การกำกับรวมอย่างมหัศจรรย์บนด้านของกราฟที่เชื่อมโยงได้และกราฟที่เชื่อมโยงไม่ได้

นางสาวศิริรัตน์ สมพงษ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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EDGE-MAGIC TOTAL LABELINGS ON
CONNECTED AND DISCONNECTED GRAPHS

Miss Sirirat Sompong

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By	Miss Sirirat Sompong
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master 's Degree

……………………………………………..Deputy Dean for Administrative Affairs

(Associate Professor Pipat Karntiang, Ph.D.) Acting Dean, Faculty of Science

THESIS COMMITTEE

…………………………………………….. Chairman

(Associate Professor Yupaporn Kemprasit, Ph.D.)

…………………………………………….. Thesis Advisor

(Associate Professor Wanida Hemakul, Ph.D.)

…………………………………………….. Member

(Krung Sinapiromsaran, Ph.D.)

ศิริรัตน์ สมพงษ์ : การกำกับรวมอย่างมหัศจรรย์บนด้านของกราฟที่เชื่อมโยงได้และกราฟที่เชื่อมโยงไม่ได้(EDGE-MAGIC TOTAL LABELINGS ON CONNECTED AND DISCONNECTED GRAPHS) อ. ที่ปรึกษา : รองศาสตราจารย์ ดร.วนิดา เหมะกุล, 65 หน้า. ISBN 974-03-0932-1.

การกำกับรวมอย่างมหัศจรรย์บนด้านของกราฟ G ที่มี $V(G)$ เป็นเซตของจุดยอด และ $E(G)$ เป็นเซตของด้าน คือฟังก์ชัน f ที่เป็นฟังก์ชันหนึ่งต่อหนึ่งจาก $V(G) \cup E(G)$ ไปทั่วถึง $\{1, 2, ..., p + q\}$ เมื่อ $p = |V(G)|$ และ $q = |E(G)|$ ที่มีสมบัติว่า สำหรับทุกด้าน xy จะได้ว่า $f(x) + f(xy) + f(y) = k$ เมื่อ k เป็นค่าคงตัวที่กำหนดให้

วิทยานิพนธ์นี้ได้ศึกษาและรวบรวมกราฟที่มีการกำกับรวมอย่างมหัศจรรย์บนด้าน นอกจากนี้เรายังพิสูจน์ว่ากราฟต่อไปนี้มีการกำกับรวมอย่างมหัศจรรย์บนด้าน กราฟว่าว n เหลี่ยมหางยาว 1 เมื่อ n เป็นจำนวนคี่ สำหรับค่า k ที่ต่าง ๆ กัน กราฟสับปะรด n เหลี่ยมจุกมี m ใบ เมื่อ n เป็นจำนวนคี่ ผลผนวกที่แยกออกจากกันของกราฟว่าวขนาด n หางยาว 1 จำนวน m ชุด เมื่อ m และ n เป็นจำนวนคี่ และ กราฟที่ประกอบด้วยผลผนวกที่แยกออกจากกันของกราฟวิถีขนาด n จำนวน m ชุด และผลผนวกที่แยกออกจากกันของกราฟบริบูรณ์ขนาด 1 จำนวน m ชุด เมื่อ m เป็นจำนวนคี่ และ n เป็นจำนวนคู่

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An edge-magic total labeling on a graph G with the vertex-set $V(G)$ and the edge-set $E(G)$ is a one-to-one function f from $V(G) \cup E(G)$ onto the set $\{ 1, 2, ..., p + q \}$ where $p = |V(G)|$ and $q = |E(G)|$ with the property that, for any edge xy, $f(x) + f(xy) + f(y) = k$ for some constant k.

This thesis surveys and collects many classes of graphs that can admit an edge-magic total labeling. Moreover, we prove that the following graphs have edge-magic total labelings: an $(n,1)$-kite when n is odd for some different values of k, an (n,m)-pineapple when n is odd, the graph $m(n,1)$-kite: the disjoint union of m copies of $(n,1)$-kite, when m and n are odd and the graph $mP_n \cup mK_1$: the graph consists of the disjoint union of m copies of P_n and the disjoint union of m copies of K_1, when m is odd and n is even.

Department **Mathematics** Student's signature..

Field of study **Mathematics** Advisor's signature..

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CHAPTER I

INTRODUCTION

An edge-magic total labeling is motivated by the idea of magic squares in number theory. In 1970 A. Kotzig and A. Rosa [6] defined an edge-magic total labeling of a graph G as a bijection from $V(G) \bigcup E(G)$ to the set of integers from 1 to $|V(G)|+|E(G)|$ such that the sum of labels on an edge and its two endpoints is the same for all edges and G is called edge-magic (graph). They proved that complete bipartite graphs $K_{n,m}$ are edge-magic for all n and m, cycles C_n are edge-magic for all $n \geq 3$ and the disjoint union of n copies of P_2 is edge-magic where n is odd. In 1996 G. Ringel and A. S. Llado [8] proved: graphs with p vertices and q edges are not edge-magic if q is even and $p + q \equiv 2(mod\ 4)$ and each vertex has odd degree and they also showed that wheels W_n are not edge-magic when $n \equiv 3(mod\ 4)$. In 1998 R. D. Godbold and P. J. Slater [5] found the maximum and minimum values of magic sums for cycles C_n. In 1999 W. D. Wallis, E. T. Baskoro, M. Miller and M. Slamin [10] enumerated every edge-magic total labeling of complete graphs K_n and proved that n-suns and (n, 1)-kites are edge-magic. R. Ichishima, R. M. Figueroa-Centeno and F. A. Muntaner-Batle [1] proved that fans f_n for all n, ladders L_n when n is odd and books B_n for all n are edge-magic. In 2000 J. Wijaya and E. T. Baskoro [11] showed that the disjoint union of m copies of C_n when m and n are odd and the disjoint union of m copies of P_n where m is odd are edge-magic. This thesis surveys, collects many classes of graphs that can admit an edge-magic total labeling and considers such a labeling applied to some classes of disconnected graphs. Also proofs of some theorems are rewritten for better understanding.

There are four chapters in this thesis. In Chapter I, we introduce some authors who have studied edge-magic total labelings on many classes of graphs.

In Chapter II, we give definitions of varieties of graphs, a lemma and propositions that will be used in this thesis. Also examples are provided.

In Chapter III, edge-magic total labelings on many classes of connected graphs are discussed and edge-magic total labelings on connected graphs: an $(n, 1)$-kite and an $(n,\, m)$-pineapple, are shown.

In Chapter IV, edge-magic total labelings on some classes of disconnected graphs are discussed and we also show edge-magic total labelings on the following disconnected graphs: the graph $m(n,\, 1)$-kites, the disjoint union of m copies of $(n,\, 1)$-kite, when m and n are odd and the graph $mP_n \bigcup mK_1$, the graph consists of the disjoint union of m copies of P_n and the disjoint union of m copies of K_1, when m is odd and n is even.

CHAPTER II

DEFINITIONS AND EXAMPLES

We first introduce the definitions, follow by examples, a lemma and propositions that are needed in the next chapters.

Definition 2.1. A *graph* G consists of a finite nonempty set $V(G)$ of elements, called *vertices*, and a set $E(G)$ of 2-element subsets of $V(G)$, called *edges*.
We call $V(G)$ as the *vertex-set* of G and $E(G)$ as the *edge-set* of G.
If $\{x, y\}$ is an edge in a graph G, then an edge $\{x, y\}$ *joins* x and y, or x and y are *adjacent*, or an edge $\{x, y\}$ is *incident* to x (or y). We usually write $\{x, y\}$ as xy.

Definition 2.2. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 2.3. A graph G is *connected* if for any given pair of vertices a and b there is a finite sequence of distinct vertices and edges of the form v_{i_0}, e_{i_1}, v_{i_1}, $\ldots$, e_{i_n}, v_{i_n} where $v_{i_0} = a$ and $v_{i_n} = b$ and $e_{i_1} = v_{i_0}v_{i_1}, e_{i_2} = v_{i_1}v_{i_2}, \ldots, e_{i_n} = v_{i_{n-1}}v_{i_n}$, and *disconnected* otherwise.

Definition 2.4. A *component* of a graph G is a connected subgraph of G that is not contained in any larger connected subgraph of G.

Definition 2.5. The *degree* of a vertex v in graph G, denoted by *deg* v, is the number of edges incident to v.

Definition 2.6. Let G_1 and G_2 be graphs with disjoint vertex-sets $V(G_1)$ and $V(G_2)$ and edge-sets $E(G_1)$ and $E(G_2)$ respectively. The *join* of G_1 and G_2, denoted by G_1+G_2, is a graph with the vertex-set $V(G_1)\bigcup V(G_2)$ and the edge-set $E(G_1)\bigcup E(G_2)$ and all edges joining vertices in $V(G_1)$ and $V(G_2)$.

Definition 2.7. A *cycle* C_n, $n \geq 3$, is a graph which the vertex-set is $\{v_1, v_2, \ldots, v_n\}$ and the edge-set is $\{e_1 = v_1v_2, e_2 = v_2v_3, \ldots, e_{n-1} = v_{n-1}v_n, e_n = v_nv_1\}$.

Definition 2.8. A *path* P_n is a cycle with an edge deleted.

Definition 2.9. A *complete graph* K_n is a graph of n vertices which every two distinct vertices are adjacent.

Definition 2.10. The *wheel* W_n, $n \geq 4$, is the graph $K_1 + C_n$.

Definition 2.11. The *fan* F_n is the graph $P_n + K_1$.

Definition 2.12. An *n-sun* is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex.

Definition 2.13. An $(n,\ t)$-*kite* is a graph which consists of a cycle C_n and a path graph P_{t+1} (the tail) attached to one vertex.

Definition 2.14. A *complete bipartite graph* $K_{n,m}$ is a graph whose the vertex-set can be partitioned into two subsets V_1 and V_2 where $|V_1| = n$ and $|V_2| = m$ and two vertices are adjacent if they lie in different sets.

Definition 2.15. A *star* is a complete bipartite graph $K_{1,n}$.

Definition 2.16. Let G_1 and G_2 be graphs with disjoint vertex-sets $V(G_1)$ and $V(G_2)$ and edge-sets $E(G_1)$ and $E(G_2)$ respectively. The *product* of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with the vertex-set $V(G_1) \times V(G_2)$ and specified by putting (u_1, u_2) adjacent to (v_1, v_2) if either $u_1 = v_1$ and $u_2v_2 \ \epsilon \ E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \ \epsilon \ E(G_1)$.

Definition 2.17. The *ladder* L_n is the graph $P_n \times P_2$.

Definition 2.18. The *book* B_n is the graph $K_{1,n} \times K_2$.

Definition 2.19. An $(n,\ m)$*-pineapple* is a graph which consists of a cycle C_n and m copies of P_2 attached to one vertex.

Definition 2.20. A *tree* is a connected graph with n vertices and $n-1$ edges.

Definition 2.21. Let $G_1, G_2, \ldots, G_m$ be graphs with disjoint vertex-sets $V(G_1), V(G_2), \ldots, V(G_m)$ and edge-sets $E(G_1), E(G_2), \ldots, E(G_m)$ respectively. The *disjoint union* of $G_1, G_2, \ldots, G_m$, denoted by $G_1 \bigcup G_2 \bigcup \ldots \bigcup G_m$, is a graph with the vertex-set $V(G_1) \bigcup V(G_2) \bigcup \ldots \bigcup V(G_m)$ and the edge-set $E(G_1) \bigcup E(G_2) \bigcup \ldots \bigcup E(G_m)$.

If $G_1 = G_2 = \ldots = G_m = G$ then $G_1 \bigcup G_2 \bigcup \ldots \bigcup G_m$ is denoted by mG and is called the disjoint union of m copies of G.

Definition 2.22. A *caterpillar* $CP_{n_1,\ldots,n_t}$ is the graph $K_{1,n_1} \bigcup \ldots \bigcup K_{1,n_t}$ in which each K_{1,n_i} shares exactly one edge with $K_{1,n_{i+1}}$ and $t-1$ is the length of the skeleton path.

Figure 2.1 and figure 2.2 show diagrams which represent C_8, W_4, K_4, P_5, F_5, 8-sun, $(4,2)$-kite, $K_{2,4}$, $K_{1,5}$, L_5, B_4, $(5,4)$-pineapple and $CP_{4,4,6,4}$.

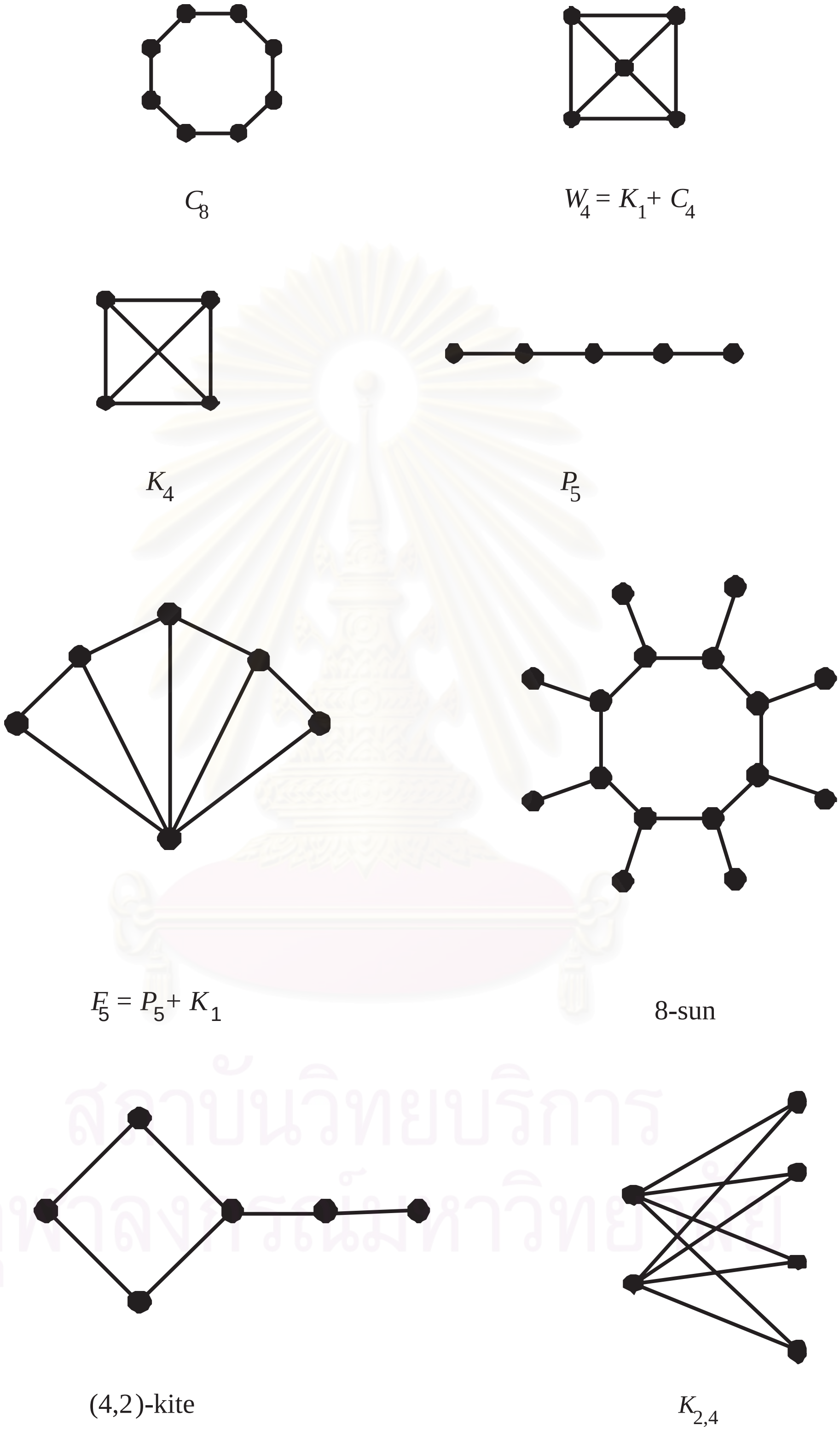


Figure 2.1: Diagrams which represent C_8, W_4, K_4, P_5, F_5, 8-sun, $(4, 2)$-kite and $K_{2,4}$.

