

E-UNITARY INVERSE SEMIGROUPS OVER SEMILATTICES

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1. Introduction. An inverse semigroup is called *E-unitary* if the equations $ea = e = e^2$ together imply $a^2 = a$. In a previous paper [4], the author showed that any *E-unitary* inverse semigroup is isomorphic to a semigroup constructed from a triple $(G, \mathcal{X}, \mathcal{Y})$ consisting of a down-directed partially ordered set \mathcal{X} , an ideal and subsemilattice \mathcal{Y} of \mathcal{X} and a group G acting on \mathcal{X} , on the left, by order automorphisms in such a way that $\mathcal{X} = G\mathcal{Y}$. This semigroup is denoted by $P(G, \mathcal{X}, \mathcal{Y})$; it consists of all pairs $(a, g) \in \mathcal{Y} \times G$ such that $g^{-1}a \in \mathcal{Y}$, under the multiplication

$$(a, g)(b, h) = (a \wedge gb, gh).$$

The aim of this paper is to give necessary and sufficient conditions on an inverse semigroup in order that it should be isomorphic to some $P(G, \mathcal{X}, \mathcal{Y})$ with \mathcal{X} a semilattice. As well, we consider those congruences ρ on an inverse semigroup $P(G, \mathcal{X}, \mathcal{Y})$ for which the quotient has the form $P(H, \mathcal{U}, \mathcal{V})$ for some triple $H, \mathcal{U}, \mathcal{V}$ as above, with \mathcal{U} a semilattice.

We shall assume familiarity with the construction and properties of $P(G, \mathcal{X}, \mathcal{Y})$ from [3], [4]. Undefined notation and terminology is that of Clifford and Preston [1]. In particular, when we are considering a partial order on an inverse semigroup, the partial order being referred to is the natural partial order; it is defined by

$$a \leq b \text{ if and only if } a = eb \text{ for some } e^2 = e \in S.$$

Throughout the paper, when the terminology “triple $(G, \mathcal{X}, \mathcal{Y})$ ” is used, it means that \mathcal{X} is a down-directed partially ordered set with \mathcal{Y} an ideal and subsemilattice of \mathcal{X} , and that G is a group acting on \mathcal{X} by order automorphisms in such a way that $\mathcal{X} = G\mathcal{Y}$.

DEFINITION 1.1. Let S be an inverse semigroup. Then we say that S is an *E-unitary inverse semigroup over a semilattice* if $S \cong P(G, \mathcal{X}, \mathcal{Y})$ for some triple $(G, \mathcal{X}, \mathcal{Y})$ with \mathcal{X} a semilattice.

In terms of Definition 1.1, the aim of this paper is therefore to characterize *E-unitary inverse semigroups over a semilattice*.

2. The general case

DEFINITION 2.1. Let S be a partially ordered set and let $\theta: S \rightarrow T$ be a mapping of S into a set T . Then θ is an *m-map* if, for each $t \in T$, the set $\{s \in S: s\theta = t\}$ has a maximum member.

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Let S be an inverse semigroup. Then Munn [7] has shown that the relation σ on S defined by

$$(a, b) \in \sigma \text{ if and only if } ea = eb \text{ for some } e^2 = e \in S$$

is the smallest congruence ρ on S for which S/ρ is a group.

The following results about σ will be used without comment in several places in the remainder of the paper.

LEMMA 2.2. (A) *Let S be an inverse semigroup and let e, f be idempotents in S . Let σ be the minimum group congruence on S . Then*

$$\sigma_{e,f} = \sigma \cap (eSf \times eSf)$$

is the minimum group congruence on eSf . Similarly

$$\sigma_e = \sigma \cap (Se \times Se)$$

is the minimum group congruence on Se .

(B) *If I is a non-empty ideal of S then $\sigma \cap (I \times I)$ is the minimum group congruence on I .*

(C) *Let $(G, \mathcal{X}, \mathcal{Y})$ be a triple and let S be an inverse subsemigroup of $P(G, \mathcal{X}, \mathcal{Y})$. Let $a = (u, g)$, $b = (v, h)$ belong to S . Then*

$$(a, b) \in \sigma \text{ if and only if } g = h.$$

Proof. (A) Let $\gamma_{e,f}$ denote the restriction of $\gamma = \sigma^h$ to a homomorphism of eSf into $G = S/\sigma$. Then, since for each $a \in S$,

$$a\gamma = e\gamma a\gamma f\gamma = (eaf)\gamma,$$

because $e\gamma = f\gamma = 1$ (the identity of G), $\gamma_{e,f}$ is a homomorphism of eSf onto G . Thus $\sigma_{e,f}$ is a group congruence on eSf .

On the other hand, suppose that ρ is a group congruence on eSf and let $(a, b) \in \sigma_{e,f}$. Then $au = bu$ for some idempotent $u \in S$. This implies

$$a(euf) = (au)ef = (bu)ef = b(euf)$$

since idempotents commute. But, since $euf \in eSf$ and ρ is a group congruence on eSf , it follows from these equalities that $(a, b) \in \rho$. Hence $\sigma_{e,f} \subseteq \rho$. In the same way it can be shown that σ_e is the minimum group congruence on Se .

