

# Edge-disjoint paths in digraphs with bounded independence number

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### Abstract

A digraph  $H$  is *infused* in a digraph  $G$  if the vertices of  $H$  are mapped to vertices of  $G$  (not necessarily distinct), and the edges of  $H$  are mapped to edge-disjoint directed paths of  $G$  joining the corresponding pairs of vertices of  $G$ . The algorithmic problem of determining whether a fixed graph  $H$  can be infused in an input graph  $G$  is polynomial-time solvable for all graphs  $H$  (using paths instead of directed paths). However, the analogous problem in digraphs is NP-complete for most digraphs  $H$ . We provide a polynomial-time algorithm to solve a rooted version of the problem, for all digraphs  $H$ , in digraphs with independence number bounded by a fixed integer  $\alpha$ . The problem that we solve is a generalization of the  $k$  edge-disjoint directed paths problem (for fixed  $k$ ).

# 1 Introduction

In [3], Fortune, Hopcroft, and Wyllie showed that the following algorithmic problem is NP-complete with  $k = 2$ :

*k Edge-Disjoint Paths* (k-EDP)

Instance: A digraph  $G$ , and  $k$  pairs  $(s_1, t_1), \dots, (s_k, t_k)$  of vertices of  $G$ .

Question: Do there exist directed paths  $P_1, \dots, P_k$  of  $G$ , mutually edge-disjoint, such that  $P_i$  is from  $s_i$  to  $t_i$  for  $i = 1, \dots, k$ ?

On the other hand, in [6], Robertson and the second author showed that the analogous problem for undirected graphs can be solved in polynomial time for any fixed natural number  $k$ . This is a common phenomenon in algorithmic graph theory, that a problem which can be solved in polynomial-time in undirected graphs is NP-complete for general digraphs. In this case it is natural to look for polynomial-time algorithms that solve the problem for restricted classes of digraphs.

The main result of this paper is a polynomial-time algorithm to solve a generalization of  $k$ -EDP in digraphs with independence number at most  $\alpha$ , for fixed  $k$  and  $\alpha$ . Before describing the generalized problem, however, we need some background and notation. Let  $G$  be a digraph (all graphs and digraphs in this paper are finite, and may have loops or parallel edges). We denote the vertex set of  $G$  by  $V(G)$  and the edge set by  $E(G)$ . For  $u, v \in V(G)$ , we let  $E_{uv}^G$  denote the set of all edges of  $G$  with tail  $u$  and head  $v$  (we may just write  $E_{uv}$  if it is clear which digraph we are referring to); and if there is an edge from  $u$  to  $v$  we say that  $u$  is *adjacent to*  $v$ , and  $v$  is *adjacent from*  $u$ . A set of vertices  $U \subseteq V(G)$  is *independent* if for all distinct  $u, v \in U$  there is no edge between  $u$  and  $v$ . The *independence number* of  $G$ , denoted by  $\alpha(G)$ , is the maximum cardinality of an independent set in  $G$ . (Note that vertices of an independent set may be incident with loops; this is non-standard, but important for us.) We say that  $P$  is a *route* of  $G$  if  $P$  is either a directed path or a directed cycle with a distinguished vertex called its *end*. For  $x, y \in V(G)$  we say that  $P$  is a route *from*  $x$  *to*  $y$  if either  $x \neq y$  and  $P$  is a directed path from  $x$  to  $y$ , or  $x = y$  and  $P$  is a directed cycle with  $x$  as its end. If  $P$  is a route from  $x$  to  $y$  then the *first edge of*  $P$  is the unique edge of  $P$  with tail  $x$  and the *last edge of*  $P$  is the unique edge of  $P$  with head  $y$ . The *interior* of a directed path  $P$  is the set of all its vertices except its ends.

A digraph is a *supertournament* if for every pair of distinct vertices  $u, v$  there is at least one edge with ends  $\{u, v\}$ . A digraph is *simple* if it has no loops, and for every pair of distinct vertices  $u, v$  there is at most one edge with tail  $u$  and head  $v$ . A *semi-complete digraph* is a simple supertournament. A digraph is a *tournament* if it is simple, and for every pair of distinct vertices  $u, v$  there is exactly one edge with set of ends  $\{u, v\}$ . Thus, every tournament is semi-complete.

In [1], Bang-Jensen showed that there exists a polynomial-time algorithm to solve 2-EDP in semi-complete digraphs. Note that our result generalizes Bang-Jensen's in two ways (the 2 is generalized to  $k$  and semi-complete digraphs are generalized to digraphs with bounded independence number).

Next, we describe a problem that is closely related to  $k$ -EDP. Again, we begin with a definition. Let  $G, H$  be digraphs. An *infusion* of  $H$  in  $G$  is a map  $\eta$  such that

- $\eta(v) \in V(G)$  for each  $v \in V(H)$
- for each edge  $e = uv$  of  $H$  (this notation means that  $e$  is directed from  $u$  to  $v$ ),  $\eta(e)$  is a route of  $G$  from  $\eta(u)$  to  $\eta(v)$

- if  $e, f \in E(H)$  are distinct, then  $\eta(e), \eta(f)$  have no edges in common, although they may share vertices.

If in addition we add the condition

- $\eta(u) \neq \eta(v)$  for distinct  $u, v \in V(H)$

then we call the relation *weak immersion*. If we also then add the condition

- if  $v \in V(H)$  and  $e \in E(H)$ , and  $e$  is not incident with  $v$  in  $H$ , then  $\eta(v)$  is not a vertex of  $\eta(e)$

we call the relation *strong immersion*.

In this paper we develop several algorithms related to infusion and immersion. First, consider the following algorithmic problem. Let  $H$  be a fixed digraph.

*H-infusion*

Instance: A digraph  $G$ .

Question: Does there exist an infusion of  $H$  in  $G$ ?

If in the question the word infusion is replaced by strong (respectively, weak) immersion, then we refer to the problem as strong (resp. weak) *H-immersion* (we will often drop the strong/weak, and in such cases everything we say holds for both versions). In [2], with Chudnovsky, we showed that there exist digraphs  $H$  for which *H-immersion* is NP-complete. On the other hand, we showed that in semi-complete digraphs, *H-immersion* is polynomial-time solvable for every fixed digraph  $H$ . The proofs of both of these results trivially extend to *H-infusion*.

Next we consider a common generalization of *H-infusion* and *k-EDP*. We say that  $(G, v_1, \dots, v_h)$  is a *rooted digraph* if  $G$  is a digraph and  $v_1, \dots, v_h \in V(G)$  (not necessarily distinct). If  $v_1, \dots, v_h$  are distinct then we say that  $(G, v_1, \dots, v_h)$  is a *distinctly-rooted digraph*. Let  $\mathbf{G} = (G, v_1, \dots, v_h)$  be a rooted digraph and  $\mathbf{H} = (H, u_1, \dots, u_h)$  be a distinctly-rooted digraph (where the number of roots is the same). We say that  $\eta$  is an *infusion* of  $\mathbf{H}$  in  $\mathbf{G}$  if  $\eta$  is an infusion of  $H$  in  $G$  with  $\eta(u_i) = v_i$  for all  $1 \leq i \leq h$ . Given a distinctly-rooted digraph  $\mathbf{H} = (H, u_1, \dots, u_h)$ , we have the following algorithmic problem:

*H-infusion*

Instance: A rooted digraph  $\mathbf{G} = (G, v_1, \dots, v_h)$ .

Question: Does there exist an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ ?

The main results of this paper are polynomial-time algorithms to solve *H-immersion* and *H-infusion* in digraphs with bounded independence number. Notice that *H-immersion* (where  $\mathbf{H}$  is a distinctly-rooted digraph), defined in the obvious way, is a close relative of these two problems, but we do not know of a polynomial-time algorithm to solve *H-immersion*.

The paper is organized as follows. In Section 2 we present both algorithms for the special case of supertournaments. In Sections 3 we prove some preliminary results that are necessary for describing and proving the correctness of the algorithms for the general case, and in Section 4 we present the algorithms for digraphs with bounded independence numbers. We note here that the algorithms for supertournaments are fixed-parameter tractable (that is, the running time is  $f(H)n^c$  for some

function  $f$  and some constant  $c$  not depending on  $H$  or  $n$ , where  $n$  is the size of the input digraph) whereas the ones for general bounded independence number are not (and have complexity  $O(n^{f(H)})$ ).

## 2 Supertournaments

In this section we describe polynomial-time algorithms to solve  $\mathbf{H}$ -infusion and  $H$ -immersion in supertournaments. Note that supertournaments are precisely the digraphs with independence number at most one. Although the algorithms of this section are special cases of the algorithms presented in Section 4, they are easier to describe and we believe that they provide a good stepping stone towards understanding the more general algorithms. Also, as mentioned in the introduction, the algorithms for supertournaments are fixed-parameter tractable, whereas the ones for digraphs with bounded independence number are not.

We begin with some definitions. Let  $\theta \geq 0$  be an integer. An enumeration  $(v_1, \dots, v_n)$  of the vertex set of a digraph has *cutwidth* at most  $\theta$  if for all  $j \in \{2, \dots, n\}$ , there are at most  $\theta$  edges  $uv$  such that  $u \in \{v_1, \dots, v_{j-1}\}$  and  $v \in \{v_j, \dots, v_n\}$ ; and a digraph has *cutwidth* at most  $\theta$  if there is an enumeration of its vertex set with cutwidth at most  $\theta$ .

Let  $P$  and  $Q$  be two directed paths of a digraph  $G$  such that  $P$  is from  $u$  to  $v$  and  $Q$  is from  $u'$  to  $v'$ . Then  $P$  and  $Q$  are *internally-disjoint* if  $V(P) \setminus \{u, v\}$  is disjoint from  $V(Q)$  and  $V(Q) \setminus \{u', v'\}$  is disjoint from  $V(P)$ . If  $P$  is a three-edge directed path in  $G$  we call the second edge of  $P$  its *body* and the other two edges its *legs*.

Let  $G$  be a digraph, let  $i \geq 0$  be an integer, let  $m \geq 0$  be real, and let  $u, v \in V(G)$  be distinct. We say that  $(u, v)$  is an  *$m$ -pair of type  $i$*  if there are at least  $m$  internally-disjoint directed paths from  $u$  to  $v$  in  $G$ , all with exactly  $i$  internal vertices (such a set of  $m$  paths is called an  *$m$ -realization of type  $i$  for  $(u, v)$* ). We say that  $(u, v)$  is an  *$m$ -useful pair* if  $(u, v)$  is an  $m$ -pair of type 1 and  $(v, u)$  is an  $m$ -pair of type 1 or 2.

Here are two results from [2] that relate the existence of  $m$ -useful pairs to the cutwidth of a semi-complete digraph:

**2.1** *Let  $T$  be a semi-complete digraph and let  $m \geq 0$  be an integer. Suppose that no pair of vertices of  $T$  is an  $m$ -useful pair. Then the cutwidth of  $T$  is at most  $72m^2 + 8m$ .*

**2.2** *There is an algorithm with running time  $O(n^4)$  which, given as input a semi-complete digraph  $T$  with  $n$  vertices and an integer  $m \geq 0$ , outputs an  $m$ -useful pair, if one exists, and otherwise outputs an enumeration of  $V(T)$  with cutwidth at most  $72m^2 + 8m$ .*

Note that 2.2 takes only semi-complete digraphs as input. To apply it to supertournaments, we need to introduce the following concept; for a digraph  $G$ , we say that a subdigraph  $S$  is the *underlying simple digraph* of  $G$  if  $S$  is simple,  $V(G) = V(S)$ , and there is an edge with tail  $u$  and head  $v$  in  $S$  if and only if  $u \neq v$  and there is an edge with tail  $u$  and head  $v$  in  $G$ . Note that a digraph and its underlying simple digraph have the same independence number. If  $G$  is a supertournament we say that  $S$  is the *underlying semi-complete digraph* of  $G$ .

Until 2.10,  $\mathbf{H} = (H, u_1, \dots, u_h)$  is a fixed distinctly-rooted digraph with  $|E(H)| = k$ ; and the goal of this section is to give a polynomial-time algorithm to test whether there is an  $\mathbf{H}$ -infusion in an input  $\mathbf{T} = (T, s_1, \dots, s_h)$  where  $T$  is a supertournament. The idea is: given as input a rooted digraph

$\mathbf{T} = (T, s_1, \dots, s_h)$ , where  $T$  is a supertournament, run the algorithm of 2.2 on its underlying semi-complete digraph with  $m = 4k$ . If the output is an  $m$ -useful pair  $(u, v)$ , identify  $u$  and  $v$  in  $T$  and run the algorithm of 2.2 on the underlying semi-complete digraph of the resulting supertournament. Repeat this process until 2.2 returns an enumeration of a semi-complete digraph  $S$  with cutwidth at most  $72m^2 + 8m$ . Then  $S$  is the underlying semi-complete digraph of a supertournament  $T'$  which can be obtained from  $T$  by successively identifying  $m$ -useful pairs. We can now use the outputted enumeration of  $V(S) = V(T')$  to solve  $\mathbf{H}$ -infusion using dynamic programming.

We need to show two things: that identifying the vertices of an  $m$ -useful pair (for sufficiently large  $m$ ) does not change the answer to the algorithmic question; and to explain the dynamic programming. We begin with the first.

