mathbf{P}^1$ will then correspond to an $O(k-2)$ bundle. We restrict our attention to the situation where the elliptic fibration is generically non-singular. (Singular fibrations are also interesting and have also been considered in the literature [7,24,25]). Non-singularity requires that the cotangent bundle of the elliptic fiber have a section over $\mathbf{P}^1$ which can happen only if $k = 1, 2$. Before discussing these two cases, we note that the choice, $k = 0$, which is not shrinkable at finite distance in moduli, corresponds to $\hat{\Sigma} = K3$, and the corresponding string is the heterotic string.

For $k = 2$, the elliptic fibration is trivial over $\mathbf{P}^1$, *i.e.* , $\hat{\Sigma} = \mathbf{P}^1 \times T^2$, and we get the tensionless string corresponding to $(4, 4)$ susy on the worldsheet (and is the same as the tensionless string that appears for type IIB theory near an A_1 singularity [4]). The choice $k = 1$ corresponds to $\hat{\Sigma} = \frac{1}{2}K3$. The reason for this terminology is that, one gets $K3$ as the elliptic fibration over $\mathbf{P}^1$ with $k = 0$, and then the modulus of the torus covers the fundamental domain 24 times (corresponding to 24 cosmic strings), the $\frac{1}{2}K3$ corresponds to an elliptic manifold over $\mathbf{P}^1$ whose modulus covers the fundamental domain 12 times. More explicitly, it can be described as

$$y^2 \;=\; x^3 + f(z)x + g(z) \;, \tag{2.1}$$

where f and g denote polynomials of degree 4 and 6 in z respectively, where z parameterizes the $\mathbf{P}^1$. The elliptic fiber is thus described by (x, y) subject to the equation (2.1). The weights are such that y belongs to an $O(3)$ bundle and x to an $O(2)$ bundle on $\mathbf{P}^1$, and so the cotangent bundle of the elliptic fiber is an $O(1)$ bundle $[dy/x]$ over $\mathbf{P}^1$.

The non-zero hodge numbers for this manifold are

$$h^{0,0} = h^{2,2} = 1 \;, \qquad h^{1,1} = 10$$

In particular this implies that $b_2^+ = 1$ (for a Kähler manifold of complex dimension 2 we have $b_2^+ = 1 + 2h^{2,0}$). This manifold can also be obtained by blowing up $\mathbf{P}^2$ at 9 points (the position of the ninth point is determined by the other eight points). For this reason the manifold $\frac{1}{2}K3$ is a del Pezzo surface sometimes denoted by $\mathcal{B}_9$, and we will use this notation interchangeably.

We will call the resulting string the E-string. The reason for this terminology is that the E-string arises in a dual description of an E_8 instanton in heterotic string theory shrinking to zero size. This connection is explained in [8,7].

Since the E-string constructed here lives in six-dimensions, and has $(0,4)$ world-sheet supersymmetry, it is natural to define the following generalized form of the elliptic genus for such a string:

$$Z \;=\; \frac{\tau_2^2}{V_4}\mathrm{Tr}\left[\,(-1)^F\;F_R^2\;q^{L_0}\;\overline{q}^{\overline{L_0}}\,\right]. \tag{2.2}$$

Here we are imagining computing the partition function on a torus with $q = \exp(2\pi i\tau)$. The factor of F_R^2 has been inserted to soak up the fermion zero-modes, and we have divided by the volume, V_4, of the four non-compact bosons representing transverse position of the string. We have also multiplied by a factor of τ_2^2, where $\tau_2 = Im(\tau)$, so as to cancel the factor of $1/\tau_2^2$ coming from the integration over the momenta of the string in the transverse directions. This partition function, Z, has (holomorphic, anti-holomorphic) modular weight $(-2,0)$: Without the factor of τ_2^2 and without the insertion of F_R^2, Z would be a modular invariant. Each factor of F_R increases the anti-holomorphic weight by one unit, and each factor of τ_2 shifts the weight $(-1,-1)$. We will discuss this partition function more extensively below.

2.3. Yang-Mills partition functions from M5-branes

Consider n parallel $M5$ branes whose world-volume is given by a six-dimensional sub-manifold of spacetime

$$\mathcal{M}^6 \;=\; T^2 \times \mathcal{N}^4$$

As is well known, compactification of n parallel $M5$ branes on T^2 yields a $U(n)$ $N=4$ supersymmetric Yang-Mills on the left-over world-volume, where the complex structure τ of T^2 gets identified with the complexified $U(n)$ Yang-Mills coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$. In this description, the Montonen-Olive duality is a consequence of the classical $SL(2,\mathbf{Z})$ symmetry of T^2 [26,4].

This relation with $N=4$ Yang-Mills implies that the partition function of n five-branes on $T^2 \times \mathcal{N}^4$ is the same as that of $U(n)$ N=4 Yang-Mills on $\mathcal{N}^4$. We can also view this slightly differently: We can think of n parallel five-branes wrapped around $\mathcal{N}^4$

as giving rise to a string, and then the partition function of this theory is the same as the partition function of the effective $(1+1)$-dimensional theory on the T^2 with modular parameter τ. This viewpoint has been suggested before in [27].

Depending upon the situation in which this arises in the string theory, one typically ends up with a topologically twisted version of $N = 4$ Yang-Mills theory [17]. There are three possible twistings for $N = 4$ [12], and we need to determine the relevant one here. For us the $\mathcal{N}^4$ is embedded in a Calabi-Yau 3-fold, and the five scalars of the five-brane, correspond to normal deformations of $\mathcal{N}^4$. The normal bundle will be trivial only in the uncompactified directions (*i.e.* the normal directions that are *not* in the Calabi-Yau) and so three of the scalars must continue to transform as scalars after twisting. The other two will be a section of the canonical bundle on $\mathcal{N}^4$, *i.e.* a section of the form $f(z_1, z_2, \overline{z}_1, \overline{z}_2)dz_1 \wedge dz_2$. The last scalar of Yang-Mills in four dimensions comes from giving the anti-symmetric 2-form of the five-brane worldvolume an expectation value on the T^2. The result is thus a periodic scalar. The three uncompactified scalars correspond to the transversal position of the left-over string of the M5 branes in uncompactified five-dimensional spacetime. This uniquely fixes the twisting of the $N = 4$ Yang-Mills and is the one already studied in detail in [12]. This also fixes the supersymmetry on the effective $(1+1)$-dimensional theory to be generically $(0,4)$ (for $K3$ the supersymmetry is $(0,8)$).

Strictly speaking, in such a supersymmetric theory the partition function of n parallel M5 branes on $T^2 \times \mathcal{N}^4$ is zero. This is because of the supersymmetric degree of freedom associated with the translations in the uncompactified space. Equivalently, in the Yang-Mills language, this is related to the zero modes coming from the $U(1) \subset U(n)$. Suppressing translations, by absorbing the correponding fermion zero modes, will result generically in a non-vanishing partition function. In the $(1+1)$-dimensional, effective theory, this means that we really need to compute the generalized form of the elliptic genus described earlier:

$$Z \;=\; \frac{\tau_2^{3/2}}{V_3} \mathrm{Tr} \left[\, (-1)^F \; F_R^2 \; q^{L_0} \; \overline{q}^{\overline{L_0}} \, \right] , \tag{2.3}$$

where $q = \exp(2\pi i\tau)$. Once again we divide by the volume of the space tranverse to the string, and introduce factors of τ_2 to cancel those coming from the integration of the corresponding momenta. This transverse space is three-dimensional since we now have the E-string in five-dimensions, or equivalently, only three of the five scalars of the M5-brane are non-compact. This partition function, Z, is expected to be a modular form of (holomorphic,anti-holomorphic) weight $(-3/2, 1/2)$ (which is $(0,2)$ shifted by $(-3/2, -3/2)$ coming from the factors of τ_2). If the corresponding $(1+1)$-dimensional theory has a discrete spectrum then one can argue that the only $\overline{\tau}$ dependence of Z comes from the bosonic zero modes. This is because the $N = 4$ supersymmetry for the right-movers implies that the right-moving oscillator contributions cancel. In the gauge language these bosonic zero-modes are related to the bosonic modes in $U(1) \subset U(n)$. In particular the $SU(n)$ part of the partition function is expected to be a function of τ only.

For $K3$ one has eight right-moving bosons and eight right-moving fermions. This means that the generalized elliptic genus (2.3) has to have an insertion of F_R^4 (as opposed to F_R^2) to absorb the fermion zero-modes. Moreover, in the compactification of M-theory on $K3$ one is in seven dimensions, with five dimensions tranverse to the string, and hence one has to divide by the volume, V_5, of this transverse space. Following the practice above, we will also multiply by $\tau_2^{5/2}$ to compensate for the transverse momentum integrations, and so the partition function Z then has a weight of $(0, 4) + (-5/2, -5/2) = (-5/2, 3/2)$.

As discussed above, the function Z has another interpretation: it is the partition function of the BPS states obtained by wrapping a string coming from n parallel M5-branes on $\mathcal{N}^4$ around another circle. In this interpretation the coefficients of q^p in Z count the BPS states which have momentum p around the circle. From the Yang-Mills point of view, p is the same as the instanton number (up to a fixed shift of $-n\chi(\mathcal{N}^4)/24$). Moreover, for n parallel M5 branes the resulting string can be viewed as bound states of n singly wound strings around the circle.

3. The partition functions for $N = 4$ Yang-Mills

The partition function, Z, of topologically twisted $N = 4$ Yang-Mills was studied for various manifolds in [12]. The most concrete results were for an $SU(2)$ gauge group on Kähler manifolds with $b_2^+ > 1$, where b_2^+ denotes the dimension of the self-dual 2-forms. Moreover, general considerations led to certain predictions for all $SU(n)$ on these manifolds. The more difficult problem of $SU(2)$ gauge theory on $\mathbf{P}^2$ was also considered in [12]. The difficulty lies in the fact that $b_2^+ = 1$, and so there are no holomorphic deformations of the canonical bundle. This means that one cannot deform the theory to give masses to the adjoint fields in a manner that preserves $N = 1$ supersymmtery. We will review $N = 4$ Yang-Mills on $\mathbf{P}^2$ later in this section and also give a more detailed discussion of Yang-Mills theories and M5-branes on manifolds lacking holomorhic deformations of their canonical bundles.

Evidence was provided in [12] that for an $SU(n)$ gauge group on an arbitrary four manifold, $\mathcal{N}^4$, with $b_2^+ > 1$, Z is a modular form of an $SL(2,Z)$ subgroup with (holomorphic, anti-holomorphic) weight $(-\chi/2, 0)$, where χ denotes the Euler characteristic of $\mathcal{N}^4$. The partition function Z has an expansion of the form

$$Z \ = \ q^{-\frac{n\chi}{24}} \ \sum_{k=0}^{\infty} \ c_k \ q^k \tag{3.1}$$

where c_k denotes the contribution of the instanton number k sector to the partition function. The overall shift of $q^{-\frac{n\chi}{24}}$ is needed to make Z a modular form. It was also noted in [12] that when the manifold is $\mathbf{P}^2$ (and conjectured more generally for Kähler manifolds with $b_2^+ = 1$), Z can only be made modular by adding $\overline{\tau}$ dependence to Z. Having done this, the expansion above is valid if one considers τ and $\overline{\tau}$ as formal variables, and τ is kept fixed while $\overline{\tau} \to \infty$.

The partition functions for $SU(n)/Z_n$ gauge groups were also studied in [12]. For such quotients we can also have non-trivial 't Hooft fluxes through the two cycles of the manifold [28]. Hence, one can write a partition function Z_α for each 't Hooft flux $\alpha \in H^2(M, Z_n)$. Furthermore, for each Z_α we get an expansion of the form (3.1), except that the instanton numbers are shifted by $\frac{\alpha^2}{2n} - \frac{\alpha^2}{2}$, and are hence generically fractional. It was also shown that under $\tau \to -1/\tau$ the Z_α mix according to

$$Z_\alpha \to \pm n^{-b_2/2} (\tau/i)^{-\chi/2} \sum_\beta \exp\left(\frac{2\pi i \alpha \cdot \beta}{n}\right) Z_\beta \ , \tag{3.2}$$

where b_2 denotes the second betti number of $\mathcal{N}^4$. In the next sub-section we discuss some of the results of [12] for $K3$. This will set the stage for the generalization to $\frac{1}{2}K3$, which is the main focus of this paper.

3.1. Partition functions for $K3$

We begin by describing the result for $K3$, with gauge group $SU(2)$, and we will generalize the result of [12] by considering a $U(2)$ gauge theory. This is straightforward as it merely involves the addition of a trivially computable free $U(1)$ theory. In fact, given that we have already suppressed the fermionic degrees of freedom, this only includes the volume factor from bosons, together with the $U(1)$ fluxes through 2-cycles. In particular we can view this theory as

$$\frac{SU(2) \times U(1)}{Z_2} \ ,$$

and so we just have to combine the partition functions of $SU(2)/Z_2$ corresponding to various ‘t Hooft fluxes with the corresponding $U(1)$ fluxes.

To get started we first consider the $U(1)$ theory by itself, which corresponds to a single five-brane. The self-dual field strength antisymmetric field B_{ij} on the five-brane leads to 19 left-moving and 3 right-moving bosonic modes, all of which are periodic. The three uncompactified scalars give 3 additional left- and right-movers. The other two scalars transform according to the canonical bundle on $K3$, which is in turn trivial, so this will give an additional two left- and two right-moving scalars. All together we have 24 left-moving and 8 right-moving bosonic degrees of freedom. Finally, there are also 8 right-moving fermionic modes. This is indeed the oscillator content of the heterotic string.

Since the effective $(1+1)$-dimensional theory has an $(0,8)$ supersymmetry, we need to consider the generalized elliptic genus

$$Z_1 \;=\; \frac{\tau_2^{5/2}}{V_5} \mathrm{Tr}\left[(-1)^{F_R} {F_R}^4 q^{L_0} \bar{q}^{L_0}\right] \tag{3.3}$$

in order to get a non-zero result. As we remarked earlier, the factor of $\tau_2^{5/2}/V_5$ comes from the five uncompactified bosons transverse to the string, and the modular weight of Z is $(-5/2, 3/2)$.

The result is then made up of two parts: (i) A lattice partition function of the $U(1)$ fluxes through each 2-cycle, and (ii) the partition function of the 24 bosonic zero modes of the single five-brane on $K3$[29]:

$$Z_1 \;=\; G(q)\; \theta_{\Gamma^{19,3}}(q,\overline{q}) \;,$$

where

$$G(q) \;=\; \frac{1}{\eta(q)^{24}} \;=\; \left(\frac{1}{q^{\frac{1}{24}}\prod_n (1-q^n)}\right)^{24}$$

and

$$\theta_{\Gamma^{19,3}}(q,\overline{q}) \;=\; \sum_{P_L, P_R \in \Gamma^{19,3}} q^{\frac{1}{2}P_L^2}\; \overline{q}^{\frac{1}{2}P_R^2} \;.$$

The lattice, $\Gamma^{19,3}$, is the even, self-dual, integral lattice with signature $(19,3)$ with a choice of polarization (the standard Narain sum). The $U(1)$ gauge fluxes of the $N=4$ gauge theory correspond to self-dual H-fluxes on the five-brane. Note that Z_1 has the desired modular weight of $(-5/2, 3/2)$ with $(19/2, 3/2)$ coming from the θ-function of the $\Gamma^{19,3}$ lattice, and $(-12, 0)$ coming from the η-functions.

