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Economics of Naturally Regenerating, Heterogeneous Forests

Olli Tahvonen

Abstract: An economic model for naturally regenerating, heterogeneous forests is specified to yield both clear-cuts and continuous cover forestry endogenously. The model includes nonconvexities and any number of state variables but is, in its simplest form, a one-state variable problem. Clear-cuts with various rotation lengths and continuous harvesting appear as locally optimal solutions. Necessary and sufficient conditions for the local and global optimality of these two forest management types are obtained. Discounting is found to increase rotation length and to favor continuous harvesting. Initial state may determine the optimality of continuous forest cover versus clear-cuts. The relative value of large trees is an important factor in the optimality of different solutions. Analytical results are demonstrated by an empirical application.

JEL Codes: Q20, Q23

Keywords: Continuous cover forestry, Forestry, Nonconvexities, Optimal harvesting, Optimal rotation, Size-structured model, Uneven-aged forestry

THE LITERATURE OF RESOURCE economics focusing on forestry relies heavily on Faustmann's (1849) rotation model. The roles of that model's simplifying assumptions about competitive markets and certainty are well understood and have been generalized in many subsequent studies (Amacher, Ollikainen, and Koskela 2009). What is seldom mentioned, however, is that Faustman's model is restricted to even-aged forests. In its purest and most commonly applied form, the model describes a forest consisting of only a single tree, and its economic problem concerns deciding

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the length of the forest's clear-cut periods. Although the model is best suited to pure plantations,¹ it is nonetheless widely applied to natural forests. By extending the plantation model beyond its narrow boundaries, resource economics may unintentionally promote plantation-type forestry at the cost of a more natural forest environment. This study shows that heterogeneous, naturally regenerating forests call for a more general model, in which the rotation-type solution is only one candidate for optimality among other forest management alternatives.

The generalized approach recognizes both the possibility of partial harvests in addition to clear-cuts and self-regeneration and the heterogeneous structure of tree populations. Adams and Ek (1974) began to explore in this direction, basing their uneven-aged discrete time setup on a biological size-structured model that is suitable for many species (e.g., trees, fish, sea mammals, birds, and herbivores). That model is a standard workhorse in population biology (Cushing 1998; Caswell 2001). Because it includes any number of state variables, its complexity goes well beyond the classical rotation setup. One strength of the rotation approach is its clear analytical demonstration of the underlying economic principles. By comparison, analytical results are lacking for uneven-aged management. Moreover, as noted by Getz and Haight (1989), numerical solutions to the complex uneven-aged model seldom offer clear economic and mathematical validity. Even when it is correctly specified, an uneven-aged model has the shortcoming of excluding the clear-cut alternative. Such a dichotomy between even- and uneven-aged management is theoretically unsatisfactory and motivates us to ask how to develop a framework that includes both alternatives as potentially optimal solutions. Haigh and Monserud (1990) and Tahvonen (2009) have presented numerical computations allowing for both alternatives, but they leave many questions open and do not develop analytical results.

My aim here is to address these shortcomings. To this end, I specify a generic model for size-structured biological resources that can be viewed as a theoretically sound version of the setup described by Adams and Ek (1974) and their followers. Mathematically, the model is a discrete-time control problem, and it includes any number of state and control variables. The forestry version of this model is complicated by the fact that for many tree species (both tropical and boreal), natural regeneration decreases as density increases. Such a negative dependence necessarily implies non-convexities and directly affects decisions whether to manage forests as successive even-aged cohorts or as continuously regenerating and harvested mixed-age class systems. Studies applying the size-structured model appear not to have addressed this complication, but it is here shown to be a key feature in a generalized forest harvesting model.

1. According to FAO (2014), about 7% of world forests are plantations.

This study will begin by specifying the general size-structured model, analyzed in its simplest form, which is somewhat surprisingly a one-state variable problem. The continuous harvesting and clear-cut alternatives may exist simultaneously as locally optimal solutions. Their local and global optimalities depend on biological features, relative values of trees in different size classes, and the interest rate. The continuous harvesting solution converges toward either constant harvesting or a cycle. Solutions with clear-cuts may fail to satisfy the necessary optimality conditions. If smaller trees do not have direct value, a higher interest rate leads to a longer rotation and favors the solution without clear-cuts: the effect of the interest rate is opposite to what is found in the classic rotation model. The global optimality of the continuous harvesting solution may, in addition, depend on the initial stand state. Furthermore, a higher relative value of small trees favors the clear-cut solution. The optimal size of harvested trees is determined by a combination of economic and biological factors and cannot be found by formulae of the Faustmann or Wicksellian type. Nonconvexities complicate the search for globally optimal solutions, but the assumption of initially bare land allows the model to be specified in a computationally efficient form. When an empirically estimated model is used, the theoretical analysis rationalizes results that cannot be explained within the classical rotation framework. These analytical results are new and reveal properties, hitherto unknown, but that are included in any realistic economic model of naturally regenerating, uneven-aged forests. Finally, this study suggests a stylized model for uneven-aged forestry that, in its simplicity, can be compared to the classic optimal rotation model.

1. THE SIZE-STRUCTURED OPTIMIZATION PROBLEM

A generic size-structured optimal harvesting problem can be specified as

$$\max_{\{h_s, s = 1, \dots, n, t = 0, 1, \dots\}} \sum_{t=0}^{\infty} U(H_t) b^{t+1}, \tag{1}$$

subject to

$$y_t = \sum_{s=1}^n \gamma_s x_{st}, \tag{2}$$

$$x_{1,t+1} = \varphi(y_t) + \beta_1 x_{1t} - h_{1t}, \tag{3}$$

$$x_{s+1,t+1} = \alpha_s x_{st} + \beta_{s+1} x_{s+1,t} - h_{s+1,t} \quad s = 1, \dots, n - 1, \tag{4}$$

$$H_t = \sum_{s=1}^n f_s h_{st}, \tag{5}$$

$$x_{st} \geq 0, \quad h_{st} \geq 0, \quad s = 1, \dots, n, \quad t = 0, 1, \dots, \tag{6}$$

$$x_{s0}, \quad s = 1, \dots, n \text{ given.} \tag{7}$$

The variables x_{st} denote the number of trees in size classes $s = 1, \dots, n$ at the beginning of the periods $t = 0, 1, \dots$ and h_{st} , $s = 1, \dots, n$, $t = 0, 1, \dots$ are the harvested trees at the ends of the periods (n is the largest size class). The utility function $U \in \mathbb{C}^1$ is increasing and linear. Total harvest per period is H_t , and f_s , $s = 1, \dots, n$ are the trees' usable volume, or market value (stumpage price). Assume $0 \leq f_s \leq f_{s+1}$, $s = 1, \dots, n-1$ and $f_n > 0$. The discount factor is $b = 1/(1+r)$ and r is the interest rate. At the end of periods before harvesting, fractions α_s , $s = 1, \dots, n$ of trees have moved to the next class, while fractions β_s , $s = 1, \dots, n$, remain in the same class. Assume $0 \leq \alpha_s < 1$, $0 \leq \beta_s < 1$, $\alpha_s + \beta_s \leq 1$, $s = 1, \dots, n$. When $\beta_s = 0$, $s = 1, \dots, n$, an age-structured model is obtained as a special case. Variable y_t measures population density (e.g., stand volume in m^3) and $\gamma_s \geq 0$, $s = 1, \dots, n$ (e.g., in m^3) is the density effects of trees in different size classes.

The function $\varphi \in \mathbb{C}^2$ denotes regeneration. For some species, regeneration is an increasing function of density (Getz and Haight 1989, 71). However, for trees, particularly shade-intolerant species, the relationship between population (or stand) density and regeneration is negative. This relationship has been described for many rain forest, hemiboreal, and boreal tree species. In a widely cited work, Buongiorno and Michie (1980) estimated a linear and decreasing relationship between regeneration and the density of the Wisconsin sugar maple. Vanclay (1992) estimated decreasing and convex relationships for several tropical rain forest tree species of North Queensland; Bollandasås, Buongiorno, and Gobakken (2008) found a similar relationship for boreal trees. This yields the assumptions:

$$\begin{aligned} \varphi(0) := \varphi_0 > 0, \varphi'(y) < 0, \varphi''(y) > 0 \text{ for } 0 \leq y < \tilde{y}, \tilde{y} \in (0, \infty), \\ \varphi \geq 0 \text{ and if } \tilde{y} \text{ is finite } \varphi(y) = \varphi'(y) = 0 \text{ for } y \geq \tilde{y}, \end{aligned} \quad (Q1)$$

that is, the function is continuous, twice continuously differentiable, strictly convex, and decreasing.² Since many rain forest trees have efficient means of dispersal, a species does not have to live within or near a site for regeneration to occur (Vanclay 1992).

Convex regeneration implies that the model (1)–(7) includes nonconvexities. To approach the problem in its simplest form (cf. Wan 1994; Salo and Tahvonen 2002) assume $n = 2$.³ Condition (3) and the restriction $h_1 \geq 0$ can now be written as

2. A decreasing linear function in Buongiorno and Michie (1980) implies that beyond some density level, regeneration must be zero (not negative), which in turn implies that their regeneration model is actually convex (but not everywhere differentiable).