We say that rooted digraphs  $\mathbf{G} = (G, s_1, \dots, s_h)$  and  $\mathbf{G}' = (G', s_1, \dots, s_h)$  are  $\mathbf{H}$ -infusion-equivalent if  $\mathbf{G}$  contains an infusion of  $\mathbf{H}$  if and only if  $\mathbf{G}'$  does. We now prove some results regarding when certain digraphs are  $\mathbf{H}$ -infusion-equivalent.

Let  $\mathbf{G} = (G, v_1, \dots, v_m)$  be a rooted digraph and let  $x, y \in V(G)$ . We say that  $\mathbf{G}' = (G', v'_1, \dots, v'_m)$  is obtained from  $\mathbf{G}$  by *identifying*  $x$  and  $y$  (into a new vertex  $z$ ) if the following conditions hold:

- $V(G') = (V(G) \setminus \{x, y\}) \cup \{z\}$
- for all  $u, v \in V(G')$  with  $u, v \neq z$ ,  $E_{uv}^{G'} = E_{uv}^G$
- for all  $u \in V(G')$  with  $u \neq z$ ,  $E_{uz}^{G'} = E_{ux}^G \cup E_{uy}^G$  and  $E_{zu}^{G'} = E_{xu}^G \cup E_{yu}^G$
- $E_{zz} = E_{xy} \cup E_{yx} \cup E_{xx} \cup E_{yy}$
- for  $1 \leq i \leq h$ , if  $v_i \neq x, y$  then  $v'_i = v_i$  and if  $v_i = x$  or  $v_i = y$  then  $v'_i = z$ .

We will show the following:

**2.3** *Let  $\mathbf{G} = (G, s_1, \dots, s_h)$  be a rooted digraph and let  $(u, v)$  be a  $4k$ -useful pair in  $G$ . If  $\mathbf{G}'$  is obtained from  $G$  by identifying the vertices  $u$  and  $v$ , then  $\mathbf{G}$  and  $\mathbf{G}'$  are  $\mathbf{H}$ -infusion-equivalent.*

For the proof we need two lemmas. If  $\eta$  is an infusion of  $H$  in  $G$ , we define

$$E_\eta = \bigcup_{e \in E(H)} E(\eta(e)).$$

**2.4** *Let  $\mathbf{G} = (G, s_1, \dots, s_h)$  be a rooted digraph and let  $(u, v)$  be a  $2k$ -pair of type 1 in  $G$ . Let  $G'$  be obtained from  $G$  by adding some edges from  $u$  to  $v$ . Then  $\mathbf{G}$  and  $\mathbf{G}' = (G', s_1, \dots, s_h)$  are  $\mathbf{H}$ -infusion-equivalent.*

**Proof.** Since  $G'$  is obtained from  $G$  by adding edges, it follows immediately that if  $\mathbf{G}$  contains an infusion of  $\mathbf{H}$  then so does  $\mathbf{G}'$ .

For the converse, suppose that  $\mathbf{G}'$  contains an infusion of  $\mathbf{H}$  and let  $\eta$  be an infusion of  $\mathbf{H}$  in  $\mathbf{G}'$  such that  $E_\eta$  contains as few edges from  $u$  to  $v$  as possible. If  $E_\eta$  contains no edges from  $u$  to  $v$  then we are done, so we may assume that for some  $e \in E(H)$  with  $e = xy$ ,  $\eta(e)$  contains an edge  $f$  from  $u$  to  $v$ . We want to show that there exists a route from  $\eta(x)$  to  $\eta(y)$  that does not use any edges from  $u$  to  $v$  and is edge-disjoint from  $\eta(g)$  for all  $g \in E(H) \setminus \{e\}$ .

Let  $A$  be a set of  $2k$  vertices such that  $u$  is adjacent to every vertex in  $A$  and  $v$  is adjacent from every vertex in  $A$  (such a set exists because  $(u, v)$  is a  $2k$ -pair of type 1). Since a route can use at most one edge with tail  $u$  and at most one edge with head  $v$ , it follows that there exists  $a \in A$  such that none of the routes  $\eta(g)$ , with  $g \in E(H) \setminus \{e\}$ , uses an edge of  $E_{ua} \cup E_{av}$ . Let  $\eta'(e)$  be obtained from  $\eta(e)$  by replacing  $f$  with a path  $u$ - $a$ - $v$  (such a path exists because both  $E_{ua}$  and  $E_{av}$  are non-empty by the definition of  $A$ ). Define  $\eta'(z) = \eta(z)$  for all  $z \in V(H)$  and  $\eta'(g) = \eta(g)$  for all  $g \in E(H) \setminus \{e\}$ . Then  $\eta'$  is an infusion of  $\mathbf{H}$  in  $\mathbf{G}'$  and  $E_{\eta'}$  contains fewer edges from  $u$  to  $v$  than  $E_{\eta}$ , contrary to our choice of  $\eta$ . This proves 2.4.  $\blacksquare$

The flavor of the following result is similar to that of 2.4, but the result is much stronger and the proof is more involved.

**2.5** *Let  $\mathbf{G} = (G, s_1, \dots, s_h)$  be a rooted digraph and let  $(u, v)$  be a  $4k$ -useful pair in  $G$ . Let  $G'$  be obtained from  $G$  by adding some edges from  $u$  to  $v$  or some edges from  $v$  to  $u$  (possibly both). Then  $\mathbf{G}$  and  $\mathbf{G}' = (G', s_1, \dots, s_h)$  are  $\mathbf{H}$ -infusion-equivalent.*

**Proof.** If both  $(u, v)$  and  $(v, u)$  are  $2k$ -pairs of type 1, then the result follows immediately from 2.4. So we may assume that  $(u, v)$  is a  $4k$ -pair of type 1 and  $(v, u)$  is a  $4k$ -pair of type 2. Furthermore, by 2.4 we may assume that there are  $k$  edges from  $u$  to  $v$  in both  $G$  and  $G'$ . Once again, it is clear that if  $\mathbf{G}$  contains an infusion of  $\mathbf{H}$  then so does  $\mathbf{G}'$ , so we just need to prove the converse.

Let  $\mathcal{P}$  be a  $4k$ -realization for  $(v, u)$  of type 2, and let  $M$  be the set of bodies of the elements of  $\mathcal{P}$ . Suppose that  $\mathbf{G}'$  contains an infusion of  $\mathbf{H}$  and let  $\eta$  be such an infusion such that  $E_{\eta}$  contains as few edges from  $v$  to  $u$  as possible, and subject to that  $M \cap E_{\eta}$  is minimal. Let

$$M' = \{e = ab : e \in M \text{ and } (E_{va} \cup E_{bu}) \cap E_{\eta} = \emptyset\}.$$

Since a route can use at most one edge with tail  $v$  and at most one edge with head  $u$ , and since  $(v, u)$  is a  $4k$ -pair of type 2, it follows that  $|M'| \geq 2k$ .

(1) *For all  $g \in E(H)$ , the route  $\eta(g)$  uses at most two edges of  $M'$ .*

Suppose that for some  $g \in E(H)$  with  $g = rs$ ,  $\eta(g)$  uses at least three edges of  $M'$ . Let  $e_1 = a_1b_1, e_2 = a_2b_2$  and  $e_3 = a_3b_3$  be three edges of  $M'$  that are used by  $\eta(g)$  in the specified order. From the definition of  $M'$  it follows that  $(E_{va_i} \cup E_{b_iu}) \cap E_{\eta} = \emptyset$  for  $1 \leq i \leq 3$ . Since there are  $k$  edges from  $u$  to  $v$  in  $G'$ , there exists  $f = uv$  such that  $f \notin E_{\eta} \setminus E(\eta(g))$ . Let  $W$  be obtained from  $\eta(g)$  by replacing the part of the route from  $b_1$  to  $a_3$  by a path  $b_1$ - $u$ - $v$ - $a_3$  (where the middle edge of the path is  $f$  and the other two are any members of  $E_{b_1u}$  and  $E_{va_3}$ , respectively). Then  $W$  is a walk. Let  $Q$  be a route from  $\eta(r)$  to  $\eta(s)$  contained in  $W$ . Then  $Q$  is disjoint from  $E_{\eta} \setminus E(\eta(g))$ . Define  $\eta'$  as follows. Let  $\eta'(z) = \eta(z)$  for all  $z \in V(H)$  and let  $\eta'(e) = \eta(e)$  for all  $e \in E(H) \setminus \{g\}$ ; let  $\eta'(g) = Q$ . Then  $M \cap E_{\eta'} \subset M \cap E_{\eta}$ , contrary to our choice of  $\eta$ . This proves (1).

(2) *For all  $g \in E(H)$ , the route  $\eta(g)$  does not use any edges from  $v$  to  $u$ .*

Suppose that for some  $e \in E(H)$  with  $e = xy$ ,  $\eta(e)$  contains an edge  $f$  from  $v$  to  $u$ . Since  $|M'| \geq 2k$ , it follows from (1) that there is an edge  $g = ab \in M'$  such that  $g \notin E_{\eta} \setminus E(\eta(e))$ . Let  $W$  be obtained from  $\eta(e)$  by replacing  $f$  by a path  $v$ - $a$ - $b$ - $u$  (where the middle edge of the path is  $g$  and the two other

edges are members of  $E_{va}$  and  $E_{bu}$  respectively). Then once again  $W$  is a walk; let  $P$  be a route from  $\eta(x)$  to  $\eta(y)$  contained in  $W$ . Then  $P$  does not use any edges from  $v$  to  $u$  and is disjoint from  $E_\eta \setminus E(\eta(e))$ . Define  $\eta'$  as follows. Let  $\eta'(z) = \eta(z)$  for all  $z \in V(H)$ ; let  $\eta'(e) = P$  and  $\eta'(g) = \eta(g)$  for all  $g \in E(H) \setminus \{e\}$ . Then  $\eta'$  is an infusion of  $\mathbf{H}$  in  $\mathbf{G}'$  and  $M \cap E_{\eta'} \subset M \cap E_\eta$ , contrary to our choice of  $\eta$ . This proves (2).

It follows from (2) that  $\eta$  is an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ . Hence,  $G$  and  $G'$  are  $\mathbf{H}$ -infusion-equivalent, as desired. This proves 2.5. ■

**Proof of 2.3.** First we show that if  $\mathbf{G}$  contains an infusion of  $\mathbf{H}$  then so does  $\mathbf{G}'$ . Let  $\eta$  be an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ . Let  $w \in V(G')$  be the vertex resulting from the identification of  $u$  and  $v$ . Define  $\eta'$  as follows. For  $x \in V(H)$ , if  $\eta(x) \neq u$  or  $v$ , then  $\eta'(x) = \eta(x)$  and if  $\eta(x) \in \{u, v\}$  then  $\eta'(x) = w$ . Next, for  $e \in E(H)$  with  $e = xy$ ,  $\eta(e)$  naturally corresponds to a walk in  $G'$  (since all edges of  $G$  are edges of  $G'$ ). This walk contains a route  $P$  from  $\eta'(x)$  to  $\eta'(y)$ ; we let  $\eta'(e) = P$ . Now  $\eta'$  is an infusion of  $\mathbf{H}$  in  $\mathbf{G}'$ .

Conversely, suppose that  $\mathbf{G}'$  contains an infusion of  $\mathbf{H}$ . By 2.5, we may assume that there are  $k$  edges from  $u$  to  $v$  and  $k$  edges from  $v$  to  $u$  in  $G$ . Let  $\eta'$  be an infusion of  $\mathbf{H}$  in  $\mathbf{G}'$ . Define  $\eta$  as follows. For  $1 \leq i \leq h$ ,  $\eta(u_i) = s_i$ . For  $x \in V(H) \setminus \{u_1, \dots, u_h\}$ , if  $\eta'(x) \neq w$ , then  $\eta(x) = \eta'(x)$ , and if  $\eta'(x) = w$  then  $\eta(x) = u$ . For  $e \in E(H)$  with  $e = xy$ ,  $\eta'(e)$  naturally corresponds to a route from  $\eta'(x)$  to  $\eta'(y)$  in  $G$  (after possibly adding an unused  $uv$  or  $vu$  edge); let  $\eta(e)$  be that route. Now  $\eta$  is an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ . This proves 2.3. ■

Before we can explain the dynamic programming in more detail, we need the following two results. The idea for them is due to Daniel Marx and greatly simplifies our dynamic programming as well as making it fixed-parameter tractable.