Now we consider the $U(2)$ theory, *i.e.* the theory of two five-branes wrapped on $K3 \times T^2$. The coupling constant of $U(2)$ relative to that of $U(1)$ is naturally halved via $\tau \to \tau/2$. This means that if we were considering the $U(1)$ theory we would replace $(q, \overline{q})$ with $(q^{\frac{1}{2}}, \overline{q}^{\frac{1}{2}})$ in the lattice theta function. However this is not quite correct because the corresponding fluxes are correlated with the $SU(2)/Z_2$ 't Hooft fluxes. These come in three categories: trivial, even and odd. The trivial flux is correlated with those lattice vectors of $\Gamma^{19,3}$ that are twice another vector. The even 't Hooft fluxes correlate with $U(1)$ fluxes with $\frac{1}{2}(P_L^2 - P_R^2)$ even, but where (P_L, P_R) is not twice another vector. The odd fluxes correlate with $U(1)$ fluxes where $\frac{1}{2}(P_L^2 - P_R^2)$ is odd. The three pieces of $U(1)$ fluxes correspond to

$$\theta_0 = \theta_{\Gamma^{19,3}}(q^2, \overline{q}^2) \ ,$$

$$\theta_{even} = \frac{1}{2}(\theta_{\Gamma^{19,3}}(q^{\frac{1}{2}}, \overline{q}^{\frac{1}{2}}) + \theta_{\Gamma^{19,3}}(-q^{\frac{1}{2}}, -\overline{q}^{\frac{1}{2}})) - \theta_{\Gamma^{19,3}}(q^2, \overline{q}^2) \ ,$$

$$\theta_{odd} = \frac{1}{2}(\theta_{\Gamma^{19,3}}(q^{\frac{1}{2}}, \overline{q}^{\frac{1}{2}}) - \theta_{\Gamma^{19,3}}(-q^{\frac{1}{2}}, -\overline{q}^{\frac{1}{2}})) \ .$$

Note that the sum of these three terms is simply $\theta_{\Gamma^{19,3}}(q^{\frac{1}{2}}, \overline{q}^{\frac{1}{2}})$ as it should be. The correlation between $SU(2)/Z_2$ ‘t Hooft fluxes and the $U(1)$ fluxes now imply that the partition function of $U(2)$ is given by

$$Z_2 = Z_0\theta_0 + Z_{even}\theta_{even} + Z_{odd}\theta_{odd}$$

where Z_0, Z_{even} and Z_{odd} denote the corresponding $SU(2)/Z_2$ partition functions. From the results in [12] we have

$$\begin{aligned} Z_0 &= \tfrac{1}{4}G(q^2) + \tfrac{1}{2}[G(q^{\frac{1}{2}}) + G(-q^{\frac{1}{2}})] \ , \\ Z_{even} &= \tfrac{1}{2}[G(q^{\frac{1}{2}}) + G(-q^{\frac{1}{2}})] \ , \qquad Z_{odd} = \frac{1}{2}[G(q^{\frac{1}{2}}) - G(-q^{\frac{1}{2}})] \ . \end{aligned}$$

Therefore we obtain:

$$Z_2 = \tfrac{1}{4}G(q^2)\theta_{\Gamma^{19,3}}(q^2, \overline{q}^2) + \tfrac{1}{2}[G(q^{\frac{1}{2}})\theta_{\Gamma^{19,3}}(q^{\frac{1}{2}}, \overline{q}^{\frac{1}{2}}) + G(-q^{\frac{1}{2}})\theta_{\Gamma^{19,3}}(-q^{\frac{1}{2}}, -\overline{q}^{\frac{1}{2}})] \ . \quad (3.4)$$

Note that the partition function Z_2 is a modular form of $SL(2, Z)$. Moreover it has the same weight as Z_1, namely it has weight $(w_L, w_R) = (-\frac{5}{2}, \frac{3}{2})$. The fact that it is a modular form of $SL(2, Z)$ is consistent with Montonen-Olive self-duality for a $U(2)$ gauge group.

We now try to interpret the Z_2 partition function from the five-brane point of view. If we have two copies of $K3 \times T^2$ in spacetime then this can be viewed as a single five-brane wrapped twice over $K3 \times T^2$. This can be done in several ways by taking coverings over

$K3$ or over T^2. As a hint we can use the fact that a five-brane wrapped around $K3$ is equivalent to the heterotic string. For two copies of the heterotic string all one has to do is double cover the worldsheet. This suggests that we should consider the two five-brane worldvolume to be $K3 \times T^2$ but with the T^2 wrapping twice around the original spacetime T^2. This means that the complex structure of the torus is now going to be different from that of the T^2 in spacetime, depending on how the world-volume T^2 wraps the spacetime T^2. There are three inequivalent ways in which a T^2 can double cover another T^2, resulting in one of the following complex moduli:

$$\tilde{\tau} = 2\tau,\ \frac{\tau}{2},\ \frac{\tau}{2} + \frac{1}{2}\ . \tag{3.5}$$

We would thus expect

$$Z_2(\tau) = \tfrac{1}{4} Z_1(2\tau) + \tfrac{1}{2} Z_1\big(\frac{\tau}{2}\big) + \tfrac{1}{2} Z_1\big(\frac{\tau}{2} + \frac{1}{2}\big) \tag{3.6}$$

where the constant in front of the Z_1 terms is fixed by modularity, up to an overall constant of proportionality which can be fixed by viewing the five-brane partition function as the counting of BPS states. This is in clear agreement with (3.4). It is now easy to generalize this to the case of n parallel fivebranes with worldvolume $K3 \times T^2$, which should be equivalent to $U(n)$ gauge theory on $K3$ (this is similar and related to the observation in [30]). We first have to recall how to enumerate the inequivalent ways in which a T^2 can cover another T^2 n times: One chooses a basis of 2-cycles on T^2, and then the inequivalent n-fold covers are given by the $GL(2)$ transformations that can be written in the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\ ,$$

with $ad = n$ and $b < d$ and $a, b, d \geq 0$. Note that for $n = 2$ this gives the three possibilities in (3.5). Thus, taking into account the modularity of the partition function we find that

$$Z_n = \frac{1}{n^2} \sum_{a,b,d} d\ Z_1\left(\frac{a\tau + b}{d}\right)\ , \tag{3.7}$$

where the sum is over a, b, d satisfying the conditions given above. The modular form Z_n in (3.7) is known as a Hecke transformation of order n [31], which maps a modular form of weight k to another modular form of weight k. For a general modular form with holomorphic/anti-holomorphic weights (w_L, w_R), the powers of n and d that appear in the Hecke transformation are $n^{w_L + w_R - 1}$ and $d^{-w_L - w_R}$.

This general structure is in agreement with what was conjectured in [12]. Namely, it was observed that for manifolds with $b_2^+ > 1$ one can deform the Yang-Mills theory by adding masses to the three chiral fields, so that locally one has an $N = 1$ theory, without changing the topological partition function[3]. In terms of $N = 1$ chiral fields the deformed superpotential takes the form

$$W \;=\; Tr[[X,Y],Z] \;-\; \frac{m}{2}\,Tr(X^2+Y^2+Z^2)$$

where X, Y and Z denote the chiral superfields in the adjoint of $SU(n)$. It was argued that for $SU(n)$ gauge theory these deformations lead to a number of inequivalent vacua, parameterized by how $SU(n)$ is broken in each vacuum. Namely the critical points of W are found by solving

$$[X,Y] = mZ, \quad [Y,Z] = mX, \quad [Z,X] = mY,$$

which after suitable rescaling, implies that X, Y and Z form an n dimensional representation of $SU(2)$. Typically, choosing such vacua results in gauge groups containing at least one unbroken $U(1)$. But these vacua contain extra fermion zero modes and so cannot contribute to the partition function. Hence, we only need to consider those vacua that have no unbroken $U(1)$ groups. If $n = ad$, then this can be done by choosing the n dimensional representation to be d copies of the a-dimensional irreducible representation of $SU(2)$. In this case we find an $N = 1$ theory with $SU(d)$ gauge symmetry. This will have d vacua, which we label by $b = 0, 1, ..., d-1$. Moreover each vacuum is expected to contribute to the total partition function and shifting τ by a constant mixes the different vacua [12]. Thus we find a partition function that matches the Hecke structure of (3.7).

The Hecke structure of (3.7) has a nice interpretation in the five-brane picture. The $GL(2)$ transformation implied by (3.7) rescales the spatial size of the string by d units and the temporal size by a units, in addition to shifting the temporal direction into the spatial direction by b units. This is equivalent to having d wrapped M5-brane strings on top of each other and results in an $SU(d)$ gauge symmetry as expected from field theory analysis. The factor of a in the temporal direction rescales the gauge coupling and the shift by b picks out one of the d vacua by shifting the θ angle.

[3] In fact $(b_2^+ - 1)/2$ is the complex dimension of moduli of holomorphic deformations of $\mathcal{N}^4$. In the five-brane description the existence of these deformations means $\mathcal{N}^4$ can be moved holomorphically in the Calabi-Yau, *i.e.* the five-branes can be moved off of each other.

As an example, consider the case $n = p$, where p is prime. In this case the partition function takes the particularly simple form

$$Z_p = \frac{1}{p^2} Z_1(p\tau) + \frac{1}{p}\Big[Z_1\big(\frac{\tau}{p}\big) + Z_1\big(\frac{\tau}{p}+\frac{1}{p}\big) + ... + Z_1\big(\frac{\tau}{p}+\frac{p-1}{p}\big)\Big] \ , \qquad (3.8)$$

which correctly counts the BPS states for p-th wound heterotic string. This also agrees with the prediction made in [12] for $SU(p)$ on $K3$.

3.2. Comments about Coincident M5 branes and Extensions

In the previous sub-section we have suggested that to compute the partition function of n M5 branes whose world-volume is $K3 \times T^2$, we simply use the result of one $M5$ brane on $K3 \times \tilde{T}^2$ where $\tilde{T}^2$ is n-times wrapped over T^2. As discussed above this is consistent with the view that heterotic string emerges as an M5 brane wrapped around $K3$. However it would be nice to try and justify this step directly from the viewpoint of M5 branes. The potential problem is that the n M5 branes could interact non-trivially and perhaps have non-trivial bound states, whereas the foregoing result implies that the M5 branes do not interact, but simply concatenate to produce an n-fold wrapping. From the perspective of the M5 branes, the main indication that the interactions will be topologically trivial is that n M5 branes on $K3$ can be holomorphically separated: the number of (supersymmetric) normal deformations of $M5$ brane in the Calabi-Yau 3-fold is $h^{2,0}(K3) = 1$. Thus if we use an element $u \in H^{2,0}$ we can holomorphically move branes off one another. It would be nice to make this argument rigorous directly in the context of branes. It is also natural to try to extend this argument to manifolds with $h^{2,0} > 1$ as was done in [12]. Once again, the deformation in normal direction to the $M5$ branes can also be done by choosing a $u \in h^{2,0}$ of the manifold. However, in general the deformations will intersect each other along some divisors D_i. So we would expect the answer for the partition function be similar to that of $K3$ modulo corrections coming from the M5 branes intersecting over D_i, leading to new field theory subsectors. This structure matches the results found in [12] for $SU(2)$, and it is natural to expect to be able to justify the results conjectured in [12] in connection with Montonen-Olive duality using the M5-branes.

3.3. Manifolds with $b_2^+ = 1$

The partition functions of $N = 4$ Yang-Mills on manifolds with $b_2^+ = 1$ are not so easy to compute. From the Yang-Mills point of view it is on such manifolds that the gauge theory cannot be deformed to an $N = 1$ theory by giving masses in a topologically invariant way. From the five-brane point of view, this is reflected by the fact that n five-branes wrapped around such a manifold cannot be seperated in a supersymmetric manner since there are no moduli for holomorphic deformations. The only such case that has been previously studied is $SU(2)/Z_2$ on $\mathbf{P}^2$. The partition function of $SU(2)$ gauge theory on $\mathbf{P}^2$ is naively expected to be a holomorphic function of the coupling, τ, however it was found in [12] that the partition function displays a holomorphic anomaly and thus also depends on $\overline{\tau}$.

There are two choices of ‘t Hooft fluxes for $\mathbf{P}^2$. We denote the partition function for each of these two choices by $G_0 = F_0/\eta^6$ (trivial flux) and $G_1 = F_1/\eta^6$ (non-trivial flux). It was found in [12] that

$$\begin{aligned} \overline{\partial}_\tau F_0 &= \frac{3}{16\pi i}\ \tau_2^{-\frac{3}{2}} \sum_{n\in\mathbf{Z}} \overline{q}^{n^2}\ , \\ \overline{\partial}_\tau F_1 &= \frac{3}{16\pi i}\ \tau_2^{-\frac{3}{2}} \sum_{n\in\mathbf{Z}} \overline{q}^{(n+\frac{1}{2})^2}\ . \end{aligned} \tag{3.9}$$

Note that the power of τ_2 in (3.9) is fixed by modular properties. The existence of this anomaly was interpreted in [12] as the contribution of reducible $SU(2)$ connections in the form of $U(1) \subset SU(2)$ gauge field configurations. Although this has not yet been verified, it seems to be a reasonable conjecture. Note that the theta-function sum on the right hand side of the anomaly has the interpretation of $U(1)$ fluxes for P^2 (which has a single non-trivial 2-cycle). From the viewpoint of two five-branes wrapped around $\mathbf{P}^2$, this would correspond to the reduction $U(2) \to U(1) \times U(1)$, which comes from seperation of the two five-branes in spacetime. Thus we interpret the existence of anomalies as a reflection of the existence of non-trivial bound states at threshold, and these are counted by the five-brane partition function. This aspect will be discussed more extensively later in this paper when we interpret the results found for the BPS states of the E-string.

From the mathematical properties of instanton moduli space on Kähler manifolds with $b_2^+ = 1$ it has become clear that there are various chambers for the partition function of topological Yang-Mills. While this has been studied mostly for $N = 2$ theories (see in particular the recent papers [32,33]), the $N = 4$ case should parallel the $N = 2$ case. Namely one expects that the topological partition function of $N = 4$ Yang-Mills will depend on the choice of the Kähler class, which is b_2-dimensional. Thus, in general we expect that the $N=4$ partition function for $SU(n)/\mathbf{Z_n}$ Yang-Mills on a Kähler manifold with $b_2^+=1$ depends not only on $\overline{\tau}$ but also on the Kähler class of the manifold.