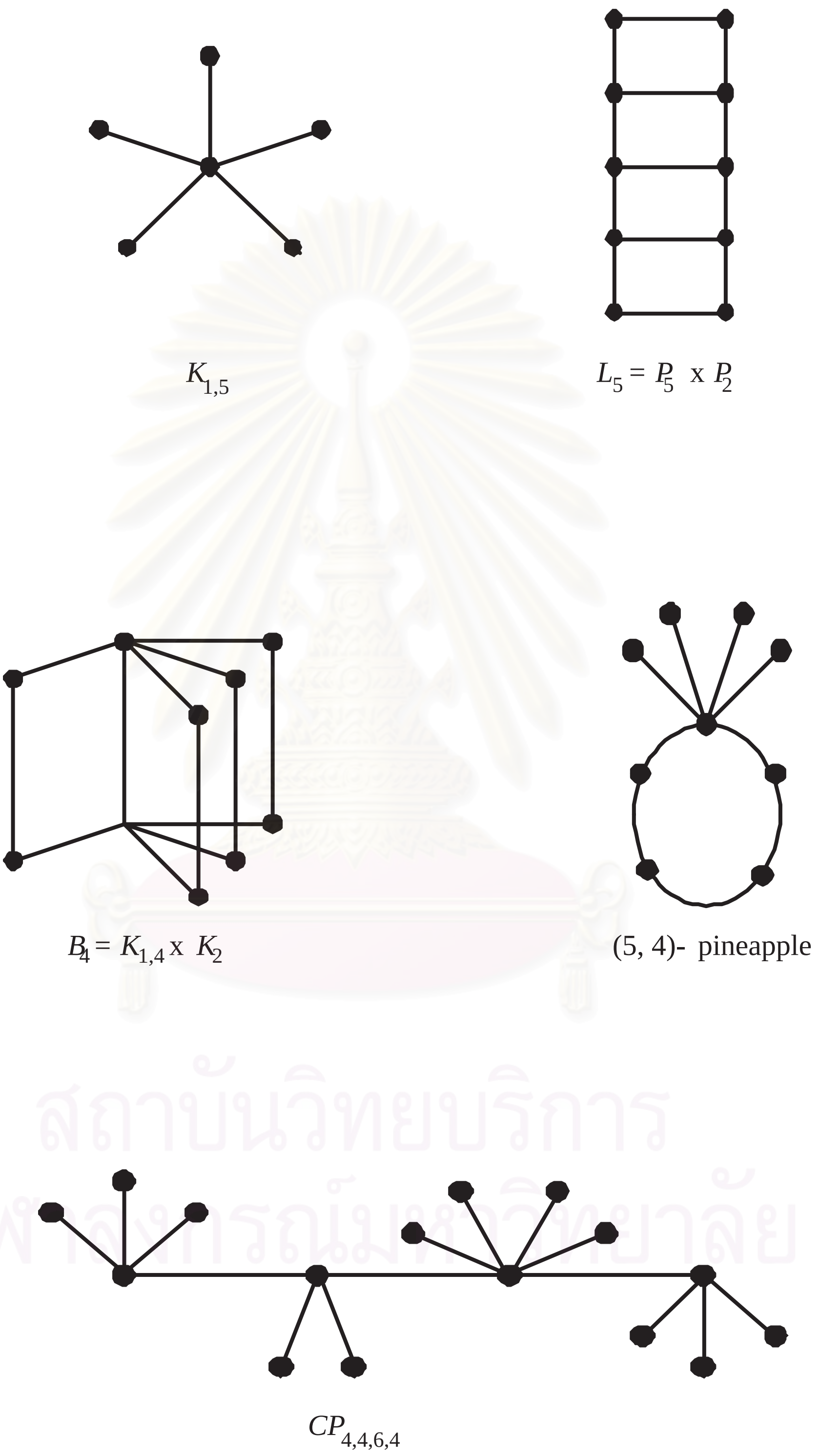


Figure 2.2: Diagrams which represent $K_{1,5}$, L_5, B_4, $(5,4)$-pineapple and $CP_{4,4,6,4}$.

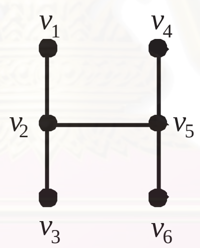
Definition 2.23. An *edge-magic total labeling* on a graph G is a one-to-one function from $V(G)\bigcup E(G)$ onto the set $\{1,2,...,p+q\}$ where $p=|V(G)|$ and $q=|E(G)|$ with the property that, for any edge xy

$$f(x)+f(xy)+f(y)=k$$

for some constant k which is called a *magic sum.*

Definition 2.24. A graph G is called *edge-magic* if it admits an edge-magic total labeling.

Example 2.25. A tree T with vertex-set $V(T) = \{v_1,v_2,v_3,v_4,v_5,v_6\}$ and edge-set $E(T) = \{v_1v_2, v_2v_3, v_4v_5, v_5v_6, v_2v_5\}$ is edge-magic with $k = 18$, that is



Define $f: V(T)\bigcup E(T) \to \{1,2,\ldots,11\}$ by $f(v_1)=3$, $f(v_2)=11$, $f(v_3)=2$, $f(v_4) = 8$, $f(v_5) = 1$, $f(v_6) = 7$, $f(v_1v_2) = 4$, $f(v_2v_3) = 5$, $f(v_4v_5) = 9$, $f(v_5v_6)=10$, $f(v_2v_5)=6$.

For edge v_1v_2, $f(v_1)+f(v_1v_2)+f(v_2)=3+4+11=18$.

For edge v_2v_3, $f(v_2)+f(v_2v_3)+f(v_3)=11+5+2=18$.

For edge v_4v_5, $f(v_4)+f(v_4v_5)+f(v_5)=8+9+1=18$.

For edge v_5v_6, $f(v_5)+f(v_5v_6)+f(v_6)=1+10+7=18$.

For edge v_2v_5, $f(v_2)+f(v_2v_5)+f(v_5)=11+6+1=18$.

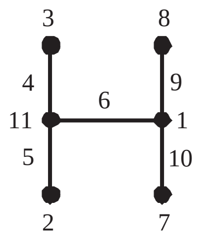


Figure 2.3: An edge-magic total labeling of a tree T.

Example 2.26. The graph in figure 2.4 is a famous graph which is disscussed by Julius Petersen and is named *Petersen graph* after him in a paper of 1898 [12]. Petersen graph is edge-magic with $k = 29$.

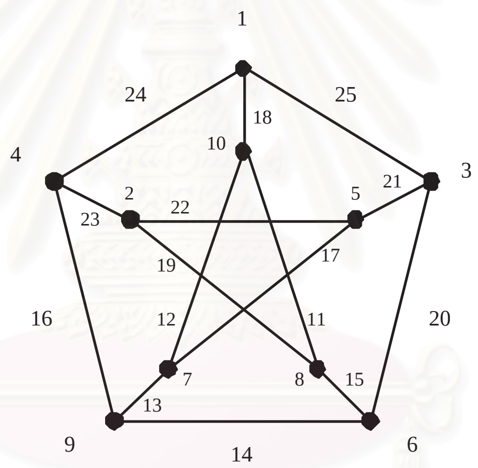


Figure 2.4: An edge-magic total labeling of Petersen graph.

Example 2.27. $3C_3$, the disjoint union of 3 copies of C_3, is edge-magic with $k = 24$.

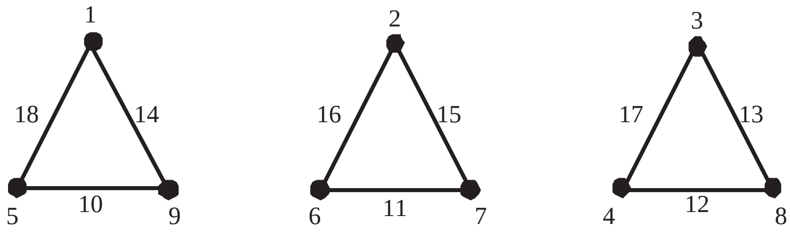


Figure 2.5: An edge-magic total labeling of $3C_3$.

From now on we assume the following:

1. G is a graph with vertex-set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and edge-set $E(G) = \{e_1, e_2, \ldots, e_q\}$ that is $|V(G)| = p$ and $|E(G)| = q$,
2. the degree of v_i is d_i,
3. if G is edge-magic, then k is a magic sum, f is an edge-magic total labeling and the label of v_i is x_i, i.e. $f(v_i) = x_i$,
4. $M = p + q + 1$,
5. $S = \{x_i : 1 \leq i \leq p\}$, and
6. $s = \sum_{i=1}^{p} x_i$.

Lemma 2.28. [10] *If G is edge-magic, then*

(a) $kq = \binom{M}{2} + \sum_{i=1}^{p} (d_i - 1)x_i$, *and*

(b) $\binom{p+1}{2} \leq s \leq pq + \binom{p+1}{2}$.

Proof. (a) Since G is edge-magic, the sum of all edge sums, the sum of labels of an edge and its two endpoints, is kq which contains each label once and each vertex label x_i an additional $d_i - 1$ times. So we obtain (a).

(b) The sum s is between the sum of 1 to p and the sum of $1+q$ to $p+q$ or $\sum_{i=1}^{p} i \leq s \leq \sum_{i=1+q}^{p+q} i$. Since $\sum_{i=1}^{p} i = \binom{p+1}{2}$ and $\sum_{i=1+q}^{p+q} i = (1+q) + (2+q) + \ldots + (p+q) = pq + \binom{p+1}{2}$, $\binom{p+1}{2} \leq s \leq pq + \binom{p+1}{2}$. $\square$

Proposition 2.29. [8] *If q is even, $p+q \equiv 2(mod\ 4)$ and each vertex has odd degree, then G is not edge-magic.*

Proof. Suppose that G is edge-magic. By lemma 2.28(a)

$$\begin{aligned} kq &= \binom{M}{2} + \sum_{i=1}^{p}(d_i - 1)x_i \\ &= \frac{(p+q+1)(p+q)}{2} + (d_1-1)x_1 + (d_2-1)x_2 + \ldots + (d_p-1)x_p. \end{aligned}$$

Since $p+q \equiv 2(mod\ 4)$, $p+q = 4t+2$ for some $t \in \mathbb{Z}$. So

$$\begin{aligned} kq &= \frac{(4t+3)(4t+2)}{2} + (d_1-1)x_1 + (d_2-1)x_2 + \ldots + (d_p-1)x_p \\ &= (4t+3)(2t+1) + (d_1-1)x_1 + (d_2-1)x_2 + \ldots + (d_p-1)x_p. \end{aligned}$$

Since q is even, 2 can divide kq. We consider only $4t+3$, because $2t+1$ is odd and 2 can divide $(d_i-1)x_i$ for all i, since d_i is odd. Since $4t$ is even, $4t+3$ is odd. So 2 can not divide kq, a contradiction. Hence G is not edge-magic. □

Definition 2.30. Let G be edge-magic. The *duality* f' of f (*or dual labeling*) is defined by $f'(v_i) = M - f(v_i)$ for any vertex v_i and $f'(e_i) = M - f(e_i)$ for any edge e_i .

Proposition 2.31. [10] *If G is edge-magic, then the duality f' of f is an edge-magic total labeling with magic sum $k' = 3M - k$ and the sum $s' = pM - s$.*

Proof. Assume G is edge-magic. Let f be an edge-magic total labeling, so f' is one-to-one and onto. Let v_iv_j be an edge in G. Then $k = f(v_i) + f(v_iv_j) + f(v_j)$. So

$$k = M - f'(v_i) + M - f'(v_iv_j) + M - f'(v_j).$$

Then

$$k' = f'(v_i) + f'(v_iv_j) + f'(v_j) = 3M - k.$$

And

$$
\begin{aligned}
s' &= \sum_{i=1}^{p} f'(v_i) \\
&= \sum_{i=1}^{p} M - f(v_i) \\
&= pM - \sum_{i=1}^{p} f(v_i) \\
&= pM - s.
\end{aligned}
$$

$\square$

Example 2.32. The graph G in figure 2.6 with the vertex-set $V(G) = \{v_1, v_2, v_3, v_4\}$ and the edge-set $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$ is edge-magic with $k = 12$.

Define $f : V(G) \bigcup E(G) \to \{1, 2, \ldots, 9\}$ by $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 4$, $f(v_4) = 3$, $f(v_1v_2) = 9$, $f(v_2v_3) = 6$, $f(v_3v_4) = 5$, $f(v_4v_1) = 8$, $f(v_1v_3) = 7$. Then $s = 10$.

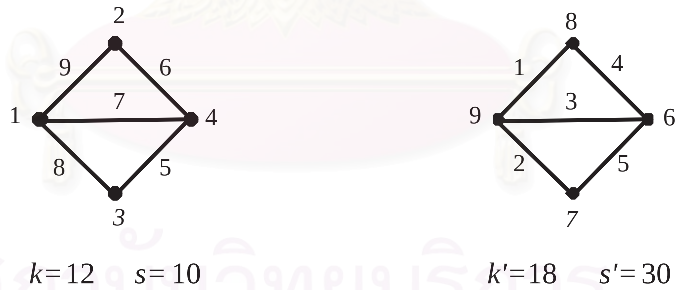


Figure 2.6: An edge-magic total labeling and its duality.

Then the duality f' is defined by $f'(v_1) = 9$, $f'(v_2) = 8$, $f'(v_3) = 6$, $f'(v_4) = 7$, $f'(v_1v_2) = 1$, $f'(v_2v_3) = 4$, $f'(v_3v_4) = 5$, $f'(v_4v_1) = 2$, $f'(v_1v_3) = 3$ with $k' = 18$ and $s' = 30$.

CHAPTER III

EDGE-MAGIC TOTAL LABELINGS ON CONNECTED GRAPHS

In this chapter, we discuss some connected graphs which are or are not edge-magic and give some examples of small cases.

Theorem 3.1. [10] *The wheel W_n when $n \equiv 3(mod\ 4)$ is not edge-magic.*

Proof. Since $n \equiv 3(mod\ 4)$, there exists $t \in \mathbb{Z}^+$ such that $n = 4t+3$. Since the number q of edges of W_n is $2n$ which is even,

$$\begin{aligned} p+q &= (n+1)+2n \\ &= 3n+1 \\ &= 3(4t+3)+1 \\ &= 12t+10 \\ &= 4(3t+2)+2. \end{aligned}$$

Thus $p+q \equiv 2(mod\ 4)$. Clearly every vertex of W_n has odd degree. By proposition 2.29, W_n is not edge-magic. □

Theorem 3.2. [8] *The complete graph K_n when $n \equiv 4$ or $6(mod\ 8)$ is not edge-magic .*

Proof. Case 1: $n \equiv 4(mod\ 8)$. There exists $t \in \mathbb{Z}^+$ such that $n = 8t+4$. Since the number q of edges of K_n is $\binom{n}{2}$ which is always even,

$$\begin{aligned} p+q &= n+\binom{n}{2} \\ &= n+\frac{n^2-n}{2} \\ &= \frac{n(n+1)}{2} \\ &= \frac{(8t+4)(8t+5)}{2} \\ &= (4t+2)(8t+5) \\ &= 32t^2+36t+10 \\ &= 4(8t^2+9t+2)+2. \end{aligned}$$

Thus $p+q \equiv 2(mod\ 4)$. Since the degree of each vertex of K_n is $n-1$ which is odd and q is even, by proposition 2.29, K_n is not edge-magic.

Case 2: $n \equiv 6(mod\ 8)$. The proof is similar to the previous case. □

Definition 3.3. [7] A *well-spread sequence* of length n is a sequence $A = (a_1, a_2, \ldots, a_n)$ of positive integers with the following properties:

1. $0 < a_1 < a_2 < \ldots < a_n$
2. $a_i + a_j \neq a_k + a_l$ whenever $i \neq j$ and $k \neq l$ (except, of course, when $\{a_i, a_j\} = \{a_k, a_l\}$).

And we define $\rho(A) = a_n + a_{n-1} - a_2 - a_1 + 1$ and $\rho^*(n) = \min\rho(A)$ where the minimum is taked over all well-spread sequences A of length n.

Remarks 3.1. The values of $\rho^*(n)$ are discussed in [6] and show that

1. $\rho^*(7) = 30$
2. $\rho^*(8) = 43$
3. $\rho^*(n) \geq n^2 - 5n + 14$ when $n > 8$.

Theorem 3.4. [10] *If G is edge-magic which contains a complete subgraph with n vertices, then the number of vertices and edges in G is at least $\rho^*(n)$.*

Proof. Assume that G is edge-magic with a magic sum k and contains a complete subgraph H with n vertices $b_1, b_2, \ldots, b_n$. Let f be an edge-magic total labeling of G and $f(b_i) = a_i$ for all i. So we can assume that $a_1 < a_2 < \ldots < a_n$. Then $A = (a_1, a_2, \ldots, a_n)$ is well-spread sequence. So $f(b_n b_{n-1}) = k - a_n - a_{n-1}$ and $f(b_2 b_1) = k - a_2 - a_1$. Then $k - a_n - a_{n-1} \geq 1$ and $k - a_2 - a_1 \leq p + q$. Therefore $p + q \geq a_n + a_{n-1} - a_2 - a_1 + 1 = \rho^*(n)$. $\square$

Theorem 3.5. [10] *No complete graph with more than 6 vertices is edge-magic.*

Proof. Suppose a complete graph K_n where $n > 6$ is edge-magic.By theorem 3.4

$$n + \binom{n}{2} \geq \rho^*(n). \tag{3.1}$$

For $n = 7$, by remarks 3.1 and the equation (3.1)

$$28 = 7 + \binom{7}{2} \geq \rho^*(7) = 30,$$

a contradiction.

For $n = 8$, by remarks 3.1 and the equation (3.1)

$$36 = 8 + \binom{8}{2} \geq \rho^*(8) = 43,$$

a contradiction.