(B) The proof of this is similar.

(C) Suppose that $(a, b) \in \sigma$. Then $ae = be$ for some idempotent $e = (f, 1) \in S$; thus $g = h$. On the other hand, suppose that $g = h$. Let $e = b^{-1}aa^{-1}b \in S$. Then $ae = be$ and so $(a, b) \in \sigma$.

Let S be an inverse semigroup and let e, f be idempotents of S . Then we shall follow the notation introduced in Lemma 2.2 and denote by $\gamma_{e,f}$ the restriction of σ^h to eSf and by γ_e the restriction of σ^h to Se ; each is a homomorphism onto $G = S/\sigma$.

THEOREM 2.3. *Let $(G, \mathcal{X}, \mathcal{Y})$ be a triple and set $S = P(G, \mathcal{X}, \mathcal{Y})$. Then \mathcal{X} is a semilattice if and only if, for each pair of idempotents $e, f \in S$, $\gamma_{e,f}: eSf \rightarrow S/\sigma$ is an m -map.*

Proof. Since $\mathcal{X} = G\mathcal{Y}$ and G acts by order automorphisms, it is easy to see that \mathcal{X} is a semilattice if and only if $a \wedge e$ exists for each $a \in \mathcal{X}$, $e \in \mathcal{Y}$.

Suppose that \mathcal{X} is a semilattice. Let $e = (u, 1)$, $f = (v, 1)$ with $u, v \in \mathcal{Y}$, and pick $g \in G$. Then $s \in eSf$ is such that $s\sigma^h = g$ if and only if $s = (b, g)$ for some $b \leq u$ with $g^{-1}b \leq v$. If this is the case, then $b \leq u$, gv and, by hypothesis, $u \wedge gv$ exists. Hence $b \leq u \wedge gv$ so that $(b, g) \leq (u \wedge gv, g) \in eSf$; but $(u \wedge gv, g)\sigma^h = g$. Thus

$$(u \wedge gv, g) = \max\{s \in eSf : s\sigma^h = g\}$$

and, since g was arbitrarily chosen in G , $\gamma_{e,f}$ is consequently an m -map.

Conversely, suppose that each $\gamma_{e,f}$ is an m -map and let $a \in \mathcal{X}$, $e \in \mathcal{Y}$; then $a = gf$ for some $g \in G$, $f \in \mathcal{Y}$. By hypothesis, the set

$$\{s \in (e, 1)S(f, 1) : s\sigma^h = g\}$$

has a maximum member (c, g) . Since $(c, g) \in (e, 1)S(f, 1)$, $c \leq e$, $g^{-1}c \leq f$, so that $c \leq e, a$. On the other hand, if $b \leq e, a$ then $(b, g) \in (e, 1)S(f, 1)$. But $(b, g)\sigma^h = g$, so that $(b, g) \leq (c, g)$; that is $b \leq c$. Hence $c = e \wedge a$ exists and \mathcal{X} is consequently a semilattice.

COROLLARY 2.4. *An inverse semigroup S is an E -unitary inverse semigroup over a semilattice if and only if S is E -unitary and each $\gamma_{e,f}$ is an m -map.*

DEFINITION 2.5 [6]. An inverse semigroup S is F -inverse if and only if $\sigma^h : S \rightarrow S/\sigma$ is an m -map.

McFadden and O'Carroll [6] showed that an F -inverse semigroup has an identity. On the other hand, it is shown in [4] that an inverse monoid is E -unitary over a semilattice if and only if it is F -inverse. This result is expressed in the context of this paper by the next theorem.

THEOREM 2.6. *Let S be an inverse semigroup. Then the following statements are equivalent:*

- (i) S is F -inverse;
- (ii) S has an identity and each $\gamma_{e,f} : eSf \rightarrow S/\sigma$ is an m -map;
- (iii) $S \cong P(G, \mathcal{X}, \mathcal{Y})$ for some triple $(G, \mathcal{X}, \mathcal{Y})$ with \mathcal{X} a semilattice and \mathcal{Y} a principal ideal of \mathcal{X} .

Proof. (i) \Rightarrow (ii). As pointed out above, McFadden and O'Carroll [6] have shown that any F -inverse semigroup has an identity; the identity is the element $e = \max\{s \in S : s\sigma^h = 1\}$, where 1 denotes the identity of S/σ .

Let u, v be idempotents of S and, for $g \in G$, let $h = \max\{s \in S : s\sigma^h = g\}$. Then $uhv \in uSv$ and $(uhv)\sigma^h = g$. If $s \in uSv$ is such that $s\sigma^h = g$ then $s \leq h$ and so $s = usv \leq uhv$. Hence $uhv = \max\{s \in uSv : s\sigma^h = g\}$; it follows that $\gamma_{u,v}$ is an m -map.