It is natural to conjecture the following form as a generalization of the holomorphic anomaly formula: Let $\theta(q,\overline{q})$ denote the partition function of $U(1)$ fluxes on a manifold with $b_2^+=1$. This can be written in the form

$$\theta(q,\overline{q}) \;=\; \sum_{P_L,P_R} q^{\frac{1}{2}P_L^2}\,\overline{q}^{\frac{1}{2}P_R^2}$$

where $P=(P_R,P_L)\in H_2(\mathcal{N},\mathbf{Z})$, and P is split to P_R and P_L by projecting H_2 on to self-dual and anti-self-dual parts using the Kähler form on $\mathcal{N}$. In particular, the topological self-intersection number is given by $P_R^2-P_L^2$. The signature of the lattice is $(1,b_2-1)$, and θ is a modular form of $\Gamma(2)$ (the lattice is integral, but not necessarily even). Once again we decompose the $U(1)$ according to fluxes based on how they are paired with the $SU(2)/Z_2$ 't Hooft fluxes. Let α denote the choices of inequivalent fluxes. They are correlated with particular subspaces Γ_α of $H_2(\mathcal{N},\mathbf{Z})$ (in a similar, but more complicated manner than $K3$). Let

$$G_\alpha \;=\; F_\alpha/\eta^{2\chi}$$

denote the partition function of $SU(2)/Z_2$ with 't Hooft flux α, where χ is the Euler characteristic of the manifold. Then there is a natural generalization for the holomorphic anomaly of P^2, which we conjecture to be

$$\overline{\partial}_\tau F_\alpha \;=\; \text{const.}\ \tau_2^{-\frac{3}{2}} \sum_{P\in\Gamma_\alpha} q^{P_L^2/4}\,\overline{q}^{P_R^2/4}\ , \tag{3.10}$$

for some constant of proportionality which may depend on the Kähler moduli of the manifold.

3.4. The partition function of $\frac{1}{2}K3$

As noted before, the interesting low energy limit occurs for E-strings when they can become tensionless, and this happens in the F-theory compactification when the Calabi-Yau has a vanishing four cycle, $\frac{1}{2}K3=\mathcal{B}_9$. This manifold has $b_2^+=1$ and is elliptic. We will be interested in the properties of multi five-branes wrapped around $\frac{1}{2}K3$, and in particular, we will show how the the BPS states of the E-string wrapped around a circle are enumerated by the partition function of $U(n)$ Yang-Mills on $\frac{1}{2}K3$.

The homology group, $H_2(\frac{1}{2}K3,\mathbf{Z})$, is generated the hyperplane section of $\mathbf{P}^2$, which we will denote by e_0, and the nine blow-ups $e_j, j = 1,\ldots,9$. (We will use the conventions of [34] in which the blow-ups are, more correctly, $-e_j$.) The intersection form is thus $e_i \cdot e_j = g_{ij}$, with $g_{ij} = \mathrm{diag}(1,-1,-1,-1,-1,-1,-1,-1,-1,-1)$, and thus $H_2(\frac{1}{2}K3,\mathbf{Z})$ is a hypercubic, Lorentzian, self-dual, integral lattice with $(1,9)$ signature, and we will denote it by $\Gamma^{9,1}$.

The lattice, $\Gamma^{9,1}$, is isomorphic to the cubic integral self-dual (but not even) $(1,1)$ lattice, $\Gamma^{1,1}$ plus the E_8 lattice Γ^8, *i.e.*

$$H_2(\tfrac{1}{2}K3,\mathbf{Z}) \;=\; \Gamma^{1,1}\oplus\Gamma^{8} \;.$$

To see this explicitly, introduce the vectors a_0, a_1 and b_i, $i = 1,\ldots,8$, defined by:

$$\begin{aligned} a_0 \;&=\; 3e_0 + \sum_{i=1}^{8} e_i \;, \qquad\qquad & a_1 \;&=\; -e_9 \;, \\ b_i \;&=\; e_i - e_{i+1} \;, \quad 1 \le i < 8 \;, \qquad\qquad & b_8 \;&=\; e_0 + e_6 + e_7 + e_8 \end{aligned} \tag{3.11}$$

We will now show that this is an integral, unimodular change of basis. To see this, first note that a_0, a_1 and b_i are integer linear combinations of the e_i. Conversely, one has $e_9 = -a_1$ and $e_8 = 3b_8 + b_7 + 3b_6 + 5b_5 + 4b_4 + 3b_3 + 2b_2 + b_1 - a_0$. The other e_i, $i > 0$, can then easily be obtained by adding integer multiples of the b_i to e_8. Using these expressions for e_6, e_7 and e_8 and subtracting them from b_8, we obtain an integer linear combination for e_0. Thus the partition function of $\Gamma^{9,1}$ can be obtained by summing over all integer linear combinations of the basis vectors (3.11).

The new basis vectors satisfy $a_0\cdot a_0 = 1$, $a_1 \cdot a_1 = -1$ and $a_i \cdot b_j = 0$. We also have that $b_i \cdot b_i = -2$, $b_i \cdot b_{i+1} = 1$ for $i = 1,\ldots,7$, and $b_5 \cdot b_8 = 1$. All other inner products are zero. Hence the b_i vectors are the Dynkin vectors for the E_8 lattice and $H_2(\frac{1}{2}K3,\mathbf{Z})$ splits into $\Gamma^{1,1}\oplus\Gamma^8$. Since $\Gamma^{9,1}$ is an odd, self-dual lattice, and Γ^8 is even self-dual, it follows that the $\Gamma^{1,1}$ lattice is an odd, self-dual lattice. For later use we note that the elliptic class $[E]$ and the base of the elliptic fibration $[B]$ can be identified with

$$[E] = 3e_0 + \sum_{i=1}^{9} e_i = a_0 - a_1$$

$$[B] = -e_9 = a_1$$

and that

$$[E]\cdot[B] = 1 \qquad [B]\cdot[B] = -1 \qquad [E]\cdot[E] = 0 \tag{3.12}$$

Once again we start by considering the partition function for a single five-brane on $\frac{1}{2}K3\times T^2$. The degrees of freedom on the string obtained by wrapping a five-brane around $\frac{1}{2}K3$ involve twelve left-moving bosonic modes ($9 = b_2 - 1$ left-moving modes coming from the anti-symmetric field and the other three coming from the three non-compact scalars) and four right-movers (three non-compact coming from the three scalars and one compact coming from the anti-symmetric field corresponding to $b_2^+ = 1$). The right movers are accompanied by four fermions giving a $(0,4)$ supersymmetry. If we compute the partition function of a single five-brane we obtain

$$Z_1 = \frac{\tau_2^{3/2}}{V_3}\ Tr\Big[(-1)^F F_R^2\ q^{L_0}\ \overline{q}^{\overline{L}_0}\Big]\ =\ \frac{\theta(q,\overline{q})}{\eta(q)^{12}}\ ,$$

where, as before, $\theta(q,\bar{q})$ is a theta function over the $H_2(\frac{1}{2}K3,\mathbf{Z})$, which *does* depend on the choice of the Kähler class of $\frac{1}{2}K3$. We have defined the partition function by dividing out the volume of the three non-compact scalars which corresponds to the transverse position of the five-dimensional string obtained by wrapping the five-brane over $\frac{1}{2}K3$. As usual we have multiplied by the factor $\tau_2^{3/2}$ to cancel that coming from the transverse momentum integrations. The modular weight of Z_1 is $(9/2,1/2)+(-6,0)=(-3/2,1/2)$ as it should be.

The metric on $\frac{1}{2}K3$ will induce a polarization on $\Gamma^{9,1}$ lattice, which is a decomposition into 'left and right momenta' as is familiar from Narain compactifications. One of the Kähler moduli on $\frac{1}{2}K3$ is $1/R$, the size of the elliptic fiber. The resulting polarization on $H_2(\frac{1}{2}K3,\mathbf{Z})$ respects the splitting to $\Gamma^{1,1}\oplus\Gamma^8$. Choosing a metric that induces a polarization on $\Gamma^{1,1}$, we write $(p_L,p_R)=\big(\frac{n+m}{2R}+\frac{n-m}{2}R, \frac{n+m}{2R}-\frac{n-m}{2}R\big)$. One can easily see that this is a basis for such an integral, odd self-dual lattice. First, ${p_L}^2-{p_R}^2=n^2-m^2$, and so the lattice is integral and odd. One can easily show that if a vector (p'_L,p'_R) is to have integer dot product with all vectors (p_L,p_R), then the former must also have the form $(p'_L,p'_R)=\Big(\frac{n'+m'}{2R}+\frac{n'-m'}{2}R, \frac{n'+m'}{2R}-\frac{n'-m'}{2}R\Big)$. Thus, the lattice is self-dual.

We therefore have:

$$\theta(q,\overline{q})\ =\ \Big[\sum_{n,m\in\mathbf{Z}} q^{\frac{1}{2}(\frac{n+m}{2R}+\frac{(n-m)R}{2})^2}\ \overline{q}^{\frac{1}{2}(\frac{n+m}{2R}+\frac{(m-n)R}{2})^2}\Big]\ \theta_{E_8}(q)\ , \tag{3.13}$$

where θ_{E_8} denotes the theta function for the E_8 lattice. Note that the contribution of $\Gamma^{1,1}$ to the θ-function does *not* have the usual Narain form for a string on a circle. The reason for the difference is that the Narain lattice is even and self-dual, while $\Gamma^{1,1}$ is only odd self-dual.

As was noted in section 2, we will be interested in the F-theory limit, where the size of the elliptic fiber shrinks to zero, $1/R \to 0$. This is the limit that corresponds to the six-dimensional tensionless string. One should note that $U(1)$ fluxes in the left-over E_8 lattice can still be non-zero and will correspond to Kähler classes of the $\frac{1}{2}K3$ for infinitesimal, but non-zero size of the elliptic fiber. In the limit $R \to \infty$ the sum over n, m reduces to a single sum over the null direction, $n = m$. This sum can be computed and gives $\frac{R}{\sqrt{2\tau_2}}$. Thus one finds that for $R \to \infty$,

$$Z_1 = \frac{R}{\sqrt{2\tau_2}} \frac{\theta_{E_8}(q)}{\eta(q)^{12}}.$$

Note that if we view R as the radius of a circle that takes us from F-theory in six dimensions down to five dimensions, then the extra factor of R is the volume factor that comes from string compactification on this extra circle. Moreover, the factor of $1/\sqrt{\tau_2}$ is manifestly coming from the continuum momentum integration in the decompactified direction. Now recall that we have adopted the convention of defining the F-theory partition function, (2.2), with a prefactor of τ_2^2/V_4 while the M theory partition function has a prefactor of $\tau_2^{3/2}/V_3$. The factor $R/\sqrt{\tau_2}$ represents precisely this difference in these prefactors. Thus dropping the factor of $R/\sqrt{\tau_2}$ should yield the partition function of a string in six dimensions. Doing this, we then view

$$Z_1 \; = \; \frac{\theta_{E_8}(q)}{\eta(q)^{12}} \; . \tag{3.14}$$

as the partition function for a six dimensional E-string. Note that this is a holomorphic modular form of weight $(-2, 0)$ as we had anticipated earlier.

We would now like to generalize this to n five-branes wrapped around $\frac{1}{2}K3$, and we start by considering $n = 2$. As already mentioned, since we are considering a Kähler manifold with $b_2^+ = 1$ there is no conjectured answer coming from the $U(n)$ Yang-Mills theory. Based on the foregoing, we should expect a holomorphic anomaly, as well as some dependence on the choices of Kähler classes. However, since we have gone to a particular degenerate limit of the Kähler class, we have effectively frozen out all the Kähler class dependence which is responsible for the existence of various chambers, and so we should only expect a holomorphic anomaly.

The $U(2)$ theory on $\frac{1}{2}K3$ requires a rescaling $\tau \to \tau/2$ as on $K3$. The partition function will decompose, as discussed above, in terms of $SU(2) \times U(1)/Z_2$ where the 't Hooft fluxes for $SU(2)/Z_2$ are correlated with the $U(1)$ fluxes. In the limit $R \to \infty$, all $U(1)$ fluxes reside in a particular null direction ($m = n$) on the $\Gamma^{1,1}$ lattice. Following the decomposition (3.13) of the $\Gamma^{9,1}$ lattice, we write the 't Hooft fluxes $\alpha = (\lambda, i, j)$ where λ is a vector on the E_8 lattice, and $i, j = 0, 1$ denote the even/odd terms along the space-like and time-like directions of $\Gamma^{1,1}$. Since we are restricting to $m = n$ on $\Gamma^{1,1}$, we are only interested in the $SU(2)/Z_2$ partition functions $Z_{\lambda,j,j}$. For a given set of $U(1)$ fluxes, both even and odd values of j give the same result in the limit $R \to \infty$, since one is simply taking the continuum limit of the sum. Therefore, we effectively get the sum of 't Hooft fluxes in this direction, and so we define:

$$\hat{Z}_\lambda = \frac{1}{2}(Z_{\lambda,0,0} + Z_{\lambda,1,1}) \ . \tag{3.15}$$

Let $\beta = (\lambda', k, \ell)$, then one has $(\lambda, j, j) \cdot \beta = \lambda \cdot \lambda' + j(k - \ell)$ (remember this is a Lorentzian inner product). Using the modular transformation (3.2) on $\hat{Z}_\lambda$ we therefore see that Z_β has an overall coefficient of $\sum_j exp(2\pi i(\lambda \cdot \lambda' + j(k - \ell))/2)$, which projects onto a sum of those Z_β with $k = \ell$. It follows that $\hat{Z}_\lambda$ maintains its form under $SL(2, Z)$ transformations, and indeed transforms with phases that depend only on the E_8 label λ. Henceforth we will drop the hats on $\hat{Z}_\lambda$ with the understanding that Z_λ means (3.15). (Similarly, for $SU(n)$ we consider $\hat{Z}_\lambda = \frac{1}{n}\sum_m Z_{\lambda,m,m}$.)

After the rescaling by the volume factor, the partition function of two five-branes on $\frac{1}{2}K3 \times T^2$, including $U(1)$ fluxes, becomes

$$Z_2 = \sum Z_\lambda \ \theta_{E_8,\lambda}(q^{\frac{1}{2}})$$

where λ denotes the inequivalent Z_2 fluxes on the the E_8 part of H_2, and $\theta_{E_8,\lambda}$ denotes the theta function for the corresponding $U(1)$ fluxes. There are three inequivalent choices for λ: 0, even and odd, exactly as there were with $K3$. One can thus readily write down the corresponding $\theta_{E_8,\lambda}$. The question remains as to how to determine Z_λ.

What properties do we expect of Z_2? The list includes: i) Z_2 is the partition function of $U(2)$ and so, using Montonen-Olive duality, we expect it to be a modular form of $SL(2, \mathbf{Z})$. ii) The total modular weight of Z_2 should be the sum of the weight $(-\chi/2, 0) = (-6, 0)$ which is the $SU(2)$ contribution and the weight $(4, 0)$ which is the contribution of the $U(1)$ fluxes. Hence, the total weight is $(-2, 0)$, which is consistent with the partition function in (2.3). iii) From [12] the smallest power in its q expansion is expected to be $q^{-2\chi/24} = q^{-1}$. iv) The instanton expansion should have a "gap" since the first non-vanishing instanton number is 2 [35] v) Z_λ should have a holomorphic anomaly in the form

$$\overline{\partial}_\tau Z_\lambda \ = \ \frac{i}{2\pi\tau_2^2} \ C_\lambda \ \frac{\theta_{E_8,\lambda}(q^{\frac{1}{2}})}{\eta^{24}} \ , \tag{3.16}$$

where the extra power of $\tau_2^{-\frac{1}{2}}$ as compared to (3.10) comes from the theta function in the limit $R \to \infty$. Note that the modular weight on both sides of (3.16) is $(-6, 2)$. These conditions go a long way towards fixing Z_2. In fact if we knew the constants C_λ, together with modularity properties and the gap for Z_2 it would fix it completely. In the next section we will derive the constants, and compute Z_λ explicitly. It turns out that the holomorphic anomaly is most conveniently stated for Z_2, and we find that

$$\overline{\partial}_\tau Z_2 \;=\; \frac{\text{const.}}{\tau_2^2}\; Z_1^2 \tag{3.17}$$

This has the interpretation of two five-branes wrapped around $\frac{1}{2}K3$ forming bound states at threshold. We will discuss this later in the paper. Also note that the modular weight on both sides of (3.17) is $(-2, 2)$.