3. Recall that the "Mittra-Wan tree farm" (Wan 1994) is a model for trees without biological connections and it actually describes forestry at a regional or market level (with endogenous price) and is not, as such, a model for uneven-aged forestry.

$$(h_{1t} =) -x_{1,t+1} + \varphi(y_t) + \beta_1 x_{1t} \geq 0. \tag{8}$$

Note that even when $x_{2,t+1} = x_{2t} = 0$ in (4), it is possible to have $h_{2t} = \alpha x_{1t} > 0$ since the fraction α_1 of x_{1t} moves to a larger size class at the end of each period. Because recruitment is decreasing in x_{0t} and x_{2t} ($\varphi' < 0$ in [Q1]), positive levels of x_{2t} will only decrease recruitment and will postpone and decrease the harvest from size class 2 ($\beta_2 < 1$), so that $x_{2t} > 0$ cannot be optimal. Thus, the two size classes problem takes the form

$$J(x_0) = \max_{\{x_{t+1,t} = 0,1,\dots\}} \sum_{t=0}^{\infty} b^{t+1} U\{f_1[-x_{t+1} + \varphi(x_t) + \beta x_t] + f_2 \alpha x_t\}, \tag{9}$$

subject to

$$x_{t+1} - \varphi(x_t) - \beta x_t \leq 0, \tag{10}$$

$$x_t \geq 0, \tag{11}$$

$$x_{t=0} \text{ given}, \tag{12}$$

where the subscripts for x_t , α , and β can be neglected without confusion and $y_t = x_t$ ($\gamma_1 = 1$). The Lagrangian and Karush-Kuhn-Tucker conditions are

$$L = \sum_{t=0}^{\infty} b^{t+1} \{U\{f_1[-x_{t+1} + \varphi(x_t) + \beta x_t] + f_2 \alpha x_t\} - \lambda_t [x_{t+1} - \varphi(x_t) - \beta x_t]\},$$

$$\begin{aligned} \frac{\partial L}{\partial x_{t+1}} b^{-t-1} &= -U'(H_t) f_1 + U'(H_{t+1}) b [f_1 \varphi'(x_{t+1}) + f_1 \beta + f_2 \alpha] - \lambda_t \\ &+ \lambda_{t+1} b [\varphi'(x_{t+1}) + \beta] \leq 0, \end{aligned} \tag{13a}$$

$$\frac{\partial L}{\partial x_{t+1}} b^{-t-1} x_{t+1} = 0, \tag{13b}$$

$$x_{t+1} \geq 0, \tag{13c}$$

$$x_{t+1} - \varphi(x_t) - \beta x_t \leq 0, \tag{14a}$$

$$[x_{t+1} - \varphi(x_t) - \beta x_t] \lambda_t = 0, \tag{14b}$$

$$\lambda_t \geq 0. \tag{14c}$$

Another possibility for obtaining the first-order necessary conditions is to apply infinite horizon discrete-time optimal control (Sydsæter et al. 2008, 441; Blot and Hayek 2014). The appendix shows that this will yield equivalent optimality conditions as (13a)–(13c) and (14a)–(14c). Since φ is convex, these conditions are not

sufficient for locally maximizing solutions. The solution strategy is to find admissible solutions that satisfy (13a)–(13c) and (14a)–(14c) and qualify as locally maximizing solutions, and then to search for the global optimum among these candidates. Variable x_t is bounded both from below and above. Given the linear utility function, the problem is linear, excluding strictly convex recruitment. If the objective function is minimized, then the first-order necessary conditions will be sufficient for minimizing solutions (see the appendix, sec. B). This holds for interior solutions in particular; thus, maximizing solutions must be identified among boundary solutions where either the lower or upper bound constraint for x_t is active in every period and satisfies (13a)–(13c) and (14a)–(14c).

2. RESULTS

2.1. Continuous Harvesting versus Two-Period Clear-Cut Cycles When $f_1 = 0$

Small trees are usually less valuable than larger trees: to obtain the simplest possible generic setup, assume that the size class 1 trees are not directly valuable. Thus, let $f_1 = 0$ and write $f_2 = f$, $x_{2t} = x_t$ for simplicity. A regime with active constraints is one in which only class 2 is harvested, x_t always equals its upper bound, and constraint (10) is binding. In this regime $H_t = fh_{2t} = f\alpha x_t$ and by (10)

$$x_{t+1} = \varphi(x_t) + \beta x_t, \quad x_0 \text{ given.} \quad (15)$$

When (15) has a stable steady state, this solution converges to $x_t = \hat{x} > 0$ and $h_2 = \alpha \hat{x}$, where $\hat{x} = \varphi(\hat{x}) + \beta \hat{x}$ (fig. 1). By (13a)–(13c)

$$\hat{\lambda} = \frac{bf\alpha}{1 - b\beta - b\varphi'(\hat{x})} > 0, \quad (16)$$

since $b \leq 1$, $\beta < 1$, $\varphi' < 0$. Thus, the solution satisfies conditions (13a)–(13c) and (14a)–(14c) for a local maximum. At the beginning of each period, the forest includes \hat{x} class 1 trees; at the period's end, before harvest, it has the same number $\hat{x} (= \varphi(\hat{x}) + \beta \hat{x})$ of class 1 trees, while $\alpha \hat{x}$ number of trees has reached size class 2. At the end of each period, only these bigger trees are harvested (cf. "thinning from above"). This is clearly a solution with continuous harvesting (and continuous forest cover) and no clear-cuts. Letting τ denote the time between clear-cuts, this solution can be referred to as $\tau = \infty$ or τ_∞ for short.

Proposition 1: Given $-1 < \varphi'(0) + \beta$ and $x_0 \geq 0$, continuous harvesting with $h_{1t} = 0$ for $t \in [0, \infty)$ is locally optimal, and the solution converges toward a globally asymptotically stable steady state.

Proof: See the appendix, section C.

Denote the two-period clear-cut cycle by τ_2 and the first (second) cycle period by $t = 0$ ($t = 1$). Assume $x_0 = 0$ in figure 1, implying by (17) and (18) that $x_1 = \varphi_0$. At $t = 1$, it holds that $h_{11} = \varphi(\varphi_0) + \beta\varphi_0$ and $h_{21} = \alpha\varphi_0$, implying $x_2 (= x_0) = 0$ and that the solution develops clockwise along the two-period clear-cut cycle (solid) line.⁴

Proposition 2: A two-period clear-cut cycle is locally optimal if and only if $1 + b[\varphi'(0) + \beta] < 0$, that is, if $\varphi'(0)$ is sufficiently low but fails to be locally optimal if the interest rate is sufficiently high or if the global stability condition of the continuous harvesting solution in proposition 1 is satisfied. Low β works in favor of local optimality for the clear-cut solution.

Proof: Given active constraints, a maximizing cycle satisfies (17), (18), $\lambda_0 > 0$, $\lambda_1 = 0$, and by (13a), $bfa - \lambda_0 = 0$ and $bfa + \lambda_0[\varphi'(0) + \beta] < 0$, that is,

$$bf\alpha\{1 + b[\varphi'(0) + \beta]\} < 0. \quad (19)$$

Given that (19) is met, the cycle can be reached optimally with any $x_0 \geq 0$, since a solution $x_1 = 0$ reaches the cycle by one step and satisfies (13a)–(13c) and (14a)–(14c) by (19). The left-hand side of (19) becomes lower as the value of $\varphi'(0)$ falls, but (19) is violated if the stability condition in proposition 1 is met or if b is sufficiently low. The left-hand side of (19) is increasing in β , implying that low β favors the local optimality of the clear-cut solution. QED

When (19) is not met, clear-cutting is not locally optimal. The interpretation is that leaving a marginal class 1 tree unharvested (instead of clear-cut) yields a next-period marginal gain bfa , because a fraction α of these trees can be harvested as class 2 trees at the end of that next period. However, regeneration two periods forward is decreased by $\varphi'(0)$, implying a marginal net loss on the class 2 harvest equal to $b^2f\alpha[\varphi'(0) + \beta]$. When this loss for the first unit of trees is lower than the next-period benefit bfa , it is optimal to leave all ($\varphi'' > 0$) the class 1 trees unharvested; that is, clear-cutting is not locally optimal. The interest rate does not favor clear-cutting because the positive effect of not cutting the small trees is realized sooner than the loss.

If condition (19) is satisfied, the clear-cut solution is locally optimal and the condition for global stability of the continuous harvesting solution is violated. However, the global stability condition is sufficient but not necessary. Thus, two locally

4. Since at the end of periods with clear-cut there are one- and two-period-old trees, I call this solution a clear-cut solution instead of an even-aged solution and uneven-aged solution as continuous cover (or harvesting) solution. For various definitions, see Pommeroy and Murphy (2004).

optimal solutions may exist simultaneously. In addition, the violation of (19) is sufficient but not necessary for the superiority of the τ_∞ solution over the τ_2 solution.

A continuous harvesting solution cannot be optimal for any x_0 if a regime switch from the continuous harvesting steady state $x_0 = \hat{x}$ increases the objective value:

Proposition 3: Given $\varphi_0/(2+r) > \hat{x}$, the τ_∞ solution is unoptimal for any x_0 , and given $x_0 = \hat{x}$, $\varphi_0/(2+r) < \hat{x}$, a switch to τ_2 is unoptimal.

Proof: Assume that $x_0 = \hat{x}$. If the solution τ_∞ is followed forever, the outcome is $J_{\tau_\infty} = bf\alpha\hat{x}/(1-b)$. A clear-cut at the end of the initial period and continuing with solution τ_2 thereafter yields $J_{\tau_2} = bf\alpha\hat{x} + b^3\alpha f\varphi_0/(1-b^2)$. The switch and clear-cut satisfy necessary optimality conditions, since $\lambda_{t+1} = bfa$ and (13a)–(13c) is satisfied by $\varphi_0/(2+r) > \hat{x}$. Maintaining τ_∞ is unoptimal, if $J_{\tau_2} > J_{\tau_\infty}$, which is equivalent to $\varphi_0/(2+r) > \hat{x}$, implying that the switch to τ_2 is unoptimal if $\varphi_0/(2+r) < \hat{x}$. QED

For interpretation, condition $J_{\tau_2} > J_{\tau_\infty}$ can be rewritten as

$$af\hat{x} < rb \left(af\hat{x} + \frac{b^2\alpha f\varphi_0}{1-b^2} \right).$$

Thus it is optimal to clear-cut if the income per period from continuous harvesting falls short of the interest earned on income from the next clear-cut plus the value of bare land. This condition can be compared to the optimality condition for the generic Faustmann model, where clear-cut is optimal when value growth falls short of interests on the income from the next clear-cut plus the value of bare land.

Again, a high interest rate favors a continuous harvesting solution and a low interest rate works against it. However, this effect is different from the local optimality condition (19), which describes the optimality of harvesting all class 1 trees at the time of clear-cutting. A high interest rate favors the continuous harvesting solution and $J_{\tau_2} < J_{\tau_\infty}$ because there is an additional time delay between harvests in the clear-cut solution that is absent in a continuous harvesting solution.