(ii) \Rightarrow (iii). Since S has an identity, it follows from Corollary 2.4 that we need only verify that S is E -unitary. Suppose that $fa = f = f^2$ for some $a \in S$. Then $a\sigma^h = 1$ so that $a \leq \max\{s \in S : s\sigma_{e,e}^h = 1\}$, where e is the identity of S . But e is a maximal element of $S = eSe$ so that $e = \max\{s \in S : s\sigma^h = 1\}$; thus $a \leq e$. This implies $a = aa^{-1}e = aa^{-1}$, so that a is idempotent. Hence S is E -unitary.

(iii) \Rightarrow (i). Suppose $S = P(G, \mathcal{X}, \mathcal{Y})$ with \mathcal{X} a semilattice and \mathcal{Y} a principal ideal of \mathcal{X} , and let e be the maximum element of \mathcal{Y} . Then, as in the proof of Theorem 2.3, $(e \wedge ge, g)$ is the maximum element s of S with $so^h = g$. Hence, S is F -inverse.

Theorem 2.6 shows that, in the presence of an identity, the condition

$$\text{each } \gamma_{e,f} \text{ is an } m\text{-map}$$

ensures that S is E -unitary. This is not the case in general.

EXAMPLE 2.7. Let M_2 be the Brandt semigroup $\mathcal{M}^0(\{1, \{1, 2\}, \{1, 2\}, \Delta)$. Then M_2 has the multiplication table

	0	a	a^{-1}	e	f
0	0	0	0	0	0
a	0	0	e	0	a
a^{-1}	0	f	0	a^{-1}	0
e	0	a	0	e	0
f	0	0	a^{-1}	0	f

with $a = (1, 2)$, $a^{-1} = (2, 1)$, $e = (1, 1)$, $f = (2, 2)$.

In M_2 , $eSe = \{e, 0\}$, $eSf = \{a, 0\}$, $fSe = \{a^{-1}, 0\}$, $fSf = \{f, 0\}$ and all other uSv with $u^2 = u$, $v^2 = v$ are $\{0\}$. Hence each $\gamma_{u,v}$ is an m -map. But $S = M_2$ is not E -unitary.

In a sense, M_2 is the only counterexample to the hypothesis:

$$\text{if each } \gamma_{e,f} \text{ is an } m\text{-map, then } S \text{ is } E\text{-unitary.}$$

Before verifying this, we prove a lemma.

DEFINITION 2.8 [9]. Let S be an inverse semigroup. Then S is E -reflexive if and only if, for $a, b \in S$, ab is idempotent if and only if ba is idempotent.

LEMMA 2.9. Let S be an inverse semigroup. Then the following statements are equivalent:

- (i) there exists $a \in S$ such that $a^2 < a$;
- (ii) S contains an isomorphic copy of M_2 ;
- (iii) S is not E -reflexive.

Proof. (i) \Rightarrow (ii). Suppose $a^2 < a$. Then $a^2 = a^2 a^{-2} a$ so that $a^3 = a^2 a^{-2} a \cdot a = a^2$. This implies that a^2 is idempotent and $a^2 = a^{-2}$. Consider the subsemigroup T of S generated by a and a^{-1} . Then, since T is a homomorphic image of the free inverse semigroup on one generator, it follows from [2] that each element of T is of the form $a^r a^{-s} a^t$ with $r, t \leq s$. Because $a^2 = a^3 = a^{-3} = a^{-2}$, one sees that T has at most five members, $a, a^{-1}, e = aa^{-1}, f = a^{-1}a$ and $0 = a^2$. Indeed, all five are distinct since, otherwise, a would belong to some subgroup of S and this would contradict $a^2 < a$. Thus T has the multiplication table in Example 2.7. That is, $T \cong M_2$.

(ii) \Rightarrow (iii). Suppose $M_2 \subseteq S$ and let $a = (1, 2)$, $b = (1, 1)$. Then $ab = 0$ is idempotent but $ba = a$ is not.

(iii) \Rightarrow (i) Let $c, d \in S$ be such that cd is idempotent but $a = dc$ is not. Then $a^2 = a^3$ so that $a^2 < a$.

THEOREM 2.10. *Let S be an inverse semigroup. Then S is an E -unitary inverse semigroup over a semilattice if and only if S is E -reflexive and each $\gamma_{e,f}: eSf \rightarrow S/\sigma$ is an m -map.*

Proof. Suppose that S is E -reflexive and that each $\gamma_{e,f}$ is an m -map. Let $ea = e = e^2$. Then, since a is a maximal member of $aa^{-1}Sa^{-1}a$,

$$a = \max\{b \in aa^{-1}Sa^{-1}a : b\sigma^h = 1\}.$$

Thus $a^2 \leq a$ so that, by Lemma 2.9, $a^2 = a$. Hence S is E -unitary and so, by Corollary 2.4, $S \cong P(G, \mathcal{X}, \mathcal{Y})$ where \mathcal{X} is a semilattice.

Conversely, let $S \cong P(G, \mathcal{X}, \mathcal{Y})$, where \mathcal{X} is a semilattice. Then, since M_2 is not E -unitary, S does not contain M_2 . Thus, by Lemma 2.9, S is E -reflexive. Further, Theorem 2.3 shows that each $\gamma_{e,f}$ is an m -map.