**2.6** *Let  $T$  be a supertournament and let  $(v_1, \dots, v_n)$  be an enumeration of  $V(T)$  with cutwidth  $\leq \theta$ . Let  $i, j$  be integers such that  $0 \leq i < j \leq n$  and  $j - i > 2\theta + 2k$ ; then  $(v_j, v_i)$  is a  $2k$ -pair of type 1.*

**Proof.** Let  $S = \{v_{i+1}, \dots, v_{j-1}\}$ ; then  $|S| \geq 2\theta + 2k$ . Since the enumeration has cutwidth  $\leq \theta$ , it follows that  $v_i$  is adjacent to at most  $\theta$  members of  $S$  and  $v_j$  is adjacent from at most  $\theta$  members of  $S$ . So there are at least  $2k$  members of  $S$  that are adjacent to  $v_i$  and adjacent from  $v_j$  and hence  $(v_j, v_i)$  is a  $2k$ -pair of type 1. This proves 2.6. ■

**2.7** *Let  $T$  be a supertournament and let  $(v_1, \dots, v_n)$  be an enumeration of  $V(T)$  with cutwidth  $\leq \theta$ . Let  $T'$  be a supertournament obtained from  $T$  by the following two operations:*

- for  $1 \leq i, j \leq n$ , if  $j - i > 2\theta + 2k$ , then add or remove  $v_j v_i$  edges so that there are  $k$  of them
- for  $u, v \in V(T)$ , if there are more than  $k$  edges from  $u$  to  $v$  in  $T$ , remove  $uv$  edges so that there are  $k$  of them.

*Then  $T$  is  $\mathbf{H}$ -infusion-equivalent to  $T'$  and the cutwidth of  $T'$  is  $\leq \theta$ .*

**Proof.** This follows from 2.6 and the fact that for all  $u, v \in V(T)$  a route of  $T$  can use at most one edge from  $u$  to  $v$ . ■



Let  $T, T'$  and  $(v_1, \dots, v_n)$  be as in the statement of 2.7. We say that  $T'$  is the  $k$ -smoothing of  $T$  with respect to  $(v_1, \dots, v_n)$  and that  $T'$  is  $k$ -smoothed with respect to  $(v_1, \dots, v_n)$ .

Throughout what follows  $\theta \geq 0$  is a fixed integer. We will describe an algorithm to test whether an input rooted supertournament  $\mathbf{T} = (T, s_1, \dots, s_h)$ , with an enumeration  $(v_1, \dots, v_n)$  of cutwidth at most  $\theta$  contains an infusion of  $\mathbf{H}$ . From 2.7 we may assume that  $T$  is  $k$ -smoothed with respect to  $(v_1, \dots, v_n)$ . Let  $S = \{s_1, \dots, s_h\}$ .

Before we describe the algorithm, we need to prove a few preliminary results. They are modifications of some results found in [2]. Let  $b = 2\theta + 2k$ . For  $b \leq i \leq n$  let  $Q_i = \{v_1, \dots, v_{i-b}\}$ ,  $R_i = \{v_{i-b+1}, \dots, v_i\}$ , and  $T_i = \{v_{i+1}, \dots, v_n\}$ .

Let  $J$  be a digraph (not necessarily a subdigraph of  $T$ ) such that

- $R_i \cup T_i \cup S \subseteq V(J) \subseteq V(T)$
- for all  $u, v \in V(J)$  with not both  $u, v \in Q_i \cup R_i$ , the number of edges from  $u$  to  $v$  in  $J$  is equal to the number of edges from  $u$  to  $v$  in  $T$
- for all distinct  $u, v \in V(J) \cap (Q_i \cup R_i)$ , there are at most  $k$  edges of  $J$  from  $u$  to  $v$ , and for each  $v \in V(J) \cap (Q_i \cup R_i)$  there are at most  $k$  loops incident with  $v$ .

We say that the rooted digraph  $\mathbf{J} = (J, s_1, \dots, s_h)$  is an  $i$ -extension (relative to  $T, s_1, \dots, s_h$ , and the enumeration  $(v_1, \dots, v_n)$ ). Let  $\mathcal{C}_i^{p,q}$  denote the set of all pairs  $(\mathbf{J}, X)$  such that

- $\mathbf{J}$  is an  $i$ -extension,  $|V(J) \cap Q_i| \leq p$ , and  $J \setminus T_i$  has at most  $q$  edges
- $X \subseteq V(J) \cap Q_i$
- there is an infusion  $\eta$  of  $\mathbf{H}$  in  $\mathbf{J}$  such that  $X = \eta(V(H)) \cap Q_i$ , where  $\eta(V(H))$  denotes  $\{\eta(v) : v \in V(H)\}$ .

**2.8** *Let  $b \leq i \leq n$  and let  $\mathbf{T}, S, \theta, Q_i, R_i$ , and  $T_i$  be as above. Let  $p \geq |V(H)| + k + 2\theta + h + 1$ , and let  $\mathbf{J}$  be an  $i$ -extension such that  $|V(J) \cap Q_i| \leq p$  and  $J \setminus T_i$  has at most  $q$  edges. Let  $X \subseteq V(J) \cap Q_i$ . Then  $(\mathbf{J}, X) \in \mathcal{C}_i^{p,q}$  if and only if either*

- $(\mathbf{J}, X) \in \mathcal{C}_i^{p-1,q}$ , or
- there exists  $v \in V(J) \cap (Q_i \setminus (S \cup X))$  such that  $(\mathbf{J} \setminus v, X) \in \mathcal{C}_i^{p-1,q}$ , or
- there exists  $e \in E(J)$  with both ends in  $Q_i \cup R_i$ , such that  $(\mathbf{J} \setminus e, X) \in \mathcal{C}_i^{p,q-1}$ , or
- there exist vertices  $v \in Q_i$  and  $u, w \in Q_i \cup R_i$ , with  $u, w \neq v$  and  $v \notin X$ , and edges  $e = uv$  and  $f = vw$  of  $J$ , such that  $(\mathbf{J}', X) \in \mathcal{C}_i^{p,q-1}$ , where  $\mathbf{J}' = (J', s_1, \dots, s_h)$  and  $J'$  denotes the digraph obtained from  $J$  by deleting  $e$  and  $f$  and adding a new edge from  $u$  to  $w$ .

**Proof.** The “if” part is clear, and holds for all  $p$ . For “only if”, suppose that  $\mathbf{J} \in \mathcal{C}_i^{p,q}$ , and let  $\eta$  be an infusion of  $\mathbf{H}$  in  $\mathbf{J}$  such that  $X = \eta(V(H)) \cap Q_i$ . Let  $K$  be the minimal subdigraph of  $J$  such that  $\eta$  is an infusion of  $\mathbf{H}$  in  $\mathbf{K} = (K, s_1, \dots, s_h)$ ; thus,  $K$  is formed by the union of  $S$  and the vertices  $\eta(v)$  ( $v \in V(H)$ ) and all the subdigraphs  $\eta(e)$  ( $e \in E(H)$ ). It follows that every vertex  $u \in K$  has outdegree at most  $k$  in  $K$ , since each  $\eta(e)$  ( $e \in E(H)$ ) uses at most one edge with tail  $u$ .

Let  $F, F' \subseteq E(J)$  be the set of edges of  $J$  from  $Q_i \cup R_i$  to  $T_i$ , and from  $T_i$  to  $Q_i \cup R_i$ , respectively. For each edge  $e$  of  $H$ ,  $\eta(e)$  is a route, and between any two members of  $F'$  in  $\eta(e)$  there is a member of  $F$ , and consequently  $|E(\eta(e)) \cap F'| \leq |E(\eta(e)) \cap F| + 1$ . Since  $|F| \leq \theta$ , by summing over all  $e \in E(H)$  we deduce that  $|E(K) \cap F'| \leq \theta + k$ . Consequently there are at most  $2\theta + k$  vertices in  $Q_i$  that are adjacent in  $K$  to or from a vertex in  $T_i$ . Moreover  $|\eta(V(H)) \cap Q_i| \leq |V(H)|$ . Now we may assume that  $|Q_i \cap V(J)| = p$ , for otherwise the first assertion of the theorem holds. Thus

$$|Q_i \cap V(J)| = p > |V(H)| + k + 2\theta + h,$$

and it follows that there exists  $v \in (Q_i \cap V(J)) \setminus S$  such that  $v$  is not adjacent in  $K$  to or from any member of  $V(T_i)$ , and  $v \notin \eta(V(H)) \cup S$ . Thus  $v \notin S \cup X$ . Now we may assume that  $v \in V(K)$ , since otherwise the second assertion of the theorem holds. From the minimality of  $K$  it follows that there is an edge  $g = ab \in E(H)$  such that  $v$  belongs to  $\eta(g)$ ; and  $v \neq \eta(a), \eta(b)$  from our choice of  $v$ . Let  $u, w$  be the vertices of  $\eta(g)$  such that  $e = uv$  and  $f = vw$  are edges of  $\eta(g)$ ; then  $u, w \in Q_i \cup R_i$ , since  $v$  is not adjacent in  $K$  to or from any member of  $V(T_i)$ . Let  $J'$  be obtained from  $J$  by deleting  $e, f$  and adding a new edge from  $u$  to  $w$ ; then there is an infusion  $\eta'$  of  $\mathbf{H}$  in  $\mathbf{J}' = (J', s_1, \dots, s_h)$  with  $X = \eta'(V(H)) \cap Q_i$ . If  $\mathbf{J}'$  is an  $i$ -extension then the fourth assertion of the theorem holds, so we may assume not; and so there are more than  $k$  edges of  $J'$  from  $u$  to  $w$ . Consequently there are more than  $k$  edges of  $J$  with tail  $u$  and head in  $Q_i \cup R_i$  (namely, at least  $k$  with head  $w$ , and one with head  $v$ ). Since  $u$  has outdegree at most  $k$  in  $K$ , as we saw earlier, it follows that there is an edge of  $J$  with tail  $u$  and head in  $Q_i \cup R_i$  that is not an edge of  $K$ . But then the third statement of the theorem holds. This proves 2.8.  $\blacksquare$

Let  $\mathcal{C}_i^p$  be the union of the sets  $\mathcal{C}_i^{p,q}$  over all  $q \in \{0, 1, \dots, (p+b)^2k\}$ . (Note that if  $J$  is an  $i$ -extension then  $J \setminus T_i$  has at most  $(p+b)^2k$  edges.) Two members  $(\mathbf{J}, X), (\mathbf{J}', X') \in \mathcal{C}_i^p$  are *equivalent* if there is an isomorphism between  $J$  and  $J'$  taking  $X$  to  $X'$  and fixing each of  $v_{i-b+1}, \dots, v_n, s_1, \dots, s_h$ .

We claim that for every integer  $p$ , the number of equivalence classes of members of  $\mathcal{C}_i^p$  is bounded above by a function of  $p, H$ , and  $\theta$ , independent of  $i$ . Let  $W_0$  be the set of vertices in  $Q_i$  that are adjacent in  $T$  to some vertex in  $R_i \cup T_i$ . Let  $W = W_0 \cup R_i \cup (\{s_1, \dots, s_h\} \cap Q_i)$ . Since  $|W_0| \leq \theta$ , and  $|R_i| = b = 2\theta + 2k$ , and  $(\{s_1, \dots, s_h\} \cap Q_i) \leq h$ , it follows that  $|W| \leq 3\theta + 2k + h$ . Now let  $(\mathbf{J}, X) \in \mathcal{C}_i^p$ ; and take an enumeration  $\mu$  of  $V(J) \setminus T_i$ . There are no edges of  $J$  from  $Q_i \setminus W$  to  $T_i$ . Moreover, since  $T$  is  $k$ -smoothed, 2.6 implies that every vertex in  $Q_i \setminus W$  is joined by  $k$  edges of  $J$  from every vertex of  $T_i$ . Consequently, to specify the equivalence class of  $(\mathbf{J}, X)$ , it suffices

- to specify the isomorphism type of the labelled digraph  $(J \setminus T_i, \mu)$  (since  $J \setminus T_i$  has at most  $p + b$  vertices and at most  $(p + b)^2k$  edges, the number of possibilities here is bounded by a function of  $p, H$ ),
- to specify whether the  $j$ th term of  $\mu$  is equal to  $w$ , for  $1 \leq j \leq |V(J) \setminus T_i|$  and each  $w \in W$  (there are at most  $(|W| + 1)^{p+b}$  possibilities here), and
- to specify whether the  $j$ th term of  $\mu$  belongs to  $X$  for  $1 \leq j \leq |V(J) \setminus T_i|$  ( $2^{p+b}$  possibilities).

This proves our claim that the number of equivalence classes of members of  $\mathcal{C}_i^p$  is bounded above by a function of  $p, H$ , and  $\theta$ .

Since the set  $\mathcal{C}_i^p$  is a union of some of these equivalence classes, we handle this set in the algorithms that follow by listing its equivalence classes. For simplicity we speak of “a knowledge of  $\mathcal{C}_i^p$ ” when

what we mean is “a knowledge of the equivalence classes of  $i$ -extensions that have union  $\mathcal{C}_i^{p''}$ ”, and so on.

**2.9** *Let  $b \leq i < n$ . Let  $p = |V(H)| + k + 2\theta + h$ . Then  $\mathcal{C}_i^p$  can be computed from a knowledge of  $\mathcal{C}_{i+1}^p$  in time that depends only on  $\theta$  and  $H$ .*

**Proof.** Starting from a knowledge of  $\mathcal{C}_{i+1}^p$ , we shall first compute  $\mathcal{C}_{i+1}^{p+1}$ , and then use this to compute  $\mathcal{C}_i^p$ , as follows.