In the continuous cover regime, trees are cut when they enter the “large trees” size class. Such a solution may be globally optimal, but will following such a regime necessarily stabilize the harvest and stand structure over time? The answer is no, because the continuous cover steady state is locally unstable when $-1 > \varphi'(\hat{x}) + \beta$, implying that the τ_∞ solution may exist as a limit cycle. This is the case when $\varphi = A/(1+ax_t)^2$, $A = 100$, $a = 1/10$, and $\beta = 1/10$, implying $\hat{x} \simeq 16.194$, $\varphi'(\hat{x}) + \beta \simeq -1.013$. Applying theorem 1.21 in Elaydi (2000, 35), define

$$z_{t+1} = \Phi(z_t) := \varphi[\varphi(z_t) + \beta z_t] + \beta[\varphi(z_t) + \beta z_t]. \tag{20}$$

Given a limit cycle of two periods, there must exist two distinct cycle points z_1 and z_2 that solve the difference equation (20). One obtains

$$z_1 \approx 12.6, z_2 \approx 20.9 \text{ and } [\varphi'(z_1) + \beta][\varphi'(z_2) + \beta] \approx 0.95,$$

showing that the cycle exists and is locally asymptotically stable. Proposition 2 suggests that this solution can be optimal only if the discount factor is low. Setting $b = 0.1$ yields $\lambda_1 \approx 0.095, \lambda_2 \approx 0.084$, that is, the cycle is locally optimal. If the conditions of proposition 2 hold, the two locally optimal cycles exist simultaneously (fig. 2). Given $b = 0.1$ and $x_0 = 0$, the clear-cut regime yields a present value equal to $J_{\tau_c} \approx 1.010$ while the continuous cover solution is globally optimal and yields $J_{\tau_c} \approx 1.013$. Thus, even when the globally optimal solution is continuous harvesting the yield is not necessarily smooth over time.

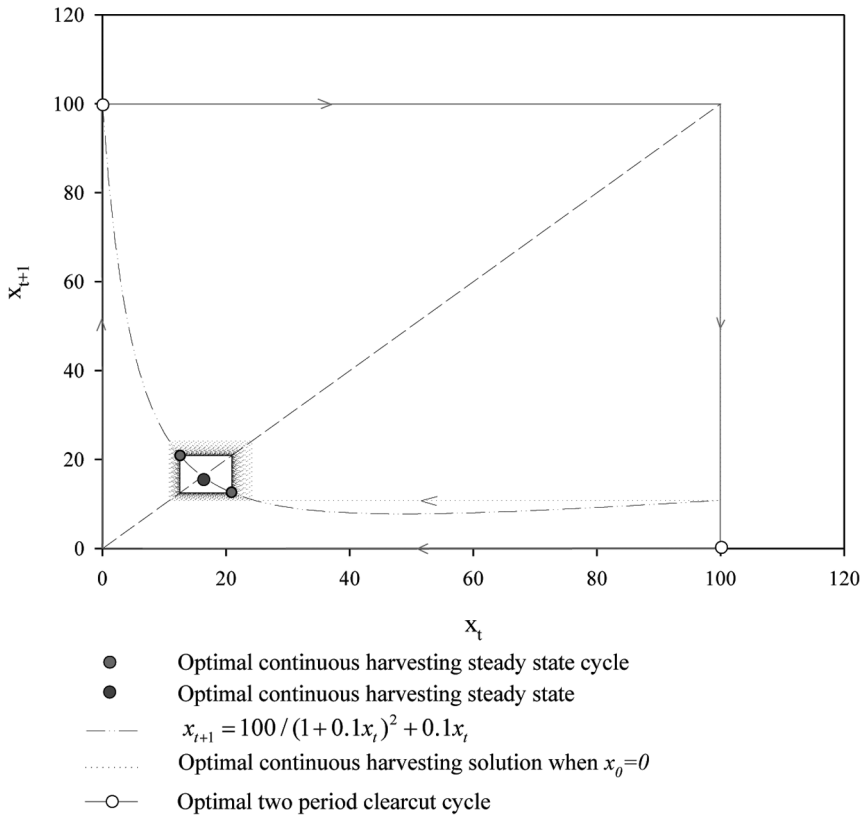


Figure 2. Globally optimal continuous harvesting cycle and locally optimal two-period clear-cut cycle.

2.2. Clear-Cut Cycles with Any Period Length

Next I will show that solutions with continuous harvesting and a two-period clear-cut cycle are extremes within the solutions in which the length of the clear-cut cycle may range from two periods to infinity. Because no partial harvests occur before the clear-cut, the two-period cycle represents a pure clear-cut solution. In the example shown in figure 3, $\hat{x} < \frac{1}{2}\varphi_0$, which implies that if $b = 1$, the continuous cover solution is not globally optimal (proposition 3). The facts $\hat{x} < \frac{1}{2}\varphi_0$ and $\varphi'' > 0$ in (Q1) imply⁵ $\varphi'(0) + \beta < -1$, that is, the τ_2 solution is locally optimal (condition [19]). Assume $x_0 = 0$ and $\alpha f = 1$. The two-period cycle produces an average yield equal to $\frac{1}{2}\varphi_0$; but in figure 3, $x_2 > \frac{1}{2}\varphi_0$, so that clear-cutting every third period increases average yield over clear-cutting every two periods. Since $x_3 < \frac{1}{2}\varphi_0$, clear-cutting every fourth period (or less frequently) decreases average yield compared to the τ_3 solution. Thus, the τ_3 clear-cut cycle with one thinning from above is globally optimal. Letting $r = \frac{2}{3}$ changes the outcome, so that the τ_∞ solution becomes globally optimal.

A locally optimal τ_3 solution satisfies $x_0 = 0, x_1 = \varphi_0, x_2 = \varphi(\varphi_0) + \beta\varphi_0, \lambda_0 > 0, \lambda_1 > 0, \lambda_2 = 0$ (and $h_{10} = h_{11} = 0, h_{12} = \varphi(x_2) + \beta x_2, h_{21} = \alpha\varphi_0, h_{22} = \alpha x_2$). By (13a), the necessary conditions include $bfa - \lambda_0 + \lambda_1 b[\varphi'(\varphi_0) + \beta] = 0, \lambda_1 = bfa$, and $bfa + \lambda_0 b[\varphi'(0) + \beta] < 0$, implying

$$\lambda_0 = bfa\{1 + b[\varphi'(\varphi_0) + \beta]\} > 0, \tag{21}$$

$$1 + b[\varphi'(0) + \beta] + b^2[\varphi'(\varphi_0) + \beta][\varphi'(0) + \beta] < 0. \tag{22}$$

Since $1 + \varphi'(\varphi_0) + \beta < 0$ implies that $\varphi(\varphi_0) + \beta\varphi_0 < 0$, a contradiction with $\varphi \geq 0$ in (Q1), condition (21) is always satisfied. If $\varphi'(0) + \beta \geq 0$, then (22) and $\varphi'' > 0$ in (Q1) rule out the local optimality of the τ_3 solution and (19) rules out the τ_2 solution, implying that the optimal solution must have $\tau > 3$. If $1 + b[\varphi'(0) + \beta] \leq 0$ but $\varphi'(\varphi_0) + \beta \geq 0$, then (22) is satisfied, and τ_2 and τ_3 solutions are both locally optimal. If $1 + b[\varphi'(0) + \beta] \leq 0, \varphi'(\varphi_0) + \beta < 0$, and (22) is not satisfied, then the τ_2 solution is locally optimal but the τ_3 solution is not. If $1 + b[\varphi'(0) + \beta] > 0, \varphi'(\varphi_0) + \beta > 0$, and (22) is satisfied, then the τ_3 solution is locally optimal but the τ_2 solution is not. In all the cases, τ_∞ is locally optimal. Thus, many locally optimal candidates exist, and the task is to find the global optimality. To this end, generalizing for any $2 \leq \tau < \infty$ yields

$$bfa - \lambda_t + \lambda_{t+1} b[\varphi'(x_{t+1}) + \beta] = 0, \quad t = 0, \dots, \tau - 3, \tag{23}$$

$$bfa - \lambda_{\tau-2} = 0, \tag{24}$$

$$bfa + \lambda_0 b[\varphi'(0) + \beta] < 0. \tag{25}$$

5. If $\varphi'(0) + \beta \geq 0$, then $\hat{x} > \frac{1}{2}\varphi_0$ by $\varphi'' > 0$.

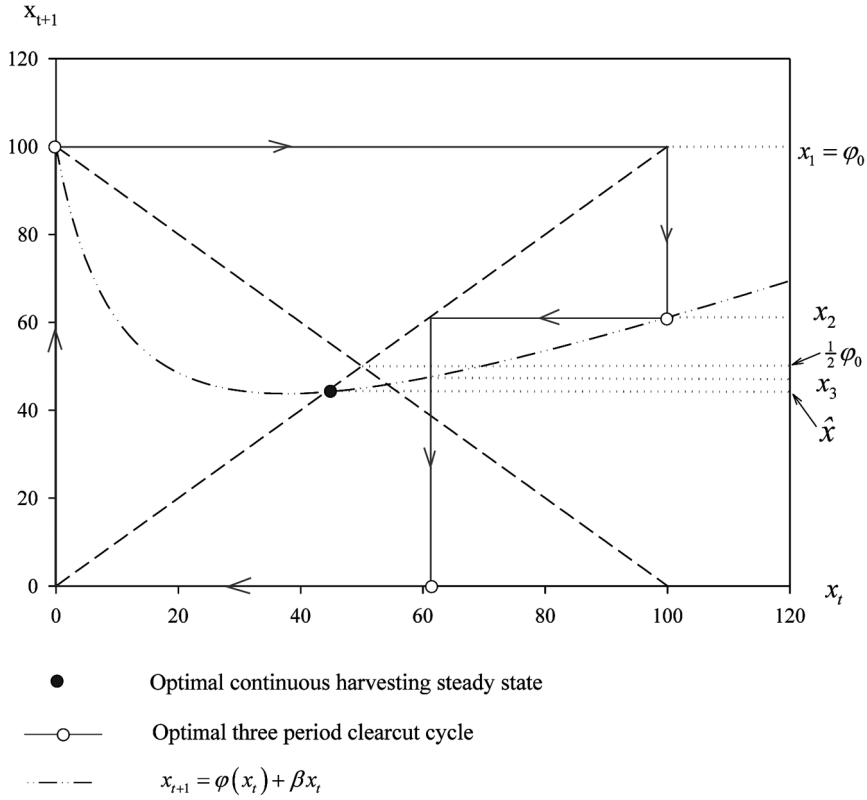


Figure 3. An optimal tree period clear-cut cycle

The solution for the nonautonomous difference equation (23), (24) reads as

$$\lambda_t = \lambda_0 \prod_{i=1}^t [\varphi'(x_i) + \beta]^{-1} - b f \alpha \sum_{i=1}^t \prod_{j=i}^t [\varphi'(x_j) + \beta]^{-1}, \quad t = 1, \dots, \tau - 2, \quad (26)$$

$$\lambda_0 = b f \alpha \left\{ 1 + \sum_{i=1}^{\tau-2} b^i \prod_{j=1}^i [\varphi'(x_j) + \beta] \right\}, \quad (27)$$

and eliminating λ_0 from (25) yields the generalization of (22)

$$1 + \sum_{i=1}^{\tau-1} b^i \prod_{j=1}^i [\varphi'(x_{j-1}) + \beta] < 0. \quad (28)$$

From (24), $\lambda_{\tau-2} > 0$, which implies by similar reasoning as in the proof for proposition 1 that a solution $\lambda_t > 0, t = 0, \dots, \tau - 2$ reaching $b f \alpha$ at $\tau - 2$ exists.