What should we expect for Z_n, the partition function for n five-branes wrapped around $\frac{1}{2}K3$? First, in taking the F-theory limit to get the six-dimensional partition function, there will now be n terms in the sum (3.15). The argument above easily generalizes to show that these sums close under the modular transformation (3.2). These coefficient functions are then to be combined with appropriate θ-functions of the suitably scaled E_8 lattice, and result is expected to be a modular form of $SL(2, \mathbf{Z})$ of weight $(-2, 0)$, whose holomorphic part (*i.e.* fixing τ and sending $\overline{\tau} \to \infty$) starts at $q^{-n/2}$. We also expect there to be a gap of n units in the q-expansion because the first non-zero instanton number is n. In other words, the next non-zero term will be the $q^{n/2}$ term. Furthermore, we expect there to be a generalization of the holomorphic anomaly corresponding to reducible connections. For $U(n)$ the natural reducible connections correspond to $U(n) \to U(k) \times U(n-k)$, so we expect $\overline{\partial}_\tau Z_n$ to involve a weighted sum over $Z_k \cdot Z_{n-k}$. As we will explicitly show, this is correct and the equation is

$$\overline{\partial}_\tau Z_n \;=\; \frac{\text{const.}}{\tau_2^2}\; \sum\; k(n-k)\; Z_k Z_{n-k} \tag{3.18}$$

From the five-brane point of view this can be interpreted as the possible ways k bound strings can bind with $n-k$ bound strings in a pairwise manner to form n bound strings.

4. Counting BPS states of the Compactified E-string

In this section we summarize the basic approach in counting the BPS states of the E-string, continuing the discussion in section 2, and sketch some of the properties of what we shall find. In this section we try to avoid the technical aspects of this computation, postponing them to the subsequent section.

4.1. Computing the pre-potential

We consider here the E-string compactified on a circle to five dimensions and count the BPS states with a given winding number and momentum. Comparing this to the structure and form of the Yang-Mills partition functions, one can read off the $\hat{Z}_\lambda$.

The basic approach that we will use for counting the BPS states was first employed in [9], and further developed in [22,10,11]. The basic idea is that if we consider type IIA theory on a Calabi-Yau threefold, the pre-potential of the four-dimensional $N = 2$ theory only receives world-sheet instanton corrections. Indeed the pre-potential can be represented as a sum over holomoprhic curves, and can thus be computed using mirror symmetry techniques. One actually needs a local version of the mirror symmetry to do this counting because the relevant part of the Calabi-Yau geometry, in the limit of turning off the gravity, is a non-compact manifold corresponding to a Calabi-Yau singularity and its immediate neighborhood. Such a non-compact specialization was used in [9,22] while more general aspects of local mirror symmetry and geometric engineering were developed in [23,36,37], where it was used to solve the Coulomb branch of $N = 2$ theories in four dimensions.

Let t_i denote the complexified Kähler moduli of Calabi-Yau threefold corresponding to the i-th 2-cycle and $N_{[n_i]}$ denote the number of holomorphic curves wrapped around the i-th cycle n_i times, then the third derivative of the pre-potential is given by

$$\partial^3_{ijk}\mathcal{F}(t_i) = \sum_{[n_i]} n_i n_j n_k N_{[n_i]} \ \frac{\prod_i q_i^{n_i}}{1-\prod_i q_i^{n_i}} \tag{4.1}$$

where $q_i = \exp(-t_i)$. Applying this to $\frac{1}{2}K3$ embedded in a Calabi-Yau manifold, and given the relation between holomorphic curves and the BPS states of the E-strings, discussed in section 2, this will amount to the computation of the BPS states of the multiply wrapped E-string around a circle. This expression also includes the contribution of multi-wound states (in the form of the denominator above) and is natural to view this function itself as the partition function of BPS states (from which the primitively wound BPS string states can be obtained). Note also that (4.1) fixes $\mathcal{F}$ up to a quadratic function of t_i, and the latter can be viewed as the classical contribution to the pre-potential.

Of the ten Kähler moduli of $\frac{1}{2}K3$, the two that index the winding state and momentum state of the compactified string can be thought of as the moduli of the base and fiber respectively of the elliptic fibration. There are in addition eight remaining Kähler moduli. The intersection matrix is an integral inner product in a ten dimensional lattice with signature $(1,9)$ and it includes a sublattice on which the intersection matrix is that of the Cartan Matrix of E_8. For future reference, we will label these moduli by ϕ, τ and $m_i, i = 1, \ldots, 8$ where ϕ and τ correspond to the base and fiber moduli and the m_i label the E_8 moduli. In the next section we will review the construction of mirror map for $\frac{1}{2}K3$ and also present a simple new derivation for it based on $R \to 1/R$ duality of tori, very much in the spirit of examples studied in [38]. The basic aspect of the new derivation is that $\frac{1}{2}K3$ arises in a self-mirror Calabi-Yau 3-fold and thus the Kähler moduli of it are mirror to its own complex moduli. In other words, roughly speaking, studying the *complex moduli* of $\frac{1}{2}K3$ amounts to obtaining the pre-potential. More precisely $\frac{1}{2}K3$ is an elliptic manifold and the corresponding Seiberg-Witten curve for the $N = 2$ theory is simply obtained from the complex structure of $\frac{1}{2}K3$ where the coordinate of the base of $\frac{1}{2}K3$ is promoted to a moduli parameter.

Recasting the mirror map in terms of periods of a torus enables one to easily study the modular properties of the partition functions as a function of τ, and by making asymptotic expansions for large $Im(\phi)$, one can explicitly obtain a series, indexed by the winding number, n, of the E-string, whose coefficients are functions of the moduli, m_i and τ. Let $\mathcal{F}$ be the pre-potential of the theory, and suppose that it has an expansion of the form:

$$\mathcal{F} \ = \ \mathcal{F}_{\rm classical} \ + \ \sum_{n=1}^{\infty} \ q^{n/2} \ Z_n(m_i;\tau) \ e^{2\pi i n\phi} \ , \tag{4.2}$$

where $q = e^{2\pi i\tau}$, and the function $\mathcal{F}_{\rm classical}$ is a cubic in the moduli, whose third derivative gives the classical intersection form. The functions $Z_n(m_i;\tau)$ count the rational curves in the $\frac{1}{2}K3$ which wrap n-times around the base. For $m_i = 0$, the functions $Z_n(0;\tau)$ are (almost) modular forms[4] of weight -2: That is, they can be written in the form:

$$Z_n(0;\tau) \ = \ \frac{1}{\eta^{12n}} \ p_n(E_2(\tau), E_4(\tau), E_6(\tau)) \ , \tag{4.3}$$

where p_n is some (quasi-) homogeneous polynomial of weight $6n-2$, and E_{2m} are the Eisenstein modular forms of weight $2m$. The function $E_2(\tau)$ is holomorphic, but transforms with the usual modular anomaly. This modular anomaly, and the fact that η^{12} changes sign under $\tau \to \tau + 1$ define the sense in which we mean that $\mathcal{F}$ is *almost* a modular form. One can remove the modular anomaly from E_2 by adding to it a suitable multiple of $1/Im(\tau)$ to yield the function $\widehat{E}_2$. In this way we can "ignore" the modular anomaly, but at the cost of introducing a rather mild anti-holomorphicity. We will denote the corresponding modified form of Z_n in this section by $\hat{Z}_n$. In the subsequent section we revert back to the Z_n notation for the partition function, including the τ_2 piece.

[4] To arrange this one has to use the somewhat unusual normalization factor of $q^{n/2}$ in (4.2).

4.2. The recursion relation

It was shown in [11,39] that the E_2 content of Z_n is, in fact, entirely determined by the recurrence relation:

$$\frac{\partial Z_n}{\partial E_2} \ = \ \frac{1}{24} \sum_{m=1}^{n-1} \ m(n-m) \ Z_m Z_{n-m} \ . \tag{4.4}$$

The corresponding equation for $\hat{Z}_n$ is

$$\frac{\partial \hat{Z}_n}{\partial \bar{\tau}} \ = \ -\frac{i}{16\pi} \ \frac{1}{(Im(\tau))^2} \ \sum_{m=1}^{n-1} \ m(n-m) \ \hat{Z}_m \hat{Z}_{n-m} \ . \tag{4.5}$$

Note also that if we kept the ϕ dependence in the partition funcation and consider $Z = \sum_n \hat{Z}_n \exp(2\pi i n\phi)$ then the foregoing equation takes the form

$$\frac{\partial Z}{\partial \overline{\tau}} = \frac{i}{64\pi^3 (Im\tau)^2} \frac{\partial Z}{\partial \phi} \frac{\partial Z}{\partial \phi}$$

It is clear from [39] that such a recurrence relation is a very general feature of an effective action of a theory with a parameter given by the modulus of a torus. In particular, the proof of [39] is easily generalized to incorporate the Kähler moduli, m_i. The non-zero parameters, m_i, mean that the Z_n transform like group characters:

$$\begin{aligned} Z_n(m_i, \tau + 1) \ &= \ (-1)^n \ Z_n(m_i, \tau) \ , \\ Z_n\Big(\frac{m_i}{\tau}, -\frac{1}{\tau}\Big) \ &= \ \tau^{-2} \ e^{\frac{i\pi}{\tau} \sum_j n \, m_j^2} \ Z_n(m_i, \tau) \ , \end{aligned} \tag{4.6}$$

where we have ignored all the modular anomalies coming from E_2 (which can be done by promoting it to $\widehat{E}_2$).

From the Yang-Mills point of view, (4.5), is the $SU(n)$ generalization of the holomorphic anomaly equation (3.10) anticipated in [12], which was conjectured to arise from reducible connections. Given that the BPS counting of E-strings is related to the counting of holomorphic curves, it is also natural to try interpreting the foregoing anomaly equation from this viewpoint. Indeed there turns out to be a simple interpretation from this angle: Recall that Z_m denotes the number of holomorphic curves which wrap m times around the base of $\frac{1}{2}K3$ and an arbitrary number of times around the elliptic fiber (which gives the τ dependence of Z_m). Given the T-duality of the fiber torus, we would expect Z_m to be a modular form of τ. However, there is a natural way in which we can obtain bound states of holomorphic curves corresponding to Z_m and Z_{n-m} and that is by joining them along a fiber at two different points along the fiber; due to translation invariance of the torus only the relative position of these two points is relevant. The resulting holomorphic curve will wind around the base of $\frac{1}{2}K3$, $m+(n-m)=n$ times. Note that we can attach the torus in m ways to the first curve, because there are m leaves of the first holomorphic curve around the sphere, and $n-m$ ways to the second one. So if we consider how we may remove a torus from Z_n so as to obtain Z_m and Z_{n-m}, we expect to have an equation of the form

$$\frac{\partial Z_n}{\partial F_1} \;=\; \sum_m m(n-m) Z_m Z_{n-m} \;, \tag{4.7}$$

where F_1 is the partition function of the torus. That is, F_1 is the number of ways we can wrap the torus around the fiber torus (modulo translation and the choice of a point on it). This function is known to be [13] $E_2/24$ (up to the holomorphic anomaly term). Using this, we see that (4.7) is precisely the recusrion relation (4.4). Thus the holomorphic anomaly here has exactly the same origin as that observed in [13][5].

4.3. The "gap"

The other key property of the Z_n is its "gap:" That is, its q-expansion ($q = e^{2\pi i\tau}$) has the form:

$$Z_n(m_i;\tau) \;=\; \frac{1}{n^3}\, q^{-\frac{n}{2}} \;+\; O(q^{+\frac{n}{2}}) \;. \tag{4.8}$$

One way of understanding the "gap" comes from recalling how $\mathcal{F}$ counts rational curves [9]. This is the equation given in (4.1) which we now specialize to the case at hand. The quantum part of $\mathcal{F}$, $\mathcal{F}_{quantum} \;=\; \mathcal{F} - \mathcal{F}_{\rm classical}$, has an expansion of the form:

$$\mathcal{F}_{quantum} \;=\; \sum_{j,k,\ell_1,\ldots,\ell_8=0}^{\infty} N_{j,k,\ell_1,\ldots,\ell_8} \sum_{r=1}^{\infty} \frac{1}{r^3}\; e^{2\pi i r(j\phi+k\tau+\vec{\ell}\cdot\vec{m})} \;, \tag{4.9}$$

where $N_{j,k,\ell_1,\ldots,\ell_8}$ is the number of irreducible rational curves with winding numbers $j,k,\ell_1,\ldots,\ell_8$ on the cycles with Kähler moduli ϕ,τ,m_i.

[5] We should recall that we have been using genus zero curves to count the holomorpic curves in the Calabi-Yau. These do seem to receive contributions which also look like higher genus curves. The reason that this is not surprising is that the genus zero partition function gives the pre-potential of the $N=2$ theory and that has the information about all the *charged* BPS states. This is also presumably why the configuration involving connecting Z_i's around a ring with T^2 fibers is not contributing to the BPS states. These would look like genuinely higher genus curves and will only modify gravitational corrections.

To see the gap property, one has to note that the intersection of two distinct holomorphic curves is always positive (since they are holomorphic the intersection points all have the same relative orientation). So let us consider the holomorphic curve corresponding to $j[B]+k[E]$ where $[B]$ denotes the class of the base and $[E]$ denotes the elliptic fiber. Consider another holomorphic curve which is $[B]$ itself. As long as $k \neq 0$ these two curves are distinct and we should thus have

$$(j[B]+k[E]) \cdot [B] \geq 0$$

Now we use the fact (3.12) that $[B] \cdot [B] = -1$ and $[B] \cdot [E] = 1$ and learn that

$$k - j \geq 0 \qquad if \quad k \neq 0$$

This establishes the existence of the gap.

One can calculate the Z_n directly by computing the expansion (at large $Im(\phi)$) of the period integrals of the appropriate torus. This will be done in the next section to obtain $Z_1(m_i,\tau)$. We have also derived $Z_2(m_i,\tau)$ in the same manner, but such direct calculations rapidly becomes unwieldy. As was seen in [11,39], the most powerful method for explicitly computing the Z_n is to use the recurrence relation (4.4) and the gap condition (4.8). That is, once one has Z_1, one can make an Ansatz for Z_n that is consistent with the modular properties (4.6) and the E_8 Weyl invariance, and then fix the coefficients using (4.4) and (4.6). This generically yields a highly overdetermined system of equations which give a unique result for Z_n. We illustrate this in some detail for Z_n in section 6 for small values of n.