**To compute  $\mathcal{C}_{i+1}^{p+1}$  from a knowledge of  $\mathcal{C}_{i+1}^p$ .** From 2.8 we can compute  $\mathcal{C}_{i+1}^{p+1,0}$  from a knowledge of  $\mathcal{C}_{i+1}^{p,0}$  and for  $q \geq 1$  we can compute  $\mathcal{C}_{i+1}^{p+1,q}$  from a knowledge of  $\mathcal{C}_{i+1}^p$  and of  $\mathcal{C}_{i+1}^{p+1,q-1}$ , all in time that depends only on  $\theta$  and  $\mathbf{H}$ ; and by repeating for  $q = 1, \dots, (b+p)^2k$  we compute  $\mathcal{C}_{i+1}^{p+1}$ .

**To compute  $\mathcal{C}_i^p$  from a knowledge of  $\mathcal{C}_{i+1}^{p+1}$ .** Let  $\mathbf{J} = (J, s_1, \dots, s_h)$  be an  $i$ -extension such that  $J \setminus (T_i \cup R_i)$  has at most  $p$  vertices, and let  $X \subseteq V(J) \setminus (T_i \cup R_i)$ . We need to determine whether  $(\mathbf{J}, X) \in \mathcal{C}_i^p$ . But  $\mathbf{J}$  is an  $(i+1)$ -extension, and therefore  $(\mathbf{J}, X) \in \mathcal{C}_i^p$  if and only if  $(\mathbf{J}, X) \in \mathcal{C}_{i+1}^{p+1}$  or  $(\mathbf{J}, X \cup \{v_{i-b+1}\}) \in \mathcal{C}_{i+1}^{p+1}$ .

This proves 2.9. ■

We are now ready to give the algorithm.

**2.10** *For each distinctly-rooted digraph  $\mathbf{H} = (H, u_1, \dots, u_h)$  with  $|E(H)| = k$  and each integer  $\theta$ , there is an algorithm with running time  $O(n)$  which, given as input a  $k$ -smoothed rooted supertournament  $\mathbf{T} = (T, s_1, \dots, s_h)$  with  $|V(T)| = n$ , and an enumeration  $(v_1, \dots, v_n)$  of  $V(T)$  with cutwidth at most  $\theta$ , outputs whether there is an infusion  $\eta$  of  $\mathbf{H}$  in  $\mathbf{T}$ .*

**Proof.** Let  $p = |V(H)| + k + 2\theta + h$ . Now  $\mathcal{C}_n^p$  can be computed in constant time, since all  $n$ -extensions have at most  $2k + 2\theta + p$  vertices and at most  $2(p+b)^2k$  edges, where  $b = 2k + 2\theta$  (so we just check them all, up to equivalence). By  $n$  applications of 2.9, we can determine  $\mathcal{C}_0^p$  in time  $O(n)$ . But the only 0-extension is  $\mathbf{T}$  itself (because of the condition that  $V(J) \subseteq V(T)$  for  $i$ -extensions), and so there is an infusion  $\eta$  of  $\mathbf{H}$  in  $\mathbf{T}$  if and only if  $\mathcal{C}_0^p \neq \emptyset$ . This proves 2.10. ■

Notice that this can easily be modified to do strong or weak immersion (just change infusion to strong (or weak) immersion in the definition of  $\mathcal{C}_i^{p,q}$  above); and it can do strong or weak infusion, with the natural definitions. The algorithm can also be modified to output an infusion if one exists, rather than just a yes/no answer (for each equivalence class in  $\mathcal{C}_i^{p,q}$ , we store one member, and a corresponding infusion of  $H$ ). We omit these details.

Finally, as claimed at the start of this section, we have:

**2.11** *For every distinctly-rooted digraph  $\mathbf{H} = (H, u_1, \dots, u_h)$  with  $|E(H)| = k$ , there is an algorithm with running time  $O(n^5)$  which, given a rooted supertournament  $\mathbf{T} = (T, s_1, \dots, s_h)$  with  $|V(T)| = n$ , outputs whether there is an infusion of  $\mathbf{H}$  in  $\mathbf{T}$ .*

**Proof.** The algorithm consists of at most  $n$  steps, each of which takes time at most  $O(n^4)$ . Let  $T_0 = T$ .

At Step  $i$ : Let  $Q_i$  be the underlying semi-complete digraph of  $T_i$ . Run 2.2 on  $Q_i$  with  $m = 4k$ .

- If the output is a  $4k$ -useful pair  $(u, v)$ , obtain  $T_{i+1}$  by identifying  $u$  and  $v$  in  $T_i$  (by 2.3, this doesn't change the outcome). Go to Step  $i + 1$ .
- If the output is an enumeration of  $V(Q_i)$  with cutwidth at most  $72m^2 + 8m$ , run the algorithm of 2.10 on the  $k$ -smoothing of  $T_i$  with this enumeration.

Notice that an enumeration of  $V(Q_i)$  with cutwidth  $\leq \theta$  is also an enumeration of the  $k$ -smoothing of  $T_i$  (with respect to the same enumeration) with cutwidth  $\leq k\theta$ . This proves 2.11.  $\blacksquare$

Once again, this can be modified to output the infusion if one exists.

The algorithm for solving  $H$ -immersion uses some of the same tools but is simpler as it has only one step and does not require any identifications of vertices. First, we need the following.

**2.12** *Let  $H$  be a digraph with  $|V(H)| = t$  and let  $T$  be a supertournament. Let  $m = 2^{t(t+2)}$  and suppose that  $T$  has an  $m$ -useful pair. Then  $T$  contains a strong immersion of  $H$ .*

An analogous result for semi-complete digraphs appears in [2] and the proof is identical. Therefore, we omit it here. Now we are ready to give the algorithm.

**2.13** *For every digraph  $H$ , there is an algorithm with running time  $O(n^4)$ , which, with input a supertournament  $T$  with  $n$  vertices, outputs whether there is a strong or weak immersion of  $H$  in  $T$ .*

**Proof.** Let  $Q$  be the underlying semi-complete digraph of  $T$ . Run 2.2 on  $Q$  with  $m = 2^{t(t+2)}$ , where  $t = |V(H)|$ . If the output is an  $m$ -useful pair of vertices, return yes, an immersion exists, by 2.12. Otherwise we obtain an enumeration of  $V(T) = V(Q)$  with cutwidth at most  $72m^2 + 8m$  in  $Q$ . Consequently this enumeration has cutwidth at most  $k(72m^2 + 8m)$  in the  $k$ -smoothing of  $T$ . Run the algorithm of 2.10 (with infusion replaced by immersion) on the  $k$ -smoothing of  $T$  with respect to this enumeration. This proves 2.13.  $\blacksquare$

As with infusion, this can be modified to output the immersion if one exists.

### 3 Useful sequences and cutwidth

The goal of the next two sections is to develop algorithms, analogous to those of Section 2, for digraphs with bounded independence number. The main result of this section is a key lemma that allows us to prove the correctness of our algorithms. Before its statement we need a few more definitions.

First, we generalize the concept of  $m$ -useful pairs to that of  $m$ -useful sequences. Let  $T$  be a digraph, let  $m \geq 0$  be real, let  $t \geq 2$  be an integer, and let  $u_1, \dots, u_t \in V(T)$  be distinct. We say that  $(u_1, \dots, u_t)$  is a *partial  $m$ -useful sequence* if  $(u_i, u_{i+1})$  is an  $m$ -pair of type 1 or 2 for all  $1 \leq i \leq t-1$ ; if in addition  $(u_t, u_1)$  is an  $m$ -pair of type 1 or 2 we say that  $(u_1, \dots, u_t)$  is an  *$m$ -useful sequence*. We say that  $(u_1, \dots, u_t)$  is an *almost  $m$ -useful sequence* if  $(u_i, u_{i+1})$  is an  $m$ -pair of type 1, 2, or 3 for  $1 \leq i \leq t$  (where  $u_{t+1} = u_1$ ). In general, for all useful or almost-useful  $m$ -sequences, all subscripts are to be read modulo the length of the sequence.

If  $G$  is a digraph, we define  $\lambda(G)$  to be the maximum  $t$  such that some vertex of  $G$  belongs to  $t$  directed cycles that are pairwise edge-disjoint, and  $\mu(G)$  the maximum  $t$  such that some vertex of  $G$  belongs to  $t$  directed cycles that are otherwise pairwise vertex-disjoint. Two vertices  $u, v$  are *k-edge-connected* if there are  $k$  pairwise edge-disjoint directed paths from  $u$  to  $v$ , and  $k$  pairwise edge-disjoint directed paths from  $v$  to  $u$ . We say  $u, v$  are *strongly k-vertex-connected* if there are  $k$  directed paths from  $u$  to  $v$ , each with an internal vertex and pairwise vertex-disjoint except for  $u, v$ , and there are  $k$  directed paths from  $v$  to  $u$ , each with an internal vertex and pairwise vertex-disjoint except for  $u, v$ .

We are now ready to state the main result of this section.

**3.1** *For every set  $\mathcal{S}$  of simple digraphs with independence number at most  $\alpha$ , the following are equivalent:*

1. *There exists  $m$  such that every member of  $\mathcal{S}$  has cutwidth at most  $m$ .*
2. *There exists  $m$  such that  $\lambda(G) \leq m$  for every  $G \in \mathcal{S}$ .*
3. *There is a digraph  $H$  such that  $H$  cannot be infused in any member of  $\mathcal{S}$ .*
4. *There exists  $m$  such that for each  $G \in \mathcal{S}$ , no two vertices of  $G$  are  $m$ -edge-connected.*
5. *There exists  $m$  such that for each  $G \in \mathcal{S}$ , there do not exist  $m$  vertices of  $G$  that are pairwise  $m$ -edge-connected.*
6. *There is a digraph  $H$  such that  $H$  cannot be weakly immersed in any member of  $\mathcal{S}$ .*
7. *There is a digraph  $H$  such that  $H$  cannot be strongly immersed in any member of  $\mathcal{S}$ .*
8. *There exists  $m$  such that for each  $G \in \mathcal{S}$ ,  $G$  does not contain an  $m$ -useful sequence of length at most  $2\alpha + 2$ .*
9. *There exists  $m$  such that for each  $G \in \mathcal{S}$ ,  $G$  does not contain an  $m$ -useful sequence.*

In [2], we showed that for  $\alpha = 1$ , the following two statements are equivalent to the statements of 3.1:

- *There exists  $m$  such that  $\mu(G) \leq m$  for every  $G \in \mathcal{S}$ .*
- *There exists  $m$  such that for each  $G \in \mathcal{S}$ , no two vertices of  $G$  are strongly  $m$ -vertex-connected.*

However for larger values of  $\alpha$ , these two statements are *not* equivalent to the statements of 3.1. To see the non-equivalence, let  $T_1, T_2, T_3$  be transitive tournaments each with  $k$  vertices and let  $H$  be obtained from  $T_1 \cup T_2 \cup T_3$  by adding three vertices  $v_1, v_2, v_3$  and for  $1 \leq i \leq 3$  making  $v_i$  adjacent to all of  $V(T_i)$  and adjacent from all of  $V(T_{i-1})$  (where the subscript are modulo 3). Then  $\lambda(G) = k$  (the vertices  $v_1, v_2, v_3$  all belong to  $k$  edge-disjoint directed cycles) but  $\mu(G) = 1$ . Also, no two vertices of  $G$  are strongly 2-vertex-connected.

This section is dedicated to the proof of 3.1. That 3.1.4 implies 3.1.5 is trivial, and so is that 3.1.6 implies 3.1.7. To see that 3.1.5 implies 3.1.6, take  $H$  to be the digraph obtained from a directed cycle of length  $m$  by replacing each edge by  $m$  parallel edges. If  $H$  can be weakly immersed in  $G$ , then the  $m$  images of the vertices of  $H$  are pairwise  $m$ -edge-connected.

To see that 3.1.2 implies 3.1.3, let  $H$  be the digraph with one vertex and  $m$  loops; if  $H$  can be infused in  $G$  then the image of the vertex of  $H$  belongs to  $m$  directed cycles that are pairwise edge-disjoint. To see that 3.1.3 implies 3.1.2, note that if some digraph  $H$  cannot be infused in  $G$ , then  $\lambda(G) < |E(H)|$ , because otherwise we could find an infusion by mapping every vertex of  $H$  to the same vertex of  $G$ . Moreover, in [2] it was shown that 3.1.1 implies 3.1.2, and 3.1.2 implies 3.1.4. Thus we have shown so far that each of the first six statements of 3.1 implies the next. It remains to show that 3.1.7 implies 3.1.8, and 3.1.8 implies 3.1.9, and 3.1.9 implies 3.1.1.

We turn first to the proof that 3.1.7 implies 3.1.8, and for that we need one lemma. Let  $(v_1, \dots, v_t)$  be an  $m$ -useful sequence and for  $1 \leq i \leq t$ , let  $\mathcal{P}_i$  be an  $m$ -realization for  $(v_i, v_{i+1})$ . Let

$$\mathcal{P} = \bigcup_{i=1}^t \mathcal{P}_i.$$

We say that  $\mathcal{P}$  is an  $m$ -realization for  $(v_1, \dots, v_t)$  and we say that  $\mathcal{P}$  is *clean* if the elements of  $\mathcal{P}$  are pairwise internally-disjoint. The sequence  $(v_1, \dots, v_t)$  is a *clean*  $m$ -useful sequence if it has a clean  $m$ -realization. The next result relates the existence of  $m$ -useful sequence to the existence of clean  $m$ -useful sequences.