Proposition 4: Given $x_0 = 0$, the continuous harvesting solution τ_∞ is globally optimal if (a) $\varphi'(0) + \beta \geq 0$, or (b) $-1 \leq \varphi'(0) + \beta < 0$ and $\varphi'(\varphi_0) + \beta \leq 0$, or (c) the discount factor b is low enough.

Proof: See the appendix, section D.

In case *c*, the high interest rate implies the global optimality of continuous harvesting, because harvesting class 1 trees is not locally optimal independently of the clear-cut cycle length (see the interpretation of proposition 1). Case *a* rules out clear-cuts as unoptimal because stand density and harvests from class 2 increase from period 1 onward. In this case the negative effect of stand density on regeneration is low, trees remain a long time in class 1, and natural mortality is low. In case *b*, the conditions for continuous cover are less favorable, but the first assumption ($-1 \leq \varphi'(0) + \beta < 0$) implies that the steady state is favorable compared to the two-period cycle (see proposition 2) and the second ($\varphi'(\varphi_0) + \beta \leq 0$) that the yields along the transition are not too high but are (cyclically) increasing. This last feature rules out the optimality of longer clear-cut cycles (see fig. 3). A different case for the global optimality of τ_∞ is as follows:

Proposition 5: If $\varphi(\varphi_0) + \beta\varphi_0 \geq \varphi_0$ and $1 + \varphi'(0) + \beta < 0$, then the globally optimal solution is regime τ_∞ or an immediate clear-cut followed by τ_∞ .

Proof: See the appendix, section E.

In figure 4, $r = 0.04$ and $\alpha f = 2$. Given $x_0 = 25$, regime τ_∞ yields $J = 2,382$ whereas the clear-cut at the end of the first period and τ_∞ thereafter yields $J = 2,444$. In this case, natural mortality is low ($\beta = 0.95$) but trees grow slowly to class 2. The clear-cut utilizes the high initial regeneration and reaches large harvest levels much sooner than the solution in which the forest is stuck at low density and harvestable size class 2 levels.

The optimality of continuous harvesting may depend on the initial state:

Proposition 6: Assume some $\tilde{\varphi}$ and φ_0 that satisfy (Q1) together with $0 < \tilde{\beta}$ and $0 < \tilde{r}$ such that $\hat{x} = \varphi_0 / (2 + \tilde{r}) = \tilde{\varphi}(\varphi_0) + \tilde{\beta}\varphi_0$. Given that \hat{x} is stable for solutions from $x_0 = 0$, there exist $\bar{\varphi}, 0 < \bar{\beta}$, and $\tilde{r} < \bar{r}$ such that $\bar{\varphi}(0) = \varphi_0$, $\hat{x} = \bar{\varphi}(\hat{x}) + \bar{\beta}\hat{x}$, $\bar{\varphi}(\varphi_0) + \bar{\beta}\varphi_0 < \hat{x}$ and that τ_∞ is optimal for $x_0 = \hat{x}$ and τ_2 is optimal for $x_0 = 0$.

Proof: See the appendix, section F.

The optimality of continuous harvesting becomes dependent on the initial state when the continuous harvesting steady state is more favorable than the switch to

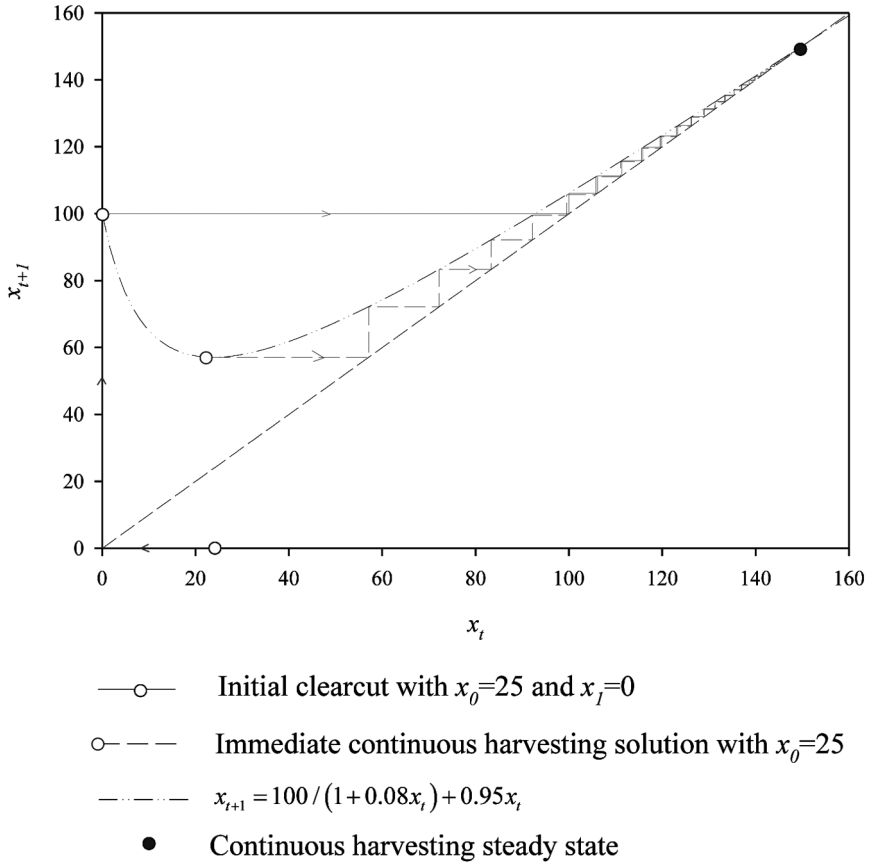


Figure 4. Optimal continuous-harvesting solution with possible initial clear-cut

clear-cuts but when the transition toward the steady state requires an initial low harvest. In this case continuous harvesting is optimal if the stand is initially heterogeneous ($x_0 = \hat{x}$), but the clear-cut solution is optimal when initial state is bare land ($x_0 = 0$). In the example given in figure 5, $\tilde{\varphi}(x_t) + \tilde{\beta}x_t = 100 / (1 + 0.040x_t) + 0.277x_t$, and $\tilde{r} = 0.102$, implying $\hat{x} \approx 47.6$ and the fact that τ_∞ and τ_2 yield equal objective functional value ($J = 464$) for both $x_0 = \hat{x}$ and $x_0 = 0$. Given $\bar{\varphi}(x_t) + \bar{\beta}x_t = 100 / (1 + 0.027x_t) + 0.08x_t$ and $\bar{r} = 0.11$, the outcome is as in proposition 6: τ_∞ is globally optimal for $x_0 = \hat{x}$ and τ_2 for $x_0 = 0$. This dependence from the initial state follows from nonconvexities.

2.3. Discounting and Rotation Length

It is possible to view the optimization problem (9)–(12) from a somewhat different angle. Assume an initial state with no trees. Recognize the large initial cohort of new trees at the end of period zero and the evolving harvest of size class 2 trees at

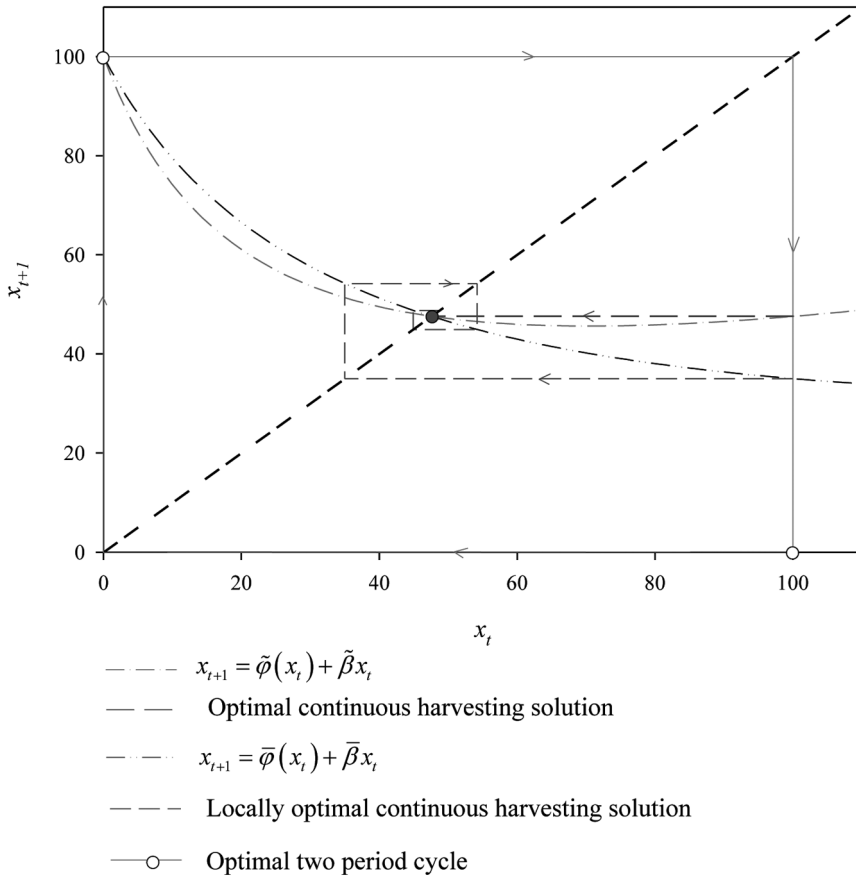


Figure 5. Dependence of solution type on initial state parameter values (see text)

the end of each period, from period 1 onward. Decide when to stop the process by a clear-cut (and restart from bare land) or whether to apply continuous harvesting forever. This view can be specified as an integer optimization problem of the form

$$\max_{\{\tau \in [1, 2, \dots, \infty)\}} \sum_{t=0}^{\tau} f\alpha x_{t-1} b^{t+1} / (1 - b^{t+1}), \text{ subject to } x_{t+1} = \varphi(x_t) + \beta x_t, x_0 = 0. \quad (29)$$

It is not optimal to clear-cut after τ periods if continuing the regime without a clear-cut for another period increases the objective function in (29), that is, if

$$\sum_{t=0}^{\tau} f\alpha x_t b^{t+1} / (1 - b^{t+1}) \geq \sum_{t=0}^{\tau-1} f\alpha x_t b^{t+1} / (1 - b^t). \quad (30)$$

Specification (29) offers an efficient procedure to compute the optimality of the continuous cover solution vis-à-vis the clear-cut solutions with different cycles (or

rotation periods). Recall that the optimal rotation τ in discrete-time Faustmann model satisfies

$$b^t x_t / (1 - b^t) \geq b^i x_i / (1 - b^i), \text{ for } t = 1, \dots, n, t \neq \tau,$$

where x_t gives the stand volume as a function of stand age t , $0 \leq \tau \leq n$, and n is the age after which the trees start to lose their value. In particular, for optimal τ , it holds that

$$u := b^\tau x_\tau / (1 - b^\tau) - b^{\tau+1} x_{\tau+1} / (1 - b^{\tau+1}) \geq 0.$$

It can be shown that $\partial u / \partial b < 0$ when $u = 0$. Thus, when the discount factor increases, u can only change its value from positive to negative, thus implying that a decrease in the discount rate can only increase the optimal rotation length.