5. Deriving the Wound E-String BPS States

In this section we spell out in a bit more detail the technique used in counting the BPS states of multiply wound E-strings. Note that as discussed before this corresponds to partition function of $N=4$ topological $U(n)$ Yang-Mills on $\frac{1}{2}K3$ where n corresponds to winding number of E-string around the circle. In the next section we apply the method to explicitly write down the BPS state for E-strings with low winding numbers.

The appropriate mirror maps were constructed in [9] for subsets of the moduli. Non-compact Calabi-Yau manifolds that modelled the non-critical string were constructed in [22] and the construction of mirror map was reduced to computing periods of a Seiberg-Witten torus. The models of [22] gave a subset of the BPS spectrum (those states with winding number equal to the momentum) and only involving the modulus ϕ. In [10] the Calabi-Yau models of [22] were extended to incorporate all the BPS states and both the τ and ϕ moduli. Moreover, after reducing the periods of the Calabi-Yau to periods of various differentials on a torus, it was shown in [10,11] how to incorporate the complete set of ten Kähler moduli. Thus, in [10,11] the construction of the complete mirror map, involving all moduli, was reduced to computing periods of a Seiberg-Witten differential on a torus. This torus had also been constructed by other authors [40] in order to get effective actions of gauge theories arising out of the E-string.

It was shown in [22,40,10,11,39] that the pre-potential of the E-string could be derived from a Seiberg-Witten curve of the form:

$$y^2 \; = \; x^3 \; - \; f(u, m_i, \tau) x \; + \; g(u, m_i, \tau) \; , \tag{5.1}$$

where u is a parameter that essentially sets the overall size of the del Pezzo surface (an overall scale has been set to one). This curve in fact describes the del-Pezzo surface B_9. The Seiberg-Witten curve depends on u, the eight mass parameters m_i and a parameter τ, which is the Kähler modulus of the elliptic fiber. The masses are inversely related to the Kähler moduli for the blow-up moduli of the B_9. That is, taking masses to infinity is equivalent to blowing down some of these points. We first present a simple new derivation of the above local mirror, and then discuss how one computes the instanton expansions from it.

5.1. The Mirror Map for $\frac{1}{2}K3$

In order to present a simple derivation of these results we need to find a Calabi-Yau threefold for which $\frac{1}{2}K3$ is embedded. Consider a Calabi-Yau threefold which is a double elliptic fibration over a sphere:

$$\begin{aligned} y_1^2 &= x_1^3 - f_1(z)x_1 + g_1(z) \\ y_2^2 &= x_2^3 - f_2(z)x_2 + g_2(z) \end{aligned} \tag{5.2}$$

where (y_i, x_i), subject to the above equations, define the two tori and z denotes the coordinate on the sphere. Moreover f_i are functions of degree 4 and g_i are functions of degree 6 in z. This is a Calabi-Yau threefold with Hodge numbers $h^{1,1} = h^{1,2} = 19$. There is a $\frac{1}{2}K3$ embedded in the Calabi-Yau of (5.2), since each of the equations in (5.2) describes a $\frac{1}{2}K3$ surface in three complex dimensions. equation. The local model, i.e. the isolated $\frac{1}{2}K3$, is obtained by enlarging the second torus, in other words by sending the Kähler class of the second torus to infinity. For a particular choice of f_i, g_i this manifold can be obtained as a $Z_2 \times Z_2$ orbifold of $T^2 \times T^2 \times T^2$: Let $\zeta_1, \zeta_2, \zeta_3$ denote the coordinates of the three tori. Consider the generators of the two Z_2 actions,

$$\alpha : \zeta_1 \to \zeta_1 + \frac{1}{2}, \quad \zeta_{2,3} \to -\zeta_{2,3} \;;$$

$$\beta : \zeta_2 \to \zeta_2 + \frac{1}{2}, \quad \zeta_{1,3} \to -\zeta_{1,3} \;;$$

where for definiteness we have taken the $d\zeta_i$ to have periods 1 and τ_i. It is easy to check using standard orbifold techniques, that this orbifold has hodge numbers $h^{1,1} = h^{1,2} = 19$. Moreover, it corresponds to a double elliptic fibration over a base $\mathbf{P}^1$. Namely, we can identify ζ_1 with the (x_1, y_1) torus, ζ_2 with the (x_2, y_2) torus and ζ_3 with z (which parameterizes $\mathbf{P}^1$). To see this note that modding out by α leads to an elliptic (ζ_1) fibration over the sphere (ζ_3 modded by Z_2) and modding out by β leads to another elliptic fibration (ζ_2) over $\mathbf{P}^1$ (again parameterized by ζ_3 modulo the Z_2 action). The double elliptic fibration, together with the equivalence of hodge numbers, identifies the orbifold with the Calabi-Yau in (5.2). Now we apply mirror symmetry as in [38], by T dualizing one circle from each of the three tori. As explained in [38] this leads to the mirror manifold if in addition one includes a non-trivial Z_2 discrete torsion $\epsilon(\alpha, \beta) = -1$. Since there are no mutual fixed points for α, β or $\alpha\beta$, this will not change the geometry of the massless modes. Hence, we obtain the same manifold (5.2) , at least as far as the pre-potential is concerned, but now with the complex moduli playing the role of what were the Kähler moduli. Note in particular that the Kähler and complex moduli of each of the two tori $\zeta_{1,2}$ are exchanged. The description of the manifold in terms of (5.2) is now quite useful because the Kähler moduli of the original Calabi-Yau now appear explicitly in the coefficients of f_i and g_i.

On the mirror side the interesting object is the holomorphic 3-form, the periods of which encode the holomorphic curves (see for example [41]). In this case, the holomorphic 3-form Ω is given by

$$\Omega = \frac{dx_1}{y_1}\frac{dx_2}{y_2}dz \tag{5.3}$$

which is regular for large z. As noted before we are interested in a large Kähler class for the second torus, which from the mirror perspective, corresponds to a degenerate complex structure for this same torus. Suppose the second torus becomes degenerate at $z = u$. Locally this means that the torus is parameterized as

$$y^2 - x^2 = \tilde{x}\tilde{y} = (z - u) \tag{5.4}$$

where $\tilde{x}$ and $\tilde{y}$ are the re-defined coordinates on the second torus, emphasizing its degenerate structure. At $z = u$ the torus is strictly degenerate since one of its cycles has shrunk to zero. Using the local model in (5.4) when $z \neq u$ means that the complex structure of the second torus remains degenerate away from $z = u$.

We now investigate the relevant periods. The holomorphic three-form, Ω, takes the form

$$\frac{dx_1}{y_1} dz \frac{d\tilde{x}}{\tilde{x}} \tag{5.5}$$

in the local limit. There are two types of 3-cycles. One type corresponds to taking a cycle of the form $C_2 \times S^1$ where C_2 is a two cycle supported on the mirror $\frac{1}{2}K3$ given by the variables (x_1, y_1, z), and S^1 is a cycle of the degenerate torus. The other type of three cycle is an S^3, which can be thought of as $S^1 \times S^1$ over an interval, where on one end of the interval the first circle shrinks to zero and on the other end the second circle shrinks to zero. We choose one of these two circles to be the small cycle of the second torus which is zero at $z = u$, and we choose the second cycle to be the a or the b cycle of the first torus. (This construction and computation of the period integrals is very similar to that of [22].)

Since we have only one non-degenerate torus we remove the subscripts from x_1, y_1 and f_1, g_1 and denote them by x, y and f, g. We first consider cycles of the form $C_2 \times S^1$. Integrating Ω over the small cycle of this torus gives a constant, multiplied by the left-over 2-form

$$\int_{1-cycle} \Omega = \Omega' = \frac{dx}{y} dz \ .$$

This 2-form is then integrated over 2-cycles of $\frac{1}{2}K3$. Even though there are ten 2-cycles on this space, Ω' vanishes on two of them, namely the 2-cycles corresponding to the fiber torus and the base sphere. So this type of cycle gives eight non-zero parameters. This is the kind of structure we expect from Seiberg-Witten geometry for masses [42], so we identify these periods with eight mass parameters m_i (These can be viewed as the dual to the Cartan subalgebra of E_8). To get cycles of the second type we have to integrate Ω' from $z = u$ to one of the points where the first torus degenerates. Call that point P. We thus have

$$\phi = \int_P^u dz \int_a \frac{dx}{y}.$$

Which point P we choose is irrelevant because the difference between two choices amounts to an integral over a 2-cycle of the first type, which means shifting ϕ by integral multiples of m_i. This is also familiar from Seiberg-Witten geometry. We can similarly define ϕ_D by integrating over a b cycle of the torus and choosing a point where the b cycle vanishes. To make contact with the usual notation of Seiberg-Witten, we re-define the dummy variable z by u and summarize the periods by

$$m_i = \int_{C_i} du \frac{dx}{y}$$

$$\phi = \int_P^u du \int_a \frac{dx}{y}$$

where

$$y^2 = x^3 - f(u)x + g(u)$$

Note also that the τ which is mirror to the Kähler class of the original $\frac{1}{2}K3$ is now mapped to the complex structure of the torus defined above. The functions f and g are now implicitly a function of $f(u, m_i, \tau)$, $g(u, m_i, \tau)$.

5.2. Computing the periods

In order to count BPS states, we need to compute the above periods for the Seiberg-Witten curve. We also want to group the BPS states into E_8 Weyl orbits. This is accomplished by turning on the mass parameters and expressing the instanton expansion in terms of E_8 characters. This was done for the special case $\tau \to i\infty$ in [10]. Now we want to consider the more general case for finite τ.

There is one major issue that needs to be addressed. For general m_i and τ the functions $f(u, m_i, \tau)$ and $g(u, m_i, \tau)$ are not known explicitly. In principle, they can be determined from the curve in [40], but in order to do so, one needs to first perform a non-trivial $SL(3)$ transformation on that curve. In [10] it was shown how to compute these functions in the $Im(\tau) \to \infty$ limit explicitly from the instanton expansion. Essentially one assumes that the instanton expansion will give E_8 characters and one then works backwards to find the curve. The key point is that one does not need really need all of the masses to be non-zero in order to find the characters, if one assumes that the characters are actually E_8 characters. At least for relatively low instanton numbers two non-zero mass parameters are enough.

The functions f and g, for two non-zero masses, were derived in [10]. In this case the Seiberg-Witten curve has the relatively simple form

$$y^2 \ = \ x^3 \ + \ \gamma u^2 x^2 \ + \ k^2 u^4 x - \ 2u(u^2 \ + \ {s_+}^2 x)(u^2 \ + \ {s_-}^2 x) \ , \tag{5.6}$$

where $\gamma = 1 + k^2$,

$$k \;=\; \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}\ , \qquad\qquad s_\pm = sn(m_\pm) \;=\; \frac{\vartheta_3(0)}{\vartheta_2(0)}\frac{\vartheta_1(m_\pm)}{\vartheta_4(m_\pm)}\ , \tag{5.7}$$

and where $m_\pm = (m_1 \pm m_2)/2$. We can then shift x in (5.6) to obtain the canonical form in (5.1). If we shift $x \to x - \frac{1}{3}\gamma u^2 + \frac{2}{3}{s_+}^2{s_-}^2 u$ and rescale u to $u \to u\vartheta_3^{12}$, one can recast (5.6) into the form of (5.1). The actual expressions for $f(u, m_1, m_2, \tau)$ and $g(u, m_1, m_2, \tau)$ are given in appendix A.

An elliptic curve with modulus $\widetilde{\tau}$, up to a rescaling has the canonical form

$$y^2 \;=\; x^3 \;-\frac{1}{3\omega^4}\widetilde{E}_4 x \;+\; \frac{2}{27\omega^6}\widetilde{E}_6 \tag{5.8}$$

where $\widetilde{E}_n = E_n(\widetilde{\tau})$ is the Eisenstein series of modular weight n and ω is the period for the elliptic curve. Comparing the curves in (5.8) and (5.1), we find that

$$\omega = \left(\frac{\widetilde{E}_4}{f(u, m_1, m_2, \tau)}\right)^{1/4}\quad . \tag{5.9}$$

The coordinate ϕ is then found by integrating ω over u, with the result

$$\phi \;=\; -\frac{1}{2\pi i}\int du\left(\frac{\widetilde{E}_4}{f(u, m_1, m_2, \tau)}\right)^{1/4}\quad , \tag{5.10}$$

while the dual coordinate ϕ_D is found by integrating $\omega\widetilde{\tau}$ over u, leaving

$$\phi_D \;=\; -\frac{1}{2\pi i}\int du\left(\frac{\widetilde{E}_4}{f(u, m_1, m_2, \tau)}\right)^{1/4}\widetilde{\tau}\ . \tag{5.11}$$

The variable $\widetilde{\tau}$ is solved by equating the series expansion of $\widetilde{E}_4^3/\widetilde{E}_6^2$ with $\frac{f(u,m_1,m_2,\tau)^3}{g(u,m_1,m_2,\tau)^2}$ and then inverting the series. The pre-potential, $\mathcal{F}$ is then obtained by integrating $\phi_D(\phi, \tau; m_i)$ with respect to ϕ.

6. BPS States of E-string With Low Winding Numbers

In this section we apply the methods of the previous section to the problem of computing the BPS states of E-strings with low winding numbers.

6.1. The singly wound E-string

Using the equations (5.10) and (5.11) we find that, to leading order in $1/u$ and up to an integration constant, $\phi = -\frac{1}{2\pi i}\log u$ and the difference of the dual coordinate with $\tau\phi$ is

$$\begin{aligned}\phi_D - \tau\phi = -\frac{54{\vartheta_4}^4}{4\pi^2\Delta\chi(m_1,m_2)}\Big[&{\vartheta_3}^6\vartheta_3(m_1)\vartheta_3(m_2) + {\vartheta_2}^6\vartheta_2(m_1)\vartheta_2(m_2) \\ &+ {\vartheta_4}^6\vartheta_4(m_1)\vartheta_4(m_2)\Big]e^{2\pi i\phi} \;+\; \mathrm{O}(e^{4\pi i\phi})\ ,\end{aligned} \tag{6.1}$$

where $\Delta = {E_4}^3 - {E_6}^2 = 1728\eta^{24}$ and

$$\chi(m_1,m_2) = {\vartheta_3}^2\vartheta_3(m_1)\vartheta_3(m_2) - {\vartheta_2}^2\vartheta_2(m_1)\vartheta_2(m_2) + {\vartheta_4}^2\vartheta_4(m_1)\vartheta_4(m_2)\ . \tag{6.2}$$

More details of this calculation can be found in appendix A. The term inside the square brackets is the contribution of the E_8 lattice with two non-zero Wilson lines. To match (6.1) to the results of [9], we should multiply this result by $16\chi(m_1,m_2)\eta^{12}/{\vartheta_4}^4$, which is equivalent to changing the integration constant in (5.10). This shift in the integration constant means that each term in the *phi*-expansion of $\phi_D - \tau\phi$ will transform with a certain modular phase, the implications of which will be discussed in the next subsection. The Yukawa coupling is now found by taking two ϕ derivatives on ϕ_D and can be written as

$$\frac{\partial^2\phi_D}{\partial\phi^2} \;=\; \sum_{n=1}^{\infty}\, n^3\ Z_n\ e^{2\pi i n\phi}\ . \tag{6.3}$$

Therefore, the first instanton contribution to the Yukawa coupling, which counts the holomorphic curves of degree one on the del Pezzo surface, is

$$Z_1 \;=\; \frac{1}{\eta^4(\tau)}\,\frac{P(m_i;\tau)}{\eta^8(\tau)} \;=\; \frac{\chi_{1,0}}{\eta^4(\tau)}\ , \tag{6.4}$$

where $P(m_i;\tau)$ is the E_8 lattice contribution for generic m_i:

$$P(m_i;\tau) \;=\; \frac{1}{2}\left[\sum_{\ell=1}^{4}\,\prod_{j=1}^{8}\,\vartheta_\ell(m_j)\right]. \tag{6.5}$$

The function $\chi_{1,0}$ is the E_8 level one character. We thus recover the proper E-string partition function (3.14).