**3.2** *Let  $m, t$  be integers, let  $G$  be a digraph and let  $(v_1, \dots, v_t)$  be a  $t(2m + 1)$ -useful sequence in  $G$ . Then  $(v_1, \dots, v_t)$  is a clean  $m$ -useful sequence in  $G$ .*

**Proof.** Since  $(v_1, \dots, v_t)$  is a  $t(2m + 1)$ -useful sequence, it follows that for each  $i$  with  $1 \leq i \leq t$  there exists a set  $\mathcal{P}_i$  of  $t(2m + 1)$  internally-disjoint directed paths from  $v_i$  to  $v_{i+1}$ , all with at most two internal vertices. Each  $\mathcal{P}_i$  contains at most  $t$  paths that use a vertex of  $\{v_1, \dots, v_t\} \setminus \{v_i, v_{i+1}\}$ . Let  $\mathcal{Q}_i \subseteq \mathcal{P}_i$  be a set of  $2mt$  paths, none of which contains a vertex of  $\{v_1, \dots, v_t\} \setminus \{v_i, v_{i+1}\}$ . Next, for  $1 \leq i \leq t$ , we choose a subset  $\mathcal{Q}'_i \subseteq \mathcal{Q}_i$ , with  $|\mathcal{Q}'_i| = m$ , inductively, as follows. Suppose that we have already chosen  $\mathcal{Q}'_1, \dots, \mathcal{Q}'_i$ . Let  $I$  be the set of interior vertices of the elements of  $\mathcal{Q}'_1 \cup \dots \cup \mathcal{Q}'_i$ . Then  $I \leq 2mi$ . Since  $\mathcal{Q}_{i+1}$  contains  $2mt$  paths, we can choose a subset  $\mathcal{Q}'_{i+1}$  of  $m$  paths, all the same length, such that no element of  $\mathcal{Q}'_{i+1}$  uses a vertex of  $I$ .

By construction, all the paths in

$$\bigcup_{i=1}^t \mathcal{Q}'_i$$

are pairwise internally vertex disjoint. Hence,  $(v_1, \dots, v_t)$  is a clean  $m$ -useful sequence in  $G$ . This proves 3.2. ■

**3.3** *For every digraph  $H$  and every integer  $\alpha \geq 0$ , there exists an integer  $m' \geq 0$  such that there is a strong immersion of  $H$  in every simple digraph  $G$  with  $\alpha(G) \leq \alpha$  that contains an  $m'$ -useful sequence of length at most  $2\alpha + 2$ . In particular, 3.1.7 implies 3.1.8.*

**Proof.** Let  $H'$  be the digraph obtained by subdividing twice every edge of  $H$  (that is, replacing each edge by a directed three-edge path joining the same pair of vertices, so that these paths have pairwise disjoint interiors). Every digraph that admits a strong immersion of  $H'$  also admits a strong immersion of  $H$ , and so it suffices to prove the result for  $H'$ . Thus we may assume that  $H$  is a subdigraph of a tournament; and indeed, by adding any missing edges, we may assume that  $H$

is a tournament. Let  $|V(H)| = r$  and let  $m$  be an integer so large that every graph on  $m$  vertices contains either an independent set of size  $\alpha + 1$  or a clique of size  $2^{r(r+2)}$  (such a value of  $m$  exists by Ramsey's theorem [5]). Let  $m' = (2m + 1)(2\alpha + 2)$ . We claim that this choice of  $m'$  satisfies (1). For let  $G$  be a digraph with  $\alpha(G) \leq \alpha$ , and let  $(u_1, \dots, u_t)$  be an  $m'$ -useful sequence in  $G$  of length at most  $2\alpha + 2$ . Then by 3.2,  $(u_1, \dots, u_t)$  is a clean  $m$ -useful sequence. Let  $\mathcal{P}$  be a clean  $m$ -realization for  $(u_1, \dots, u_t)$  and let  $\{P_1, \dots, P_m\} \subseteq \mathcal{P}$  be an  $m$ -realization for  $(u_1, u_2)$ . Let  $X \subseteq V(G)$  be the set of second vertices of the paths  $P_1, \dots, P_m$ . Thus  $|X| = m$ .

From the definition of a clean  $m$ -useful sequence, it follows (since  $m \geq 2^{r(r+2)}$ ) that there is a set  $\{Q_e : e \in E(H)\}$  of directed, pairwise edge-disjoint paths from  $u_2$  to  $u_1$ . Now by Ramsey's theorem [5], the subdigraph of  $G$  induced on  $X$  (say  $G|X$ ) contains a tournament of size at least  $2^{r(r+2)}$  (since  $\alpha(G|X) \leq \alpha$ ). Also, every tournament with  $2^n$  vertices contains a transitive tournament with  $n$  vertices. Thus we may assume that there exist  $x_1, x_2, \dots, x_{r^2+2r} \in X$ , such that  $x_i$  is adjacent to  $x_j$  for  $1 \leq i < j \leq r^2 + 2r$ . Let  $V(H) = \{h_1, \dots, h_r\}$ , and for  $1 \leq i \leq r$  define  $\eta(h_i) = x_{i(r+1)}$ . For each edge  $e = h_i h_j$  of  $H$ , we define  $\eta(e)$  as follows. Let  $p = i(r + 1)$  and  $q = j(r + 1)$ . Then (in the obvious notation)  $\eta(e)$  is the directed path

$$\eta(h_i) = x_p - x_{p+j} - P_{p+j} - u_2 - Q_e - u_1 - x_{q-i} - x_q = \eta(h_j).$$

It is easy to check that  $\eta$  is a strong immersion of  $H$  in  $G$ . This proves 3.3. ■

Now we turn to the proof that 3.1.9 implies 3.1.1. We need two lemmas, the following.

**3.4** *Let  $G$  be a simple digraph with  $n$  vertices and independence number  $\alpha \geq 1$ . Then there exists a vertex in  $V(G)$  with outdegree at least  $\frac{n-\alpha}{2\alpha}$ . Similarly, there exists a vertex in  $V(G)$  with indegree at least  $\frac{n-\alpha}{2\alpha}$ .*

**Proof.** Let  $H$  be the underlying undirected graph of  $G$ . Then  $\overline{H}$  has no clique of size  $\alpha + 1$  and so by Turan's theorem [7],  $\overline{H}$  has at most  $\frac{1}{2}n^2(\frac{\alpha-1}{\alpha})$  edges. It follows that  $H$  has at least  $\frac{n^2-n\alpha}{2\alpha}$  edges, and hence so does  $G$ . So the average outdegree of a vertex in  $G$  is at least  $\frac{n-\alpha}{2\alpha}$ , and so there is a vertex in  $V(G)$  with outdegree at least  $\frac{n-\alpha}{2\alpha}$ . The proof for indegree is identical. This proves 3.4. ■

Let  $G$  be a digraph. For every enumeration  $(v_1, \dots, v_n)$  of the vertex set of  $G$ , we define the *converse-degree* of this enumeration to be the maximum over all  $j \in \{1, \dots, n\}$  of the larger of

- the number of edges with head  $v_j$  and tail in  $\{v_1, \dots, v_{j-1}\}$
- the number of edges with tail  $v_j$  and head in  $\{v_{j+1}, \dots, v_n\}$ .

We define the *converse-degree* of  $G$  to be the smallest  $k$  such that some enumeration of  $V(G)$  has converse-degree  $k$ . Thus, the converse-degree of  $G$  is at most the cutwidth of  $G$ . We need:

**3.5** *Let  $G$  be a simple digraph with independence number at most  $\alpha \geq 1$ , and let  $m \geq 0$ . If  $G$  has no  $m$ -useful sequence then the converse-degree of  $G$  is at most  $\alpha(2m + 1)$ .*

**Proof.** Since  $G$  has no  $m$ -useful sequence, there is an enumeration  $(v_1, \dots, v_n)$  of  $V(G)$  such that for all distinct  $i, j \in \{1, \dots, n\}$ , if  $(v_i, v_j)$  is an  $m$ -pair of type 1 or 2 then  $j < i$ . We claim this enumeration has converse-degree at most  $\alpha(2m + 1)$ . For let  $1 \leq j \leq n$ , and let

$$X = \{v_i : 1 \leq i < j, v_i \text{ is adjacent to } v_j\}, Y = \{v_i : j < i \leq n, v_i \text{ is adjacent from } v_j\}.$$

We claim that  $|X| < \alpha(2m + 1)$ . Thus we may assume that  $X \neq \emptyset$ , and so, since the independence number of  $G$  is at most  $\alpha$ , by 3.4 some vertex  $v_i \in X$  is adjacent to at least  $\frac{|X| - \alpha}{2\alpha}$  other members of  $X$ . Since  $i < j$  (because  $v_i \in X$ ), it follows that  $(v_i, v_j)$  is not an  $m$ -pair of type 1 or 2, and so  $\frac{|X| - \alpha}{2\alpha} < m$ , that is  $|X| < \alpha(2m + 1)$ . Similarly,  $|Y| < \alpha(2m + 1)$ , and so  $G$  has converse-degree at most  $\alpha(2m + 1)$ .  $\blacksquare$

We deduce:

**3.6** *Let  $G$  be a simple digraph with independence number at most  $\alpha \geq 1$  and let  $m \geq 0$  be an integer. Suppose that  $G$  has no  $m$ -useful sequence. Then the cutwidth of  $G$  is at most  $4\alpha^3(m + 1)(2m + 1)$ . In particular, 3.1.9 implies 3.1.1.*

**Proof.** As in 3.5 we consider the enumeration  $(v_1, \dots, v_n)$  of  $V(G)$  such that for all distinct  $i, j \in \{1, \dots, n\}$ , if  $(v_i, v_j)$  is an  $m$ -pair of type 1 or 2 then  $j < i$ . By 3.5, this enumeration has converse-degree at most  $\alpha(2m + 1)$ . We claim that this enumeration has cutwidth at most  $4\alpha^3(m + 1)(2m + 1)$ . For let  $2 \leq j \leq n$ ; let  $A = \{v_j, v_{j+1}, \dots, v_n\}$  and  $B = \{v_1, \dots, v_{j-1}\}$ . We must show that there are at most  $4\alpha^3(m + 1)(2m + 1)$  edges with tail in  $B$  and head in  $A$ . Let  $F$  be the set of all such edges. Since the enumeration has converse-degree at most  $\alpha(2m + 1)$ , we have immediately

(1) *Every vertex of  $G$  is incident with at most  $\alpha(2m + 1)$  edges in  $F$ .*

Suppose that there are at least  $4\alpha^3(m + 1)(2m + 1)$  edges in  $F$ . These edges form the edge set of a bipartite graph (with bipartition  $(B, A)$ ) with maximum degree at most  $\alpha(2m + 1)$ ; and so every set of vertices that meets every edge of this bipartite graph has cardinality at least  $\frac{4\alpha^3(m+1)(2m+1)}{\alpha(2m+1)} = 4\alpha^2(m + 1)$ . By König's theorem it follows that this bipartite graph has a matching  $M$  of cardinality  $4\alpha^2(m + 1)$ ; and so there exist distinct  $a_1, \dots, a_{4\alpha^2(m+1)} \in A$  and distinct  $b_1, \dots, b_{4\alpha^2(m+1)} \in B$  such that  $b_i$  is adjacent in  $G$  to  $a_i$  for  $1 \leq i \leq 4\alpha^2(m + 1)$ .

Let  $A' = \{a_1, \dots, a_{4\alpha^2(m+1)}\}$ . By 3.4, some vertex  $a \in A'$  is adjacent from at least  $\frac{|A'| - \alpha}{2\alpha}$  other members of  $A'$ . Let  $A'' \subseteq A'$  be the set of in-neighbors of  $a$  in  $A'$ . Then  $|A''| \geq \frac{|A'| - \alpha}{2\alpha} = 2\alpha(m + 1) - \frac{1}{2}$ . Let  $B'' = \{b_i : a_i \in A''\}$ . Again by 3.4, some vertex  $b \in B''$  is adjacent to at least  $\frac{|B''| - \alpha}{2\alpha}$  other members of  $B''$ . Since  $(b, a)$  is not an  $m$ -pair of type 1 or 2 (because  $b \in B$  and  $a \in A$ ), it follows that  $\frac{|B''| - \alpha}{2\alpha} < m$ . Hence, since  $|B''| = |A''|$ ,

$$m > \frac{|B''| - \alpha}{2\alpha} \geq \frac{2\alpha(m + 1) - 1 - \alpha}{2\alpha} = (m + 1) - \frac{\alpha + 1}{2\alpha} \geq m,$$

a contradiction. We conclude that there are fewer than  $4\alpha^3(m + 1)(2m + 1)$  edges in  $F$ , and therefore the cutwidth of  $G$  is at most  $4\alpha^3(m + 1)(2m + 1)$ . This proves 3.6.  $\blacksquare$



Finally, we need to show that 3.1.8 implies 3.1.9. This needs several lemmas.