Proposition 7: Given problem (29), (Q1), decreasing x_t , and a finite optimal rotation length, a decrease in the discount rate can only decrease the rotation length.

Proof: Write (30) in the form

$$g := \sum_{i=1}^{\tau-1} x_{i-1} b^i / (1 - b^{\tau-1}) - \sum_{i=1}^{\tau} x_{i-1} b^i / (1 - b^i) \leq 0,$$

where rotation τ is finite, x_t is the end of period yield from harvesting the size class 2 trees, and $fa = 1$ without losing generality. Differentiation and $x_0 = 0$ yields

$$\left. \frac{\partial g}{\partial b} \right|_{g=0} = \frac{b^2 x_2}{1 + b - b^3 - b^4} > 0 \text{ for } \tau = 3,$$

$$\left. \frac{\partial g}{\partial b} \right|_{g=0} = \frac{b^3 [b^2 x_2 + x_3 (1 - b^2)]}{1 + b + b^2 - b^4 - b^5 - b^6} > 0 \text{ for } \tau = 4,$$

and in general for $\tau = 5, 6, \dots$

$$\left. \frac{\partial g}{\partial b} \right|_{g=0} = \frac{b^{\tau-1} \left[b^2 x_2 + x_{\tau-1} (1 - b^2) + \sum_{i=3}^{\tau-2} (\tau - i) b^{i-1} (x_{\tau-i+2} - x_{\tau-1}) \right]}{\sum_{i=1}^{\tau-1} b^{i-1} - \sum_{i=\tau}^{2(\tau-1)} b^i} > 0. \tag{31}$$

In (31), the numerator and denominator (by decreasing x_t) are positive. Thus, when the discount factor increases, the sign of g can change only from negative to positive, that is, a decrease in the interest rate decreases the length of the optimal rotation. QED

In proposition 7, it is assumed that x_t decreases over time, because with monotonically increasing or constant x_t , the globally optimal solution is continuous harvesting (propositions 4 and 5). A higher interest rate may not only cause the clear-cut solutions to become locally unoptimal (proposition 4), but increase rotation length, as the delay without harvest after the clear-cut becomes costly. In figure 6,

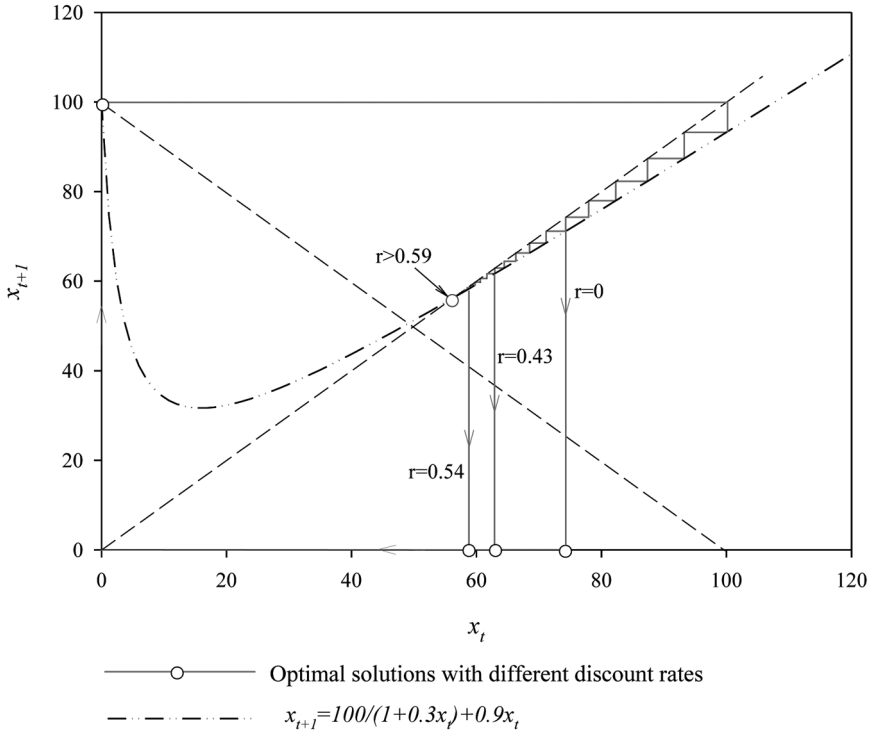


Figure 6. Effect of discounting on optimal clear-cut cycle length

the development of x_t is computed forward from $x_0 = 0$ and the objective function is evaluated with all clear-cut cycles from τ_2 onward by applying (29). Since $\hat{x} > \frac{1}{2}\varphi_0$, the τ_∞ solution yields a higher average yield than τ_2 , but because $\varphi'(0) + \beta < -1$, both these solutions are locally optimal under $r = 0$ (propositions 1 and 2). However, the highest average yield per period (73.6) is obtained by τ_7 . Increasing the interest rate increases the length of the globally optimal clear-cut cycle, and when $r > 0.6$, the continuous harvesting solution is globally optimal.

2.4. Trees in Class 1 Have Immediate Value: $f_1 > 0$

Given that small trees are directly valuable and $f_1 > 0$, it may be optimal to harvest the whole cohort at one period old. The necessary condition for the τ_1 solution from (13a)–(13c) to (14a)–(14c) is

$$f_1 - baf_2 - bf_1[\varphi'(0) + \beta] > 0. \tag{32}$$

If $f_1 - baf_2 > 0$, the direct present value from harvesting class 1, is higher than from harvesting class 2. A strong recruitment effect, $\varphi'(0) + \beta < 0$, favors harvesting trees from size class 1 as does a high discount rate and a high $f_1/(f_2\alpha)$ ratio.

By (13a)–(13c) to (14a)–(14c), the steady-state solution of τ_∞ is locally optimal if $\lambda > 0$, that is, if

$$f_1 - b\alpha f_2 - bf_1[\varphi'(\hat{x}) + \beta] < 0, \tag{33}$$

$$\Leftrightarrow \frac{\frac{\alpha f_2}{1 - b[\beta + \varphi'(\hat{x})]} - f_1}{f_1} > r. \tag{34}$$

For the interpretation, set $\beta = \varphi' = 0$. Now (34) reads: $(\alpha f_2 - f_1)/f_1 > r$. Thus it is optimal to cut trees from class 2 if the relative value growth exceeds the interest rate. If $\alpha = 1$, this is the Wicksellian wine-aging case. When $\beta > 0$, saving an individual in class 1 causes an infinitely lasting increase in the flow from class 1 to class 2 with the present value equal to $f_2 \alpha \sum_{t=0}^\infty (b\beta)^t = f_2 \alpha / (1 - b\beta)$. Positive β increases the rate of return and favors harvesting trees from class 2. Given $\varphi' < 0$, the marginal effect and the infinitely lasting flow is lower. The marginal rate of return in (34) is a combination of biological and economic parameters and may be labeled the “bioeconomic rate of return.” Higher levels of α , β , f_2 increase the size of harvested trees, whereas the effects of r , φ' , and f_1 are to reduce the size of harvested trees. This shows that the optimal size of harvested trees cannot be found by formulae of the Wicksellian or Faustmann type that circumvent combining the economic and biological parameters through the time delays of the size-structured growth process.

Next, it is possible to generalize the necessary optimality condition (28) for $f_1 > 0$ and any τ_n , $n \geq 2$. Given $\sigma(x) := -f_1 + b\alpha f_2 + bf_1[\varphi'(x) + \beta]$, this yields

$$\sigma(0) + \sum_{i=1}^{\tau-1} \sigma(x_i) b^i \prod_{j=1}^i [\varphi'(x_{j-1}) + \beta] \leq 0, \tag{35}$$

which collapses to (28) when $f_1 = 0$.

Proposition 8: Given any $x_0 \geq 0$ and (a) $\sigma(\hat{x}) < 0$, a clear-cut solution with some period length is globally optimal, or (b) $\sigma(0) > 0$ and $\varphi'(0) + \beta \geq 0$, the continuous cover solution τ_∞ is globally optimal.

Proof: When $\sigma(\hat{x}) < 0$, condition (33) is violated, that is, the τ_∞ solution is not locally optimal. Since at least condition (32) is satisfied, some $\tau < \infty$ is optimal. When $\sigma(0) > 0$, the convexity of φ implies that $\sigma(x_i) > 0$ for all $x_i > 0$, and $\varphi'(0) + \beta \geq 0$ that (35) is positive for all $\tau \geq 1$. Thus, necessary optimality conditions are violated for all $\tau < \infty$, but by $\sigma(0) > 0$, condition (33) is satisfied and τ_∞ is globally optimal. QED

In case *a* continuous harvesting is not optimal because the bioeconomic rate of return from saving small trees is lower than the interest rate (cf. eqs. [6], [34]). In case *b* the reverse holds and the transition path is favorable since harvest from class 2 is continuously increasing (cf. proposition 4a).