6.2. The doubly wound E-string; U(2) Yang-Mills on $\frac{1}{2}K3$

We could find the contribution to the doubly wound E-string by carrying out the u-expansions in (5.10) and (5.11) to next order. However, this turns out to be laborious. Instead we will utilize the recursion relation (4.4) and the existence of gaps. From the recursion relation we see that

$$Z_2 \;=\; \tfrac{1}{8}\, Z_2^{(0)} \;+\; \tfrac{1}{24}\, E_2\, (Z_1)^2 \;, \tag{6.6}$$

where $Z_2^{(0)}$ has no E_2 dependence. The first term in the right hand side of (6.6) is modular invariant with weight -2, except for a modular phase coming from the non-zero m_i. Recall that ϑ-functions with non-zero arguments pick up extra phases under modular transformations. Hence, under a modular transformation $P(m_i;\tau)$ transforms as:

$$P\left(\frac{m_i}{c\tau+d}\,;\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^4\ \exp\left(\frac{i}{c\tau+d}\sum_i m_i^2\right)\ P(m_i;\tau)\ . \tag{6.7}$$

The modular "phase" can be compensated for in the Yukawa coupling if, under the modular transformation, ϕ is shifted by $\sum m_i^2/(c\tau+d)$. Indeed, one should recall from the last subsection that the origin of the modular phase is precisely because of the choice of a "constant of integration" in ϕ. This means that the modular phase for $Z_2^{(0)}$ has to be the same as the modular phase for $(Z_1)^2$. More generally, this is one way of seeing that the n^{th}-instanton contribution, Z_n, must transform as in (4.6).

We now look for generic expressions for $Z_2^{(0)}$ that have the correct modular weight and modular phase.

Notice that $P(nm_i;n\tau)$ has the same modular phase as $(P(m_i;\tau))^n$. Ignoring the modular phase, $P(nm_i;n\tau)$ is obviously not a modular form of $SL(2,\mathbb{Z})$, but it can be viewed as a modular form of weight four for the subgroup $\Gamma_1(n)$ of $SL(2,\mathbb{Z})$. This subgroup is comprised of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad-bc=1$ and $a,d=1 \bmod n$ and $c=0 \bmod n$. If n is prime, then under the full modular group $P(nm_i;n\tau)$ can be transformed to $P(m_i;\tau/n+\ell/n)$ where ℓ is an integer with $0\le\ell<n$. $P(m_i;\tau/n)$ is not a $\Gamma_1(n)$ form, but it is a form for the smaller subgroup $\Gamma(n)$, which has the additional requirement that $b=0 \bmod n$.

Hence, for the second instanton, we take the Ansatz for $Z_2^{(0)}$:

$$\begin{aligned} Z_2^{(0)} = \frac{1}{\eta^{24}}\ \Big(f(\tau)\ P(2m_i;2\tau)\ &+\ \frac{1}{16\tau^6}\ f(-1/\tau)\ P(m_i;\tau/2) \\ &+\ \frac{1}{16(\tau-1)^6}\ f(-1/(\tau-1))\ P(m_i;\tau/2+1/2)\Big)\ . \end{aligned} \tag{6.8}$$

In order that $Z_2^{(0)}$ have modular weight -2, $f(\tau)$ must have weight six. The factors of $1/16$ appear from the transformation of $P(2m_i; 2\tau)$ while the factors of τ^{-6} come from transforming $f(\tau)$. Since the instanton expansion has integer coefficients and since Z_2 is a modular form (up to the modular phase), the function $f(\tau)$ must be a weight six form of $\Gamma_1(2)$. The $\Gamma_1(2)$ forms are generated by the weight two form ${\vartheta_3}^4 + {\vartheta_4}^4$ and the weight four form ${\vartheta_2}^8$. Hence $f(\tau)$ is determined up to two coefficients and has the form

$$f(\tau) = \left({\vartheta_3}^4 + {\vartheta_4}^4\right)\left(a_1\left({\vartheta_3}^4 + {\vartheta_4}^4\right)^2 + a_2{\vartheta_2}^8\right) \ , \tag{6.9}$$

while the other terms in the orbit are given by

$$\begin{aligned} f(-1/\tau) &= -\tau^6\left({\vartheta_3}^4 + {\vartheta_2}^4\right)\left(a_1\left({\vartheta_3}^4 + {\vartheta_2}^4\right)^2 + a_2{\vartheta_4}^8\right) \\ f(-1/(\tau-1)) &= -(\tau-1)^6\left({\vartheta_4}^4 - {\vartheta_2}^4\right)\left(a_1\left({\vartheta_4}^4 - {\vartheta_2}^4\right)^2 + a_2{\vartheta_3}^8\right) \ . \end{aligned} \tag{6.10}$$

The q expansion of Z_2 has the form $Z_2 = q^{-1} + \mathrm{O}(q)$, that is, there is a gap for the q^0 term. The coefficients a_1 and a_2 in (6.8) can be adjusted to match the first two terms in the expansion. The unique choice for these coefficients is $a_1 = \frac{1}{12}$, $a_2 = -\frac{1}{12}$. Thus, $f(\tau)$ is

$$f(\tau) = \frac{1}{3}{\vartheta_3}^4{\vartheta_4}^4\left({\vartheta_3}^4 + {\vartheta_4}^4\right) \ . \tag{6.11}$$

We have also checked this result by making the direct computation of the second order term in the pre-potential.

The ${Z_1}^2$ term that appears in Z_2 can be linearized in terms of invariant Weyl orbits. The number of inequivalent Weyl orbits is equal to the number of level 2 E_8 characters, which is three. As with $K3$, the three orbits are the trivial, even and odd, with

$$\begin{aligned} P_0(m_i,\tau) &= P(2m_i; 2\tau) \ , \\ P_{even}(m_i,\tau) &= \frac{1}{2}\Big(P(m_i;\tau/2) + P(m_i;\tau/2+1/2)\Big) - P(2m_i;2\tau) \ , \\ P_{odd}(m_i,\tau) &= \frac{1}{2}\Big(P(m_i;\tau/2) - P(m_i;\tau/2+1/2)\Big) \ , \end{aligned} \tag{6.12}$$

where the trivial orbit is the contribution from lattice vectors that are twice another lattice vector, the even orbit is the contribution from lattice vectors that have length squared 0 mod 4 and which are not twice another lattice vector, and the odd orbit is the contribution from lattice vectors with length squared 2 mod 4. The ${Z_1}^2$ term is basically the tensor product of two level one characters, hence this term is expressible in terms of the level two Weyl orbits. The result has the very simple form

$$
\begin{aligned}
{Z_1}^2 \; &= \; \frac{1}{\eta^{24}(\tau)}\Big(P_0(0,\tau)\;P_0(m_i,\tau) + \frac{1}{135}P_{even}(0,\tau)\;P_{even}(m_i,\tau) \\
&\qquad\qquad\qquad\qquad\qquad + \frac{1}{120}P_{odd}(0,\tau)\;P_{odd}(m_i,\tau)\Big)\;.
\end{aligned}
\tag{6.13}
$$

Notice that (6.13) has the structure of a reducible connection of $U(2) \to U(1)\times U(1)$. The P_α denote the three types of $U(1)$ fluxes on the del Pezzo surface, with a P_α factor for each $U(1)$.

The $Z_2^{(0)}$ term in (6.8) can also be expressed in terms of the invariant Weyl orbits. Hence, the entire Z_2 term can be written as

$$
Z_2 = Z_0(\tau)P_0(m_i,\tau) + Z_{even}P_{even}(m_i,\tau) + Z_{odd}P_{odd}(m_i,\tau)\;, \tag{6.14}
$$

where

$$
\begin{aligned}
Z_0(\tau) &= \frac{1}{24\eta^{24}}\left(\widehat{E}_2P_0(0,\tau) + \left({\vartheta_3}^4{\vartheta_4}^4 - \frac{1}{8}{\vartheta_2}^8\right)\left({\vartheta_3}^4 + {\vartheta_4}^4\right)\right) \\
Z_{even}(\tau) &= \frac{1}{24\eta^{24}}\left(\frac{1}{135}\widehat{E}_2P_{even}(0,\tau) - \frac{1}{8}{\vartheta_2}^8\left({\vartheta_3}^4 + {\vartheta_4}^4\right)\right) \\
Z_{odd}(\tau) &= \frac{1}{24\eta^{24}(\tau)}\left(\frac{1}{120}\widehat{E}_2P_{odd}(0,\tau) - \frac{1}{8}{\vartheta_2}^4E_4\right).
\end{aligned}
\tag{6.15}
$$

Thus, we obtain the partition function of $N=4$ $SU(2)$ Yang-Mills on $\frac{1}{2}K3$, corresponding to three inequivalent types of 't Hooft flux on the E_8 part of $H_2(\frac{1}{2}K3)$ and subject to the Kähler limit described in section 3. We now learn from (6.15) that the holomorphic anomaly in this case is

$$
\overline{\partial} Z_\lambda = \frac{i}{2\pi\tau_2^2}\frac{C_\lambda}{\eta^{24}(\tau)}P_\lambda(0,\tau), \tag{6.16}
$$

where

$$
C_0 = 1 \qquad\qquad C_{even} = \frac{1}{135} \qquad\qquad C_{odd} = \frac{1}{120}. \tag{6.17}
$$

$P_\lambda(0,\tau)$ corresponds to the class of $U(1)\subset SU(2)$ fluxes in $H_2(\frac{1}{2}K3)$ corresponding to the 't Hooft flux denoted by λ. Note that there are 2^8 distinct 't Hooft fluxes and that ${C_\lambda}^{-1}$ is the number of fluxes for each class λ. We thus conclude that the weight factors C_λ are the inverses of the number of 't Hooft fluxes in each class.[6] Each Z_λ is the contribution to the

[6] The holomorphic anomaly for $SU(2)/Z_2$ on $\mathbf{P}^2$ has a similar dependence on the flux degeneracy. In this case there are two classes, each with a prefactor of 1. These correspond to the two possible choices of 't Hooft flux, one trivial and one nontrivial.

$SU(2)/Z_2$ partition function for one particular flux in the class λ. To find the full partition function, one should sum over each flux in the class. This will cancel out the degeneracy factors C_λ in (6.16), and so for the full partition function each class comes with an equal weight in the anomaly equation and is a natural setup to explain the above result.

The three functions $P_0(m_i,\tau), P_{even}(m_i,\tau)$ and $P_{even}(m_i,\tau)$ can also be written in terms of level two characters of E_8. To determined the characters we note that

$$E_8^{(1)} \times E_8^{(1)} \;=\; (\textit{Ising model})\; E_8^{(2)} \;, \tag{6.18}$$

where the superscript in parentheses denotes the level of the current algebra. The branching functions are given by the chracters of the Ising models. After some straight forward algebra it is easy to check that the level two characters for E_8 are given by

$$\begin{aligned}
\chi_{248} &= \frac{1}{2}\frac{\eta(\tau)}{\eta(2\tau)}\left(\frac{(P(m_i,\tau))^2}{\eta(\tau)^{16}} - \frac{P(2m_i,2\tau)}{\eta(2\tau)^8}\right) \\
\chi_0 &= \frac{1}{2}\frac{\eta(\tau)}{\eta(\tau/2)}\left(\frac{P(m_i,\tau)^2}{\eta(\tau)^{16}} - \frac{P(m_i/2,\tau/2)}{\eta(\tau/2)^8}\right) + \\
&\quad \frac{1}{2}\frac{\eta(\tau+1)}{\eta(\tau/2+1/2)}\left(\frac{P(m_i,\tau)^2}{\eta(\tau+1)^{16}} - \frac{P(m_i/2+1/2,\tau/2+1/2)}{\eta(\tau/2+1/2)^8}\right) \\
\chi_{3875} &= \frac{1}{2}\frac{\eta(\tau)}{\eta(\tau/2)}\left(\frac{P(m_i,\tau)^2}{\eta(\tau)^{16}} - \frac{P(m_i/2,\tau/2)}{\eta(\tau/2)^8}\right) - \\
&\quad \frac{1}{2}\frac{\eta(\tau+1)}{\eta(\tau/2+1/2)}\left(\frac{P(m_i\tau)^2}{\eta(\tau+1)^{16}} - \frac{P(m_i/2+1/2,\tau/2+1/2)}{\eta(\tau/2+1/2)^8}\right) .
\end{aligned} \tag{6.19}$$

Inverting these equations we have

$$\begin{aligned}
\frac{P_0(m_i,\tau)}{\eta(\tau)^8} &= \frac{\eta(2\tau)^8}{\eta(\tau)^8}\,(\chi_0\chi_{1,1} - \chi_{248}\chi_{1,2} + \chi_{3875}\chi_{2,1}) \\
\frac{P_{even}(m_i,\tau)}{\eta(\tau)^8} &= \frac{\eta(\tau/2)^8}{\eta(\tau)^8}((\chi_0+\chi_{3875})(\chi_{1,1}+\chi_{2,1}) + \chi_{248}\chi_{1,2}) - \\
&\quad \frac{\eta(\tau/2+1/2)^8}{\eta(\tau)^8}((\chi_0-\chi_{3875})(\chi_{1,1}-\chi_{2,1}) + \chi_{248}\chi_{1,2}) + \\
&\quad \frac{\eta(2\tau)^8}{\eta(\tau)^8}\,(\chi_0\chi_{1,1} - \chi_{248}\chi_{1,2} + \chi_{3875}\chi_{2,1}) \\
\frac{P_{odd}(m_i,\tau)}{\eta(\tau)^8} &= \frac{\eta(\tau/2)^8}{\eta(\tau)^8}((\chi_0+\chi_{3875})(\chi_{1,1}+\chi_{2,1}) + \chi_{248}\chi_{1,2}) + \\
&\quad \frac{\eta(\tau/2+1/2)^8}{\eta(\tau)^8}((\chi_0-\chi_{3875})(\chi_{1,1}-\chi_{2,1}) + \chi_{248}\chi_{1,2}) \;,
\end{aligned} \tag{6.20}$$

where $\chi_{1,1}, \chi_{1,2}$ and $\chi_{2,1}$ are the characters of the Ising model.