For $n > 8$, by remarks 3.1 and the equation (3.1)

$$n + \binom{n}{2} \geq \rho^*(n) \geq n^2 - 5n + 14.$$

So

$$\frac{1}{2}n^2 - \frac{11}{2}n + 14 \leq 0.$$

If $\frac{1}{2}n^2 - \frac{11}{2}n + 14 = 0$, then $n = \frac{11 \pm \sqrt{11^2 - 8(48)}}{2} = 4$ or 7.

If $\frac{1}{2}n^2 - \frac{11}{2}n + 14 < 0$, then $4 < n < 7$.

All cases are contradicted. Therefore K_n is not edge-magic when $n > 6$. $\square$

n	s	k
odd	$\frac{1}{2}n(n+1)$	$\frac{1}{2}(5n+3)$
	$\frac{1}{2}n(n+3)$	$\frac{1}{2}(5n+5)$
	$\vdots$	$\vdots$
	$\frac{1}{2}n(n+2i-1)$	$\frac{1}{2}(5n+2i+1)$
	$\vdots$	$\vdots$
	$\frac{1}{2}n(3n+1)$	$\frac{1}{2}(7n+3)$
even	$\frac{1}{2}n^2+n$	$\frac{5}{2}n+2$
	$\frac{1}{2}n^2+2n$	$\frac{5}{2}n+3$
	$\vdots$	$\vdots$
	$\frac{1}{2}n^2+in$	$\frac{5}{2}n+i+1$
	$\vdots$	$\vdots$
	$\frac{3}{2}n^2$	$\frac{7}{2}n+1$

Table 3.1: The possible values s with corresponding magic sums k of cycles C_n.

Next we will consider some graphs which are edge-magic. A. Kotzig and A. Rosa [6] proved that all cycles are edge-magic and R. D. Godbold and P. J. Slater [5] can find the minimum and maximum values of magic sums k; later, W. D. Wallis and others [10] can find many possible values of magic sums k. So we start with the way to find magic sums k.

Proposition 3.6. [10] *If cycle C_n is edge-magic, then possible values s with corresponding magic sums k are in table 3.1.*

Proof. Assume C_n is edge-magic. Since the degree of the cycle C_n is 2, by lemma 2.28(a)

$$kn = \binom{2n+1}{2} + s\,.$$

So

$$kn = \frac{2n(2n+1)}{2} + s\,. \tag{3.2}$$

The equation (3.2) is possible if n divides s. So $s = n(k - 2n - 1)$.

By lemma 2.28(b),

$$\frac{n(n+1)}{2} \leq s \leq n^2 + \frac{n(n+1)}{2}\,.$$

So

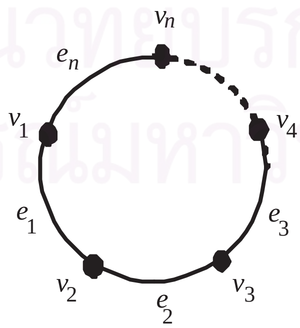
$$\frac{n(n+1)}{2} \leq s \leq \frac{2n^2 + (n+1)n}{2}\,.$$

Therefore

$$\frac{n(n+1)}{2} \leq s \leq \frac{n(3n+1)}{2}\,. \tag{3.3}$$

By the equations (3.2) and (3.3) we can know all possible values s, and also can get magic sums k which are corresponded to s. $\square$

We are going to show that C_n is edge-magic by giving the notations for C_n as follows: $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ where $e_1 = v_1v_2$, $e_2 = v_2v_3, \ldots, e_{n-1} = v_{n-1}v_n, e_n = v_nv_1$, that is



Theorem 3.7. [10] *Every odd cycle C_n is edge-magic with $k = \frac{1}{2}(5n+3)$.*

Proof. Let $n = 2t+1$ for some $t \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+1}{2} & \text{for } i = 1, 3, \ldots, 2t+1, \\ t + \frac{i+2}{2} & \text{for } i = 2, 4, \ldots, 2t; \end{cases}$$

and

$$f(e_i) = \begin{cases} 4t+2-i & \text{for } i = 1, 2, \ldots, 2t, \\ 4t+2 & \text{for } i = 2t+1. \end{cases}$$

The labeling f is shown in figure 3.1.

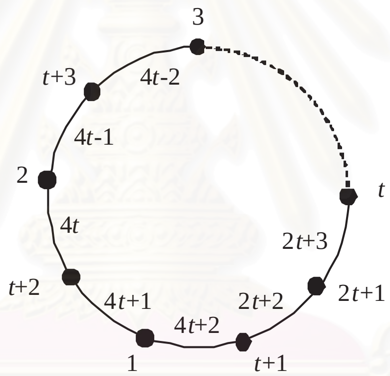


Figure 3.1: An edge-magic total labeling of C_{2t+1} with $k = 5t+4$ for some $t \in \mathbb{Z}^+$.

The numbers 1, 2, 3, …, $t+1$ are labels of v_1, v_3, v_5, …, v_{2t+1}. The numbers $t+2$, $t+3$, $t+4$, …, $2t+1$ are labels of v_2, v_4, v_6, …, v_{2t}. The numbers $2t+2, 2t+3, \ldots, 4t+1$ are labels of $e_{2t}, e_{2t-1}, \ldots, e_1$ and the number $4t+2$ is a label of e_{2t+1}. So all numbers 1 through $2n = 4t+2$ are used exactly once. Observe that

for e_i; $i = 1, 3, \ldots, 2t-1$,

$f(v_i) + f(e_i) + f(v_{i+1}) = (\frac{i+1}{2}) + (4t+2-i) + (t + \frac{i+1+2}{2}) = 5t+4 = \frac{1}{2}(5n+3)$,

for e_i; $i = 2, 4, \ldots, 2t$,

$f(v_i) + f(e_i) + f(v_{i+1}) = (t + \frac{i+2}{2}) + (4t+2-i) + (\frac{i+1+1}{2}) = 5t+4 = \frac{1}{2}(5n+3)$, and $f(v_1) + f(e_{2t+1}) + f(v_{2t+1}) = 1 + (4t+2) + (t+1) = 5t+4 = \frac{1}{2}(5n+3)$. Therefore f is an edge-magic total labeling with $k = \frac{1}{2}(5n+3)$ (this case is for the smallest magic sum k in proposition 3.6). □

By duality, we have the following corollary.

Corollary 3.2. [10] *Every odd cycle C_n is edge-magic with $k = \frac{1}{2}(7n+3)$.* □

Theorem 3.8. [10] *Every odd cycle C_n is edge-magic with $k = 3n+1$.*

Proof. Let $n = 2t+1$ for some $t \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} i & \text{for } i = 1, 3, \ldots, 2t+1, \\ 2t+i+1 & \text{for } i = 2, 4, \ldots, 2t; \end{cases}$$

and

$$f(e_i) = \begin{cases} 4t-2i+2 & \text{for } i = 1, 2, \ldots, 2t, \\ 4t+2 & \text{for } i = 2t+1. \end{cases}$$

The proof is similar to the previous theorem. Therefore f is an edge-magic total labeling with $k = 3n+1$. □

By duality, we have the following corollary.

Corollary 3.3. [10] *Every odd cycle C_n is edge-magic with $k = 3n+2$.* □

Theorem 3.9. [10] *Every even cycle C_n is edge-magic with $k = \frac{1}{2}(5n+4)$.*

Proof. Assume $n = 2t$ for some $t \in \mathbb{Z}^+$.

Case 1: *t is even.* Let $t = 2t'$ for some $t' \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+1}{2} & \text{for } i = 1, 3, \ldots, 2t'+1, \\ 6t' & \text{for } i = 2, \\ \frac{4t'+i}{2} & \text{for } i = 4, 6, \ldots, 2t', \\ \frac{i+2}{2} & \text{for } i = 2t'+2, 2t'+4, \ldots, 4t', \\ \frac{4t'+i-1}{2} & \text{for } i = 2t'+3, 2t'+5, \ldots, 4t'-1; \end{cases}$$

and

$$f(e_i) = \begin{cases} 4t'+1 & \text{for } i = 1, \\ 4t' & \text{for } i = 2, \\ 8t'-i+1 & \text{for } i = 3, 4, \ldots, 2t' \text{ and } i = 2t'+2, 2t'+3, \ldots, 4t'-1, \\ 8t'-1 & \text{for } i = 2t'+1, \\ 8t' & \text{for } i = 4t'. \end{cases}$$

From the given labeling f, the numbers $1, 2, \ldots, t'+1$ are labels of v_1, v_3, $\ldots$, $v_{2t'+1}$. The numbers $t'+2, t'+3, \ldots, 2t'+1$ are labels of $v_{2t'+2}$, $v_{2t'+4}$, $\ldots$, $v_{4t'}$. The numbers $2t'+2, 2t'+3, \ldots, 3t'$ are labels of v_4, v_6, $\ldots$, $v_{2t'}$. The numbers $3t'+1, 3t'+2, \ldots, 4t'-1$ are labels of $v_{2t'+3}$, $v_{2t'+5}$, $\ldots$, $v_{4t'-1}$. And the numbers $4t'$ and $4t'+1$ are labels of e_2 and e_1. The numbers $4t'+2$, $4t'+3$, $\ldots$, $6t'-1$ are labels of $e_{4t'-1}$, $e_{4t'-2}$, $\ldots$, $e_{2t'+2}$. The number $6t'$ is a label of v_2. The numbers $6t'+1$, $6t'+2$, $\ldots$, $8t'-2$ are labels of $e_{2t'}$, $e_{2t'-1}$, $\ldots$, e_3. And the numbers $8t'-1$ and $8t'$ are labels of $e_{2t'+1}$ and $e_{4t'}$. So all numbers 1 through $2n = 4t = 8t'$ are used exactly once. Observe that

$f(v_1) + f(e_1) + f(v_2) = 1 + (4t'+1) + 6t' = 10t' + 2 = \frac{1}{2}(5n+4)$,

$f(v_2) + f(e_2) + f(v_3) = 6t' + (4t') + 2 = 10t' + 2 = \frac{1}{2}(5n+4)$,

for e_i; $i = 3, 5, \ldots, 2t'-1$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{i+1}{2} + (8t' + 1 - i) + \frac{4t'+i+1}{2} = 10t' + 2 = \frac{1}{2}(5n + 4)$,

for e_i; $i = 4, 6, \ldots, 2t'$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{4t'+i}{2} + (8t' + 1 - i) + \frac{i+1+1}{2} = 10t' + 2 = \frac{1}{2}(5n + 4)$,

$f(v_{2t'+1}) + f(e_{2t'+1}) + f(v_{2t'+2}) = t' + 1 + (8t' - 1) + t' + 2 = 10t' + 2 = \frac{1}{2}(5n + 4)$,

for e_i; $i = 2t' + 2, 2t' + 4, \ldots, 2t' - 2$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{i+2}{2} + (8t' - i + 1) + \frac{4t'+i+1-1}{2} = 10t' + 2 = \frac{1}{2}(5n + 4)$,

for e_i; $i = 2t' + 3, 2t' + 5, \ldots, 2t' - 1$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{4t'+i-1}{2} + (8t' - i + 1) + \frac{i+1+2}{2} = 10t' + 2 = \frac{1}{2}(5n + 4)$,

and $f(v_{4t'}) + f(e_{4t'}) + f(v_1) = 2t' + 1 + (8t') + 1 = 10t' + 2 = \frac{1}{2}(5n + 4)$.

Case 2: *t is odd.* Let $t = 2t' + 1$ for some $t' \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+1}{2} & \text{for } i = 1, 3, \ldots 2t' + 1, \\ 6t' + 3 & \text{for } i = 2, \\ \frac{4t'+i+4}{2} & \text{for } i = 4, 6, \ldots 2t', \\ \frac{2t'+4}{2} & \text{for } i = 2t' + 2, \\ \frac{i+3}{2} & \text{for } i = 2t' + 3, 2t' + 5, \ldots 4t' + 1, \\ \frac{4t'+i+2}{2} & \text{for } i = 2t' + 4, 2t' + 6, \ldots 4t', \\ 2t' + 3 & \text{for } i = 4t' + 2; \end{cases}$$

and

$$f(e_i) = \begin{cases} 4t'+3 & \text{for } i = 1, \\ 4t'+2 & \text{for } i = 2, \\ 8t'-i+4 & \text{for } i = 3,4,\ldots,2t' \text{ and } i = 2t'+3, 2t'+4, \ldots, 4t', \\ 8t'+4 & \text{for } i = 2t'+1, \\ 8t'+2 & \text{for } i = 2t'+2, \\ 6t'+2 & \text{for } i = 4t'+1, \\ 8t'+3 & \text{for } i = 4t'+2. \end{cases}$$

We can verify similarly to the case 1. Therefore f is an edge-magic total labeling with $k = \frac{1}{2}(5n+4)$. □

By duality, we have the following corollary.

Corollary 3.4. [10] *Every cycle C_n when 4 divides n is edge-magic with $k = \frac{1}{2}(7n+2)$.* □

Theorem 3.10. [10] *Every cycle C_n when 4 divides n is edge-magic with $k = 3n$.*

Proof. For $n = 4$ we use the same labeling from theorem 3.9. So assume $n \geq 8$ and $n = 4t$ for some $t \in \mathbb{Z}^+$. Define a labeling f as follows:

$$f(v_i) = \begin{cases} i & \text{for } i = 1,3,\ldots 2t-1, \\ 4t+i+1 & \text{for } i = 2,4,\ldots,2t-2, \\ i+1 & \text{for } i = 2t, 2t+2, \ldots, 4t-2, \\ 4t+i & \text{for } i = 2t+1, 2t+3, \ldots 4t-3, \\ 2 & \text{for } i = 4t-1, \\ 2n-2 & \text{for } i = 4t; \end{cases}$$

and

$$f(e_i) = \begin{cases} 8t-2i-2 & \text{for } i = 1, 2, \ldots, 2t-2 \text{ and } i = 2t, 2t+1, \ldots, 4t-3, \\ 8t & \text{for } i = 2t-1, \\ 8t-1 & \text{for } i = 4t-2, \\ 4t & \text{for } i = 4t-1, \\ 4t+1 & \text{for } i = 4t. \end{cases}$$

We can verify similarly to the previous theorem. Therefore f is an edge-magic total labeling with $k = 3n$. □

By duality, we have the following corollary.

Corollary 3.5. [10] *Every cycle C_n when 4 divides n is edge-magic with $k = 3n + 3$.* □

Table 3.2 shows all possible edge-magic total labelings for C_n when $n \leq 6$.

Theorem 3.11. [10] *Every path graph P_n is edge-magic.*

Proof. Let f be an edge-magic total labeling from theorem 3.7, 3.8, 3.9 and 3.10 where the number $2n$ is a label of an edge. If we delete the edge which label $2n$, then a path graph P_n is edge-magic. □

cycle	k	s	label x_1,x_2,...,x_n
C_3	9	6	1,2,3
	10	9	1,3,5
	11	12	2,4,6
	12	15	4,5,6
C_4	12	12	1,3,2,6
	13	16	1,4,6,5
			1,5,2,8
C_5	14	15	1,4,2,5,3
	16	25	1,5,9,3,7
			1,7,3,4,10
C_6	17	24	1,5,2,3,6,7
			1,6,7,2,3,5
			1,5,4,3,2,9
	18	30	1,8,4,2,5,10
	19	36	1,6,11,3,7,8
			1,7,3,12,5,8
			1,8,7,3,5,12
			1,8,9,4,3,11
			2,7,11,3,4,9
			3,4,5,6,11,7

Table 3.2: All possible edge-magic total labelings for C_n when $n \leq 6$.

Theorem 3.12. [8] *A caterpillar* $CP_{n_1,\ n_2,\ \ldots,\ n_t}$ *is edge-magic.*

Proof. Define f by mapping consecutively (we start at the number 1) the non-center vertices of the stars K_{1,n_1}, K_{1,n_3}, K_{1,n_5}, ... and then the non-center vertices of the stars K_{1,n_2}, K_{1,n_4}, K_{1,n_6}, ... And we map the edges of the stars K_{1,n_t}, $K_{1,n_{t-1}}$, $K_{1,n_{t-2}}$, ... by starting at the edge which is incident to the vertex with the highest label. Then all vertices and edges are labeled.

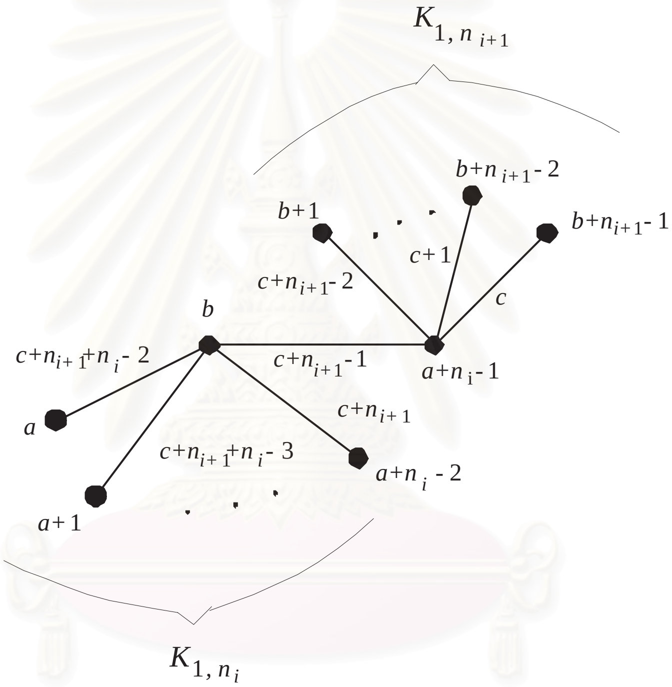


Figure 3.2: A labeling of caterpillar in the star K_{1,n_i} and the star $K_{1,n_{i+1}}$.