Applying steps analogous to proposition 3, it is better to switch to solution τ_2 from the τ_∞ steady state if

$$\hat{x} < \frac{\alpha f_2 \varphi_0 + f_1 [\varphi(\varphi_0) + \beta \varphi_0 + r \varphi(\hat{x})(r + 2)]}{(r + 2)(\alpha f_2 - \beta f_1 r)}. \tag{36}$$

Since this condition collapses to $\hat{x} < \varphi_0 / (2 + r)$ if $f_1 = 0$ and since the right-hand side increases with f_1 , it follows that a positive level of f_1 works against the global optimality of the continuous cover solution. The effect of interest rate is a priori indeterminate.

Assuming $x_0 = 0$ and $f_1 > 0$, problem (29) can be extended to

$$\max_{\{\tau \in [1, \infty)\}} \frac{\sum_{t=0}^{\tau-1} f_2 \alpha x_t b^{t+1} + \{f_2 \alpha x_\tau + f_1 [\varphi(x_\tau) + \beta x_\tau]\} b^{\tau+1}}{1 - b^{\tau+1}}, \text{ subject to (3) and } x_0 = 0, \tag{37}$$

where the numerator is the present-value yield per cycle from harvesting classes 2 and 1 (the latter only at the clear-cut). If harvesting class 2 trees before the clear-cut is ruled out, the model collapses back to the Faustman model and the optimal rotation period will be finite by construction. If harvesting of size class 2 trees is included but continuous regeneration is replaced by costly investment in regeneration, the model collapses back to a rotation model with optimized intermediate harvesting, and again the rotation period is finite by construction. Specification (37) can be extended to include various kinds of management costs, but I leave these complications for future studies.

2.5. Any Number of Size Classes

An extension of problem (37) to include any number of size classes can be given as

$$J_\tau = \max_{\{\tau, b_{n,s}=1, \dots, n, t=0, 1, \dots, \tau\}} \frac{\sum_{t=0}^{\tau} U(H_t) b^{t+1}}{1 - b^{\tau+1}}, \tag{38}$$

subject to (2)–(6) and the stand’s initial state. Computation of this problem can be performed by applying the empirically estimated growth model of Bollandsås et al. (2008). Their model differs from (2)–(7) since their $\alpha_s = 1, \dots, n - 1$, and $\beta_s = 1, \dots, n$ depend on stand density. The dependence is statistically weak and could be removed, but the results here will be qualitatively the same when it is maintained. In Bollandsås et al. (2008), the period length is 5 years, and the recruitment function for Norway spruce (per hectare) is

$$\varphi(y_t) = \frac{147.809(y_t + \sigma)^{-0.157}}{1 + e^{-0.599 + 0.018y_t}},$$

where y_t is the cross section (basal) area (in m^2) of trees measured at breast height. This function is decreasing and strictly convex in y_t (cf. [Q1]). In Bollandsås et al. (2008), $\sigma = 0$, which implies that $\varphi(y) \rightarrow \infty$ as $y \rightarrow 0$. Bollandsås et al. (2008) do not discuss this unrealistic feature and here various values for σ are used to obtain finite values for $\varphi(0)$. This causes only negligible effects in the regeneration with typical levels of y . The fractions of trees moving to the next size classes are estimated as

$$\alpha_s = 0.02(17.839 + 0.0476\delta_s - 11.585 \times 10^{-5}\delta_s^2 - 0.3412y_{st} + 0.906\zeta - 0.024y_t - 0.268l),$$

where $s = 1, \dots, n - 1$, δ_s is the diameter (cm), y_{st} is the basal area of trees with larger diameters than trees in size class s , $\zeta (= 15)$ is the site index, and $l (= 60)$ is latitude. Write $\beta_{st} = 1 - \alpha_{st} - \eta_{st}$, $s = 1, \dots, n$, $t = 0, 1, \dots$, where η_{st} is natural mortality given as

$$\eta_{st} = [1 + e^{2.492+0.020\delta_s+3.2 \times 10^{-5}\delta_s^2+0.031y_{st}}]^{-1}.$$

Table A1 in the appendix, section G, provides tree volumes. The objective function (38) is given as

$$\max_{\{\tau, h_s, s=1, \dots, n, t=0, 1, \dots, \tau\}} \frac{\sum_{t=0}^{\tau} b^{t+1} \sum_{s=1}^n h_{st}(v_{1s}p_1 + v_{2s}p_2)}{1 - b^{\tau+1}}, \tag{39}$$

where v_{1s} and v_{2s} are the volumes per tree for sawn timber and smaller diameter logs in m^3 and $p_1 = €55.5/m^3$ and $p_2 = 23.7€/m^3$ are the stumpage prices.

Figure 7 shows that when $\varphi_0 \leq 825$, continuous harvesting solutions are optimal with interest rates (per annum) $0 \leq r \leq 0.1$, whereas when $\varphi_0 \geq 1,025$, clear-cut solutions become optimal with interest rates $0 \leq r \leq 0.1$. Between these boundaries, the continuous harvesting solution is optimal if the interest rate is high and vice versa. Although the numerical model includes a couple of extensions, these results are in line with the analytical results: high interest rate supports and high initial regeneration φ_0 works against the optimality of the continuous cover solution.

Figure 7 shows that given $\varphi_0 = 900$, the continuous cover solution is optimal for $r = 0.03$ but clear-cut is optimal for $r = 0$. Figure 8 compares the development of various variables for these two solutions. The length of optimal rotation under $r = 0$ is 145 years, and in intermediate cuttings the trees are harvested from diameter classes 32.5 cm and 37.5 cm. Given $r = 0.03$, trees are harvested only from diameter class 27.5 cm. The development of all the variables is intuitive and empirically realistic.

In figure 9, trees with diameters less than 37.5 cm are valueless. This setup is an extension of specification (29). Clear-cut solutions are unoptimal at any interest rate level when $\varphi_0 < 2,000$ (fig. 9a). When $\varphi_0 \geq 2,000$, their optimality depends on discounting. Assuming $\varphi_0 = 2,500$, the optimal rotation is 255 years under $r = 0$ but continuous harvesting is optimal when $r \geq 0.002$. In each case, it becomes optimal to

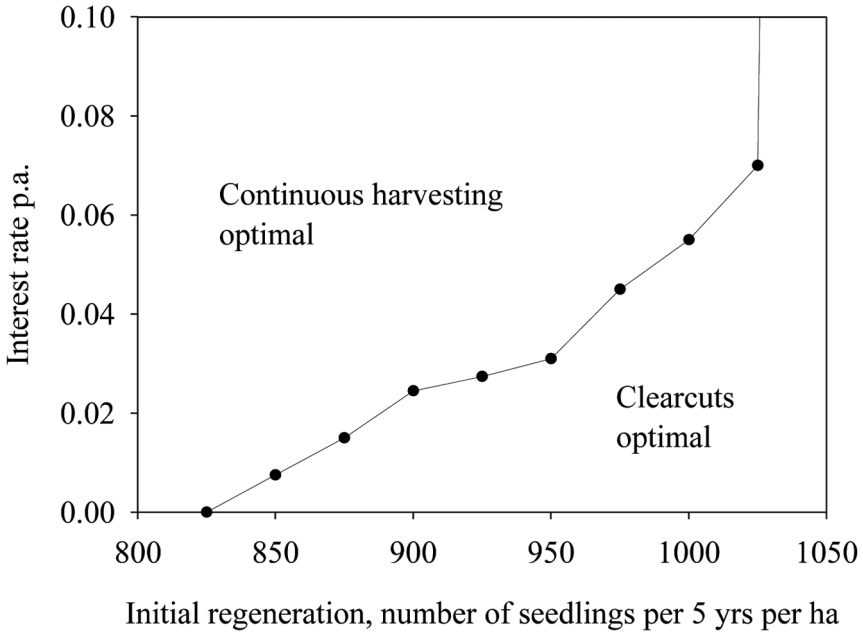


Figure 7. Optimal solution type with different regeneration and interest rate levels data (see text).

harvest trees when they enter the diameter class 37.5 cm. Figure 9b shows how optimized annual revenues develop over time with various regeneration levels.

2.6. A Stylized Model for Continuous Harvesting

An important strength of the classic rotation model is its simplicity, which, however, has a cost. The most generic version (Samuelson 1976) describes optimal rotation of a single tree or a number of similar trees without biological connections. After adding density dependence (even with no internal structure), the solution without intermediate cuttings becomes a special case (Clark 1976, 263), but in the generic version, intermediate cuttings are ruled out by construction.

This study as well as some previous ones (Getz and Haight 1989, 301; Tahvonen 2011) shows that it is characteristic of continuous harvesting solutions to target cuttings to the largest trees. Specifying model (37) in continuous time leads to the problem

$$\max_{\{b(t), \tau \in [0, \infty)\}} J = \frac{\int_0^\tau p b(t) e^{-r t} dt + e^{-r \tau} p x(\tau)}{1 - e^{-r \tau}}, \text{ subject to } x = g(t) f(x) - b, x(0) = x_0, \quad (40)$$

where $b(t)$ denotes intermediate cutting of largest trees, $x(\tau)$ is the clear-cut volume, p the stumpage price, $g(t)$ denotes aging of trees regenerated after clear-cut,