One can now substitute this into (6.15) and (6.14), and expand to obtain an explicit expression for Z_2 in terms of level 2 characters of E_8 and "branching functions" of some conformal field theory with central charge $c = 9/2$. Similarly, for the Z_n case we end up with an E_8 current algebra at level n, 4 transverse bosonic oscillator and a leftover conformal system with central charge $c = 12n - 4 - \frac{248n}{n+30}$.

Note that the combinations of η-functions that multiplies the E_8 characters can be written in terms $SU(2)_1$ characters $\chi_0^{su(2)}$ and $\chi_1^{su(2)}$:

$$\frac{\eta(2\tau)^8}{\eta(\tau)^8} \; = \; (\chi_0^{su(2)} - \chi_1^{su(2)})^4 \; . \tag{6.21}$$

6.3. Expressions for Z_3 and Z_4

We now continue the foregoing computations to obtain the partition functions of the three- and four-times wrapped E-string, corresponding to $SU(3)$ and $SU(4)$ Yang-Mills on $\frac{1}{2}K3$. In this section we give the explicit formulae for Z_3 and Z_4, and discuss the Ansatz that is appropriate at higher orders. Even though the techniques we have discussed yield answers for all n it becomes more and more laborious to write the explicit form for higher n's. As we have seen, the higher functions Z_n have the form $Z_n(m_i;\tau) = g_n(m_i;\tau)/\eta^{12n}$, where $g_n(m_i;\tau)$ is a modular form of weight $6n-2$ (but with a modular phase of the form (4.6)). The functions $g_n(m_i;\tau)$ have two types of component: E_8 root lattice terms, containing the m_i dependence and coefficient functions for each of these terms. To get Z_n for $n \geq 3$ we once again make an Ansatz and use the recurrence relation (4.4) and the gap condition (4.8). Here we will describe the first Ansatz that we made, present the results, and in the next sub-section we will justify why the Ansatz is correct, and improve upon it.

The E_2-dependent parts of g_n are determined by (4.4), and so we need only make an Ansatz for the E_2-independent part. As in the last section, we start with the E_8 building blocks: $P(nm_i; n\tau)$, and $P(m_i; (\tau+j)/n)$, $j = 0, 1, \ldots, (n-1)$, since these functions have the proper periodicity, and the correct m_i-dependent phases under modular inversion. The coefficient functions must be modular forms of $\Gamma(n)$, and have weight $6(n-1)$. Moreover, since $P(nm_i; n\tau)$ is invariant under $\tau \to \tau+1$, its coefficient must be a modular form of $\Gamma_1(n)$. We therefore make an Ansatz for this coefficient function, and then determine all the other terms by taking the orbit under the full modular group. For g_2, such an Ansatz was sufficient, but not for the higher g_n. One must also allow E_8-building blocks of the form:

$$\prod_k P(n_k m_i; n_k \tau) \quad \prod_\ell P(m_i; (\tau + j_\ell)/n_\ell) \ , \tag{6.22}$$

where $\sum n_k + \sum n_\ell = n$. When all such terms are included the system is highly overdetermined for almost all the coefficients, but there are apparently one or two underdetermined constants in the final result. Upon closer inspection, it turns out that these undetermined coefficients actually multiply an expression that is identically zero by virtue of some peculiar identity.

We find the following results for Z_3:

$$\begin{aligned} Z_3 \ = \ & \frac{1}{864\ \eta^{36}} \Big[\ A_0(\tau)\ P(3m_i; 3\tau) \ + \ A_1(\tau)\ P\big(m_i; \frac{\tau}{3}\big) \ + \ A_1(\tau+1)\ P\big(m_i; \frac{(\tau+1)}{3}\big) \\ & + \ A_1(\tau+2)\ P\big(m_i; \frac{(\tau+2)}{3}\big) \ - \ 3\ E_4(\tau)\ (P(m_i;\tau))^3 \Big] \\ & + \ \frac{1}{6}\ E_2(\tau)\ Z_1(m_i;\tau) Z_2(m_i;\tau) \ - \ \frac{1}{288}\ (E_2(\tau))^2\ (Z_1(m_i;\tau))^3 \ , \end{aligned} \tag{6.23}$$

where

$$\begin{aligned} A_0(\tau) \ &= 20\ \frac{\eta^{36}(\tau)}{\eta^{12}(3\tau)} \ + \ 972\ \eta^{24}(\tau) \ , \\ A_1(\tau) \ &= 12 \left[\ 15\ \frac{\eta^{36}(\tau)}{\eta^{12}(\tau/3)} \ + \ \eta^{24}(\tau) \ \right] . \end{aligned} \tag{6.24}$$

For Z_4 we obtain:

$$\begin{aligned} Z_4 \ = \ & \frac{1}{1152\ \eta^{48}} \Big[\ B_0(\tau)\ P(4m_i; 4\tau) \ + \ B_1(\tau)\ E\big(m_i; \frac{\tau}{4}\big) \ + \ B_1(\tau+1)\ E\big(m_i; \frac{(\tau+1)}{4}\big) \\ & + \ B_1(\tau+2)\ E\big(m_i; \frac{(\tau+2)}{4}\big) \ + \ B_1(\tau+3)\ E\big(m_i; \frac{(\tau+3)}{4}\big) \\ & + \ B_2(\tau)\ E\big(2m_i; \tau + \frac{1}{2}\big) \ + \ E_6(\tau)\ \eta^{24}(\tau)\ P(2m_i;\tau) \ \Big] \\ & + \ \frac{1}{12}\ E_2(\tau)\ (3\ Z_1(m_i;\tau)\ Z_3(m_i;\tau) \ + \ 2\ (Z_2(m_i;\tau))^2) \\ & - \ \frac{1}{36}\ ((E_2(\tau))^2 \ + \ E_4(\tau))\ (Z_1(m_i;\tau))^2\ Z_2(m_i;\tau) \\ & + \ \frac{1}{2592}\ ((E_2(\tau))^3 \ + \ 3\ E_2(\tau)\ E_4(\tau) \ - \ E_6(\tau))\ (Z_1(m_i;\tau))^4 \ , \end{aligned} \tag{6.25}$$

where

$$
\begin{aligned}
B_0(\tau) \; &= \vartheta_3(0|2\tau)^4 \; \vartheta_4(0|2\tau)^{24} \; \big[32 \; \vartheta_3(0|2\tau)^8 \\
&\qquad\qquad - \; 20 \; \vartheta_3(0|2\tau)^4 \; \vartheta_4(0|2\tau)^4 \; - \; \vartheta_4(0|2\tau)^8 \; \big] \; , \\
B_1(\tau) \; &= \; - \, \frac{1}{2^{26}} \; \vartheta_3(0|\tau/2)^4 \; \vartheta_2(0|\tau/2)^{24} \; \big[11 \; \vartheta_3(0|\tau/2)^8 \\
&\qquad\qquad + \; 22 \; \vartheta_3(0|2\tau)^4 \; \vartheta_4(0|\tau/2)^4 \; - \; \vartheta_4(0|\tau/2)^8 \; \big] \; , \\
B_2(\tau) \; &= \; \frac{1}{16} \; \vartheta_2(0|2\tau)^4 \; \vartheta_4(0|2\tau)^{24} \; \big[11 \; \vartheta_2(0|2\tau)^8 \\
&\qquad\qquad + \; 22 \; \vartheta_2(0|2\tau)^4 \; \vartheta_3(0|2\tau)^4 \; - \; \vartheta_3(0|2\tau)^8 \; \big] \; .
\end{aligned}
\tag{6.26}
$$

6.4. The Structure of the Partition Functions: Weyl Orbits

To understand and generalize the Ansätze above, we note several properties of the partition functions. First, as functions of the m_i, the parammeters of the torus are doubly periodic under translations $m_i \to m_i + \alpha_i$ and $m_i \to m_i + \tau\alpha_i$, where α_i is a root of E_8. It follows that the m_i dependence of the partition functions must appear as a sum over a scaled version of the E_8-root lattice. The precise scale is set by the modular phase and the modular invariance. We also know that the function $Z_n(m_i;\tau)$ must contain a part that is $Z_1(nm_i;n\tau) = P(nm_i;n\tau)$, coming from the multiple windings of the base of the elliptic fibration of $\frac{1}{2}K3$. This leads fairly unambiguously to a generic E_8 root lattice term in g_n of the form:

$$
P_{n,\lambda}(m_i;\tau) \; = \; \sum_{w\in W(E_8)} \sum_{\alpha\in\Lambda(E_8)} q^{\frac{1}{2n} \; (\lambda+n\alpha)^2} \; e^{2\pi i \vec{m}\cdot w(\vec{\lambda}+n\vec{\alpha})} \; , \tag{6.27}
$$

where $\Lambda(E_8)$ and $W(E_8)$ are, respectively, the root lattice and Weyl group of E_8. The sum over the Weyl group is included since the rational curves of $\frac{1}{2}K3$ have an E_8 Weyl invariance, and so the partition functions must have this invariance. While the individual $P_{n,\lambda}$ are not modular invariant, they transform into one another with a modular weight of 4 and a modular phase of the form (4.6).

To count the number of independent functions $P_{n,\lambda}$, one views them as the partition functions of the diagonal $U(1)^8$ in a product of n copies E_8, or equivalently, one thinks of them as the $U(1)^8$ part of the partition function of the a representation of the current algebra $E_8^{(n)}$. The number of independent functions, $P_{n,\lambda}$, is thus equal to the number of characters of $E_8^{(n)}$. For $n = 1,2,3,4$ this number is $1,3,5,10$.

The function g_n is then given by:

$$
g_n(m_i;\tau) \; = \; \sum_{\lambda} \; Z_\lambda(\tau) \; P_{n,\lambda}(m_i;\tau) \; , \tag{6.28}
$$

where the functions $Z_\lambda(\tau)$ are modular forms of weight $6(n-1)$ in $\Gamma(n)$. In particular Z_0 must be a modular form of $\Gamma_1(n)$.

One can now understand the role of the general building blocks of the form (6.22). There are $n+1$ functions of the form $P(nm_i; n\tau)$, and $P(m_i; (\tau+j)/n)$, but for $n \geq 3$ there are more than $n+1$ functions $P_{n,\lambda}$. We therefore need to find ways to weight the characters of independent Weyl orbits in different ways. This can be done explicitly as in (6.28), or implicitly using expressions like (6.22).

Rewriting the partition functions as weighted sums over Weyl orbits is very natural from the Yang-Mills perspective, and is also natural in finding universal properties of the characters [10]. One can thus rewrite the partition function Z_3 as follows.

Introduce the functions:

$$
\begin{aligned}
h_0(\tau) &= \sum_{n_1,n_2=-\infty}^{\infty} q^{n_1^2+n_2^2-n_1 n_2} \ , \qquad h_1(\tau) \ = \ \frac{1}{2}(h_0(\tau/3) - h_0(\tau)) \ , \\
h_2(\tau) &= \frac{\eta^9(\tau)}{\eta^3(3\tau)} \ ; \qquad h_3(\tau) \ = \ 27\,\frac{\eta^9(3\tau)}{\eta^3(\tau)} \ .
\end{aligned}
\tag{6.29}
$$

Note that: (i) h_0, h_2 and h_3 are modular forms of $\Gamma_1(3)$ of weights $1, 3$ and 3 respecively; (ii) h_0 and $h_0(\tau/3)$ are, respectively, the partition functions of the root and weight lattice of $SU(3)$; and (iii) one has $h_0^3 = h_2 + h_3$. Define the functions Q_j as follows:

$$
\begin{aligned}
Q_0(m_i,\tau) &= P(3m_i; 3\tau) \ = \ P_{3,\lambda=0} \ , \\
Q_j(m_i,\tau) &= \frac{1}{3}\,\Big(P\big(m_i;\frac{\tau}{3}\big) \ + \ \omega^{j-1}\ P\big(m_i;\frac{(\tau+1)}{3}\big) \ + \ \omega^{2(j-1)} P\big(m_i;\frac{(\tau+2)}{3}\big)\Big) \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1,2,3 \ , \\
Q_4(m_i,\tau) &= \sum_{\beta\in\Delta(E_8)} \ \sum_{\alpha\in\Lambda(E_8)} q^{\frac{1}{6}\ (\beta+3\alpha)^2}\ e^{2\pi i \vec{m}\cdot(\vec{\beta}+3\vec{\alpha})} \ ,
\end{aligned}
\tag{6.30}
$$

where $\omega = e^{2\pi i/3}$, and $\Delta(E_8)$ and $\Lambda(E_8)$ are, respectively, the roots and root lattice of E_8. Note that Q_j represents the projection of the root lattice onto those vectors of $(length)^2 \equiv 2(j-1)\ mod\ 6$. The functions $P_{3,\lambda}$ are then: $Q_0, Q_4, Q_3, (Q_1 - Q_0)$ and $(Q_2 - Q_4)$.

One then has:

$$\begin{aligned} Z_3(m_i;\tau) \ = \ \frac{1}{864\ \eta^{36}} \Big[& \ h_2^2\ \big[(17\ h_2^2+6h_2h_3-27h_3^2)\ +\ 12\ E_2\ h_0h_2\ +\ 3\ E_2^2\ h_0^2\ \big]\ Q_0 \\ & +\ h_3\ \big[(\tfrac{2}{3}\ h_2^3+57\ h_2^2h_3+204h_2h_3^2+153h_3^3) \\ & -\ 12\ E_2\ h_0\ (\tfrac{1}{9}\ h_2^2+\tfrac{7}{3}\ h_2h_3+3\ h_3^2)\ +\ 3\ E_2^2\ h_0^2\ (\tfrac{2}{9}h_2+h_3)\ \big]\ Q_1 \\ & +\ h_3h_1^2\ \big[\ h_0\ (24\ h_2^3+153\ h_2h_3+153\ h_3^2) \\ & -\ 4\ E_2\ h_0^2\ (4\ h_2+9\ h_3)\ +\ E_2^2\ (\tfrac{8}{3}\ h_2+3\ h_3)\ \big]\ Q_2 \\ & +\ h_3h_1\ \big[\ h_0^2\ (5\ h_2^3+102\ h_2h_3+153\ h_3^2) \\ & -\ \tfrac{4}{3}\ E_2\ (5\ h_2^2+30h_2h_3+27\ h_3^2)\ +\ \tfrac{1}{3}\ E_2^2\ h_0\ (5\ h_2+9\ h_3)\ \big]\ Q_3 \\ & -\ h_2^2h_1^2\ \big[\ h_0\ (h_2+9\ h_3)\ -\ E_2^2\ \big]\ Q_4\ \Big]\ . \end{aligned} \tag{6.31}$$

In terms of the Weyl orbits, the function Z_3 decomposes as:

$$\begin{aligned} Z_3(m_i;\tau) \ = \ \frac{1}{864\ \eta^{36}} \Big[& \big[(17\ h_2^4+\tfrac{20}{3}h_2^3h_3+30h_2^2h_3^2+204h_2h_3^3+153h_3^4) \\ & +\ 12\ E_2\ h_0(h_2^3-\tfrac{1}{9}h_2^2h_3-\tfrac{7}{3}h_2h_3^2-3h_3^3) \\ & +\ 3\ E_2^2\ h_0^2(h_2^2+\tfrac{2}{9}h_2h_3+h_3)^2\ \big]\ P_{3,\lambda_0} \\ & -\ h_1^2\ \big[\ h_0\ (h_2^3-15\ h_2^3h_3-153\ h_2h_3^2+153\ h_3^3) \\ & -\ 4\ E_2\ h_0^2\ (4\ h_2h_3+9\ h_3^2) \\ & -\ E_2^2\ (h_2^2+\tfrac{8}{3}\ h_2h_3+3h_3^2)\ \big]\ P_{3,\lambda_1} \\ & +\ h_3h_1\ \big[\ h_0^2\ (5\ h_2^3+102\ h_2h_3+153\ h_3^2) \\ & -\ \tfrac{4}{3}\ E_2\ (5\ h_2^2+30h_2h_3+27\ h_3^2) \\ & +\ \tfrac{1}{3}\ E_2^2\ h_0\ (5\ h_2+9\ h_3)\ \big]\ P_{3,\lambda_2} \\ & +\ h_3\ \big[(\tfrac{2}{3}\ h_2^3+57\ h_2^2h_3+204h_2h_3^2+153h_3^3) \\ & -\ 12\ E_2\ h_0\ (\tfrac{1}{9}\ h_2^2+\tfrac{7}{3}\ h_2h_3+3\ h_3^2) \\ & +\ 3\ E_2^2\ h_0^2\ (\tfrac{2}{9}h_2+h_3)\ \big]\ P_{3,\lambda_3} \\ & +\ h_3h_1^2\ \big[\ h_0\ (24\ h_2^3+153\ h_2h_3+153\ h_3^2) \\ & -\ 4\ E_2\ h_0^2\ (4\ h_2+9\ h_3)\ +\ E_2^2\ (\tfrac{8}{3}\ h_2+3\ h_3)\ \big]\ P_{3,\lambda_4}\ \Big]\ . \end{aligned} \tag{6.32}$$

7. The String Interpretation

We have seen how to compute the number of BPS states for E-strings wrapped n-times around a circle. In this section we would like to examine the nature of the low-energy modes propagating on the E-string. Before doing this, it would be helpful to review the relationship between multi-wound and singly-wound heterotic strings. It is natural to compare the E-string to the heterotic string since the heterotic string arises by wrapping an M5 brane around $K3$, whereas E-string arises by wrapping an M5 brane around $\frac{1}{2}K3$.