From the given labeling f, we consider the star K_{1,n_i} and the star $K_{1,n_{i+1}}$ in figure 3.2. If $a, a+1, \ldots, a+n_i-1$ are labels of the non-center vertices of the star K_{1,n_i} and $b, b+1, \ldots, b+n_{i+1}-1$ are labels of the non-center vertices of the star $K_{1,n_{i+1}}$ and $c, c+1, \ldots, c+n_{i+1}+n_i-2$ are labels of all edges of the star K_{1,n_i} and the star $K_{1,n_{i+1}}$, then the sum of labels of each edge and its two vertices which are adjacent is $a+b+c+n_{i+1}+n_i-2$ that is the same for all edges. $\square$

Conjecture [8] *Every tree is edge-magic.*

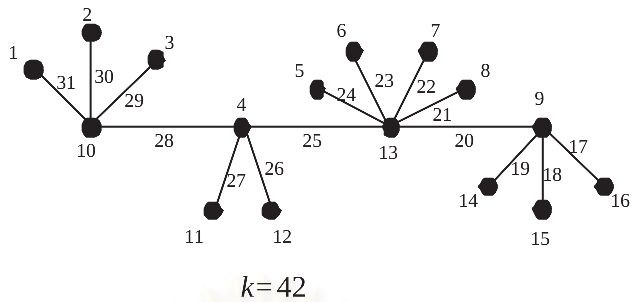
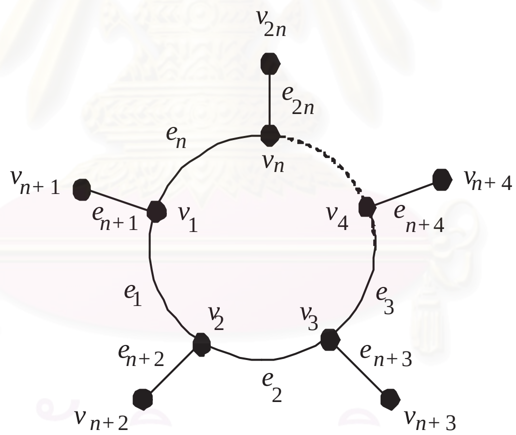


Figure 3.3: An edge-magic total labeling of $CP_{4,4,6,4}$ with a magic sum $k = 42$.

We are going to show that an n-sun is edge-magic by giving the notations for an n-sun as follows: $V(n\text{-sun}) = \{v_1, v_2, \ldots, v_{2n}\}$ and $E(n\text{-sun}) = \{e_1, e_2, \ldots, e_{2n}\}$ where $e_1 = v_1v_2, e_2 = v_2v_3, \ldots, e_{n-1} = v_{n-1}v_n, e_n = v_nv_1$ and $e_{n+i} = v_{n+i}v_i$ for $i = 1, 2, \ldots, n$, that is



Theorem 3.13. [10] *Every n-sun is edge-magic with $k = \frac{1}{2}(11n + 3)$ when n is odd.*

Proof. Let $n = 2t + 1$ for some $t \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+4t+3}{2} & \text{for } i = 1, 3, \ldots 2t+1, \\ \frac{i+4}{2} + 3t & \text{for } i = 2, 4, \ldots, 2t, \\ \frac{i+2}{2} & \text{for } i = 2t+2, 2t+4, \ldots, 4t, \\ \frac{i-2t+1}{2} & \text{for } i = 2t+3, 2t+5, \ldots, 4t+1, \\ 1 & \text{for } i = 4t+2; \end{cases}$$

and

$$f(e_i) = \begin{cases} 6t - i + 3 & \text{for } i = 1, 2, \ldots, 2t, \\ 6t+3 & \text{for } i = 2t+1, \\ 10t - i + 5 & \text{for } i = 2t+2, 2t+3, \ldots, 4t+1, \\ 8t+4 & \text{for } i = 4t+2. \end{cases}$$

From the given labeling f, the numbers 1, 2, ..., t+1 are labels of v_{4t+2}, v_{2t+3}, v_{2t+5}, ..., v_{4t+1}. The numbers $t+2$, $t+3$, ..., $2t+1$ are labels of v_{2t+2}, v_{2t+4}, ..., v_{4t}. The numbers $2t+2$, $2t+3$, ..., $3t+2$ are labels of v_1, v_3, ..., v_{2t+1}. The numbers $3t+3$, $3t+4$, ..., $4t+2$ are labels of v_2, v_4, ..., v_{2t}. The numbers $4t+3$, $4t+4$, ..., $6t+2$ are labels of e_{2t}, e_{2t-1}, ..., e_1. The numbers $6t+3$ is a label of e_{2t+1}. The numbers $6t+4$, $6t+5$, ..., $8t+3$ are labels of e_{4t+1}, e_{4t}, ..., e_{2t+2}. The number $8t+4$ is a label of e_{4t+2}. So all numbers 1 through $4n = 8t+4$ are used exactly once. Observe that

for e_i; $i = 1, 3, \ldots, 2t-1$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{i+4t+3}{2} + (6t - i + 3) + \frac{i+1+4}{2} + 3t = 11t + 7 = \frac{1}{2}(11n+3)$,

for e_i; $i = 2, 4, \ldots, 2t$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{i+4}{2} + 3t + (6t - i + 3) + \frac{i+1+4t+3}{2} = 11t + 7 = \frac{1}{2}(11n+3)$,

for e_{2t+1},

$f(v_{2t+1}) + f(e_{2t+1}) + f(v_1) = \frac{2t+1+4t+3}{2} + (6t+3) + \frac{1+4t+3}{2} = 11t + 7 = \frac{1}{2}(11n+3)$, for e_i; $i = 2t+2, 2t+4, \ldots, 4t$,

$$\begin{aligned} f(v_i) + f(e_i) + f(v_{i-(2t+1)}) &= \tfrac{i+2}{2} + (10t - i + 5) + \tfrac{i-2t-1+4t+3}{2} \\ &= 11t + 7 = \tfrac{1}{2}(11n+3), \end{aligned}$$

for e_i; $i = 2t+3, 2t+5, \ldots, 4t+1$,

$$\begin{aligned} f(v_i) + f(e_i) + f(v_{i-(2t+1)}) &= \tfrac{i-2t+1}{2} + (10t - i + 5) + \tfrac{i-2t-1+4}{2} + 3t \\ &= 11t + 7 = \tfrac{1}{2}(11n+3), \end{aligned}$$

and $f(v_{2t+1}) + f(e_{4t+2}) + f(v_{4t+2}) = 3t + 2 + (8t+4) + 1 = 11t + 7 = \frac{1}{2}(11n+3)$. Therefore f is an edge-magic total labeling with $k = \frac{1}{2}(11n+3)$ when n is odd. □

Theorem 3.14. [10] *Every n-sun is edge-magic with $k = \frac{1}{2}(11n+4)$ when n is even.*

Proof. Let $n = 2t$ for some $t \in \mathbb{Z}^+$.

Case 1: t is even. Let $t = 2t'$ for some $t' \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+8t'+1}{2} & \text{for } i = 1, 3, \ldots, 2t'+1, \\ 10t' & \text{for } i = 2, \\ \frac{i+12t'}{2} & \text{for } i = 4, 6, \ldots, 2t' \\ \frac{i+8t'+2}{2} & \text{for } i = 2t'+2, 2t'+4, \ldots, 4t' \\ \frac{i+12t'-1}{2} & \text{for } i = 2t'+3, 2t'+5, \ldots, 4t'-1 \\ \frac{i+1}{2} & \text{for } i = 4t'+1, 4t'+3, \ldots, 6t'+1, \\ 1 & \text{for } i = 4t'+2, \\ \frac{i}{2} - 2t' + 1 & \text{for } i = 4t'+4, 4t'+6, \ldots, 6t', \\ \frac{i+2}{2} & \text{for } i = 6t'+2, 6t'+4, \ldots, 8t'-2, \\ 2 & \text{for } i = 8t', \\ \frac{i+1}{2} - 2t' & \text{for } i = 6t'+3, 6t'+5, \ldots, 8t'-1; \end{cases}$$

and

$$f(e_i) = \begin{cases} 8t'+1 & \text{for } i = 1, \\ 8t' & \text{for } i = 2, \\ 12t'-i+1 & \text{for } i = 3, 4, \ldots, 2t' \\ & \text{and } i = 2t'+2, 2t'+3, \ldots, 4t'-1, \\ 12t'-1 & \text{for } i = 2t'+1, \\ 12t' & \text{for } i = 4t', \\ 20t'-i+1 & \text{for } i = 4t'+1, 4t'+3, \ldots, 6t'+1, \\ 12t'+1 & \text{for } i = 4t'+2, \\ 20t'-i+1 & \text{for } i = 4t'+4, 4t'+6, \ldots, 6t', \\ 20t'-i & \text{for } i = 6t'+2, 6t'+4, \ldots, 8t'-2, \\ 20t'-i+2 & \text{for } i = 6t'+3, 6t'+5, \ldots, 8t'-1, \\ 16t'-1 & \text{for } i = 8t'. \end{cases}$$

From the given labeling f, the numbers 1 and 2 are labels of $v_{4t'+2}$ and $v_{8t'}$. The numbers 3, 4, $\ldots$, $t'+1$ are labels of $v_{4t'+4}$, $v_{4t'+6}$, $\ldots$, $v_{6t'}$. The numbers $t'+2$, $t'+3$, $\ldots$, $2t'$ are labels of $v_{6t'+3}$, $v_{6t'+5}$, $\ldots$, $v_{8t'-1}$. The numbers $2t'+1$, $2t'+2$, $\ldots$, $3t'+1$ are labels of $v_{4t'+1}$, $v_{4t'+3}$, $\ldots$, $v_{6t'+1}$. The numbers $3t'+2$, $3t'+3$, $\ldots$, $4t'$ are labels of $v_{6t'+2}$, $v_{6t'+4}$, $\ldots$, $v_{8t'-2}$. The numbers $4t'+1$, $4t'+2$, $\ldots$, $5t'+1$ are labels of v_1, v_3, $\ldots$, $v_{2t'+1}$. The numbers $5t'+2$, $5t'+3$, $\ldots$, $6t'+1$ are labels of $v_{2t'+2}$, $v_{2t'+4}$, $\ldots$, $v_{4t'}$. The numbers $6t'+2$, $6t'+3$, $\ldots$, $7t'$ are labels of v_4, v_6, $\ldots$, $v_{2t'}$. The numbers $7t'+1$, $7t'+2$, $\ldots$, $8t'-1$ are labels of $v_{2t'+3}$, $v_{2t'+5}$, $\ldots$, $v_{4t'-1}$. The numbers $8t'$ and $8t'+1$ are labels of e_2 and e_1. The numbers $8t'+2$, $8t'+3$, $\ldots$, $10t'-1$ are labels of $e_{4t'-1}$, $e_{4t'-2}$, $\ldots$, $e_{2t'+2}$. The number $10t'$ is a label of v_2. The numbers $10t'+1$, $10t'+2$, $\ldots$, $12t'-2$ are labels of $e_{2t'}$, $e_{2t'-1}$,

$\ldots$, e_3. The numbers $12t'-1$, $12t'$ and $12t'+1$ are labels of $e_{2t'+1}$, $e_{4t'}$ and $e_{4t'+2}$. The numbers $12t'+2$, $12t'+3$, $12t'+4$, $12t'+5$, $\ldots$, $14t'-2$, $14t'-1$ are labels of $e_{8t'-2}$, $e_{8t'-1}$, $e_{8t'-4}$, $e_{8t'-3}$, $\ldots$, $e_{6t'+2}$, $e_{6t'+3}$. The numbers $14t'$, $14t'+1$, $\ldots$, $16t'-2$ are labels of $e_{6t'+1}$, $e_{6t'}$, $\ldots$, $e_{4t'+3}$. The numbers $16t'-1$ and $16t'$ are labels of $e_{8t'}$ and $e_{4t'+1}$. So all numbers 1 through $4n = 8t = 16t'$ are used exactly once. Observe that

for e_1,

$f(v_1)+f(e_1)+f(v_2) = 4t'+1+(8t'+1)+10t' = 22t'+2 = 11t+2 = \frac{1}{2}(11n+4)$,

for e_2,

$f(v_2)+f(e_2)+f(v_3) = 10t'+(8t')+4t'+2 = 22t'+2 = 11t+2 = \frac{1}{2}(11n+4)$,

for e_i; $i = 3, 5, \ldots, 2t'+1$,

$$\begin{aligned} f(v_i)+f(e_i)+f(v_{i+1}) &= \tfrac{i+8t'+1}{2} + (12t'+1-i) + \tfrac{i+1+12t'}{2} \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for e_i; $i = 4, 6, \ldots, 2t'$,

$$\begin{aligned} f(v_i)+f(e_i)+f(v_{i+1}) &= \tfrac{i+12t'}{2} + (12t'+1-i) + \tfrac{i+1+8t'+1}{2} \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for e_i; $i = 2t'+2, 2t'+4, \ldots, 4t'$,

$$\begin{aligned} f(v_i)+f(e_i)+f(v_{i+1}) &= \tfrac{i+8t'+2}{2} + (12t'+1-i) + \tfrac{i+1+12t'+1}{2} \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for e_i; $i = 2t'+3, 2t'+5, \ldots, 4t'-1$,

$$\begin{aligned} f(v_i)+f(e_i)+f(v_{i+1}) &= \tfrac{i+12t'-1}{2} + (12t'+1-i) + \tfrac{i+1+8t'+2}{2} \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for $e_{4t'+1}$,

$$\begin{aligned} f(v_1)+f(e_{4t'+1})+f(v_{4t'+1}) &= 4t'+1+(16t')+2t'+1 \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for $e_{4t'+2}$,

$f(v_2)+f(e_{4t'+2})+f(v_{4t'+2}) = 10t'+(12t'+1)+1 = 22t'+2 = 11t+2 = \frac{1}{2}(11n+4)$,

for e_i; $i = 4t'+3, 4t'+5, \ldots, 6t'+1$,

$$\begin{aligned} f(v_{i-4t'}) + f(e_i) + f(v_i) &= \tfrac{i-4t'+8t'+1}{2} + (20t'-i+1) + (\tfrac{i+1}{2}) \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for e_i; $i = 4t'+4, 4t'+6, \ldots, 6t'$,

$$\begin{aligned} f(v_{i-4t'}) + f(e_i) + f(v_i) &= \tfrac{i-4t'+12t'}{2} + (20t'-i+1) + \tfrac{i}{2} - 2t' + 1 \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for e_i; $i = 6t'+2, 6t'+4, \ldots, 8t'-2$,

$$\begin{aligned} f(v_{i-4t'}) + f(e_i) + f(v_i) &= \tfrac{i-4t'+8t'+2}{2} + 20t' - i + \tfrac{i+2}{2} \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for e_i; $i = 6t'+3, 6t'+5, \ldots, 8t'-1$,

$$\begin{aligned} f(v_{i-4t'}) + f(e_i) + f(v_i) &= \tfrac{i-4t'+12t'-1}{2} + 20t' - i + 2 + \tfrac{i+1}{2} - 2t' \\ &= 22t'+2 = 11t+2 = \tfrac{1}{2}(11n+4), \end{aligned}$$

for $e_{8t'}$,

$f(v_{4t'})+f(e_{8t'})+f(v_{8t'}) = 6t'+1+(16t'-1)+2 = 22t'+2 = 11t+1 = \frac{1}{2}(11n+4)$.