It is an easy matter to see that if an inverse semigroup T is E -reflexive and each $\gamma_{e,f}: eTf \rightarrow T/\sigma$ is an m -map, then the same is true for each ideal of T . In particular, if an inverse semigroup S is embedded as an ideal in an F -inverse semigroup T then S is E -unitary over a semilattice. Example 2.13 shows that the converse need not be the case.

LEMMA 2.11. *Let $(G, \mathcal{U}, \mathcal{V})$ be a triple and let \mathcal{Y} be an ideal of \mathcal{V} , thus of \mathcal{U} , and set $\mathcal{X} = G\mathcal{Y}$. Suppose that $\mathcal{X} \cap \mathcal{V} = \mathcal{Y}$. Then $(G, \mathcal{X}, \mathcal{Y})$ is a triple and $P(G, \mathcal{X}, \mathcal{Y})$ is an ideal of $P(G, \mathcal{U}, \mathcal{V})$; if \mathcal{U} is a semilattice, so is \mathcal{X} .*

Conversely, if S is an ideal of $P(G, \mathcal{U}, \mathcal{V})$ then $\mathcal{Y} = \{a \in \mathcal{V} : (a, 1) \in S\}$ is an ideal of \mathcal{V} such that $G\mathcal{Y} \cap \mathcal{V} = \mathcal{Y}$. Further, $S = P(G, G\mathcal{Y}, \mathcal{Y})$.

Proof. This is straightforward.

THEOREM 2.12. *Let S be an inverse semigroup. Then the following statements are equivalent:*

- (1) *each $\gamma_e: Se \rightarrow S/\sigma$ is an m -map, for $e^2 = e \in S$;*
- (2) *the translational hull $\Omega(S)$ of S is F -inverse;*
- (3) *S can be embedded as an ideal in an F -inverse semigroup.*

Proof. (1) \Rightarrow (2). Suppose that (1) holds. We first show that S is E -unitary. Suppose that $ea = e = e^2$. Then $a \in Sa^{-1}a$ is such that $a\sigma^h = 1$. By hypothesis, the set $\{s \in Sa^{-1}a : s\sigma^h = 1\}$ has a maximum member t ; thus $a, a^{-1}a \leq t$. But $a, a^{-1}a$ are maximal in $Sa^{-1}a$, from which it follows that $t = a = a^{-1}a$. Thus $a^2 = a$ and so S is E -unitary.

We may therefore suppose that $S = P(G, \mathcal{X}, \mathcal{Y})$ for some triple $(G, \mathcal{X}, \mathcal{Y})$. Let \mathcal{X}^* and \mathcal{Y}^* denote the set of all non-empty order ideals of \mathcal{X} and \mathcal{Y} , respectively, under inclusion, and let G act on \mathcal{X}^* by $gA = \{ga : a \in A\}$ for each $A \in \mathcal{X}^*$. Let $\mathcal{X}^* = G\mathcal{Y}^*$. Then $(G, \mathcal{X}^*, \mathcal{Y}^*)$ is a triple and we may regard S as being embedded in $P(G, \mathcal{X}^*, \mathcal{Y}^*)$ by $(a, g) \mapsto (\bar{a}, g)$, where, for $a \in \mathcal{X}$, $\bar{a} = \{x \in \mathcal{X} : x \leq a\}$. Assume that this has been done. Then it is shown in [5, Section 3] that $\Omega(S)$ is isomorphic to the idealizer of S in $P(G, \mathcal{X}^*, \mathcal{Y}^*)$. Further, it is shown there that $\hat{S} = P(G, \mathcal{X}^*, \mathcal{Y}^*)$ is an F -inverse semigroup in which, for each $g \in G$, (I_g, g) is the maximum element t of \hat{S} with $t\sigma^h = g$; here $I_g = \mathcal{Y} \cap g\mathcal{Y}$.

It follows from these remarks and Lemma 2.2 that in order to show that $\Omega(S)$ is F -inverse, it suffices to show that (I_g, g) is in $\Omega(S)$ for each $g \in G$. It is shown in [5, Theorem 3.9] that $\Omega(S)$ consists of all pairs $(A, g) \in P(G, \mathcal{X}^*, \mathcal{Y}^*)$ such that, for each $e \in \mathcal{Y}$, $\{x \in A : x \leq e\}$ has a maximum member.

Let $e \in \mathcal{Y}$, $g \in G$. Then, by hypothesis, the set $\{s \in S(e, 1) : s\sigma^h = g^{-1}\}$ has a maximum member $t = (g^{-1}c, g^{-1})$ for some $c \in \mathcal{Y}$. Since $t \in S(e, 1)$, $t^{-1}t \leq (e, 1)$; that is, $c \leq e$. Hence $c \in \{x \in \mathcal{Y} \cap g\mathcal{Y} : x \leq e\}$. On the other hand, if $x \in \mathcal{Y} \cap g\mathcal{Y}$, $x \leq e$ then $(g^{-1}x, g^{-1}) \in S(e, 1)$ and $(g^{-1}x, g^{-1})\sigma^h = g^{-1}$. This means that $(g^{-1}x, g^{-1}) \leq (g^{-1}c, g^{-1})$ so that $g^{-1}x \leq g^{-1}c$ and so $x \leq c$. It follows that $c = \max\{x \in I_g : x \leq e\}$. Hence $(I_g, g) \in \Omega(S)$ and then $(I_g, g) = \max\{s \in \Omega(S) : s\sigma = g\}$. Consequently $\Omega(S)$ is F -inverse.