7.1. Multi-wound strings in general

It is one of the remarkable properties of M5 branes that when we wrap n of them around $K3 \times T^2$ we simply obtain the n-wound heterotic string on the torus: One has started out with n independent objects, but the process of wrapping them results in the Hilbert space of a single string. The multi-wrapping has effectively resolved itself into a concatenation of strings. As is well known, the structure of the Hilbert space for multi-wound heterotic strings can be easily obtained from the Hilbert space of singly wound strings. In particular if we wish to find BPS states for the heterotic string with momentum p and winding number n we should consider

$$pn = N_L - 1$$

For a singly-wound string we have $p = N_L - 1$, while for a string wrapped n times we have

$$p = \frac{N_L - 1}{n}.$$

Given that p is an integer, we obtain states with oscillator numbers that are a multiple n of the basic unit.

Thus, thinking of the heterotic string as arising from an M5-brane wrapping $K3$, we see that a "doubled" heterotic string coming from two M5-branes wrapping $K3$ is in some sense not a new object but is the same as the original heterotic string merely doubly wound. Moreover, the BPS states of the doubly wound heterotic string are obtained from those of the singly-wound string by scaling the torus and doing appropriate projections. We interpret this as being due to the mild interaction between contiguous heterotic strings. However, for other types of strings we may not find such simple behavior if there are strong dynamics between strings (and their parent M5-branes) as they get close to one another.

It is also true that if a heterotic string has some level one group G current algebra as a symmetry, then the spectrum of BPS states with winding number n has a natural action under the level n current algebra. This is so because the level one G currents get projected onto the subset of modes that are 0 mod n in the n-winding sector. It is elementary to verify that such modes, J^i_{pn} , $p \in \mathbb{Z}$, satisfy a level n current algebra.

Now what do our results tell us about the E-string and its dynamics when two or more strings lie close to one another? For a singly wound E-string the spectrum of BPS states in (6.4) is consistent with a level one E_8 current algebra. Moreover, the low energy dynamics seems to be that of a free theory, corresponding to the position of the string in transverse space. The string picture also readily predicts that all BPS partition functions should have weight $(-2, 0)$. This is because the n bound E-strings have 4 non-compact transverse modes for their center of mass motion. We also saw that the number of inequivalent flux classes for n five-branes is equal to the number of inequivalent orbits of $\Gamma^8/n\Gamma^8$ under the E_8 Weyl group (we mod out by the E_8 Weyl group because fluxes which map to each other under the Weyl group are diffeomorphically equivalent). This matches the number of level n E_8 characters. This suggests that the corresponding n string bound state carries a level n E_8 current algebra. This too agrees with the string picture in the sense that a level n current algebra appears in the n-wound heterotic string.

However, this is where the similarities end. Namely the partition function we have found for the BPS states of n-wound E-strings coming from n M5 branes wrapping $\frac{1}{2}K3$ cannot be viewed as coming from a simple multi-winding of a one E-string, suggesting that there are non-trivial strong dynamics between E-strings. In particular our results suggest that n copies of an E-string can form *new bound states*, that are new strings in their own right. We call these $E^{(n)}$-strings. These should be viewed as bound states of $E^{(i)}$-strings at threshold with $i < n$.

We know that the low energy modes on the $E^{(n)}$-string seem to have a level n E_8 current algebra, along with some other left-over degrees of freedom. Of course, this is only for the left-movers which couple to the right-moving ground state. The rest of the modes remain unknown.

7.2. Holomorphic Anomaly and $E^{(n)}$-String

We have already given an interpretation of holomorphic anomaly from the viewpoint of $N = 4$ Yang-Mills in terms of reducible connections $U(n) \to U(k) \times U(n-k)$. We also have given an interpretation of the anomaly in terms of holomorphic curve counting. We would also like to interpret the anomaly from the viewpoint of the $E^{(n)}$-string. Though a bit less precise than the other versions, this nevertheless teaches us something about the spectrum of the Hilbert space of $E^{(n)}$.

It should be clear from the previous discussions that the anomaly is related to the decomposition $E^{(n)} \to E^{(k)} + E^{(n-k)}$. In fact we can see this as follows: The formal argument that $Tr\left[(-1)^F F_R^2 q_0^L \overline{q}^{\overline{L}_0}\right]$ is holomorphic in τ (for non-compact bosons) relies on the cancellation of supersymmetric states in pairs to remove all non-holomorphic contributions. However given the existence of a channel $E^{(n)} \to E^{(k)} + E^{(n-k)}$, one expects the Hilbert space of the $E^{(n)}$ string to be gapless. When there is no gap the formal arguments for holomorphicity can have potential anomalies, basically because the density of states in a non-compact space may differ between bosons and their fermionic partners. Examples of this were found in [44]. Thus we find a natural string interpretation for the anomaly.

8. Suggestions for Further Study

We have studied the BPS states of E-strings from various perspectives and thus have connected the counting of holomorphic curves with the partition functions of $N = 4$ topological Yang-Mills on $\frac{1}{2}K3$. We have found that the holomorphic anomaly first observed for topological strings [13] is related to the holomorphic anomaly for BPS state counting [11,39] which in turn is related to the holomorphic anomaly for topological $N = 4$ Yang-Mills on manifolds with $b_2^+ = 1$ [12].

The BPS state partition functions can be completely determined using four inputs: modular invariance, the holomorphic anomaly, the gap, and the winding number 1 partition function Z_1. It is amusing that each of these facts can perhaps be understood from different perspectives: The holomorphic anomaly is natural from the curve counting viewpoint. The presence of the gap and $SL(2, Z)$ invariance are natural from both the curve counting and the $N = 4$ Yang-Mills viewpoint and the partition function Z_1 with a level one E_8 current algebra is natural from the free string description of the single E-string.

There are many possible extensions of this work. We have found the partition functions for left-moving excitations responsible for the BPS states of the E-strings. The partition functions contain a level n E_8 current algebra, along with a computable but not so easily identifiable piece. It would be nice to associate this extra piece with a specific two-dimensional quantum field theory.

We have also computed the partition function for $N = 4$ topological $U(n)$ Yang-Mills on $K3$, extending the results in [12]. The derivation has a simple interpretation in terms of M5 branes, but it is not rigorous, since we assumed that coincident points of M5 branes in target space do not contribute anything extra to the partition function. It would be nice to make this statement rigorous by finding a topological deformation that separates the M5 branes while preserving at least $N = 1$ supersymmetry. It is also natural to ask if we can extend these results to other groups. We probably have enough ingredients to actually do the calculation. It should also be possible to extend the results for $K3$ to manifolds with $h^{2,0} > 1$. We gave a heuristic argument in terms of M5 branes that this should be possible.

We have computed the partition function of $N = 4$ Yang-Mills on a specific manifold with $b_2^+ = 1$. There are other manifolds (for example P^2 blown up at $n \neq 9$ points) for which we do not know the answer. In fact the case with $n < 9$ is interesting for counting BPS states for five-dimensional versions of E-strings [45,46,10]. The count in [9] applies only to electric M2 branes wrapped inside these del Pezzo surfaces, whereas the N=4 Yang-Mills theory on these spaces computes the magnetic states, which are obtained from the wrapped 5-brane. Only for $\frac{1}{2}K3$, which is *elliptic*, are these two the same, because the $SL(2, Z)$ on the fiber generates electric/magnetic self-duality. We can also turn this around. In particular the computation of the $SU(2)$ partition function on $\mathbf{P}^2$ discussed in [12] (see also [47]) gives the number of BPS bound states of two M5-branes wrapped around $P^2 \times S^1$. Note that this partition function is not related in a simple way to the partition function of a single M5 brane wrapped around $\mathbf{P}^2 \times S^1$. This shows that also here new bound strings are appearing at threshold, very much like the E-string story. Also the fact that the moduli space of instantons on $\mathbf{P}^2$ blown up at up to 8 points varies dramatically as we change the Kähler metric, implies that the $N = 4$ topological Yang-Mills partition function on these spaces, which computes the number of BPS states of the corresponding string, will exhibit similar phenomena. This is then interpreted from the $N = 2$ field theory perspective in four dimension as the decay of BPS states as we vary the vector multiplet moduli of $N = 2$ theories. It would thus be very interesting to develop techniques to compute the $N = 4$ topological $U(n)$ Yang-Mills on these spaces as well.

Note added: After completing this work K. Yoshioka has managed to prove the formula derived in (6.15) for the partition function of $SU(2)$ on $\frac{1}{2}K3$ [48] using [47] and results relating differences of Donaldson invariants to Jacobi forms [49].

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Appendix A. Computing the BPS States For Singly Wound E-string

Taking the curve in (5.6) and shifting x by $x \to x - \frac{1}{3}\gamma u^2 + \frac{2}{3}{s_+}^2{s_-}^2 u$ and rescaling u to $u \to u{\vartheta_3}^{12}$, one is left with a curve in the form of (5.1) with

$$
\begin{aligned}
f(u,m_1,m_2,\tau) &= \frac{1}{3}\left(E_4u^4 + (6A-4\gamma B)u^3/\vartheta_3^4 + 4B^2u^2/\vartheta_3^{16}\right) = \frac{1}{3}\overline{E}_4 \\
g(u,m_1,m_2,\tau) &= \frac{2}{27}\Big(E_6u^6 - 27u^5 + 9k^2Bu^5 + 9\gamma Au^5 - 6\gamma^2Bu^5 + 12\gamma B^2/\vartheta_3^{12} \\
&\qquad - 18ABu^4/\vartheta_3^{12} + 8B^3u^3/\vartheta_3^{24}\Big) = \frac{2}{27}\bar{E}_6\ .
\end{aligned}
\tag{A.1}
$$

The Eisenstein functions, E_n, can be written in terms of k:

$$
E_4 \ = \ (1-k^2+k^4)\ \vartheta_3^8\ , \qquad\qquad E_6 \ = \ \frac{1}{2}(1+k^2)(1-2k^2)(2-k^2)\ \vartheta_3^{12}\ . \tag{A.2}
$$

The coefficients A and B are given by:

$$
A = {s_+}^2 + {s_-}^2 \qquad\qquad B = {s_+}^2{s_-}^2\ . \tag{A.3}
$$

To leading order in $1/u$, $\left(\frac{\widetilde{E}_4}{f(u,m_1,m_2,\tau)}\right)^{1/4} = 1$, however to lowest order $\widetilde{\tau}-\tau = 0$. To find the next to leading order contribution to this difference, we equate $\widetilde{E}_4^3/\widetilde{E}_6^2$ with f^3/g^2. Up to the next to leading order we find that:

$$
\frac{\widetilde{E}_4^3}{\widetilde{E}_6^2} \ = \ \frac{{E_4}^3}{{E_6}^2} + \left((12A-4\gamma)\frac{{E_4}^2}{{E_6}^2{\vartheta_3}^4} + (54-(18k^2+6\gamma^2)B - 18\gamma A)\frac{{E_4}^3}{{E_6}^3}\right)\frac{1}{u} \ + \ \mathrm{O}(u^{-2}) \tag{A.4}
$$

Taking a u derivative on both sides of the equation in (A.4) we find

$$
\frac{\Delta {E_4}^2}{{E_6}^3}\frac{1}{q}\frac{\partial\widetilde{q}}{\partial u} = -\left((12A-4\gamma)\frac{{E_4}^2}{{E_6}^2{\vartheta_3}^4} + (54-(18k^2+6\gamma^2)B - 18\gamma A)\frac{{E_4}^3}{{E_6}^3}\right)\frac{1}{u^2} \ + \mathrm{O}(u^{-3}), \tag{A.5}
$$

where $\widetilde{q} = e^{2\pi i\widetilde{\tau}}$. Thus,

$$
\widetilde{\tau}-\tau = \frac{1}{2\pi i\Delta}\left((12A-4\gamma)\frac{E_6}{{\vartheta_3}^4} + (54-(18k^2+6\gamma^2)B - 18\gamma A)E_4\right)\frac{1}{u} + + \mathrm{O}(u^{-2}). \tag{A.6}
$$

If we now substitute (A.6) and (A.3) into (5.11), use the identities

$$
\begin{aligned}
A \ &= \ 2\ \frac{{\vartheta_3}^2}{{\vartheta_2}^2}\ \frac{{\vartheta_2}^2\vartheta_3(m_1)\vartheta_3(m_2) - {\vartheta_3}^2\vartheta_2(m_1)\vartheta_2(m_2)}{\chi(m_1,m_2)} \\
B \ &= \ \frac{{\vartheta_3}^4}{{\vartheta_2}^4}\ \frac{{\vartheta_3}^2\vartheta_3(m_1)\vartheta_3(m_2) - {\vartheta_2}^2\vartheta_2(m_1)\vartheta_2(m_2) - {\vartheta_4}^2\vartheta_4(m_1)\vartheta_4(m_2)}{\chi(m_1,m_2)}\ ,
\end{aligned}
\tag{A.7}
$$

integrate with respect to u, and then make the lowest order approximation $u = e^{-2\pi i\phi}$, we find the equation in (6.1).

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