Case 2: t is odd. Let $t = 2t'+1$ for some $t' \in \mathbb{Z}^+$ and define a labeling f as follows:

$$
f(v_i) = \begin{cases}
\frac{i+8t'+5}{2} & \text{for } i = 1, 3, \ldots, 2t'+1, \\
10t'+5 & \text{for } i = 2, \\
\frac{i+12t'+8}{2} & \text{for } i = 4, 6, \ldots, 2t', \\
5t'+4 & \text{for } i = 2t'+2, \\
\frac{i+8t'+7}{2} & \text{for } i = 2t'+3, 2t'+5, \ldots, 4t'+1, \\
\frac{i+12t'+6}{2} & \text{for } i = 2t'+4, 2t'+6, \ldots, 4t', \\
6t'+5 & \text{for } i = 4t'+2, \\
\frac{i+1}{2} & \text{for } i = 4t'+3, 4t'+5, \ldots, 6t'+3, \\
1 & \text{for } i = 4t'+4, \\
\frac{i}{2} - 2t' + 1 & \text{for } i = 4t'+6, 4t'+8, \ldots, 6t'+2, \\
3t'+3 & \text{for } i = 6t'+4, \\
\frac{i+3}{2} & \text{for } i = 6t'+5, 6t'+7, \ldots, 8t'+1, \\
2 & \text{for } i = 8t'+3, \\
\frac{i}{2} - 2t' & \text{for } i = 6t'+6, 6t'+8, \ldots, 8t'+2, \\
3 & \text{for } i = 8t'+4;
\end{cases}
$$

and

$$f(e_i) = \begin{cases} 8t' + 5 & \text{for } i = 1, \\ 8t' + 4 & \text{for } i = 2, \\ 12t' - i + 6 & \text{for } i = 3, 4, \ldots, 2t', \\ & \text{and for } i = 2t' + 3, 2t' + 4, \ldots, 4t', \\ 12t' + 6 & \text{for } i = 2t' + 1, \\ 12t' + 4 & \text{for } i = 2t' + 2, \\ 10t' + 4 & \text{for } i = 4t' + 1, \\ 12t' + 5 & \text{for } i = 4t' + 2, \\ 20t' - i + 11 & \text{for } i = 4t' + 3, \\ & \text{and for } i = 4t' + 5, 4t' + 7, \ldots, 6t' + 3, \\ & \text{and for } i = 6t' + 6, 6t' + 8, \ldots, 8t' + 2, \\ 12t' + 7 & \text{for } i = 4t' + 4, \\ 14t' + 6 & \text{for } i = 6t' + 4, \\ 20t' - i + 9 & \text{for } i = 4t' + 6, 4t' + 8, \ldots, 6t' + 2, \\ & \text{and for } i = 6t' + 5, 6t' + 7, \ldots, 8t' + 1, \\ 16t' + 7 & \text{for } i = 8t' + 3, \\ 16t' + 5 & \text{for } i = 8t' + 4. \end{cases}$$

We can verify similarly to the previous case. Therefore f is an edge-magic total labeling with $k = \frac{1}{2}(11n + 4)$ when n is even. □

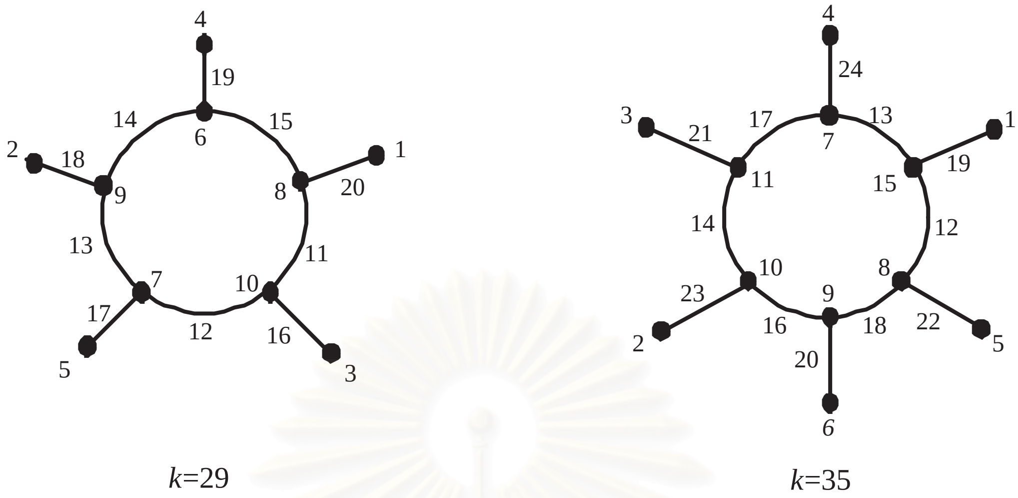
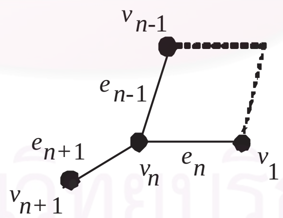


Figure 3.4: Edge-magic total labelings of 5-sun and 6-sun with magic sums $k = 29$ and $k = 35$ respectively.

We are going to show that an $(n, 1)$-kite is edge-magic by giving the notations as follows: $V((n, 1)\text{-kite}) = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ and $E((n, 1)\text{-kite}) = \{e_1, e_2, \ldots, e_n, e_{n+1}\}$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $e_n = v_n v_1$ and $e_{n+1} = v_n v_{n+1}$, that is



Theorem 3.15. *An $(n, 1)$-kite is edge-magic with $k = \frac{1}{2}(7n + 9)$ when n is odd.*

Proof. Let $n = 2t + 1$ for some $t \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} 4t + \frac{7-i}{2} & \text{for } i = 1, 3, \ldots, 2t+1, \\ 3t + \frac{6-i}{2} & \text{for } i = 2, 4, \ldots, 2t, \\ 4t + 4 & \text{for } i = 2t+2; \end{cases}$$

and

$$f(e_i) = \begin{cases} i+2 & \text{for } i = 1, 2, \ldots, 2t, \\ 2 & \text{for } i = 2t+1, \\ 1 & \text{for } i = 2t+2. \end{cases}$$

The numbers 1 and 2 are labels of e_{2t+2} and e_{2t+1}. The numbers 3, 4, ..., 2t+2 are labels of e_1, e_2, ..., e_{2t}. The numbers $2t+3$, $2t+4$, ..., $3t+2$ are labels of v_{2t}, v_{2t-2}, ..., v_2. The numbers $3t+3$, $3t+4$, ..., $4t+3$ are labels of v_{2t+1}, v_{2t-1}, ..., v_1. And the number $4t+4$ is a label of v_{2t+2}. So all numbers 1 through $2n = 4t+4$ are used exactly once. Observe that

for e_i; $i = 1, 3, \ldots, 2t-1$,

$f(v_i) + f(e_i) + f(v_{i+1}) = 4t + \frac{7-i}{2} + (i+2) + 3t + \frac{6-(i+1)}{2} = 7t + 8 = \frac{1}{2}(7n+9)$,

for e_i; $i = 2, 4, \ldots, 2t$,

$f(v_i) + f(e_i) + f(v_{i+1}) = 3t + \frac{6-i}{2} + (i+2) + 4t + \frac{7-(i+1)}{2} = 7t + 8 = \frac{1}{2}(7n+9)$,

for e_{2t+1},

$f(v_{2t+1}) + f(e_{2t+1}) + f(v_1) = 4t + \frac{7-2t-1}{2} + (2) + 4t + 3 = 7t + 8 = \frac{1}{2}(7n+9)$,

for e_{2t+2},

$f(v_{2t+2}) + f(e_{2t+2}) + f(v_{2t+1}) = 4t + 4 + (1) + 3t + 3 = 7t + 8 = \frac{1}{2}(7n+9)$.

Therefore f is an edge-magic total labeling with $k = \frac{1}{2}(7n+9)$ when n is odd. □

By duality, we have the following corollary.

Corollary 3.16. [10] *An $(n, 1)$-kite is edge-magic with $k = \frac{1}{2}(5n+9)$ when n is odd.* □

Theorem 3.17. *An $(n, 1)$-kite is edge-magic with $k = 3n + 4$ when n is odd.*

Proof. Let $n = 2t+1$ for some $t \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} i+1 & \text{for } i = 1, 3, \ldots, 2t+1, \\ 2t+i+2 & \text{for } i = 2, 4, \ldots, 2t, \\ 4t+4 & \text{for } i = 2t+2; \end{cases}$$

and

$$f(e_i) = \begin{cases} 4t-2i+3 & \text{for } i = 1, 2, \ldots, 2t, \\ 4t+3 & \text{for } i = 2t+1, \\ 1 & \text{for } i = 2t+2. \end{cases}$$

It is easy to verify that f is an edge-magic total labeling with $k = 3n+4$ when n is odd. □

By duality, we have the following corollary.

Corollary 3.18. *An $(n, 1)$-kite is edge-magic with $k = 3n+5$ when n is odd.* □

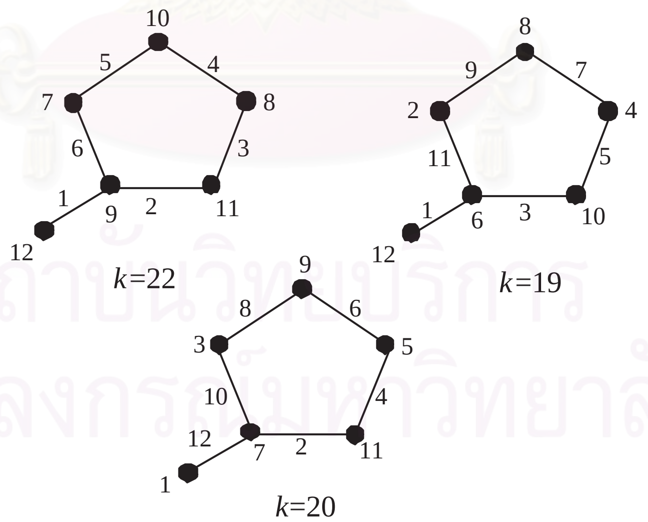


Figure 3.5: Edge-magic total labelings of (5, 1)-kites with magic sums $k = 22$, $k = 19$ and $k = 20$..

Theorem 3.19. [10] *An $(n, 1)$-kite is edge-magic with $k = \frac{1}{2}(5n + 10)$ when n is even.*

Proof. Let $n = 2t$ for some $t \in \mathbb{Z}^+$.

Case 1: *t is even.* Let $t = 2t'$ for some $t' \in \mathbb{Z}^+$ and define a labeling f as follows:

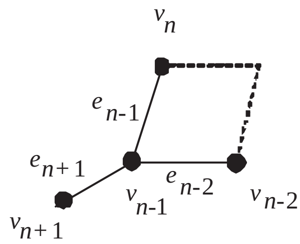
$$f(v_i) = \begin{cases} \frac{i+3}{2} & \text{for } i = 1, 3, \ldots, 2t' + 1, \\ 6t' + 1 & \text{for } i = 2, \\ \frac{4t'+i+2}{2} & \text{for } i = 4, 6, \ldots, 2t', \\ \frac{i+4}{2} & \text{for } i = 2t' + 2, 2t' + 4, \ldots, 4t', \\ \frac{4t'+i+1}{2} & \text{for } i = 2t' + 3, 2t' + 5, \ldots, 4t' - 1, \\ 8t' + 2 & \text{for } i = 4t' + 1; \end{cases}$$

and

$$f(e_i) = \begin{cases} 4t' + 2 & \text{for } i = 1, \\ 4t' + 1 & \text{for } i = 2 \\ 8t' - i + 2 & \text{for } i = 3, 4, \ldots, 2t' \text{ and } i = 2t' + 2, 2t' + 3, \ldots, 4t' - 1, \\ 8t' & \text{for } i = 2t' + 1, \\ 8t' + 1 & \text{for } i = 4t', \\ 1 & \text{for } i = 4t' + 1. \end{cases}$$

It is easy to verify that f is an edge-magic total labeling with $k = \frac{1}{2}(5n + 10)$ when $n = 2t$ where t is even.

Case 2: *t is odd.* Let $v_1, v_2, \ldots, v_{n+1}$ be the vertices and for $i = 1, 2, \ldots, n - 1$, $e_i = v_i v_{i+1}$ and $e_n = v_n v_1$ and $e_{n+1} = v_{n-1} v_{n+1}$, that is



Let $t = 2t' + 1$ for some $t' \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+3}{2} & \text{for } i = 1, 3, \ldots, 2t' + 1, \\ 6t' + 4 & \text{for } i = 2, \\ \frac{4t'+i+6}{2} & \text{for } i = 4, 6, \ldots, 2t', \\ \frac{2t'+6}{2} & \text{for } i = 2t' + 2, \\ \frac{i+5}{2} & \text{for } i = 2t' + 3, 2t' + 5, \ldots, 4t' + 1, \\ \frac{4t'+i+4}{2} & \text{for } i = 2t' + 4, 2t' + 6, \ldots, 4t', \\ 2t' + 4 & \text{for } i = 4t' + 2, \\ 8t' + 6 & \text{for } i = 4t' + 3; \end{cases}$$

and

$$f(e_i) = \begin{cases} 4t' + 4 & \text{for } i = 1, \\ 4t' + 3 & \text{for } i = 2, \\ 8t' - i + 5 & \text{for } i = 3, 4, \ldots, 2t', \\ & \text{and for } i = 2t' + 3, 2t' + 4, \ldots, 4t', \\ 8t' + 5 & \text{for } i = 2t' + 1, \\ 8t' + 3 & \text{for } i = 2t' + 2, \\ 6t' + 3 & \text{for } i = 4t' + 1, \\ 8t' + 4 & \text{for } i = 4t' + 2, \\ 1 & \text{for } i = 4t' + 3. \end{cases}$$

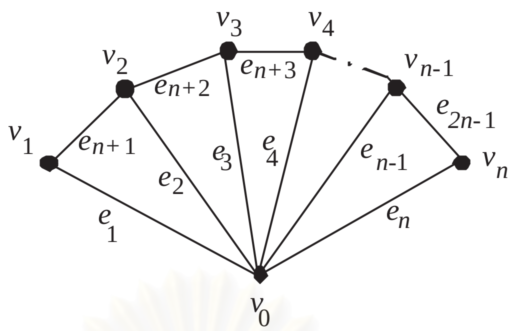
It is easy to verify that f is an edge-magic total labeling with $k = \frac{1}{2}(5n + 10)$ when $n = 2t$ where t is odd. $\square$

By duality, we have the following corollary.

Corollary 3.20. *An $(n, 1)$-kite is edge-magic with $k = \frac{1}{2}(7n + 8)$ when n is odd.* $\square$

Theorem 3.21. [3] *The fan F_n is edge-magic with $k = 3n + 3$ for all positive integer n.*

Proof. Let v_i be the vertex of F_n for $i = 0, 1, \ldots, n$ and $e_i = v_0v_i$ for $i = 1, \ldots, n$ and $e_{n+i} = v_iv_{i+1}$ for $i = 1, \ldots, n-1$, that is



Define a labeling f as follows:

$$f(v_i) = \begin{cases} 1 & \text{for } i = 0, \\ \frac{1-5(-1)^i+6i}{4} & \text{for } i = 1, \ldots, n; \end{cases}$$

and

$$f(e_i) = \begin{cases} \frac{12n+7+5(-1)^i-6i}{4} & \text{for } i = 1, \ldots, n, \\ 6n-3i+1 & \text{for } i = n+1, \ldots, 2n-1. \end{cases}$$

Case 1: *n is even.* Let $n = 2t$ for some $t \in \mathbb{Z}^+$. From the given labeling f, the number 1 is a label of v_0. The numbers 2, 5, 8, ..., $3t-1$ are labels of v_2, v_4, v_6, ..., v_{2t}. The numbers 3, 6, 9, ..., 3t are labels of v_1, v_3, v_5, ..., v_{2t-1}. The numbers 4, 7, 10, ..., $3t+1$, $3t+4$, ..., $6t-2$ are labels of e_{4t-1}, e_{4t-2}, ..., e_{3t+2}, e_{3t+1}, ..., e_{2t+1}. And the numbers $3t+2$ and $3t+3$ are labels of e_{2t-1} and e_{2t}. The numbers $3t+5$ and $3t+6$ are labels of e_{2t-3} and e_{2t-2}. Until the numbers $6t-4$ and $6t-3$ are labels of e_3 and e_4. And the numbers $6t-1$ and $6t$ are labels of e_1 and e_2. Observe that

for e_i; $i = 1, 2, \ldots, 2t$,

$f(v_0) + f(e_i) + f(v_i) = 1 + (\frac{24t+7+5(-1)^i-6i}{4}) + \frac{1-5(-1)^i+6i}{4} = 6t+3 = 3n+3$,

for e_i; $i = 2t+1, 2t+2, \ldots, 4t-1$,

$$f(v_{i-2t}) + f(e_i) + f(v_{i+1-2t}) = \frac{1-5(-1)^{i-2t}+6(i-2t)}{4} + (12t-3i+1) + \frac{1-5(-1)^{i+1-2t}+6(i+1-2t)}{4}$$
$$= 6t+3 = 3n+3.$$

Case 2: *n is odd.* The proof is similar to the previous case.

Therefore f is an edge-magic total labeling of F_n . $\square$

By duality, we have the following corollary.