(2) \Rightarrow (3) is immediate, since S is an ideal of $\Omega(S)$.

(3) \Rightarrow (1). Suppose that S is an ideal of an F -inverse semigroup T . By Theorem 2.6 we may assume $T = P(G, \mathcal{U}, \mathcal{V})$, where \mathcal{U} is a semilattice and \mathcal{V} is a principal ideal with greatest element v . Let $e = (f, 1) \in S$ and pick $g \in G$. Then $g^{-1}v \wedge f \leq f$ so that $g^{-1}v \wedge f \in \mathcal{Y}$, where $\mathcal{Y} = \{u \in \mathcal{U} : (u, 1) \in S\}$, and $g(g^{-1}v \wedge f) \geq v$ so that $g(g^{-1}v \wedge f) \in \mathcal{V} \cap G\mathcal{Y} = \mathcal{Y}$, by Lemma 2.11. Now by Lemma 2.11, $S = P(G, G\mathcal{Y}, \mathcal{Y})$. Hence $(g^{-1}v \wedge f, g^{-1}) \in S$ and so, consequently, $(g^{-1}v \wedge f, g^{-1})^{-1} = (v \wedge gf, g) \in S$. Indeed, since $g^{-1}(gf \wedge v) \leq f$, $(v \wedge gf, g) \in S$ and $(v \wedge gf, g)\sigma^h = g$.

On the other hand, suppose that $(a, g) \in S$. Then $g^{-1}a \leq f$ and $a \in \mathcal{Y} \subseteq \mathcal{V}$ so that $a \leq v$. Hence $a \leq v \wedge gf$ and consequently $(a, g) \leq (v \wedge gf, g)$. It follows that $(v \wedge gf, g) = \max\{s \in S : s\sigma^h = g\}$; hence σ_e is an m -map.

EXAMPLE 2.13. Let \mathbb{Q}^+ be the set of positive rationals under the reverse of the usual ordering and let $\Upsilon = \{x \in \mathbb{Q}^+ : x^2 > 2\}$. Then the group G of positive rationals acts on \mathbb{Q}^+ by multiplication. Let $S = P(G, \mathbb{Q}^+, \Upsilon)$ and let $e = (2, 1) \in S$, $g = \frac{1}{3} \in G$. Then $(f, g) \in S$ for all $f \in \Upsilon$. Thus, as ordered sets,

$$\{s \in S : s\sigma^h = g\} \cong \Upsilon.$$

But Υ has no maximum member ($\sqrt{2}$ is irrational) so that σ_e is not an m -map. Hence S cannot be embedded as an ideal in an F -inverse semigroup.

From the characterization of $\Omega(S)$ in [5, Theorem 3.9], one can show that $\Omega(S) = S^1$.

3. Some special cases. In this section, we consider some special cases in which it is possible to improve on the result in Theorem 2.10.

PROPOSITION 3.1. *Let S be an inverse semigroup and suppose that the semilattice of idempotents of S is up-directed. Then S is an E -unitary inverse semigroup over a semilattice if and only if eSe is F -inverse for each $e^2 = e \in S$.*

Proof. Suppose that S is E -unitary over a semilattice and let $e^2 = e \in S$. Then, by Theorem 2.3, $\gamma_{e,e} : eSe \rightarrow G = S/\sigma$ is an m -map. Since eSe is an inverse semigroup, it follows, from Lemma 2.2(A), that eSe is F -inverse. On the other hand, suppose that each eSe is F -inverse. We show first that S cannot contain M_2 . Thus, by Lemma 2.9, S is E -reflexive.

For any $a \in S$, there exists $e^2 = e$ such that $e \geq aa^{-1}$, $a^{-1}a$. Then a, a^{-1} belong to eSe which is E -unitary, being F -inverse. Since M_2 is not E -unitary, it follows that $T = \langle a, a^{-1} \rangle$ is not isomorphic to M_2 . Hence S does not contain any copy of M_2 .

Next, let u, v be idempotents and let $e^2 = e \geq u, v$. Then $uSv \subseteq eSe$ and, by hypothesis, eSe is F -inverse. Let $g \in G = S/\sigma$. Then, as in the proof of Theorem 2.6,

$$u(\max\{s \in eSe : s\sigma^h = g\})v = \max\{s \in uSv : s\sigma^h = g\}.$$

Hence $\gamma_{u,v}$ is an m -map for all idempotents $u, v \in S$.

It now follows from Theorem 2.10 that S is E -unitary over a semilattice.

The results in the next two propositions are similar to Proposition 3.1. However, they depend on the algebraic structure of the semigroup S rather than on the order structure of the idempotents of S .