**3.7** *Let  $G$  be a simple digraph with  $n$  vertices and independence number  $\alpha \geq 1$ . Then there exists a vertex in  $V(G)$  with outdegree and indegree at least  $\frac{n-2\alpha}{4\alpha}$ .*

**Proof.** Let  $V_1 \in V(G)$  be the set of vertices with at least as many in-neighbors as out-neighbors and let  $V_2 \in V(G)$  be the set of vertices with at least as many out-neighbors as in-neighbors. Then either  $|V_1| \geq \frac{n}{2}$  or  $|V_2| \geq \frac{n}{2}$ . Suppose that  $|V_1| \geq \frac{n}{2}$ . Then by 3.4, there exists a vertex  $v \in V_1$  with at least  $\frac{|V_1|-\alpha}{2\alpha}$  out-neighbors. It follows that  $v$  has outdegree and indegree at least  $\frac{n-2\alpha}{4\alpha}$ . The case where  $|V_2| \geq \frac{n}{2}$  is analogous. This proves 3.7.  $\blacksquare$

**3.8** *Let  $G$  be a digraph with independence number  $\alpha$  and let  $u, v \in V(G)$ . Suppose that there are  $m$  internally-disjoint directed paths from  $u$  to  $v$ , all with exactly three internal vertices. Then there exists a vertex  $w \in V(G)$  such that there are at least  $\frac{m-2\alpha}{4\alpha}$  internally-disjoint directed paths from  $u$  to  $w$  and at least  $\frac{m-2\alpha}{4\alpha}$  internally-disjoint directed paths from  $w$  to  $v$ , all with exactly two internal vertices.*

**Proof.** Let  $P_1, \dots, P_m$  be internally-disjoint directed paths from  $u$  to  $v$ , all with exactly three internal vertices. Let  $x_i$  be the middle vertex of the path  $P_i$  and let  $M = \{x_i : 1 \leq i \leq m\}$ . By 3.7, applied to the underlying simple digraph induced on  $M$ , there exists a vertex  $w \in M$  that has at least  $\frac{m-2\alpha}{4\alpha}$  out-neighbors and at least  $\frac{m-2\alpha}{4\alpha}$  in-neighbors. It is easy to see that  $w$  satisfies the conditions of the conclusion of 3.8.  $\blacksquare$

Let  $G$  be a digraph and let  $X, Y \subseteq V(G)$ . A *pairing* from  $X$  to  $Y$  is a set of vertices and edges  $\{v_1, \dots, v_l, e_{l+1}, \dots, e_m\}$  with  $e_i = x_i y_i$ , such that  $v_1, \dots, v_l \in X \cap Y$ ,  $x_i \in X \setminus Y$  and  $y_i \in Y \setminus X$  for all  $i$  with  $l+1 \leq i \leq m$ , and  $v_1, \dots, v_l, x_{l+1}, \dots, x_m, y_{l+1}, \dots, y_m$  are all distinct. We call

$$\{v_1, \dots, v_l, x_{l+1}, \dots, x_m, y_{l+1}, \dots, y_m\}$$

the *support* of the pairing. If  $X$  and  $Y$  are disjoint, then a pairing from  $X$  to  $Y$  is referred to as a *matching* from  $X$  to  $Y$ .

**3.9** *Let  $G$  be a digraph with independence number at most  $\alpha$  and let  $m \geq 0$  be an integer. Then given  $X_1, \dots, X_{\alpha+1} \subseteq V(G)$  with  $|X_i| \geq 2m\alpha + 1$  for  $1 \leq i \leq \alpha + 1$ , there is a pairing of size  $m$  from  $X_i$  to  $X_j$  for some  $i \neq j$ .*

**Proof.** Suppose that for all  $i \neq j$  there is no pairing of size  $m$  from  $X_i$  to  $X_j$ . Then for all choices of  $i \neq j$ , by König's theorem [4], there is a set  $R_{ij}$  of at most  $2m$  vertices that hits all edges from  $X_i$  to  $X_j$  and from  $X_j$  to  $X_i$  and such that  $X_i \cap X_j \subseteq R_{ij} \subseteq X_i \cup X_j$ . Let

$$R = \bigcup_{i \neq j} R_{ij}.$$

Then for each  $i \in \{1, \dots, \alpha + 1\}$ ,  $|X_i \cap R| \leq 2m\alpha$  and so  $X_i \setminus R \neq \emptyset$ . For  $1 \leq i \leq \alpha + 1$ , let  $v_i \in X_i \setminus R$  (notice that the  $v_i$ 's are all distinct). Then  $\{v_1, \dots, v_{\alpha+1}\}$  is an independent set of size  $\alpha + 1$  in  $G$ , a contradiction. This proves 3.9.  $\blacksquare$

The next result relates the existence of almost  $m$ -useful sequences to the existence of  $m$ -useful sequences.

**3.10** *Let  $G$  be a digraph with independence number at most  $\alpha$ , let  $m \geq 0$  be an integer, and let  $m' = \lceil \frac{m-2\alpha}{4\alpha} \rceil$ . Suppose that  $G$  has an almost  $m$ -useful sequence of length  $t$ . Then  $G$  has a  $m'$ -useful sequence of length  $\leq 2t$ .*

**Proof.** It follows immediately from 3.8 that  $G$  has a sequence  $(u_1, \dots, u_r)$  of length  $\leq 2t$ , such that  $(u_i, u_{i+1})$  is an  $m'$ -pair of type 1 or 2 for all  $1 \leq i \leq r$  (note however, that  $u_1, \dots, u_r$  might not be distinct). The sequence  $(u_1, \dots, u_r)$  then has a subsequence of length at most  $r \leq 2t$  such that all of its terms are distinct and every pair of consecutive terms (modulo its length) is an  $m'$ -pair of type 1 or 2. Therefore,  $G$  has an  $m'$ -useful sequence of length  $\leq 2t$ .  $\blacksquare$

We need one more lemma:

**3.11** *Let  $h, \alpha \geq 1$  be integers, and let  $m = 1 + 2\alpha h$ . Let  $G$  be a digraph with independence number at most  $\alpha$  and let  $(u_1, \dots, u_t)$  be a partial  $m$ -useful sequence in  $G$ . Suppose that  $(u_t, u_1)$  is an  $\frac{h}{3}$ -pair of type 1, 2 or 3. Then there exists an  $m'$ -useful sequence in  $G$  of length at most  $2\alpha + 2$ , where  $m' \geq \frac{m}{24\alpha^2} - 1$ .*

**Proof.** The proof is by induction on  $t$ . For  $1 \leq i \leq t-1$ , let  $\mathcal{P}_i$  be an  $m$ -realization for  $(u_i, u_{i+1})$  and let  $A_i$  be the set of second vertices of the members of  $\mathcal{P}_i$ . For  $1 \leq i, j \leq t$ , we write  $i \Rightarrow j$  if there is a pairing of size  $h$  from  $A_i$  to  $A_j$ .

(1) *If  $1 \leq i, j \leq t-1$  and  $j \neq i+1$ , and  $j \Rightarrow i$ , then  $(u_j, u_{i+1})$  is an  $\frac{h}{3}$ -pair of type 1, 2, or 3.*

If at least  $h/3$  vertices in  $A_j$  are adjacent to  $u_{i+1}$ , then  $(u_j, u_{i+1})$  is a  $\frac{h}{3}$ -pair of type 1, as required. So we may assume there is a pairing of size at least  $2h/3$  from  $A_j$  to  $A_i$ , such that no vertex in its support belongs to  $A_j$  and is adjacent to  $u_{i+1}$ . Consequently there are at least  $2h/3$  directed paths from  $u_j$  to  $u_{i+1}$ , pairwise vertex-disjoint except for  $u_j$  and  $u_{i+1}$ , and each with three or four edges. At least  $h/3$  of these have the same length, and so  $(u_j, u_{i+1})$  is an  $\frac{h}{3}$ -pair of type 2 or 3. This proves (1).

(2) *We may assume that there do not exist  $i, j$  with  $1 \leq i, j \leq t-1$  and  $j \geq i+1$  such that  $j \Rightarrow i$ .*

For suppose that for some such  $i, j$  there is a pairing of size  $h$  from  $A_j$  to  $A_i$ . Suppose first that  $j \geq i+2$ . By (2),  $(u_j, u_{i+1})$  is an  $\frac{h}{3}$ -pair of type 1, 2, or 3. Hence,  $(u_{i+1}, \dots, u_j)$  is a sequence that satisfies the assumptions of 3.11 and is shorter than  $(u_1, \dots, u_t)$ , and so we win by induction. Thus we may assume that  $j = i+1$ . Let  $A'$  be the intersection of  $A_i$  with the support of this pairing. Thus  $|A'| = h$ . Let  $m' = \frac{h-2\alpha}{4\alpha}$ . By 3.7, there exists  $v \in A_i$  such that  $v$  is adjacent to at least  $m'$  members of  $A'$  and adjacent from at least  $m'$  members of  $A'$ . Consequently,  $(v, u_{i+1})$  is an  $m'$ -useful sequence, and since  $m' \geq \frac{m}{24\alpha^2} - 1$ , in this case the result holds. This proves (2).

For  $1 \leq i \leq t-1$ , we say that  $j$  is the *follower* of  $i$  if  $i < j \leq t$  and  $i \Rightarrow j-1$ , and subject to these conditions  $j$  is as large as possible. (Note that  $i \Rightarrow i$ , and so  $i$  has a follower, for  $1 \leq i \leq t-1$ )

. Let  $i_1 = 1$ , and inductively if we have defined  $i_s$  and  $i_s \neq t$ , let  $i_{s+1}$  be the follower of  $i_s$ . Since  $i_{s+1} > i_s$  for all  $s$ , there exists  $k$  such that  $i_k = t$ .

(3)  $k \leq \alpha + 1$ .

If  $1 \leq r < s < k$ , then  $i_r \not\neq i_s$  by our choice of  $i_1, \dots, i_k$ ; and  $i_s \not\neq i_r$  by (2). Hence, it follows that  $\{A_{i_1}, \dots, A_{i_{k-1}}\}$  is a set of  $k - 1$  sets, each of size  $m = 2h\alpha + 1$ , such that there is no pairing of size  $h$  between any two of them. Therefore, by 3.9,  $k \leq \alpha + 1$ . This proves (3).

By (1),  $(u_{i_j}, u_{i_{j+1}})$  is an  $\frac{h}{3}$ -pair of type 1, 2 or 3, for  $1 \leq j < k$ . Also, by hypothesis,  $(u_{i_k}, u_{i_1}) = (t, 1)$  is an  $\frac{h}{3}$ -pair of type 1, 2, or 3. Hence,  $(u_{i_1}, \dots, u_{i_k})$  is an almost  $\frac{h}{3}$ -useful sequence of length at most  $\alpha + 1$  in  $G$ , and so by 3.10,  $G$  has an  $m'$ -useful sequence of length at most  $2\alpha + 2$  with

$$m' \geq \frac{\frac{h}{3} - 2\alpha}{4\alpha} = \frac{\frac{m-1}{6\alpha} - 2\alpha}{4\alpha} \geq \frac{m}{24\alpha^2} - 1.$$

This proves 3.11. ▀

**3.12** *For all integers  $m', \alpha \geq 0$ , there exists an integer  $m \geq 0$  such that every digraph  $G$  with independence number at most  $\alpha$  that contains an  $m$ -useful sequence contains an  $m'$ -useful sequence of length at most  $2\alpha + 2$ . In particular, 3.1.8 implies 3.1.9.*

**Proof.** Let  $G$  be a digraph with independence number at most  $\alpha$  and let  $m' \geq 0$  be an integer. Let  $m = 24\alpha^2(m' + 1)$  and let  $(u_1, \dots, u_t)$  be an  $m$ -useful sequence. Then  $(u_1, \dots, u_t)$  satisfies the hypothesis of 3.11 and so  $G$  contains an  $m'$ -useful sequence of length at most  $2\alpha + 2$ . This proves 3.12. ▀

This completes the proof of 3.1.

We can convert 3.6, 3.11, and 3.12 to an algorithm as follows:

**3.13** *There is an algorithm with running time  $O(n^4)$ , which, given as input a simple digraph  $G$  with  $n$  vertices and integers  $m', \alpha \geq 0$ , outputs one of the following:*

- an  $m'$ -useful sequence of length  $\leq 2\alpha + 2$  if one exists, or
- a true statement that  $\alpha(G) > \alpha$ , or
- an enumeration of  $V(G)$  with cutwidth at most  $4\alpha^3(m + 1)(2m + 1)$ , where  $m = 24\alpha^2(m' + 1)$ .

**Proof.** For every pair of distinct vertices  $u, v$ , we determine whether  $(u, v)$  is an  $m$ -pair of type 1 or 2 as follows. We find the set  $A$  of out-neighbors of  $u$ , and the set  $B$  of in-neighbors of  $v$ . If  $|A \cap B| \geq m$  we conclude that  $(u, v)$  is an  $m$ -pair of type 1. Otherwise, we run a bipartite matching algorithm on the graph formed by the edges of  $G$  from  $A$  to  $B$  (note that such an edge might have ends in  $A \cap B$ ) and if we find a matching of size  $m$  then  $(u, v)$  is an  $m$ -pair of type 2; and otherwise  $(u, v)$  is not an  $m$ -pair of type 1 or 2. For each pair this takes time  $O(n^2)$  (since  $m$  is a constant) and so the total time for all pairs is  $O(n^4)$ .