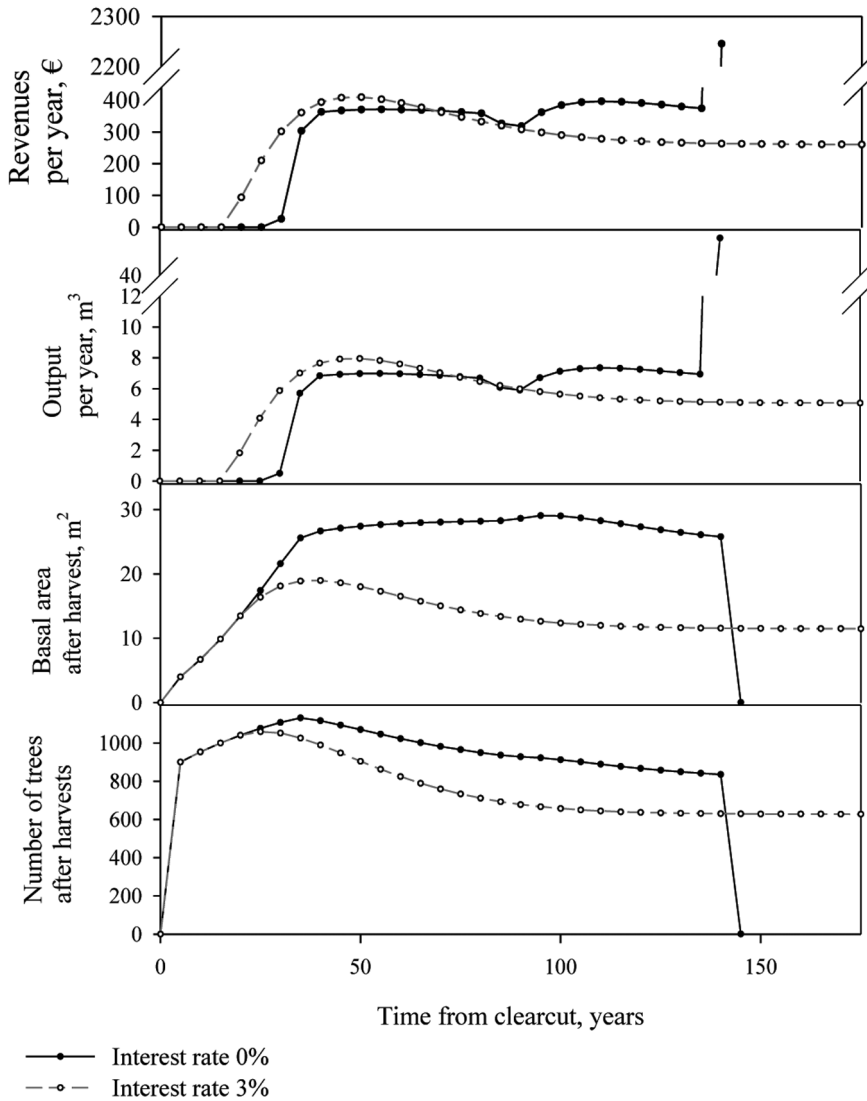


Figure 8. Optimal solutions when regeneration after clear-cut is 900 seedlings per 5 years. All figures per hectare.

and $f(x)$ a component of stand growth that depends on density. This stylized model is close to Clark's (1976, 263), but there is no reason to assume a pure even-aged plantation case where $g(t) \rightarrow 0$ as the regenerated trees age. Thus, whereas Clark's specification yields clear-cuts by construction, specification (40) with the appropriate growth model ($g(t) > 0, g'(t) \leq 0, t \in [0, \infty)$) and related solution for the intermediate cuttings leaves the choice of forest management type

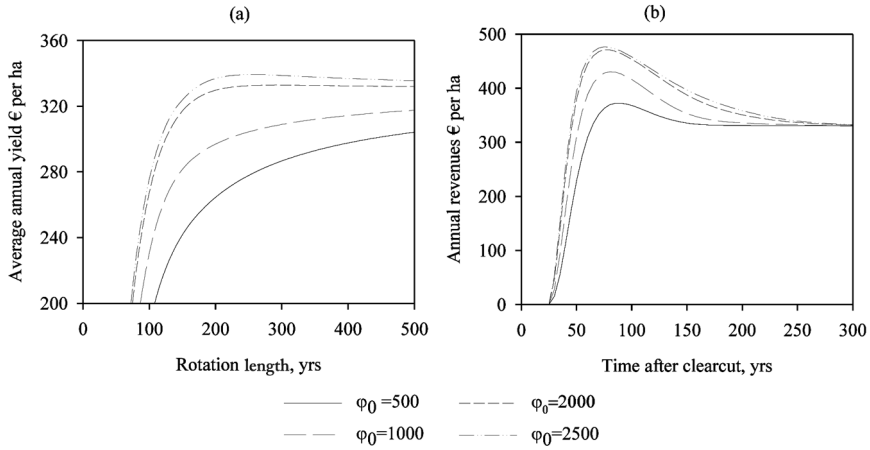


Figure 9. Average (a) and annual (b) revenues with different regeneration levels when only trees with diameters above or equal to 37.5 cm are valuable.

open a priori. The first-order necessary condition for the rotation period in (40) is $ph(\tau) + px'(\tau) - rpx(\tau) - rJ = 0$, if locally optimal τ is finite, and $\lim_{\tau \rightarrow \infty} [ph(\tau) + px'(\tau) - rpx(\tau) - rJ] \geq 0$ if locally optimal τ is infinite. The optimal intermediate harvests can be mimicked by solving a singular solution as in Clark (1976), or by utilizing the output from size-structured models (figs. 8 and 9b). Although model (40) has a somewhat loose and ad hoc connection with empirical models of forest growth, and although it cannot capture various features such as a size-structured forest's optimal density or the size of optimally harvested trees, it nonetheless characterizes the choice between forestry with and without clear-cuts. Thus, it is not optimal to clear-cut if the steady-state present value of revenues from continuous harvesting integrated over infinite horizon exceeds the sum of the clear-cut revenues and the value of bare land.

3. DISCUSSION AND CONCLUSIONS

Although previous analytical results are nonexistent, the basic understanding described here can be compared with some empirical/numerical studies that have applied economically coherent optimization setups. The intermediate cuttings in my study remove large trees and increase the length of optimal rotation in a manner similar to empirical numerically computed structured models of even-aged management (e.g., Tahvonen, Pihlainen, and Niinimäki 2013). The typical (steady-state) solution in empirical/numerical uneven-aged models (Getz and Haight 1989, 301) is the cutting of largest trees, similar to the analytical solutions obtained here. Most uneven-aged studies present only the continuous cover solution and are silent about the possibility that nonconvexities in regeneration may produce optimal clear-cuts

endogenously. Exceptions are Haight and Monserud (1990) and Tahvonen (2009), who present numerical examples for cyclical close-to-clear-cut solutions based on natural regeneration. Nonconvexities require that numerical results should be computed with care, as most algorithms have serious difficulties in finding solutions such as the clear-cut cycle. This study presents an effective computation scheme for finding the globally optimal solution: write the uneven-aged problem as a rotation model with (optimized) intermediate cuttings and vary the rotation length up through infinity.

My analytical investigation produced several results that have no counterpart in existing studies. Clear-cuts may not satisfy necessary optimality conditions, because saving small size class trees increases yields from intermediate cuttings, and the negative density dependence effect that reduces future yields may be smaller and is realized later. This is one reason why the interest rate works against clear-cutting. Clear-cuts are followed by periods with no harvest. Such periods are costly and the cost increases with interest rate, implying that it becomes optimal to clear-cut less frequently or abandon clear-cuts completely. These results are surprising in the light of the conventional Faustmann approach but very natural in the size-structured setup analyzed here. The initial stand state matters in two ways: depending on initial state, the optimal long-term solution may be either continuous cover solution or clear-cuts (and short rotation). In addition, depending on the initial state, it may be optimal to clear-cut the initial stand, even if the optimal long-run solution is to maintain continuous forest cover. The optimal size of harvested trees is determined by a bioeconomic rate of return that combines the economic and biological parameters through time delays in the stand growth process and thus deviates from the Wicksellian and Faustmann optimal tree-cutting formulae.

I have here followed most numerical uneven-aged studies by applying stumpage prices without explicit harvest costs. Lower harvest costs per tree in clear-cuts favors this regime. However, in naturally regenerated heterogeneous forests, continuous cover harvesting may yield a much higher average volume of harvested trees, thus decreasing the cost per harvested volume. The cost difference also depends strongly on both site circumstances and harvesting equipment.

The choice between clear-cuts and continuous forest cover is greatly influenced by the relative values of different-size trees, a fact that explains the historical alternation of these management practices in Nordic countries. Continuous cover forestry has been favored when demand for small-diameter trees has been low, and vice versa. The current trend is a shift in pulpwood production from boreal regions to tropical plantations. A resulting increase in the relative prices of large-diameter trees in the North would favor continuous cover forestry. In addition, including continuous cover forestry in the palette of management alternatives has potential benefits for biodiversity, landscape esthetics, and mitigating climate change everywhere, from tropics to boreal forests.

APPENDIX

A. The Infinite Horizon Discrete-Time Optimal Control Formulation

Write the problem for $n = 2$ as

$$\begin{aligned} & \max_{\{h_t := 0, 1, \dots\}} \sum_{t=0}^{\infty} b^{t+1} (f_1 h_t + f_2 \alpha x_t) \\ & \text{subject to } x_{t+1} = \varphi(x_t) + \beta x_t - h_t, \\ & \quad x_0 \text{ given, } h_t \geq 0, \\ & \quad -\varphi(x_t) - \beta x_t + h_t \leq 0. \end{aligned} \tag{A1}$$

Applying theorem 12.4.1 in Sydsæter et al. (2008, 441) and its extension, theorem 2.20 in Blot and Hayek (2014, 57),

$$\begin{aligned} H &= b^{t+1} (f_1 h_t + f_2 \alpha x_t) + \tilde{p}_t [\varphi(x_t) + \beta x_t - h_t], \\ L &= H - \tilde{\mu}_t [-\varphi(x_t) - \beta x_t + h_t], \end{aligned}$$

where H is the Hamiltonian and L is the Lagrangian. The first-order necessary conditions include (A1) and

$$\frac{\partial L}{\partial h_t} = b^{t+1} f_1 - \tilde{p}_t - \tilde{\mu}_t \leq 0, \tag{A2}$$

$$[b^{t+1} f_1 - \tilde{p}_t - \tilde{\mu}_t] h_t = 0, \tag{A3}$$

$$h_t \geq 0, \tag{A4}$$

$$-\varphi(x_t) - \beta x_t + h_t \leq 0, \tag{A5}$$

$$\tilde{\mu}_t [-\varphi(x_t) - \beta x_t + h_t] = 0, \tag{A6}$$

$$\tilde{\mu}_t \geq 0, \tag{A7}$$

$$\tilde{p}_{t-1} = b^{t+1} f_2 \alpha + (\tilde{p}_t + \tilde{\mu}_t) [\varphi'(x_t) + \beta]. \tag{A8}$$

Let $\tilde{p}_t := p_t b^{t+1}$, $\tilde{\mu}_t := \mu_t b^{t+1}$, and $\lambda_t := -f_1 + p_t + \mu_t$. Thus (A2)–(A4) can be written as

$$\lambda_t \geq 0, \tag{A9}$$

$$\lambda_t [x_{t+1} - \varphi(x_t) - \beta x_t] = 0, \tag{A10}$$

$$x_{t+1} - \varphi(x_t) - \beta x_t \leq 0. \tag{A11}$$

From (A8), $p_{t-1} = b f_2 \alpha + b(p_t + \mu_t) [\varphi'(x_t) + \beta]$. Letting $t - 1 \rightarrow t$, $t \rightarrow t + 1$ and applying the fact that $p_t = \lambda_t + f_1 - \mu_t$, it follows that (A8) takes the form:

$$b f_2 \alpha - \lambda_t - f_1 + (\lambda_{t+1} + f_1) [\varphi'(x_{t+1}) + \beta] = -\mu_t \leq 0. \tag{A12}$$

Since $-x_{t+1} = -\varphi(x_t) - \beta x_t + h_t$ conditions (A5)–(A7) can be written as

$$x_{t+1} \geq 0, \tag{A13}$$

$$\Phi x_{t+1} = 0, \tag{A14}$$

$$\Phi \leq 0, \tag{A15}$$

where $\Phi := bf_2\alpha - \lambda_t - f_1 + (\lambda_{t+1} + f_1)[\varphi'(x_{t+1}) + \beta]$. Conditions (A9)–(A11) coincide with conditions (14a)–(14c), and conditions (A13)–(A15) coincide with conditions (13a)–(13c).