PROPOSITION 3.2. *Let S be an inverse semigroup which is a semilattice of groups. Then S is an E -unitary inverse semigroup over a semilattice if and only if eSe is F -inverse for each $e^2 = e \in S$.*

Proof. As with Proposition 3.1, the condition is necessary. On the other hand, suppose that eSe is F -inverse for each $e^2 = e \in S$. Then, since S cannot contain M_2 , we need only show that each $\gamma_{u,v}$ is an m -map. But $uSv = uvSuv$ since idempotents are central, and then, by definition, $\sigma_{u,v} = \sigma_{uv,uv}$, so that the hypothesis in the statement of the proposition implies that each $\gamma_{u,v}$ is an m -map.

PROPOSITION 3.3. *Let S be a simple E -unitary inverse semigroup. Then $S \cong P(G, \mathcal{X}, \mathcal{Y})$ for some triple $(G, \mathcal{X}, \mathcal{Y})$, with \mathcal{X} a semilattice, if and only if, for some idempotent $e \in S$, eSe is F -inverse.*

Proof. We need only show that the condition is sufficient. Suppose eSe is F -inverse for some idempotent e and let u, v be idempotents of S . Since S is simple, there exist b, c such that $bb^{-1} = u$, $b^{-1}b \leq e$, $cc^{-1} = v$, $c^{-1}c \leq e$. Suppose that $x \in uSv$ is such that $x\sigma^h = g \in G$. Then $b^{-1}xc$ is in eSe since

$$b^{-1}xccc^{-1}x^{-1}b \leq b^{-1}b \leq e \quad \text{and} \quad c^{-1}x^{-1}bb^{-1}xc \leq c^{-1}c \leq e.$$

Further $(b^{-1}xc)\sigma^h = (b^{-1}\sigma^h)g(c\sigma^h) = h$, say, so that, by the hypothesis on eSe , $b^{-1}xc \leq z = \max\{x \in eSe : x\sigma^h = h\}$. Then $x = bb^{-1}xccc^{-1} \leq bzc^{-1}$, and $bzc^{-1} \in uSv$ is such that $(bzc^{-1})\sigma^h = g$. Hence $bzc^{-1} = \max\{s \in uSv : s\sigma^h = g\}$, and so $\gamma_{u,v}$ is an m -map.

The requirement, in Proposition 3.3, that S should be E -unitary is necessary as the following example shows.

EXAMPLE 3.4. Let E be the ω -tree with the Hasse diagram shown in Fig. 1 and let $S = T_E$ be the inverse semigroup of order isomorphisms between principal ideals of E . Then E is uniform so that S is bisimple.

For each $a \in E$, denote by ε_a the identity mapping on the principal ideal $\bar{a} = \{x \in E : x \leq a\}$. Then

$$\varepsilon_0 T_E \varepsilon_0 = \{\alpha \in T_E : \Delta\alpha \cup \nabla\alpha \subseteq \bar{0}\}.$$

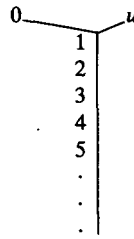


Figure 1

Thus $\varepsilon_0 S \varepsilon_0$ is the semigroup of isomorphisms between principal ideals of $\{0, 1, \dots\}$. Consequently $\varepsilon_0 S \varepsilon_0 \cong B$, where B is the bicyclic semigroup. McFadden and O’Carroll [6] have shown that B is F -inverse.

On the other hand, let $\alpha \in T_E$ be defined by $\Delta\alpha = \bar{0}$ and

$$x\alpha = \begin{cases} u & \text{if } x = 0, \\ x & \text{if } x > 0. \end{cases}$$

Then $\alpha^2 = \varepsilon_1 < \alpha$ so that S contains a copy of M_2 . Hence S is not E -unitary.

The last result is different in type from the earlier ones in this section since on this occasion we impose a restriction on the idempotents of S and deduce a result about the maps $\gamma_{e,f}$.

DEFINITION 3.5. A *locally finite tree* is a partially ordered set X in which the following three conditions are satisfied:

- (i) X is down-directed;
- (ii) if $a, b \leq c$ where $a, b, c \in X$, then $a \leq b$ or $b \leq a$;
- (iii) the set $\{x \in X : a \leq x \leq b\}$ is finite for all $a, b \in X$ with $a \leq b$.

PROPOSITION 3.6. Let S be an inverse semigroup whose idempotents form a locally finite tree. Then the following conditions on S are equivalent:

- (i) S is E -unitary;
- (ii) S is E -unitary over a semilattice;
- (iii) $\Omega(S)$ is F -inverse.

Proof. (i) \Rightarrow (ii). Suppose that S is E -unitary. Then $S \cong P(G, \mathcal{X}, \mathcal{Y})$ for some triple $(G, \mathcal{X}, \mathcal{Y})$, where \mathcal{X} is down-directed and G acts on \mathcal{X} in such a way that $\mathcal{X} = G\mathcal{Y}$. Since $P(G, \mathcal{X}, \mathcal{Y})$ has semilattice of idempotents $\mathcal{Y} \times 1$, \mathcal{Y} is a locally finite tree. In fact, since $\mathcal{X} = G\mathcal{Y}$ and \mathcal{X} is down-directed, it is easy to see that \mathcal{X} is a locally finite tree. Hence \mathcal{X} is a semilattice.