Corollary 3.22. *The F_n is edge-magic with $k = 6n - 3$.* $\square$

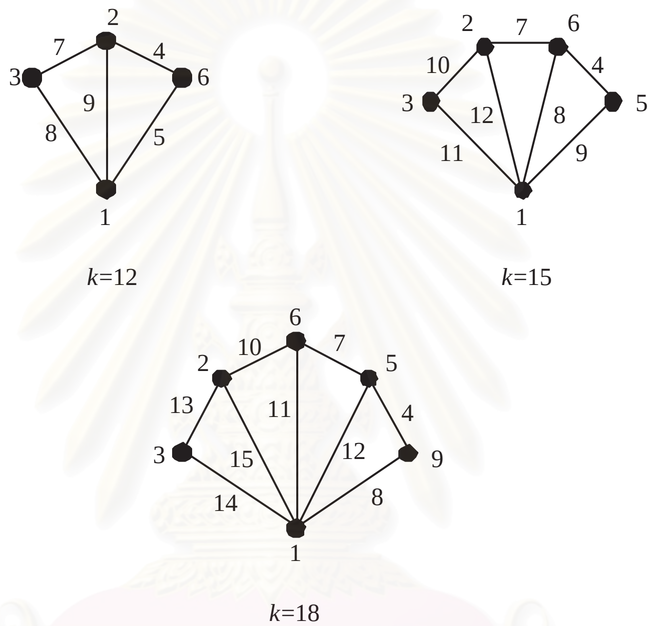
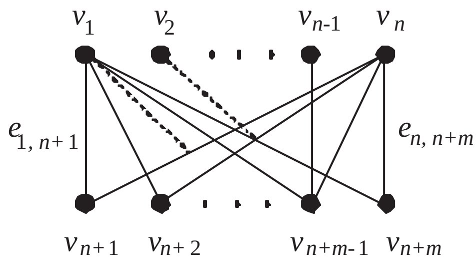


Figure 3.6: Edge-magic total labelings of F_3, F_4 and F_5 with magic sums $k = 12$, $k = 15$ and $k = 18$ respectively.

Theorem 3.23. [10] *The complete bipartite graph $K_{n,m}$ is edge-magic for any n and m with $k = (m + 2)(n + 1)$.*

Proof. Let v_1, v_2, $\ldots$, v_n be the vertices in the set V_1 and v_{n+1}, v_{n+2}, $\ldots$, v_{n+m} the vertices in the set V_2. And $e_{i,j} = v_i v_j$ for $i = 1, 2, \ldots, n$ and $j = n + 1$, n+2, $\ldots$, $n + m$ are edges of $K_{n,m}$, that is



Define a labeling f as follows:

$$f(v_i) = \begin{cases} i & \text{for } i = 1, 2, \ldots, n, \\ (i-n)(n+1) & \text{for } i = n+1, n+2, \ldots, n+m; \end{cases}$$

and for $i = 1, 2, \ldots, n$ and $j = n+1, n+2, \ldots, n+m$,

$$f(e_{i,j}) = (m+n-j+2)(n+1) - i.$$

From the given labeling f, the numbers 1, 2, $\ldots$, n are labels of v_1, v_2, $\ldots$, v_n. The numbers $(n+1)$, $2(n+1)$, $\ldots$, $m(n+1)$ are labels of v_{n+1}, v_{n+2}, $\ldots$, v_{n+m}. The numbers $n+2$, $n+3$, $\ldots$, $2n+1$ are labels of $e_{n,n+m}$, $e_{n-1,n+m}$, $\ldots$, $e_{1,n+m}$. The numbers $2n+3$, $2n+4$, $\ldots$, $3n+2$ are labels of $e_{n,n+m-1}$, $e_{n-1,n+m-1}$, $\ldots$, $e_{1,n+m-1}$. Until the numbers $m(n+1)+1$, $m(n+1)+2$, $\ldots$, $m(n+1)+n$ are labels of $e_{n,n+1}$, $e_{n-1,n+1}$, $\ldots$, $e_{1,n+1}$. So all numbers 1 through $m(n+1)+n$ are used exactly once. Observe that

for $e_{i,j}$; $i = 1, 2, \ldots, n$ and $j = n+1, n+2, \ldots, n+m$,

$$\begin{aligned} f(v_i) + f(e_{i,j}) + f(v_j) &= i + (m+n-j+2)(n+1) - i + (j-n)(n+1) \\ &= (m+2)(n+1). \end{aligned}$$

Therefore f is an edge-magic total labeling with $k = (m+2)(n+1)$. $\square$

By duality, we have the following corollary.

Corollary 3.24. *The complete bipartite graph $K_{n,m}$ is edge-magic with $k = (2m+1)(n+1)$.* $\square$

W.D. Wallis and the others [10] enumerated every edge-magic total labeling for $K_{2,3}$ in the case $14 \leq k \leq 22$ and $K_{3,3}$ for the case $18 \leq k \leq 30$ and k must be even which are shown in table 3.3.

$K_{m,n}$	k	labeling for V_1	labeling for V_2
$K_{2,3}$	14	no solutions	
	15	1,2	3,6,9
	16	1,2	5,8,11
	17	5,6	1,4,9
	18	1,5	9,10,11
	19	6,7	3,8,11
	20	10,11	1,4,7
	21	10,11	3,6,9
	22	no solutions	
$K_{3,3}$	18	no solutions	
	20	1,2,3	4,8,12
	22	1,2,3	7,11,15
	24	no solutions	
	26	1,5,9	13,14,15
	28	4,8,12	13,14,15
	30	no solutions	

Table 3.3: Edge-magic total labelings of $K_{2,3}$ in the case $14 \leq k \leq 22$ and $K_{3,3}$ for the case $18 \leq k \leq 30$ and k must be even.

Lemma 3.6. [10] *If a star $K_{1,n}$ is edge-magic, then the center receives label 1, $n+1$ or $2n+1$.*

Proof. Assume that the center receives label x. Then by lemma 2.28(a)

$$kn = \binom{2n+2}{2} + (n-1)x. \tag{3.4}$$

Then

$$(n-1)x = kn - (n+1)(2n+1).$$

So

$$x - \frac{x}{n} = k - 2n - 3 - \frac{1}{n}\ .$$

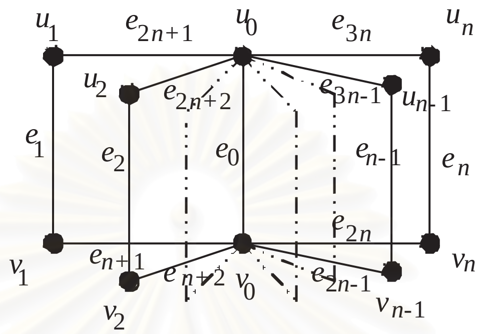
Then $x \equiv 1(mod\ n)$. Since all labels of star are $1, 2, \ldots, 2n+1$, x can be $1, n+1$ or $2n+1$. $\square$

Theorem 3.25. [10] *There are $3 \cdot 2^n$ edge-magic total labelings of star $K_{1,n}$ up to equivalence.*

Proof. Let f be the edge-magic total labeling of star $K_{1,n}$ and c the label of the center v_0 and x_i the label of the edge $e_i = v_0v_i$ where v_i is the peripheral vertex for $i = 1, 2, \ldots, n$. By lemma 3.6 and the equation (3.4), we have $k = 2n+4$, $3n+3$ and $4n+2$ if c is 1, $n+1$ and $2n+1$ respectively. Since f is an edge-magic total labeling, $x_i + y_i = k - c = T$ where $y_i = f(e_i)$ and e_i is an edge which is incident to x_i. So $T = 2n+3$, $2n+2$ or $2n+1$. Then there is exactly one way to partition the $2n+1$ integers 1, 2, …, $2n+1$ into $n+1$ set. i.e.$\{c\}$, $\{a_1, b_1\}$, $\{a_2, b_2\}$,…, $\{a_n, b_n\}$ where $a_i + b_i = T$. For convenience, we choose the labels so that $a_i < b_i$ for every i and $a_1 < a_2 < \ldots < a_n$. Then up to isomorphism, one can assume that $\{x_i, y_i\}$=$\{a_i, b_i\}$. Each of these n equations provides two choices, according as x_i=a_i or b_i. So each of the three values of c gives 2^n edge-magic total labelings of star $K_{1,n}$. $\square$

Theorem 3.26. [3] *The book B_n is edge-magic with $k = 7n+6$ for all positive integer n.*

Proof. Let u_i and v_i be the vertices of B_n for $i = 0, 1, \ldots, n$ and $e_i = u_iv_i$ for $i = 0, 1, \ldots, n$ and $e_{n+i} = v_0v_i$ for $i = 1, \ldots, n$ and $e_{2n+i} = u_0u_i$ for $i = 1, \ldots, n$, that is



Define a labeling f as follows:

$$f(u_i) = \begin{cases} 1 & \text{for } i = 0, \\ 2n + i + 2 & \text{for } i = 1, \ldots, n; \end{cases}$$

and

$$f(v_i) = \begin{cases} 5n + 3 & \text{for } i = 0, \\ 2n - 2i + 2 & \text{for } i = 1, \ldots, n; \end{cases}$$

and

$$f(e_i) = \begin{cases} 2n + 2 & \text{for } i = 0, \\ 3n + i + 2 & \text{for } i = 1, \ldots, n, \\ 2i + 1 - 2n & \text{for } i = n + 1, \ldots, 2n, \\ 7n - i + 3 & \text{for } i = 2n + 1, \ldots, 3n. \end{cases}$$

From the given labeling f, the number 1 is a label of u_0. The numbers 2, 4, ..., $2n-2, 2n$ are labels of $v_n, v_{n-1}, \ldots, v_2, v_1$. The numbers 3, 5, ..., 2n-1, 2n+1 are labels of $e_{n+1}, e_{n+2}, \ldots, e_{2n}$. The number $2n+2$ is a label of e_0. The numbers 2n+3, $2n+4, \ldots, 3n+2$ are labels of $u_1, u_2, \ldots, u_n$. The numbers $3n+3, 3n+4, \ldots, 4n+2$ are labels of $e_1, e_2, \ldots, e_n$. The numbers $4n+3, 4n+4, \ldots, 5n+1, 5n+2$ are labels of $e_{3n}, e_{3n-1}, \ldots, e_{2n+2}, e_{2n+1}$. And the number $5n+3$ is a label of v_0. Then all numbers 1 through $5n+3$ are used exactly once. Observe that, for $i = 1, 2, \ldots, n$

$f(u_0) + f(e_{2n+i}) + f(u_i) = 1 + (7n - (2n+i) + 3) + 2n + i + 2 = 7n + 6,$

$f(v_0) + f(e_{n+i}) + f(v_i) = 5n + 3 + (2(n+i) + 1 - 2n) + 2n - 2i + 2 = 7n + 6,$

$f(u_i) + f(e_i) + f(v_i) = 2n + i + 2 + (3n + i + 2) + 2n - 2i + 2 = 7n + 6,$

and $f(u_0) + f(e_0) + f(v_0) = 1 + (2n+2) + 5n + 3 = 7n + 6.$

Therefore f is an edge-magic total labeling with $k = 7n + 6$. $\square$

By duality, we have the following corollary.

Corollary 3.27. *The book B_n is edge-magic with $k = 8n$.* $\square$

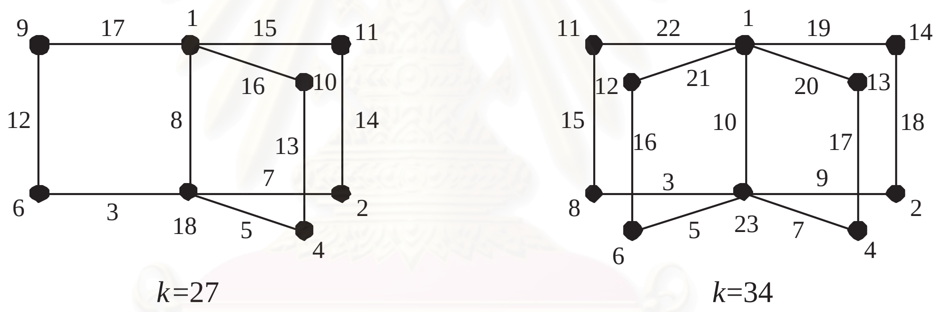
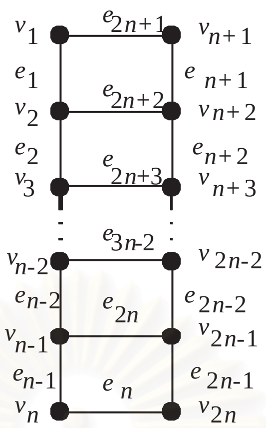


Figure 3.7: Edge-magic total labelings of B_3 and B_4 with magic sums $k = 27$ and $k = 34$ respectively.

Theorem 3.28. [3] *The ladder L_n is edge-magic with $k = \frac{1}{2}(11n+1)$ when n is odd.*

Proof. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of L_n and $e_1 = v_1v_2, e_2 = v_2v_3, \ldots,$ $e_{n-1} = v_{n-1}v_n, e_n = v_nv_{2n}, e_{n+1} = v_{n+1}v_{n+2}, e_{n+2} = v_{n+2}v_{n+3}, \ldots, e_{2n-1} = v_{2n-1}v_{2n},$ $e_{2n} = v_{n-1}v_{2n-1}, e_{2n+1} = v_1v_{n+1}, e_{2n+2} = v_2v_{n+2}, \ldots, e_{3n-2} = v_{n-2}v_{2n-2}$ edges of L_n, that is



Let $n = 2t+1$ for some $t \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+1}{2} & \text{for } i = 1, 3, \ldots, 2t+1, \\ \frac{2t+i+2}{2} & \text{for } i = 2, 4, \ldots, 2t, \\ \frac{4t+i+2}{2} & \text{for } i = 2t+2, 2t+4, \ldots, 4t+2, \\ \frac{2t+i+1}{2} & \text{for } i = 2t+3, 2t+5, \ldots, 4t+1; \end{cases}$$

and

$$f(e_i) = \begin{cases} 10t - i + 4 & \text{for } i = 1, 2, \ldots, 2t, \\ 6t+3 & \text{for } i = 2t+1, \\ 8t - i + 4 & \text{for } i = 2t+2, 2t+3, \ldots, 4t+1, \\ 6t+4 & \text{for } i = 4t+2, \\ 12t - i + 6 & \text{for } i = 4t+3, 4t+4, \ldots, 6t+1. \end{cases}$$

From the given labeling f, the numbers 1, 2, $\ldots$, $t+1$ are labels of v_1, v_3, $\ldots$, v_{2t+1}. The numbers $t+2$, $t+3$, $\ldots$, $2t+1$ are labels of v_2, v_4, $\ldots$, v_{2t}. The numbers $2t+2$, $2t+3$, $\ldots$, $3t+1$ are labels of v_{2t+3}, v_{2t+5}, $\ldots$, v_{4t+1}. The numbers $3t+2$, $3t+3$, $\ldots$, $4t+2$ are labels of v_{2t+2}, v_{2t+4}, $\ldots$, v_{4t+2}. The numbers $4t+3$, $4t+4$, $\ldots$, $6t+2$ are labels of e_{4t+1}, e_{4t}, $\ldots$, e_{2t+2}. The numbers $6t+3$ and $6t+4$ are labels of e_{2t+1} and e_{4t+2}. The numbers $6t+5$, $6t+6$, $\ldots$, $8t+3$ are labels of e_{6t+1},

$e_{6t}, \ldots, e_{4t+3}$. The numbers $8t+4, 8t+5, \ldots, 10t+3$ are labels of $e_{2t}, e_{2t-1}, \ldots, e_1$. So all numbers 1 through $5n-2 = 10t+3$ are used exactly once. Observe that

for e_i; $i = 1, 3, \ldots, 2t-1$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{i+1}{2} + (10t - i + 4) + \frac{2t+2+i+1}{2} = \frac{22t+12}{2} = \frac{1}{2}(11n+1)$,

for e_{i+4t+2}; $i = 1, 3, \ldots, 2t-1$,

$$\begin{aligned} f(v_i) + f(e_{i+4t+2}) + f(v_{i+2t+1}) &= \tfrac{i+1}{2} + (12t + 6 - (i + 4t + 2)) + \tfrac{4t+2+i+(2t+1)}{2} \\ &= \tfrac{22t+12}{2} = \tfrac{1}{2}(11n+1), \end{aligned}$$

for e_i; $i = 2, 4, \ldots, 2t$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{2t+i+2}{2} + (10t - i + 4) + \frac{i+1+1}{2} = \frac{22t+12}{2} = \frac{1}{2}(11n+1)$,

for e_{i+4t+2}; $i = 2, 4, \ldots, 2t$,

$$\begin{aligned} f(v_i) + f(e_{i+4t+2}) + f(v_{i+2t+1}) &= \tfrac{2t+i+2}{2} + (12t + 6 - (i + 4t + 2)) + \tfrac{2t+1+i+(2t+1)}{2} \\ &= \tfrac{22t+12}{2} = \tfrac{1}{2}(11n+1), \end{aligned}$$

for e_n, $f(v_n) + f(e_n) + f(v_{2n}) = t + 1 + (6t+3) + 4t + 2 = 11t + 6 = \frac{1}{2}(11n+1)$,

for e_i; $i = 2t+2, 2t+4, \ldots, 4t$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{4t+i+2}{2} + (8t - i + 4) + \frac{2t+1+i+1}{2} = \frac{22t+12}{2} = \frac{1}{2}(11n+1)$,

for e_i; $i = 2t+3, 2t+5, \ldots, 4t+1$,

$f(v_i) + f(e_i) + f(v_{i+1}) = \frac{2t+i+1}{2} + (8t - i + 4) + \frac{4t+2+i+1}{2} = \frac{22t+12}{2} = \frac{1}{2}(11n+1)$.