Next, let  $H$  be the auxiliary digraph defined as follows:  $V(H) = V(G)$  and  $uv$  is an edge of  $H$  if and only if  $(u, v)$  is an  $m$ -pair of type 1 or 2 in  $G$ . We determine whether  $H$  has a directed cycle (this takes time  $O(n^2)$ ). If  $H$  does not have a directed cycle, we construct an acyclic ordering of  $H$  (this takes time  $O(n^2)$ ). This enumeration has cutwidth at most  $4\alpha^3(m+1)(2m+1)$  in  $G$ . We then output this enumeration.

If  $H$  has a directed cycle, then define another auxiliary digraph  $H'$  as follows:  $V(H') = V(G)$  and  $uv$  is an edge of  $H'$  if and only if  $(u, v)$  is an  $m'$ -pair of type 1 or 2 in  $G$ . We find the shortest directed cycle in  $H'$  in time  $O(n^3)$ . This cycle corresponds to the shortest  $m'$ -useful sequence  $(u_1, \dots, u_t)$  in  $G$ . If  $t \leq 2\alpha + 2$ , then we output the sequence  $(u_1, \dots, u_t)$ . Otherwise, by 3.12,  $\alpha(G) > \alpha$ , and so we output this fact. This proves 3.13.  $\blacksquare$

## 4 The algorithms

As already mentioned earlier, the algorithm to solve  $\mathbf{H}$ -infusion in digraphs with bounded independence number is similar to the one for supertournaments. The idea is as follows: given as input a digraph  $G$  with independence number at most  $\alpha$ , run the algorithm of 3.13 on its underlying simple digraph with some appropriate value of  $m$ . If the output is an  $m$ -useful sequence  $(u_1, \dots, u_t)$  of length at most  $2\alpha + 2$ , then identify all the vertices of that sequence and run the algorithm of 3.13 on the underlying simple digraph of the resulting digraph. Repeat this process until 3.13 returns an enumeration of the vertex set of a simple digraph  $S$  with bounded cutwidth. Finally, as with supertournaments, we then use the outputted enumeration of  $V(S)$  to solve  $\mathbf{H}$ -infusion using dynamic programming.

There are two steps of the algorithm that need to be explained in more detail. First, we need to show that identifying the vertices of an  $m$ -useful sequence of bounded length (for sufficiently large  $m$ ) does not affect the feasibility of the  $\mathbf{H}$ -infusion problem. Second, we need to explain the dynamic programming. We begin with a result that justifies the former.

**4.1** *Let  $\mathbf{H} = (H, u_1, \dots, u_h)$  be a distinctly-rooted digraph with  $|E(H)| = k$ . Let  $\mathbf{G} = (G, s_1, \dots, s_h)$  be a rooted digraph and let  $(v_1, \dots, v_t)$  be an  $(8kt + t)$ -useful sequence in  $G$ . If  $G'$  is obtained from  $G$  by identifying  $v_1, \dots, v_t$ , then  $\mathbf{G}' = (G', s_1, \dots, s_h)$  contains an infusion of  $\mathbf{H}$  if and only if  $\mathbf{G}$  does.*

**Proof.** By 3.2,  $(v_1, \dots, v_t)$  is a clean  $4k$ -useful sequence in  $G$ . Let  $\mathcal{P}$  be a clean  $4k$ -realization for  $(v_1, \dots, v_t)$  and let  $\mathcal{P}_i \subseteq \mathcal{P}$  be a  $4k$ -realization for  $(v_i, v_{i+1})$ .

First we show that if  $\mathbf{G}$  contains an infusion of  $\mathbf{H}$  then so does  $\mathbf{G}'$ . Let  $\eta$  be an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ . Let  $w \in V(G')$  be the vertex resulting from the identification of  $v_1, \dots, v_t$ . Define  $\eta'$  as follows. For  $x \in V(H)$ , if  $\eta(x) \notin \{v_1, \dots, v_t\}$ , then  $\eta'(x) = \eta(x)$  and if  $\eta(x) \in \{v_1, \dots, v_t\}$  then  $\eta'(x) = w$ . Next, for  $e \in E(H)$  with  $e = xy$ ,  $\eta(e)$  naturally corresponds to a walk in  $G'$  (since all edges of  $G$  are edges of  $G'$ ). This walk contains a route  $P$  from  $\eta'(x)$  to  $\eta'(y)$ ; we let  $\eta'(e) = P$ . Now  $\eta'$  is an infusion of  $\mathbf{H}$  in  $\mathbf{G}'$ .

It remains to prove the converse. Let  $L = \{i : (v_i, v_{i+1}) \text{ is an } m\text{-pair of type 2}\}$ . For  $i \in L$ , let  $B_i$  be the set of bodies of the elements of  $\mathcal{P}_i$  and let  $B = \bigcup_{i \in L} B_i$ . For an infusion  $\eta'$  of  $\mathbf{H}$  in  $\mathbf{G}'$  and for  $i \in L$ , define  $B_i^{\eta'} \subseteq B_i$  as follows:  $b \in B_i^{\eta'}$  if  $b$  is the body of a path  $P \in \mathcal{P}_i$  with legs  $a, c$  such that  $a, c \notin E_{\eta'}$ . Define

$$B^{\eta'} = \bigcup_{i \in L} B_i^{\eta'}.$$

Now suppose that there is an infusion of  $\mathbf{H}$  in  $\mathbf{G}'$  and let  $\eta'$  be such an infusion with  $E_{\eta'} \cap B_{\eta'}$  minimal (this makes sense because  $E(G) = E(G')$ ).

(1) For all  $g \in E(H)$ , the route  $\eta'(g)$  uses at most two edges of  $B_{\eta'}$ .

Suppose that for some  $g \in E(H)$  with  $g = rs$ ,  $\eta'(g)$  uses at least three edges of  $B_{\eta'}$ . Let  $f_1 = a_1b_1, f_2 = a_2b_2$  and  $f_3 = a_3b_3$  be three edges of  $B_{\eta'}$  that are used by  $\eta'(g)$  in the specified order. From the definition of  $B_{\eta'}$ , it follows that there exist edges  $f_4 = b_1w$  and  $f_5 = wa_3$  such that  $f_4, f_5 \notin E_{\eta'}$ . Let  $W$  be obtained from  $\eta'(g)$  by replacing the part of the route from  $b_1$  to  $a_3$  by a path  $b_1-w-a_3$  (where the two edges of the path are  $f_4$  and  $f_5$ ). Then  $W$  is a walk in  $G'$ . Let  $Q$  be a route of  $G'$  from  $\eta'(r)$  to  $\eta'(s)$  contained in  $W$ . Then  $Q$  is disjoint from  $E_{\eta'} \setminus E(\eta'(g))$ . Define  $\eta$  as follows. Let  $\eta(z) = \eta'(z)$  for all  $z \in V(H)$  and let  $\eta(e) = \eta'(e)$  for all  $e \in E(H) \setminus \{g\}$ ; let  $\eta(g) = Q$ . Then  $B_{\eta} \cap E_{\eta} \subset B_{\eta'} \cap E_{\eta'}$ , contrary to our choice of  $\eta'$ . This proves (1).

Now since all the edges of  $G'$  are also edges of  $G$ , it follows that in  $G$  each  $\eta'(e)$  is either a route or the disjoint union of two routes (each of which is a directed path). Our goal is to show that we can modify  $\eta'$  to obtain an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ .

Let  $E(H) = \{e_1, \dots, e_k\}$ . We want to show that for each  $1 \leq i \leq t$  there exist edge-disjoint paths  $P_i^1, \dots, P_i^k \in \mathcal{P}_i$  such that  $P_i^j$  is disjoint from  $E_{\eta'} \setminus E(\eta'(e_j))$  for all  $1 \leq j \leq k$ . If  $i \notin L$ , then there exist  $2k$  members of  $\mathcal{P}_i$  that are disjoint from  $E_{\eta'}$ , because for each  $e \in E(H)$ ,  $\eta'(e)$  meets at most two members of  $\mathcal{P}_i$ ; so choose  $P_i^1, \dots, P_i^k \in \mathcal{P}_i$  from among those  $2k$ . Suppose now that  $i \in L$ . Since a route can use at most one edge with tail  $v_i$  and at most one edge with head  $v_{i+1}$ , and since  $(v_i, v_{i+1})$  is a  $4k$ -pair of type 2, it follows that  $|B_{\eta'}^i| \geq 2k$ . Now by (1), we can choose disjoint  $M_1, \dots, M_k \subseteq B_{\eta'}^i$  such that  $|M_j| = 2$  and  $\eta'(e_j)$  is disjoint from  $B_{\eta'}^i \setminus M_j$ . For  $1 \leq j \leq k$ , choose  $P_i^j$  such that its body is in  $M_j$ . Now for  $1 \leq j \leq k$ , let

$$C_j = \bigcup_{i=1}^t P_i^j.$$

Then  $C_1, \dots, C_k$  are edge-disjoint cycles of  $G$  such that for each  $j$ ,  $\{v_1, \dots, v_t\} \subseteq V(C_j)$  and  $C_j$  is disjoint from  $E_{\eta'} \setminus E(\eta'(e_j))$ .

We are now ready to construct an infusion  $\eta$  of  $\mathbf{H}$  in  $\mathbf{G}$ . For  $1 \leq l \leq h$ , let  $\eta(u_l) = s_l$ . For  $z \in V(H) \setminus \{u_1, \dots, u_h\}$ , if  $\eta'(z) \neq w$ , then let  $\eta(z) = \eta'(z)$ , and if  $\eta'(z) = w$  then let  $\eta(z) = v_1$ . For  $e_j \in E(H)$  with  $e_j = x_j y_j$ ,  $\eta'(e_j) \cup C_j$  contains a route from  $\eta'(x_j)$  to  $\eta'(y_j)$ ; let  $\eta(e_j)$  be that route. Now  $\eta$  is an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ . This proves 4.1.  $\blacksquare$

Next, we discuss the dynamic programming for general digraphs with bounded cutwidth. First we set up and prove a result for all digraphs.

For a digraph  $G$ , a *weak component* is a component of the underlying undirected graph of  $G$ . Let  $P$  be a route and let  $Q$  be a subdigraph of  $P$ , and let  $q$  be the number of weak components of  $Q$  that have at least one edge. Define  $c(P, Q)$  as follows: if  $P$  is a directed cycle with end  $v$  and some component of  $Q$  is a path with  $v$  in its interior, then  $c(P, Q) = q + 1$ , and otherwise  $c(P, Q) = q$ .

Let  $G$  be a digraph, and let  $(e_1, \dots, e_m)$  be some enumeration of its edge set. For  $0 \leq i \leq m$ , let  $G_i$  be the digraph with  $V(G_i) = V(G)$  and  $E(G_i) = \{e_{i+1}, \dots, e_m\}$ . If  $P_1, \dots, P_k$  are pairwise

edge-disjoint routes of  $G$ , we say that  $(P_1, \dots, P_k)$  is *t-cohesive* (with respect to this enumeration) if

$$\sum_{j=1}^k c(P_j, P_j \cap G_i) \leq t$$

for for  $0 \leq i \leq m$ .

Let  $a_1, b_1, \dots, a_k, b_k$  be vertices of a digraph  $G$  (not necessarily distinct). We say that the sequence of pairs  $((a_1, b_1), \dots, (a_k, b_k))$  is *feasible* in  $G$  if there exist edge-disjoint routes  $P_1, \dots, P_k$  such that for  $1 \leq i \leq k$ ,  $a_i$  and  $b_i$  are the first and last vertices of  $P_i$ , respectively. We say that the sequence of pairs is *t-cohesively feasible* (with respect to a given enumeration of  $E(G)$ ) if in addition  $(P_1, \dots, P_k)$  is *t-cohesive*. Note that if a list of pairs is *t-cohesively feasible* then it has at most  $t$  terms.

**4.2** For all  $t$  there is an algorithm as follows:

- **Input:** A digraph  $G$  with  $|V(G)| = n$ , and an enumeration  $(e_1, \dots, e_m)$  of its edge set.
- **Output:** The set of all sequences of pairs of vertices of  $G$  that are *t-cohesively feasible* in  $G$  (with respect to the given enumeration).
- **Running time:**  $O(m^{2t+1})$ .

**Proof.** Let  $G_i$  be defined as above; then  $(e_{i+1}, \dots, e_m)$  is an enumeration of the edge set of  $G_i$ . Let  $C_i$  be the set of all sequences of pairs of vertices of  $G$  that are *t-cohesively feasible* in  $G_i$  with respect to the enumeration  $(e_{i+1}, \dots, e_m)$ . Thus  $C_m = \{\emptyset\}$  since  $G_m$  has no edges, and  $C_0$  is the desired output. Moreover, each  $C_i$  has at most  $2n^{2t}$  members. Some notation: if  $S$  is a sequence with  $k$  terms and  $1 \leq j \leq k$ ,  $S^j$  denote the sequence obtained from  $S$  by omitting the  $j$ th term. Also, if  $S = ((a_1, b_1), \dots, (a_k, b_k))$  then  $S + (a, b)$  denotes the sequence  $((a_1, b_1), \dots, (a_{k+1}, b_{k+1}))$  where  $(a_{k+1}, b_{k+1}) = (a, b)$ .