(ii) \Rightarrow (iii). Again, we may suppose that $S = P(G, \mathcal{X}, \mathcal{Y})$ and, as in the proof of Theorem 2.12, it suffices to show that, for each $g \in G, e \in \mathcal{Y}$, the set $\{x \in I_g : x \leq e\}$ has a maximum member.

For each $x \in I_g$ such that $x \leq e$, the set $\{y \in I_g : x \leq y \leq e\}$ has a maximum member m_x since \mathcal{Y} is locally finite. Fix such an x and let $z \in I_g$ with $z \leq e$. Then either $z \leq x$ or $x < z$. In the first case $z \leq x \leq m_x$ while, in the second, $x < z \leq e$, $z \in I_g$ imply $z \leq m_x$. Hence $m_x = \max\{y \in I_g : y \leq e\}$. This shows that $(I_g, g) \in \Omega(S)$, so that $\Omega(S)$ is F -inverse.

(iii) \Rightarrow (i). This is immediate.

4. Congruences. In this section, we characterize those congruences on an E -unitary inverse semigroup over a semilattice whose quotient is also of this type.

DEFINITION 4.1 [11]. Let S be an inverse semigroup with semilattice of idempotents E . Then a *normal partition on E* is an equivalence π on E such that

- (i) if $e \pi f$, $u \in E$ then $eu \pi fu$;
- (ii) if $e \pi f$ then $a^{-1}ea \pi a^{-1}fa$ for all $a \in S$.

If ρ is a congruence on S then $\pi_\rho = \rho \cap (E \times E)$ is a normal partition on E and is called the normal partition *induced by ρ* .

If π is a normal partition on E then Reilly and Scheiblich [11] show that the smallest congruence ρ_π on S which induces π is given by the following rule: $(a, b) \in \rho_\pi$ if and only if $ea = eb$ for some $e^2 = e \in S$ such that $aa^{-1} \pi e \pi bb^{-1}$.

The result in the next lemma is due to Reilly and Munn [10]. They derived it by giving an explicit construction for S/ρ_π in the form $P(G, \mathcal{X}, \mathcal{Y})$. We shall derive it as a direct consequence of Corollary 2.4.

LEMMA 4.2. *Let S be an E -unitary inverse semigroup over a semilattice and let π be a normal partition on the idempotents of S . Then S/ρ_π is an E -unitary inverse semigroup over a semilattice.*

Proof. We verify that the conditions of Corollary 2.4 hold for $T = S/\rho_\pi$. First, suppose that $(e, ea) \in \rho_\pi$ for some $e^2 = e$. Then $fe = fea$ for some idempotent f . Since S is E -unitary, this implies that $a^2 = a$. Hence T is E -unitary.

Next, let \bar{X} be a σ -class of T and let \bar{e}, \bar{f} be idempotents of T , say with $\bar{e} = e\rho_\pi$, $\bar{f} = f\rho_\pi$, $e^2 = e$, $f^2 = f$. Suppose $\bar{x} \in \bar{X} \cap \bar{e}T\bar{f}$ and let $x \in eSf$ be such that $x\rho_\pi = \bar{x}$. By hypothesis, $x \leq z$, where $z = \max\{s \in eSf : s\gamma_{e,f} = x\gamma_{e,f}\}$. Hence $\bar{x} \leq \bar{z} = z\rho_\pi$. But, since $x\gamma_{e,f} = z\gamma_{e,f}$, $uz = ux$ for some $u^2 = u \in S$; thus $\bar{x}\sigma\bar{z}$. By the choice of e, f , $\bar{z} \in \bar{X} \cap \bar{e}T\bar{f}$. Hence \bar{z} is the maximum element of $\bar{X} \cap \bar{e}T\bar{f}$. It follows that $\gamma_{\bar{e}, \bar{f}}$ is an m -map for each pair \bar{e}, \bar{f} of idempotents of T . Therefore, by Corollary 2.4, T is an E -unitary inverse semigroup over a semilattice.

THEOREM 4.3 [10]. *Let S be an inverse semigroup. Then S is an idempotent-separating homomorphic image of an E -unitary inverse semigroup over a semilattice.*

Proof. Let $\theta: FI_X \rightarrow S$ be a homomorphism from a free inverse semigroup onto S . Then (cf. [8]) FI_X^1 is F -inverse and FI_X is an ideal of FI_X^1 . Hence, by Lemma 2.11, FI_X is E -unitary over a semilattice.

Let $\pi = \theta \circ \theta^{-1} \cap (E \times E)$ where E is the set of idempotents of FI_X . Then π is a normal partition and S is an idempotent-separating homomorphic image of S/ρ_π . The result now follows immediately from Lemma 4.2.

The following corollary strengthens Theorem 4.2 of [4].

COROLLARY 4.4. *Let E be a semilattice. Then E can be embedded as an ideal in a semilattice F with the following property: each isomorphism between principal ideals of F can be extended to an automorphism of F .*

Proof. This follows from Theorem 4.3, using the argument in [4, Theorem 4.2].