B. Interior Solutions as Minimizers

To find interior solutions that are minimizers for the original problem, write

$$\begin{aligned} & \max_{\{h_t, t = 0, 1, \dots\}} \sum_{t=0}^{\infty} b^{t+1}(-f_1 h_t - f_2 \alpha x_t) \\ & \text{subject to } x_{t+1} = \varphi(x_t) + \beta x_t - h_t, \quad x_0 \text{ given.} \end{aligned}$$

The Hamiltonian is $H = b^{t+1}(-f_1 h_t - f_2 \alpha x_t) + q_t[\varphi(x_t) + \beta x_t - h_t]$ and the necessary optimality conditions are given as

$$-b^{t+1}f_1 - q_t = 0, \tag{A16}$$

$$q_{t-1} = -b^{t+1}f_2\alpha + q_t[\varphi'(x_t) + \beta]. \tag{A17}$$

Condition (A16) implies that $q_t = -b^{t+1}f_1 \leq 0$, where the equality holds if $f_1 = 0$. The fact $q_t \leq 0$ implies that Hamiltonian is concave. By the theorem 12.5.2 in Sydsæter et al. (2008, 447) the solutions satisfying (A16) and (A17) are optimal because they in addition satisfy for all admissible solutions

$$\lim_{t \rightarrow \infty} q_t(x_t - x_t^*) \geq 0$$

since $\lim_{t \rightarrow \infty} q_t = \lim_{t \rightarrow \infty} -b^{t+1}f_1 = 0$ and $x_t - x_t^*$ are bounded by assumptions in (Q1). Letting $\tilde{p}_t = -q_t$ shows that the interior solutions for (A2)–(A8) are equivalent to (A16) and (A17); that is, the interior solutions for (A2)–(A8) yield minimizers for the original problem (9)–(12).

C. Proof of Proposition 1

Given $-1 < \varphi'(0) + \beta$, the steady state for (15) is globally asymptotically stable by $\varphi' < 0$, $\varphi'' \geq 0$ in (Q1), $0 \leq \beta < 1$, and theorem 11.7.1 and note 1 in Sydsæter et al. (2008, 419–20). Along a solution toward the steady state \hat{x} , it must hold that $\lambda_t > 0$, and by (13a)–(13c) and (15)

$$\lambda_{t+1} = (\lambda_t - bf\alpha) / \{b[\varphi'(\varphi(x_t) + \beta x_t) + \beta]\}. \tag{A18}$$

The characteristic equation for system (15) and (A18) is

$$r^2 - r\{\varphi'(\hat{x}) + \beta + \{b[\varphi'(\hat{x}) + \beta]\}^{-1}\} + b^{-1} = 0.$$

Given $-1 < \varphi'(\hat{x}) + \beta < 1$, the absolute value of one root is always above 1 while the absolute value of the other is below 1; that is, the steady state is a saddle point. Since $\hat{\lambda} > 0$ at the steady state, $\lambda_t > 0$ must hold in the vicinity of the steady state as well. Assume that the denominator in (A18) is positive and write $y(x_t) := \varphi'(\varphi(x_t) + \beta x_t) + \beta > 0$. Thus, given that $\lambda_{t+1} > 0$ holds for some t , it is possible to choose some $0 < b\alpha < \lambda_t$ and proceed backward toward x_0 and maintain $\lambda_t > 0$. If $y(x_t) < 0$, the same argument holds, given that λ_t satisfies $0 < \lambda_t < b\alpha$. If $y(x_t)$ changes its sign, similar reasoning applies (as well as in the cases where $y(x_t) = 0$ for some x_t). Thus, given $x_t \rightarrow \hat{x}$ as $t \rightarrow \infty$, there exists some positive sequence $\lambda_t \rightarrow \hat{\lambda}$; that is, the solution τ_∞ represents a local maximum. QED

D. Proof of Proposition 4

(a) Since $\lambda_0 > 0$ is necessary, the case $\varphi'(0) + \beta \geq 0$, equation (25) or (28) and convexity of φ rule out local optimality of all $\tau < \infty$ solutions and imply that τ_∞ is optimal.

(b) The conditions $-1 \leq \varphi'(0) + \beta < 0$ and $\varphi'(\varphi_0) + \beta \leq 0$ imply that $\frac{1}{2}\varphi_0 < \hat{x}$, $\varphi'(\hat{x}) + \beta < 0$ (by $\varphi'' > 0$), and that $x \rightarrow \hat{x}$ cyclically. The left-hand side of (28), (29) can be rewritten as

$$1 + b[\varphi'(0) + \beta] + b[\varphi'(0) + \beta]\{b[\varphi'(x_1) + \beta] + b^2[\varphi'(x_1) + \beta][\varphi'(x_2) + \beta] + \dots\}. \tag{A19}$$

Given the first assumption in b , the term $1 + b[\cdot]$ in (A19) is nonnegative. The term $\{\cdot\}$ is negative since the products with an odd number of square bracket terms are negative and the subsequent terms with an even number of square bracket terms are positive but always lower than the absolute value of the preceding negative terms. Thus, $[\cdot]\{\cdot\} > 0$, implying that (A19) and (28) are positive and that the local optimality of all $\tau < \infty$ solutions is ruled out: τ_∞ is optimal.

(c) Condition (28) implies directly that when b is sufficiently low, no candidates with $\tau < \infty$ satisfy the necessary optimality conditions, and thus that τ_∞ is globally optimal. QED

E. Proof of Proposition 5

If $x_0 = 0$, the τ_∞ regime yields $J_{\tau_\infty} = \sum_{t=1}^{\infty} b^{t-1} f a x_t$ where $x_t \geq \varphi_0$ for all t by $\varphi(\varphi_0) + \beta\varphi_0 \geq \varphi_0$, implying τ_∞ dominates all clear-cut cycles. Given $x_0 > 0$, a clear-cut at the end of period $t = 0$ satisfies conditions (13a)–(13c) and (14a)–(14c) by $1 + \varphi'(0) + \beta < 0$, because $\lambda_1 > b\alpha$ and because $\varphi(\varphi_0) + \beta\varphi_0 \geq \varphi_0$ implies $\varphi'(\varphi_0) +$

$\beta > 0$. Since $1 + \varphi'(0) + \beta < 0$ and $\hat{x} \geq \varphi_0$, the function $\varphi(x) + \beta x$ has a minimum (satisfying $\varphi(x) + \beta x > x$) that can be made arbitrarily low without violating (Q1). Thus, if τ_∞ is chosen, the increase of x_t toward φ_0 from some arbitrarily low $x_0 = \arg \min \{\varphi(x) + \beta x\}$ becomes arbitrarily slow and regime τ_∞ is dominated by a clear-cut at the end of period $t = 0$ followed by regime τ_∞ . If the clear-cut at the end of $t = 0$ and regime τ_∞ thereafter is dominated by a later clear-cut, then it is never optimal to clear-cut, because increasing x_t implies that the net gain from postponing the clear-cut can only increase. Thus, the optimal solution is either regime τ_∞ from $t = 0$ or a clear-cut at $t = 0$ followed by τ_∞ . QED

F. Proof of Proposition 6

Given $\hat{x} = \varphi_0 / (2 + \tilde{r})$ and $x_0 = \hat{x}$, the objective values for all $\tau_n, n \geq 2$ solutions are equal and the same holds for $x_0 = 0$ by $\hat{x} = \bar{\varphi}(\varphi_0) + \beta\varphi_0$. Given $\bar{\varphi}, 0 < \bar{\beta}$, $\hat{x} = \bar{\varphi}(\hat{x}) + \bar{\beta}\hat{x}$, and $r = \tilde{r}$, the solutions τ_∞ and τ_2 for $x_0 = \hat{x}$ are unaltered, but by $\tilde{r} > 0$, τ_2 yields a higher objective value than any $\tau_n, n > 2$ for $x_0 = 0$ if

$$\hat{x} - x_t > x_{t+1} - \hat{x} \text{ for all } t = 2, 4, \dots \tag{A20}$$

Since $\hat{x} = \bar{\varphi}(\hat{x}) + \bar{\beta}\hat{x}$ and $x_{t+1} = \bar{\varphi}(x_t) + \bar{\beta}x_t$ for all $t \geq 2$, equation (A20) can be written as $g := \hat{x} - x_t - \bar{\varphi}(x_t) - \bar{\beta}x_t + \bar{\varphi}(\hat{x}) + \bar{\beta}\hat{x} > 0$. The assumption $\bar{\varphi}(\varphi_0) + \bar{\beta}\varphi_0 < \hat{x}$ implies that $x_t < \hat{x}$ for all $t = 2, 4, \dots$. Differentiating yields $\partial g / \partial x_t = -1 - \bar{\varphi}'(x_t) - \bar{\beta} < 0$, where the sign holds by the stability of \hat{x} , convexity of φ , and $x_t < \hat{x}$. Thus, $g > 0$ and by proposition 2, there exists $\bar{r} > \tilde{r}$ such that τ_∞ is optimal for $x_0 = \hat{x}$ and τ_2 for $x_0 = 0$. QED

G. Tree Volumes

Table A1. Parameter Values for Model Specifications (39), (2)–(7)

	Size Class					
	1	2	3	4	5	6
Diameter ^a	75	125	175	225	275	325
v_1^b	0	0	0	.234	.446	.684
v_2^b	.014	.067	.167	.081	.065	.060
	7	8	9	10	11	12
Diameter ^a	375	425	475	525	575	625
v_1^b	.963	1.253	1.574	1.900	2.214	2.564
v_2^b	.0498	.050	.043	.0392	.033	.031

^a Unit: mm.

^b Unit: m³; v_1 = sawn timber volume; v_2 = pulp volume.

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