Therefore f is an edge-magic total labeling with $k = \frac{1}{2}(11n+1)$ when n is odd. □

By duality, we have the following corollary.

Corollary 3.29. *The ladder L_n is edge-magic with $k = \frac{1}{2}(19n-13)$ when n is odd.* □

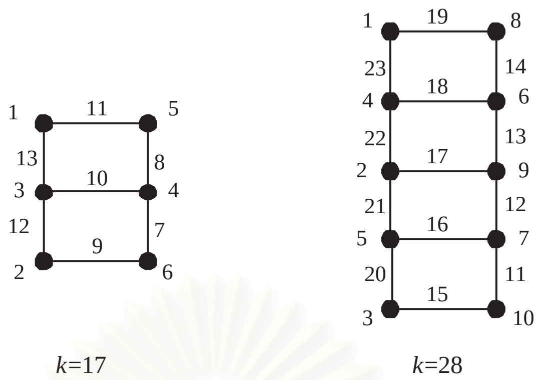
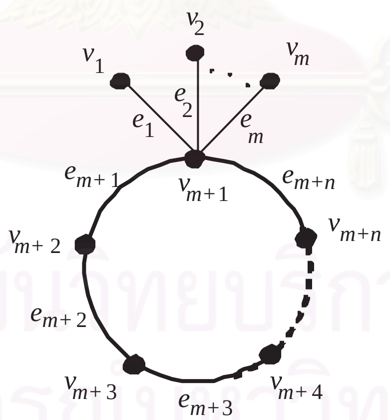


Figure 3.8: Edge-magic total labelings of L_3 and L_5 with magic sums $k = 17$ and $k = 28$ respectively.

Theorem 3.30. *An* $(n,\ m)$*-pineapple is edge-magic with* $k = 3m + 3n + 1$ *when* n *is odd.*

Proof. Let $v_1, v_2, \ldots, v_{m+n}$ be the vertices and $e_i = v_i v_{m+1}$ for $i = 1, 2, \ldots, m$ and $e_i = v_i v_{i+1}$ for $i = m+1, m+2, \ldots, m+n-1$ and $e_{m+n} = v_{m+n} v_{m+1}$, that is



Let $n = 2t + 1$ for some $t \in \mathbb{Z}^+$ and define a labeling f as follows:

$$f(v_i) = \begin{cases} i & \text{for } i = 1, 2, \ldots, m, \\ 2t + i & \text{for } i = m+1, m+3, m+5, \ldots, m+2t+1, \\ i - 1 & \text{for } i = m+2, m+4, m+6, \ldots, m+2t; \end{cases}$$

and

$$f(e_i) = \begin{cases} 2m + 4t - i + 3 & \text{for } i = 1, 2, \ldots, m, \\ 3m + 4t - 2i + 4 & \text{for } i = m + 1, m + 2, \ldots, m + 2t + 1. \end{cases}$$

From the given labeling f, the numbers 1, 2, $\ldots$, m are labels of v_1, v_2, $\ldots$, v_m. The numbers $m + 1$, $m + 3$, $\ldots$, $m + 2t - 1$ are labels of v_{m+2}, v_{m+4}, $\ldots$, v_{m+2t}. The numbers $m + 2$, $m + 4$, $\ldots$, $m + 2t$, $m + 2t + 2$, $\ldots$, $m + 4t + 2$ are labels of e_{m+2t+1}, e_{m+2t}, $\ldots$, e_{m+t+2}, e_{m+t+1}, $\ldots$, e_{m+1}. The numbers $m+2t+1$, $m+2t+3$, $\ldots$, $m + 4t + 1$ are labels of v_{m+1}, v_{m+3}, $\ldots$, v_{m+2t+1}. The numbers $m + 4t + 3$, $m + 4t + 4$, $\ldots$, $2m + 4t + 2$ are labels of e_m, e_{m-1}, $\ldots$, e_1. Observe that

for e_i; $i = 1, 2, \ldots, m$,

$f(v_i)+f(e_i)+f(v_{m+1}) = i+(2m+4t-i+3)+m+2t+1 = 3m+6t+4 = 3m+3n+1$,

for e_i; $i = m + 1, m + 3, \ldots, m + 2t - 1$,

$$\begin{aligned} f(v_i) + f(e_i) + f(v_{i+1}) &= 2t + i + (3m + 4t - 2i + 4) + i + 1 - 1 \\ &= 3m + 6t + 4 = 3m + 3n + 1, \end{aligned}$$

for e_i; $i = m + 2, m + 4, \ldots, m + 2t$,

$$\begin{aligned} f(v_i) + f(e_i) + f(v_{i+1}) &= i - 1 + (3m + 4t - 2i + 4) + 2t + i + 1 \\ &= 3m + 6t + 4 = 3m + 3n + 1, \end{aligned}$$

for e_{m+2t+1},

$$\begin{aligned} f(v_{m+2t+1}) + f(e_{m+2t+1}) + f(v_{m+1}) &= m + 4t + 1 + (m + 2) + m + 2t + 1 \\ &= 3m + 6t + 4 = 3m + 3n + 1, \end{aligned}$$

Therefore f is an edge-magic total labeling with $k = 3m + 3n + 1$ when n is odd. □

By duality, we have the following corollary.

Corollary 3.31. *An (n, m)-pineapple is edge-magic with $k = 3m + 3n + 2$ when n is odd.* □

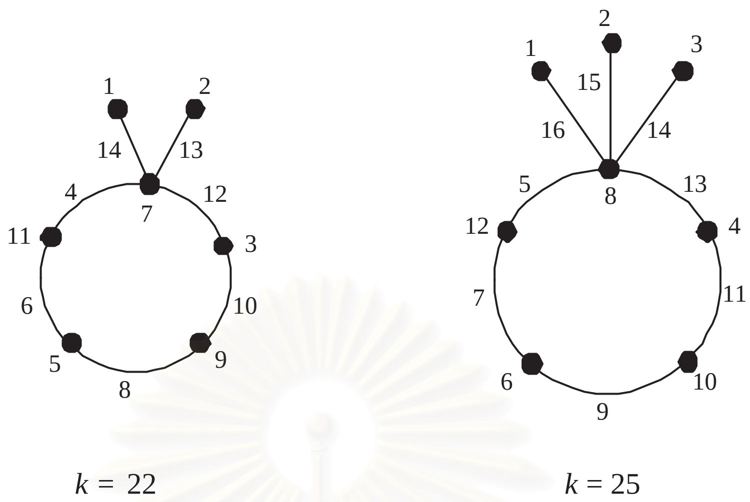


Figure 3.9: Edge-magic total labelings of (5, 2)-pineapple and (5, 3)-pineapple with magic sums $k = 22$ and $k = 25$ respectively.

CHAPTER IV

EDGE-MAGIC TOTAL LABELINGS ON DISCONNECTED GRAPHS

In this chapter, we discuss the disconnected graph which are or are not edge-magic. Moreover, some examples are shown.

Theorem 4.1. *The graph mK_n is not edge-magic when m is odd and $n \equiv 4(mod\ 8)$.*

Proof. Since $n \equiv 4(mod\ 8)$, there exists $t \in \mathbb{Z}^+$ such that $n = 8t + 4$. Let $m = 2t' + 1$ for some $t' \in \mathbb{Z}^+$. Then

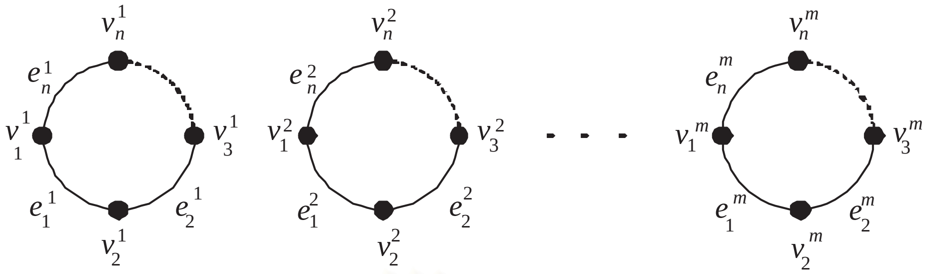
$$\begin{aligned} p + q &= mn + m\binom{n}{2} \\ &= (2t' + 1)(8t + 4 + \frac{(8t + 4)^2 - (8t + 4)}{2}) \\ &= (2t' + 1)(32t^2 + 36t + 10) \\ &= 4(2t' + 1)(8t^2 + 9t + 2) + 2. \end{aligned}$$

Thus $p + q \equiv 2(mod\ 4)$. Since each vertex of mK_n has $n - 1$ degree which is odd and q is even, by proposition 2.29, K_n is not edge-magic. □

Theorem 4.2. *The graph mW_n is not edge-magic when m is odd and $n \equiv 3(mod\ 4)$.*

Proof. The proof is similar to the previous theorem. □

We are going to show that the graph mC_n is edge-magic by giving the notations as follows: $V(mC_n) = V_1 \bigcup \ldots \bigcup V_m$ where $V_i = \{v_1^i, v_2^i, \ldots, v_n^i\}$ and $E(mC_n) = E_1 \bigcup \ldots \bigcup E_m$ where $E_i = \{e_1^i, e_2^i, \ldots, e_n^i\}$ and $e_1^i = v_1^i v_2^i$, $e_2^i = v_2^i v_3^i$, $\ldots$, $e_{n-1}^i = v_{n-1}^i v_n^i$, $e_n^i = v_n^i v_1^i$, that is



Theorem 4.3. [11] *The graph mC_n is edge-magic with $k = \frac{1}{2}(5nm+3)$ when m (>1) and n are odd.*

Proof. Let m (>1) and n be odd.

For $i = 1, \ldots, \frac{m-1}{2}$, define a labeling f as follows:

$$f(v_j^i) = \begin{cases} i & \text{for } j = 1, \\ tm + i & \text{for } j = 2t+1;\ t = 1, \ldots, \frac{n-3}{2}, \\ \frac{1}{2}[(n+2t)m + 1 + 2i] & \text{for } j = 2t;\ t = 1, \ldots, \frac{n-1}{2}, \\ \frac{1}{2}(n+1)m + 1 - 2i & \text{for } j = n; \end{cases}$$

and

$$f(e_j^i) = \begin{cases} (2n-1)m + 1 - 2i & \text{for } j = 1, \\ (2n-2t-1)m + 1 - 2i & \text{for } j = 2t+1;\ t = 1, 2, \ldots, \frac{n-3}{2}, \\ (2n-2t)m + 1 - 2i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n-3}{2}, \\ nm + i & \text{for } j = n-1, \\ \frac{1}{2}[(4n-1)m + 1 + 2i] & \text{for } j = n. \end{cases}$$

For $i = \frac{m+1}{2}, \dots, m$, define a labeling f as follows:

$$f(v_j^i) = \begin{cases} i & \text{for } j = 1, \\ tm + i & \text{for } j = 2t+1;\ t = 1, \dots, \frac{n-3}{2}, \\ \frac{1}{2}[(n+2t-2)m + 1 + 2i] & \text{for } j = 2t;\ t = 1, \dots, \frac{n-1}{2}, \\ \frac{1}{2}(n+3)m + 1 - 2i & \text{for } j = n; \end{cases}$$

and

$$f(e_j^i) = \begin{cases} 2nm + 1 - 2i & \text{for } j = 1, \\ (2n-2t)m + 1 - 2i & \text{for } j = 2t+1;\ t = 1, 2, \dots, \frac{n-3}{2}, \\ (2n-2t+1)m + 1 - 2i & \text{for } j = 2t;\ t = 1, 2, \dots, \frac{n-3}{2}, \\ nm + i & \text{for } j = n-1, \\ \frac{1}{2}[(4n-3)m + 1 + 2i] & \text{for } j = n. \end{cases}$$

Let $n = 2z+1$ and $m = 2z'+1$ for some $z, z' \in \mathbb{Z}^+$. The numbers $1, 2, \dots, 2z'+1, 2z'+2, \dots, 2zz'+z$ are labels of $v_1^1, v_1^2, \dots, v_1^m, v_3^1, v_3^2, \dots, v_3^m, \dots, v_{n-2}^1, \dots, v_{n-2}^m$. The numbers $2zz'+z+1, \dots, 2zz'+z+2z'+1$ are labels of v_n^i for $i = m, \frac{m-1}{2}, m-1, \frac{m-1}{2}-1, \dots, \frac{m+1}{2}$. The numbers $2zz'+z+2z'+2, \dots, 4zz'+2z+2z'+1$ are labels of $v_2^{\frac{m+1}{2}}, v_2^{\frac{m+3}{2}}, \dots, v_2^m$ $v_4^{\frac{m+1}{2}}, v_4^{\frac{m+3}{2}}$ $\dots, v_4^m, \dots, v_{n-1}^m$. The numbers $4zz'+2z+2z'+2, \dots, 4zz'+2z+4z'+2$ are labels of $e_{n-1}^1, e_{n-1}^2, \dots, e_{n-1}^m$. The numbers $4zz'+2z+4z'+3, \dots, 8zz'+4z+2z'+1$ are labels of $e_{n-2}^m, e_{n-2}^{\frac{m-1}{2}}, e_{n-2}^{m-1}, \dots, e_{n-2}^{\frac{m+1}{2}}, e_{n-3}^m, e_{n-3}^{\frac{m-1}{2}}, e_{n-3}^{m-1}, \dots, e_{n-3}^{\frac{m+1}{2}}, \dots, e_1^{\frac{m+1}{2}}$. The numbers $8zz'+4z+2z'+2, \dots, 8zz'+4z+3z'+2$ are labels of $e_n^{\frac{m+1}{2}}, e_n^{\frac{m+3}{2}}, \dots, e_n^m$. The numbers $8zz'+4z+3z'+3, \dots, 8zz'+4z+4z'+2$ are labels of $e_n^1, e_n^2, \dots, e_n^{\frac{m-1}{2}}$. So all numbers 1 through $2nm = 8zz'+4z+4z'+2$ are used exactly once.

Observe that, for $i = 1, \dots, \frac{m-1}{2}$, for e_1^i,

$f(v_1^i) + f(e_1^i) + f(v_2^i) = i + ((2n-1)m + 1 - 2i) + \frac{(n+2)m+1+2i}{2} = \frac{1}{2}(5nm+3)$,

for e_j^i; $j = 2t$ where $t = 1, 2, \ldots, \frac{n-3}{2}$,

$f(v_j^i) + f(e_j^i) + f(v_{j+1}^i) = \frac{(n+2t)m+1+2i}{2} + (2n-2t)m + 1 - 2i + tm + i = \frac{1}{2}(5nm+3)$,

for e_j^i; $j = 2t+1$ where $t = 1, 2, \ldots, \frac{n-3}{2}$,

$$f(v_j^i) + f(e_j^i) + f(v_{j+1}^i) = tm + i + (2n-2t-1)m + 1 - 2i + \frac{(n+2(t+1))m+1+2i}{2}$$
$$= \frac{1}{2}(5nm+3),$$

for e_{n-1}^i,

$f(v_{n-1}^i) + f(e_{n-1}^i) + f(v_n^i) = \frac{(2n-1)m+1+2i}{2} + nm + i + \frac{(n+1)m}{2} + 1 - 2i = \frac{1}{2}(5nm+3)$,

for e_n^i,

$f(v_n^i) + f(e_n^i) + f(v_1^i) = \frac{(n+1)m}{2} + 1 - 2i + \frac{(4n-1)m+1+2i}{2} + i = \frac{1}{2}(5nm+3)$.

For $i = \frac{m+1}{2}, \ldots, m$, we can verify similarly. Therefore f is an edge-magic total labeling with $k = \frac{1}{2}(5nm+3)$ when m and n are odd. □

By duality, we have the following corollary.