We now turn to consider the idempotent-separating congruences on an inverse semigroup $P = P(G, \mathcal{X}, \mathcal{Y})$, where \mathcal{X} is a semilattice. The next result is related to some of those in [10], and the proof is omitted.

LEMMA 4.5. *Let $(G, \mathcal{X}, \mathcal{Y})$ be a triple and let N be a normal subgroup of G such that*

$$a \in \mathcal{Y}, na \in \mathcal{Y} \text{ imply } a = na \text{ for each } a \in \mathcal{Y}, n \in N. \tag{*}$$

Then the relation ρ_N defined by

$$(a, g) \rho_N (b, h) \text{ if and only if } a = b \text{ and } g^{-1}h \in N$$

is an idempotent-separating congruence on $S = P(G, \mathcal{X}, \mathcal{Y})$ such that S/ρ_N is E -unitary.

Conversely, suppose that ρ is an idempotent-separating congruence on S such that S/ρ is E -unitary. Then

$$N = \{g \in G : (a, g) \rho (a, 1) \text{ for some } a \in \mathcal{Y}\}$$

is a normal subgroup of G which satisfies condition (). Further $\rho = \rho_N$.*

The next proposition gives necessary and sufficient conditions on N in order that $T = S/\rho_N$ should be E -unitary over a semilattice. In the proof of the result, we shall denote the elements of T by $[a, g]$ where $(a, g) \in S$. If $[a, g] = (a, g)\rho_N \in T$, it is easy to see that the mapping $[a, g] \rightarrow Ng$ is a homomorphism of T onto G/N which induces σ . Hence we may identify T/σ with G/N by means of this mapping.

PROPOSITION 4.6. *Let $S = P(G, \mathcal{X}, \mathcal{Y})$, where \mathcal{X} is a semilattice, and let N be a normal subgroup of G which satisfies condition (*). Then $T = S/\rho_N$ is E -unitary over a semilattice if and only if, for each $a, b \in \mathcal{X}$,*

$$\{a \wedge nb : n \in N\}$$

has a maximum member.

Proof. First note that, by Lemma 4.5, T is E -unitary. Suppose that, for all $a, b \in \mathcal{X}$, $\{a \wedge nb : n \in N\}$ has a maximum member. Let u, v be idempotents of T and let $X = Ng \in T/\sigma$; then $u = [e, 1], v = [f, 1]$ for some $e, f \in \mathcal{Y}$. By hypothesis, $\bar{e} = \max\{e \wedge ngf : n \in N\}$ exists; say $\bar{e} = e \wedge n_1gf$. Then $[\bar{e}, n_1g] \in uTv$ and $[\bar{e}, n_1g]\sigma = Ng$.

On the other hand, suppose that $[a, h] \in uTv$ is such that $[a, h]\sigma^h = Ng$. Then, because ρ_N is idempotent-separating, $a \leq e, h^{-1}a \leq f$ and, further, $h = ng$ for some $n \in N$. Thus

$a \leq e \wedge n g f \leq \bar{e}$ so that $[a, h] \leq [\bar{e}, n_1 g]$. Hence

$$[\bar{e}, n_1 g] = \max\{t \in u T v : t \sigma^h = N g\}.$$

Thus, by Corollary 2.4, T is E -unitary over a semilattice.

Conversely, suppose that T is E -unitary over a semilattice and let $a = g e$, $b = h f$, where $e, f \in \mathcal{Y}$ and $g, h \in G$. By the hypothesis that T is over a semilattice,

$$x = \max\{t \in [e, 1] T [f, 1] : t \sigma^h = N g^{-1} h\}$$

exists. Then $x = [u, k]$, where $u \leq e$, $k^{-1} u \leq f$ and $N k = N g^{-1} h$; thus $k = g^{-1} n_1 h$ and $u \leq e \wedge g^{-1} n_1 h f$ for some $n_1 \in N$, so that $g u \leq a \wedge n_1 b$.

On the other hand, for any $n \in N$, $t = [e \wedge g^{-1} n h f, g^{-1} n h]$ is in $[e, 1] T [f, 1]$ and $t \sigma^h = N g^{-1} h$. Hence $t \leq x$ which, since ρ_N is idempotent-separating, implies $e \wedge g^{-1} n h f \leq u$; that is, $a \wedge n b \leq g u$. But, as we have seen, $g u \leq a \wedge n_1 b$, so we must have

$$g u = a \wedge n_1 b = \max\{a \wedge n b : n \in N\}.$$

COROLLARY 4.7. *Let $S = P(G, \mathcal{X}, \mathcal{Y})$, where \mathcal{X} is a semilattice, and let N be a normal subgroup of G such that $n a = a$ for each $n \in N$, $a \in \mathcal{X}$. Then S/ρ_N is E -unitary over a semilattice.*

It was shown in [5] that every idempotent-separating congruence on $S = P(G, \mathcal{X}, \mathcal{Y})$, where \mathcal{X} is a semilattice, can be extended to an idempotent-separating congruence on $P(G, \mathcal{X}, \mathcal{X})$. It is easy to see that ρ_N can be extended to an E -unitary congruence if and only if $N \subseteq \{g \in G : g a = a \text{ for all } a \in \mathcal{X}\}$.

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