We claim that for  $1 \leq i \leq m$ ,  $C_{i-1}$  can be computed from a knowledge of  $C_i$  in time  $O(m^{2t})$ . To see this, let  $S = ((a_1, b_1), \dots, (a_k, b_k))$  be a sequence of pairs of vertices of  $G$  with  $k \leq t$ . Let us show how to decide whether  $S \in C_{i-1}$  (assuming we can test membership of  $C_i$  in constant time). Let  $e_i$  have tail  $u$  and head  $v$ . We claim that  $S \in C_{i-1}$  if and only if either:

- $S \in C_i$ , or
- there exists  $j$  with  $1 \leq j \leq k$  such that  $a_j = u$  and  $b_j = v$  and  $S^j \in C_i$ , or
- there exists  $j$  with  $1 \leq j \leq k$  such that  $a_j = u$  and  $b_j \neq v$  and  $S^j + (v, b_j) \in C_i$ , or
- there exists  $j$  with  $1 \leq j \leq k$  such that  $a_j \neq u$  and  $b_j = v$  and  $S^j + (a_j, u) \in C_i$ , or
- $k < t$  and there exists  $j$  with  $1 \leq j \leq k$  such that  $a_j \neq u$  and  $b_j \neq v$  and  $(S^j + (a_j, u)) + (v, b_j) \in C_i$ .

To see this, it is easy to check the “if” part; let us show “only if”. Suppose then that  $S \in C_{i-1}$ , and let  $P_1, \dots, P_k$  be the corresponding routes of  $G_{i-1}$ . If none of  $P_1, \dots, P_k$  contain the edge  $e_i$ , then  $S \in C_i$ , so we may assume that  $P_j$  contains  $e_i$  say. (This value of  $j$  is unique since  $P_1, \dots, P_k$  are edge-disjoint.) If  $e_i$  is the first or last edge of  $P_j$  then the second, third or fourth bulleted statement

holds; so we assume that  $a_j \neq u$  and  $b_j \neq v$ . It follows that  $c(P_j, P_j \cap G_i) = 2$ , and  $c(P_h, P_h \cap G_i) = 1$  for  $1 \leq h \leq k$  with  $h \neq j$  (note that this is true even if  $a_j = b_j$ ). Since  $(P_1, \dots, P_k)$  is  $t$ -cohesive, it follows that

$$t \geq \sum_{h=1}^k c(P_h, P_h \cap G_i) = k + 1,$$

and so the fifth bulleted statement holds. This proves our claim.

All the bulleted statements can be checked in constant time, and so we can check whether  $S \in C_{i-1}$  in constant time, as claimed. Since there are at most  $m^{2t}$  choices of  $S$  to check, we can compute  $C_{i-1}$  from  $C_i$  in time  $O(m^{2t})$ . This has to be run for each of the  $m$  values of  $i$ , so the total running time is  $O(m^{2t+1})$ . This proves 4.2.  $\blacksquare$

Let  $G$  be a digraph and let  $(v_1, \dots, v_n)$  be an enumeration of  $V(G)$ . Let  $(e_1, \dots, e_m)$  be an enumeration of  $E(G)$  obtained as follows: first list all the edges with head  $v_1$ , next all the edges with head  $v_2$ , etc. We say that  $(e_1, \dots, e_m)$  is a *head-first* enumeration of  $E(G)$  derived from  $(v_1, \dots, v_n)$ .

**4.3** *Let  $G$  be a digraph and let  $(v_1, \dots, v_n)$  be an enumeration of  $V(G)$  with cutwidth at most  $\theta$ . Let  $(e_1, \dots, e_m)$  be a head-first enumeration of  $E(G)$  derived from  $(v_1, \dots, v_n)$ , and let  $P_1, \dots, P_k$  be edge-disjoint routes in  $G$ . Then  $(P_1, \dots, P_k)$  is  $(k + 2\theta)$ -cohesive with respect to the enumeration  $(e_1, \dots, e_m)$ .*

**Proof.** Let  $G_i$  be the digraph with  $V(G_i) = V(G)$  and  $E(G_i) = \{e_{i+1}, \dots, e_m\}$ . We need to show that

$$\sum_{j=1}^k c(P_j, P_j \cap G_i) \leq k + 2\theta,$$

where  $c(P_j, P_j \cap G_i)$  is the same as in the definition of  $t$ -cohesive.

Let  $v_h$  be the head of  $e_{i+1}$ . Let  $F$  be the set of edges with tail in  $\{v_1, \dots, v_{h-1}\}$  and head in  $\{v_h, \dots, v_n\}$ , and let  $F'$  be the set of edges with tail in  $\{v_1, \dots, v_h\}$  and head in  $\{v_{h+1}, \dots, v_n\}$ . Fix  $j$  with  $1 \leq j \leq k$  for the moment.

$$(1) \quad c(P_j, P_j \cap G_i) \leq 1 + |(F \cup F') \cap E(P_j)|.$$

If  $P_j$  is a subdigraph of  $G_i$  then this is clear, so we may assume that  $P_j$  is not a subdigraph of  $G_i$ . Let  $Q_1, \dots, Q_q$  be the maximal weakly-connected subdigraphs of  $P_j \cap G_i$  that have at least one edge, numbered in their order in  $P_j$  if  $P_j$  is a path, and such that the end of  $P_j$  does not belong to  $Q_2, \dots, Q_q$  if  $P_j$  is a cycle. It follows that each  $Q_s$  is a directed path (since  $P_j$  is not a subdigraph of  $G_i$ ). We claim that for each  $s > 1$ , the first edge  $f_s$  of  $Q_s$  belongs to  $F \cup F'$ . For let  $e_s$  be the edge of  $P_j$  with head the tail of  $f_s$ ; and let  $e_s, f_s$  be  $uv, vw$  say. From the maximality of  $Q_s$ ,  $e_s \notin E(G_i)$ , and so  $v = v_a$  for some  $a \leq h$ . Let  $w = v_b$ . Since  $f_s \in E(G_i)$ , it follows that  $b \geq h$ . But  $v \neq w$  since  $f_s$  is not a loop, and so either  $a < h$  or  $b > h$ . Consequently  $f_s \in F \cup F'$ . This proves our claim. It follows that  $q - 1 \leq |(F \cup F') \cap E(P_j)|$ . If  $c(P_j, P_j \cap G_i) = q$ , then this proves (1), so we may assume that  $P_j$  is a cycle and its end is an internal vertex of  $Q_1$ , and  $c(P_j, P_j \cap G_i) = q + 1$ . But then the same argument shows that the first edge of  $Q_1$  is in  $F \cup F'$ , and so  $q \leq |(F \cup F') \cap E(P_j)|$ ; and so again (1) follows. This proves (1).

Since  $|F \cup F'| \leq 2\theta$ , by summing the inequality of (1) over all  $j$ , we deduce that

$$\sum_{j=1}^k c(P_j \cap G_i) \leq k + 2\theta.$$

It follows that  $(P_1, \dots, P_k)$  is  $(k + 2\theta)$ -cohesive with respect to the enumeration  $(e_1, \dots, e_m)$ . This proves 4.3.  $\blacksquare$

Let  $\eta$  be an infusion of  $H$  in  $G$ , and choose an enumeration  $\{e_1, \dots, e_k\}$  of  $E(H)$ . For  $1 \leq i \leq k$ , let  $a_i, b_i$  be the first and last vertices of  $\eta(e_i)$  respectively. The sequence  $((a_1, b_1), \dots, (a_k, b_k))$  is called a *frame* of  $\eta$ . (Thus all frames of  $\eta$  can be obtained one from another by reordering the terms.) We now use 4.2 and 4.3 for the following:

**4.4** *For each digraph  $H$  with  $|E(H)| = k$  and integer  $\theta$ , there exists an algorithm as follows:*

- **Input:** *A digraph  $G$  and an enumeration of its vertex set  $(v_1, \dots, v_n)$  with cutwidth  $\leq \theta$ .*
- **Output:** *The set of all frames of infusions of  $H$  in  $G$ .*
- **Running time:**  *$O(|E(G)|^{2k+4\theta+3})$ .*

**Proof.** Let  $(e_1, \dots, e_{|E(G)|})$  be the head-first enumeration of  $E(G)$  derived from  $(v_1, \dots, v_n)$ . Run the algorithm of 4.2 with  $t = k + 2\theta$  and input  $G$  and this enumeration. For each sequence of pairs outputted by the algorithm we can determine whether it is the frame of an infusion of  $H$  in constant time. Moreover, by 4.3, every frame of an infusion of  $H$  will arise in such a way. This proves 4.4.  $\blacksquare$

Finally, we have the main result of this paper. If  $G$  is a digraph, its  *$k$ -pruning* is the maximal subdigraph  $G'$  of  $G$  such that every vertex of  $G'$  is incident with at most  $k$  loops, and for all distinct vertices  $u, v$ , there are at most  $k$  edges from  $u$  to  $v$ .

**4.5** *For every distinctly-rooted digraph  $\mathbf{H} = (H, u_1, \dots, u_h)$  with  $|E(H)| = k$ , and every integer  $\alpha \geq 0$ , there is an algorithm with running time  $O(n^{4k+8\theta+6})$ , where  $\theta = 4\alpha^3(m+1)(2m+1)$ ,  $m = 24\alpha^2(m'+1)$ , and  $m' = 8k(2\alpha+2) + (2\alpha+2)$ , which, given as input a rooted digraph  $\mathbf{G} = (G, s_1, \dots, s_h)$  with  $|V(G)| = n$  outputs one of the following:*

- *a true statement that there exists an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ , or*
- *a true statement that there does not exist an infusion of  $\mathbf{H}$  in  $\mathbf{G}$ , or*
- *a true statement that  $\alpha(G) > \alpha$ .*

**Proof.** As mentioned earlier, the idea is very similar to the one for supertournaments. However, the steps are slightly more involved. We may assume that  $H$  has no isolated vertices except for possibly  $u_1, \dots, u_h$ . Let  $G_0 = G$  and start at Step 1.

At Step  $i$ : Let  $Q_{i-1}$  be the underlying simple digraph of  $G_{i-1}$ . Run the algorithm of 3.13 on  $Q_{i-1}$  with  $m' = (8k+1)(2\alpha+2)$ .



- If the output is a  $m'$ -useful sequence  $(v_1, \dots, v_t)$  with  $t \leq 2\alpha + 2$ , obtain  $G_i$  by identifying  $v_1$  and  $v_2$  in  $G_{i-1}$  (by 4.1, this does not change the outcome). Go to Step  $i + 1$ .
- If the output is an enumeration of  $V(Q_{i-1})$  with cutwidth at most  $\theta = 4\alpha^3(m+1)(2m+1)$ , where  $m = 24\alpha^2(m' + 1)$ , run the algorithm of 4.4 on the  $k$ -pruning of  $G_{i-1}$  with this enumeration. For each frame  $\phi$  of an infusion returned by 4.4 determine whether  $\phi(u_i) = s_i$  for all  $1 \leq i \leq h$ , and if such a frame is found, return that an infusion exists. Otherwise, return that an infusion does not exist. This can be done in constant time for each frame. Hence, the running time for this step is  $O((kn^2)^{2k+4\theta+3}) = O(n^{4k+8\theta+6})$ .
- If the output is a true statement that  $\alpha(Q_{i-1}) > \alpha$ , output this statement.

Notice that an enumeration of  $V(Q_i)$  with cutwidth  $\leq \theta$  is also an enumeration of the  $k$ -pruning of  $G_i$  with cutwidth  $\leq k\theta$ . The running time is dominated by the last step (the dynamic programming). This proves 4.5. ■

Once again, this can be modified to output the infusion if one exists.

Next, we describe the algorithm to solve  $H$ -immersion. First, we note that 4.2 and 4.4 can be modified to output all frames of immersions instead of infusions. For that, in 4.2 the pairs of edges would come with sets of forbidden vertices that the paths are not allowed to go through. We omit the details.

**4.6** *For every digraph  $H$  and every integer  $\alpha \geq 0$ , there exists an algorithm, with running time  $O(n^c)$  where  $c$  depends on  $H$  and  $\alpha$ , which, given as input a digraph  $G$  with  $n$  vertices, outputs one of the following:*

- a true statement that there exists a strong (or weak) immersion of  $H$  in  $G$ , or
- a true statement that there does not exist a strong (or weak) immersion of  $H$  in  $G$ , or
- a true statement that  $\alpha(G) > \alpha$ .

**Proof.** Run the algorithm of 3.13 on the underlying simple digraph of  $G$  with  $m'$  as in 3.3.

- If the output is an  $m'$ -useful sequence, return that the immersion exists (this is justified by 3.3).
- If the output is an enumeration of  $V(G)$  with cutwidth at most  $4\alpha^3(m+1)(2m+1)$ , where  $m = 24\alpha^2(m' + 1)$ , run a version of the algorithm of 4.4 on the  $k$ -pruning of  $G$  with this enumeration, modified to search for immersions of  $H$  in  $G$  instead of searching for infusions.
- If the output is a true statement that  $\alpha(G) > \alpha$ , output this statement.

Once again, the running time is dominated by the dynamic programming. ■

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