Corollary 4.1. [11] *The graph mC_n is edge-magic with $k = \frac{1}{2}(7nm+3)$ when m (> 1) and n are odd.* □

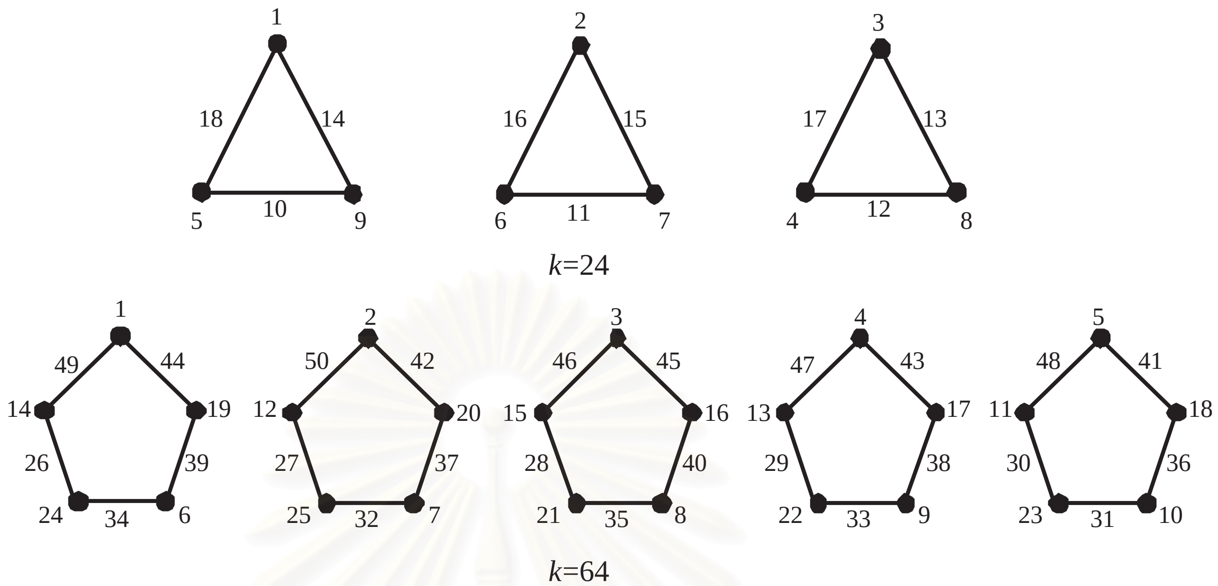
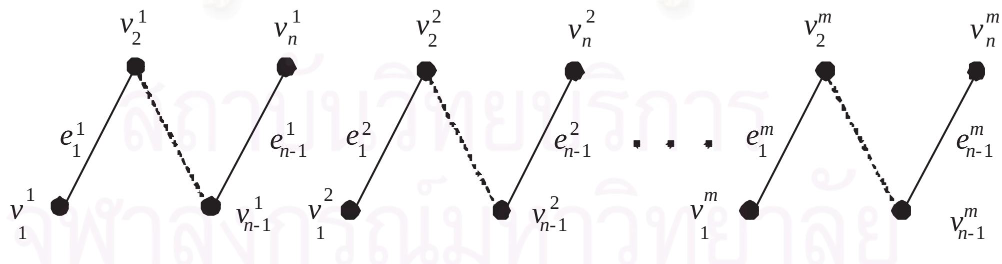


Figure 4.1: Edge-magic total labelings of $3C_3$ and $5C_5$ with magic sums $k = 24$ and $k = 64$ respectively.

We are going to show that the graph mP_n is edge-magic by giving the notations as follows: $V(mP_n) = V_1 \bigcup \ldots \bigcup V_m$ where $V_i = \{v_1^i, v_2^i, \ldots, v_n^i\}$ and $E(mP_n) = E_1 \bigcup \ldots \bigcup E_m$ where $E_i = \{e_1^i, e_2^i, \ldots, e_{n-1}^i\}$ and $e_1^i = v_1^i v_2^i$, $e_2^i = v_2^i v_3^i$, $\ldots$, $e_{n-1}^i = v_{n-1}^i v_n^i$, that is



Theorem 4.4. [11] *The graph mP_n is edge-magic with $k = \frac{1}{2}(5nm + 3)$ when m (> 1) and n are odd.*

Proof. By the edge-magic total labeling f in theorem 4.3, the m highest labels appear as edge labels with one in each component. We delete those edges and their labels. Then we have a graph mP_n which is edge-magic with $k = \frac{1}{2}(5nm+3)$. $\square$

By duality, we have the following corollary.

Corollary 4.2. [11] *The graph mP_n is edge-magic with $k = \frac{1}{2}[(7n-6)m+3]$ when m (>1) and n are odd.* $\square$

Theorem 4.5. [11] *The graph mP_n is edge-magic with $k = \frac{1}{2}[(5n-2)m+3]$ when m (>1) and n are odd.*

Proof. Let $m > 1$ and n be odd.

For $i = 1, \ldots, \frac{m-1}{2}$, define a labeling f as follows:

$$f(v_j^i) = \begin{cases} \frac{1}{2}(nm+1+2i) & \text{for } j = 1, \\ \frac{1}{2}[(n+2t)m+1+2i] & \text{for } j = 2t+1;\ t = 1, \ldots, \frac{n-1}{2}, \\ (t-1)m+i & \text{for } j = 2t;\ t = 1, \ldots, \frac{n-1}{2}; \end{cases}$$

and

$$f(e_j^i) = \begin{cases} (2n-1)m+1-2i & \text{for } j = 1, \\ (2n-2t-1)m+1-2i & \text{for } j = 2t+1;\ t = 1, 2, \ldots, \frac{n-3}{2}, \\ (2n-2t)m+1-2i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n-1}{2}. \end{cases}$$

For $i = \frac{m+1}{2}, \ldots, m$, define a labeling f as follows:

$$f(v_j^i) = \begin{cases} \frac{1}{2}[(n-2)m+1+2i] & \text{for } j = 1, \\ \frac{1}{2}[(n+2t-2)m+1+2i] & \text{for } j = 2t+1;\ t = 1, \ldots, \frac{n-1}{2}, \\ (t-1)m+i & \text{for } j = 2t;\ t = 1, \ldots, \frac{n-1}{2}; \end{cases}$$

and

$$f(e_j^i) = \begin{cases} 2nm+1-2i & \text{for } j = 1, \\ (2n-2t)m+1-2i & \text{for } j = 2t+1;\ t = 1, 2, \ldots, \frac{n-3}{2}, \\ (2n+1-2t)m+1-2i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n-1}{2}. \end{cases}$$

It is easy to verify that all numbers 1 through $m(2n-1)$ are used exactly once. Observe that, for $i = 1, 2, \ldots, \frac{m-1}{2}$,

for e_1^i,

$f(v_1^i) + f(e_1^i) + f(v_2^i) = \frac{nm+1+2i}{2} + (2n-1)m + 1 - 2i + i = \frac{1}{2}(5nm - 2m + 3)$,

for e_j^i; $j = 2t$ where $t = 1, 2, \ldots, \frac{n-1}{2}$,

$$\begin{aligned} f(v_j^i) + f(e_j^i) + f(v_{j+1}^i) &= (t-1)m + i + (2n-2t)m + 1 - 2i + \tfrac{(n+2t)m+1+2i}{2} \\ &= \tfrac{1}{2}(5nm - 2m + 3), \end{aligned}$$

for e_j^i; $j = 2t+1$ where $t = 1, 2, \ldots, \frac{n-3}{2}$,

$$\begin{aligned} f(v_j^i) + f(e_j^i) + f(v_{j+1}^i) &= \tfrac{(n+2t)m+1+2i}{2} + (2n-2t-1)m + 1 - 2i + (t+1-1)m + i \\ &= \tfrac{1}{2}(5nm - 2m + 3). \end{aligned}$$

For $i = \frac{m+1}{2}, \ldots, m$, we can verify similarly. Therefore f is an edge-magic total labeling with $k = \frac{1}{2}[(5n-2)m+3]$ when m and n are odd. $\square$

By duality, we have the following corollary.

Corollary 4.3. [11] *The graph mP_n is edge-magic with $k = \frac{1}{2}[(7n-4)m+3]$ when m (>1) and n are odd.* $\square$

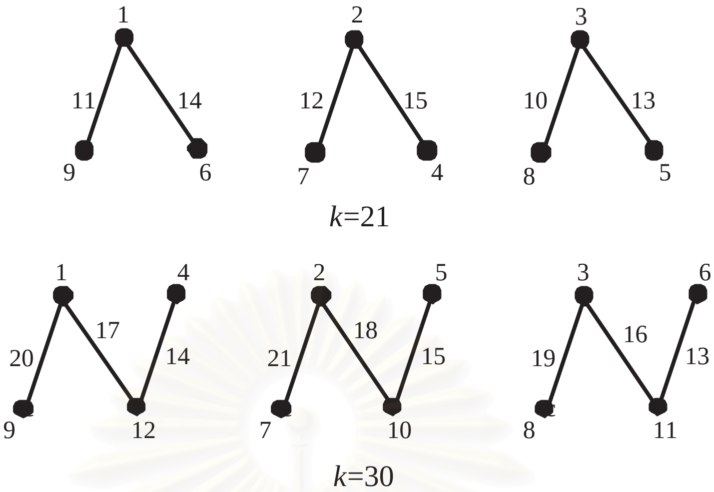


Figure 4.2: Edge-magic total labelings of $3P_3$ and $3P_4$ with magic sums $k = 21$ and $k = 30$ respectively.

Theorem 4.6. [11] *The graph mP_n is edge-magic with $k = \frac{1}{2}[(5n-1)m+3]$ when m (> 1) is odd and n is even.*

Proof. Let $m > 1$ be odd and n even.

For $i = 1, \ldots, \frac{m-1}{2}$, define a labeling f as follows:

$$f(v_j^i) = \begin{cases} \frac{1}{2}[(n+1)m+1+2i] & \text{for } j = 1, \\ \frac{1}{2}[(n+2t+1)m+1+2i] & \text{for } j = 2t+1;\ t = 1,\ 2, \ldots,\ \frac{n-2}{2}, \\ (t-1)m+i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n}{2}; \end{cases}$$

and

$$f(e_j^i) = \begin{cases} (2n-1)m+1-2i & \text{for } j = 1, \\ (2n-2t-1)m+1-2i & \text{for } j = 2t+1;\ t = 1, 2, \ldots, \frac{n-2}{2}, \\ (2n-2t)m+1-2i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n-2}{2}. \end{cases}$$

For $i = \frac{m+1}{2}, \ldots, m$, define a labeling f as follows:

$$f(v_j^i) = \begin{cases} \frac{1}{2}[(n-1)m + 1 + 2i] & \text{for } j = 1, \\ \frac{1}{2}[(n+2t-1)m + 1 + 2i] & \text{for } j = 2t+1;\ t = 1,2,\ \ldots,\ \frac{n-2}{2}, \\ (t-1)m + i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n}{2}; \end{cases}$$

and

$$f(e_j^i) = \begin{cases} 2nm + 1 - 2i & \text{for } j = 1, \\ (2n-2t)m + 1 - 2i & \text{for } j = 2t+1;\ t = 1, 2, \ldots, \frac{n-2}{2}, \\ (2n-2t+1)m + 1 - 2i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n-2}{2}. \end{cases}$$

It is easy to verify that f is an edge-magic total labeling with $k = \frac{1}{2}[(5n-1)m+3]$ when m is odd and n is even. $\square$

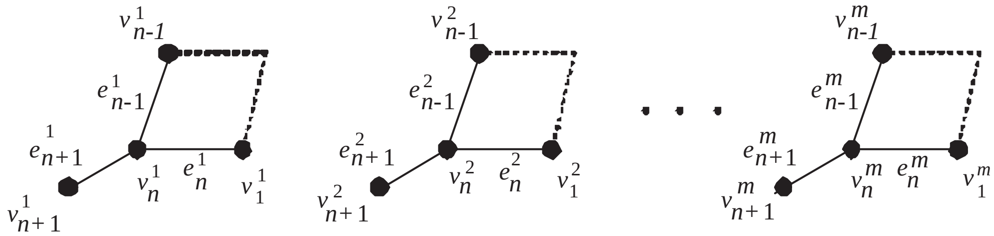
By duality, we have the following corollary.

Corollary 4.4. [11] *The graph mP_n is edge-magic with $k = \frac{1}{2}[(7n-5)m+3]$ when m (> 1) is odd and n is even.* $\square$

Theorem 4.7. *The graph $mP_n \bigcup mK_1$, the graph consists of the disjoint union of m copies of P_n and the disjoint union of m copies of K_1, is edge-magic with $k = \frac{1}{2}[(5n-2)m+3]$ when $m > 1$ and m is odd and n is even.*

Proof. By the edge-magic total labeling f in theorem 4.5, the m highest labels appear as edge labels with one in each component. We delete those edges and their labels. Then we have the graph $mP_n \bigcup mK_1$ which is edge-magic with the same magic sum. $\square$

We are going to show that the graph $m(n,1)$-kite is edge-magic by giving the notations as follows: $V(m(n,1)\text{-kite}) = V_1 \bigcup \ldots \bigcup V_m$ where V_i=$\{v_1^i, v_2^i, \ldots, v_n^i, v_{n+1}^i\}$ and $E(m(n,1)\text{-kite}) = E_1 \bigcup \ldots \bigcup E_m$ where E_i=$\{e_1^i, e_2^i, \ldots, e_n^i, e_{n+1}^i\}$ and $e_1^i = v_1^i v_2^i$, $e_2^i = v_2^i v_3^i$, $\ldots$, $e_{n-1}^i = v_{n-1}^i v_n^i$, $e_n^i = v_n^i v_1^i$, $e_{n+1}^i = v_n^i v_{n+1}^i$, that is



Theorem 4.8. *The graph $m(n,1)$-kite is magic with $k = \frac{1}{2}[m(5n+6)+3]$ when m (>1) and n are odd.*

Proof. Let m (>1) and n be odd.

For components $i = 1, \ldots, \frac{m-1}{2}$, define a labeling f as follows:

$$f(v^i_j) = \begin{cases} i+m & \text{for } j=1, \\ (t+1)m+i & \text{for } j=2t+1;\ t=1,2,\ldots,\frac{n-3}{2}, \\ \frac{1}{2}[(n+2t+2)m+1+2i] & \text{for } j=2t;\ t=1,2,\ldots,\frac{n-1}{2}, \\ \frac{1}{2}(n+3)m+1-2i & \text{for } j=n, \\ \frac{1}{2}[(4n+3)m+1]+i & \text{for } j=n+1; \end{cases}$$

and

$$f(e^i_j) = \begin{cases} 2mn+1-2i & \text{for } j=1, \\ (2n-2t)m+1-2i & \text{for } j=2t+1;\ t=1,2,\ldots,\frac{n-3}{2}, \\ (2n-2t+1)m+1-2i & \text{for } j=2t;\ t=1,2,\ldots,\frac{n-3}{2}, \\ (n+1)m+i & \text{for } j=n-1, \\ \frac{1}{2}[(4n+1)m+1+2i] & \text{for } j=n, \\ i & \text{for } j=n+1. \end{cases}$$

For components $i = \frac{m+1}{2}, \ldots, m$, define a labeling f as follows:

$$f(v_j^i) = \begin{cases} i + m & \text{for } j = 1, \\ (t+1)m + i & \text{for } j = 2t+1;\ t = 1,\ \ldots,\ \frac{n-3}{2}, \\ \frac{1}{2}[(n+2t)m + 1 + 2i] & \text{for } j = 2t;\ t = 1,\ \ldots,\ \frac{n-1}{2}, \\ \frac{1}{2}(n+5)m + 1 - 2i & \text{for } j = n, \\ \frac{1}{2}[(4n+1)m + 1] + i & \text{for } j = n+1; \end{cases}$$

and

$$f(e_j^i) = \begin{cases} (2n+1)m + 1 - 2i & \text{for } j = 1, \\ (2n-2t+1)m + 1 - 2i & \text{for } j = 2t+1;\ t = 1, 2, \ldots, \frac{n-3}{2}, \\ (2n-2t+2)m + 1 - 2i & \text{for } j = 2t;\ t = 1, 2, \ldots, \frac{n-3}{2}, \\ (n+1)m + i & \text{for } j = n-1, \\ \frac{1}{2}[(4n-1)m + 1 + 2i] & \text{for } j = n, \\ i & \text{for } j = n+1. \end{cases}$$

It is easy to verify that f is an edge-magic total labeling with $k = \frac{1}{2}[m(5n+6)+3]$ when m and n are odd. $\square$

By duality, we have the following corollary.

Corollary 4.5. *The graph $m(n,1)$-kite is edge-magic with $k = \frac{1}{2}[m(7n+6)+3]$ when m (>1) and n are odd.* $\square$

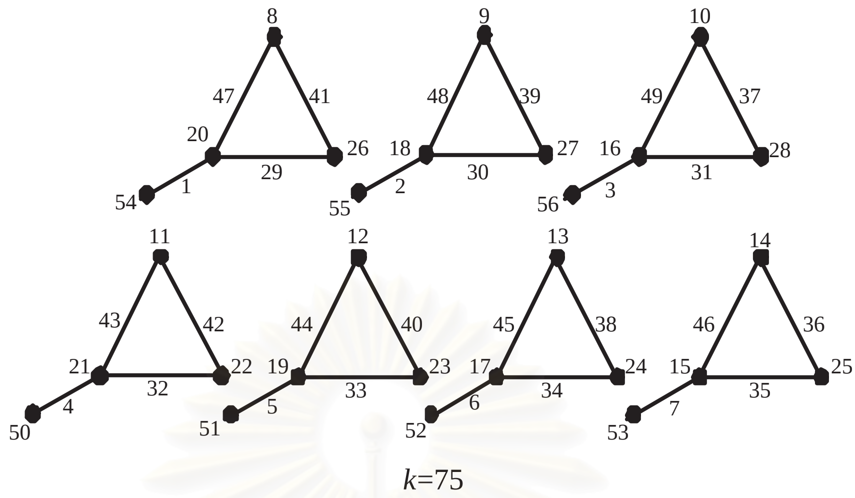


Figure 4.3: An edge-magic labeling of $7(3,1)$-kite with a magic sum $k = 75$.

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VITA

Name : Miss. Sirirat Sompong

Degree obtained : Bachelor of Science, 1999, Khon Kaen University, Khon Kaen, Thailand.

Scholarship : The Development and Promotion of Science and Technology Talents Project. (DPST)