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EDGEWORTH EXPANSION OF A FUNCTION OF SAMPLE MEANS[1]

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Many important statistics can be written as functions of sample means of vector variables. A fundamental contribution to the Edgeworth expansion for functions of sample means was made by Bhattacharya and Ghosh. In their work the crucial Cramér c-condition is assumed on the joint distribution of all the components of the vector variable. However, in many practical situations, only one or a few of the components satisfy (conditionally) this condition while the rest do not (such a case is referred to as satisfying the partial Cramér c-condition). The purpose of this paper is to establish Edgeworth expansions for functions of sample means when only the partial Cramér c-condition is satisfied.

1. Introduction. Suppose that $\{\mathbf{Z}_j\}$ is a sequence of iid (independent and identically distributed) random k-vectors and H is a real-valued Borel measurable function defined on R^k. Consider the statistic

$$W_n = \sqrt{n}\,\sigma^{-1}\big(H(\overline{\mathbf{Z}}_n) - H(\boldsymbol{\mu})\big),$$

where

$$\overline{\mathbf{Z}}_n = \frac{1}{n}\sum_{j=1}^{n}\mathbf{Z}_j = \big(\overline{Z}_1, \dots, \overline{Z}_k\big), \qquad \boldsymbol{\mu} = E\mathbf{Z}_1,$$

and $\sigma > 0$ is a normalizing factor which will be specified later.

It is well known that many statistics have the form $H(\overline{\mathbf{Z}})$, for instance, the sample mean, variance, correlation coefficient and studentized t-statistic and so on. If $\mathbf{Z}_1$ has finite second moment and H is continuously differentiable in a neighborhood of its mean $\boldsymbol{\mu}$, then W_n is asymptotically $\mathbb{N}(0,1)$ if we choose

$$\sigma^2 = \mathbf{l}\Sigma\mathbf{l}', \tag{1.1}$$

where $\mathbf{l} = (l_1, \dots, l_k) = \text{grad}[H(\boldsymbol{\mu})] \neq 0$ and Σ is the covariance matrix of $\mathbf{Z}_1$. A fundamental contribution to the Edgeworth expansion of W_n was made by Bhattacharya and Ghosh (1978) and Bhattacharya (1985). Recently, Hall (1987) considered the Edgeworth expansion of student's t-statistic under weaker moment conditions. In all these papers, a basic assumption is made that the underlying distribution (or a certain number of convolutions of itself) has an absolute continuous component in its Lebesgue decomposition. However, this assumption is not satisfied in many practical situations. For exam-

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ple, in the area of survival analysis, one or more components of $\mathbf{Z}_1$ are counting numbers (such as those dead or alive at a given time). Babu and Singh (1989a, b) considered such a case and got a one-term Edgeworth expansion. In this paper, we shall consider the general form of Edgeworth expansion of W_n. We do not assume the Cramér c-condition on the distribution of $\mathbf{Z}_1$ as a whole; instead we make assumptions on the conditional c.f. (characteristic function) of only one component of $\mathbf{Z}_1$. It is easy to see that this assumption is generally weaker than the Cramér c-condition on the joint distribution of all the components of $\mathbf{Z}_1$.

Suppose σ^2 is chosen according to (1.1) and denote by F_n the distribution function of W_n. In the present paper we shall prove the validity of the Edgeworth expansion of F_n under the following assumptions.

ASSUMPTION 1. H is $m-1$ times continuously differentiable in a neighborhood of $\boldsymbol{\mu}$, where $m \geq 3$ is an integer.

ASSUMPTION 2. $\mathbf{l} = (l_1, \dots, l_k) = \operatorname{grad}[H(\boldsymbol{\mu})] \neq \mathbf{0}$. Without loss of generality, we assume that $l_1 > 0$.

ASSUMPTION 3. $\mathbf{Z}_1$ has finite mth moment, where $m \geq 3$ is a known integer.

Write

$$v_j(t) = v_j(t, Z_{2j}, \dots, Z_{kj}) = E\left[\exp\{itZ_{1j}\} | Z_{2j}, \dots, Z_{kj}\right], \tag{1.2}$$

where $Z_{ij}, Z_{2j}, \dots, Z_{kj}$ are components of $\mathbf{Z}_j$.

ASSUMPTION 4.

$$\limsup_{|t| \to \infty} E|v_1(t)| < 1. \tag{1.3}$$

Let

$$\alpha_j^* = E(Z_{1j} | Z_{2j}, \dots, Z_{kj}),$$

$$\beta_j^* = \operatorname{Var}(Z_{1j} | Z_{2j}, \dots, Z_{kj}),$$

$$B_n^2 = \sum_{j=1}^{n} \beta_j^* \quad \text{and} \quad \beta = E\beta_j^*.$$

REMARK 1.1. From Assumption 4, it is not difficult to prove that $\beta > 0$, where $\beta = E\beta_1^*$.

In this paper we prove the following main theorem.

THEOREM 1. *Suppose that the Assumptions* 1–4 *are true. Then we have*

$$\|F_n - U_{mn}\| = \sup_x |F_n(x) - U_{mn}(x)| = o(n^{-(m-2)/2}), \tag{1.4}$$

where

$$U_{mn}(x) = \Phi(x) + \sum_{\nu=1}^{m-2} n^{-\nu/2} Q_\nu(x)\varphi(x),$$

$\Phi(x)$ and $\varphi(x)$ are the distribution and density functions of the standard normal variable, $Q_\nu(x)$ is a polynomial of degree not exceeding 3ν whose coefficients are determined by the first $\nu + 2$ moments of $\mathbf{Z}_1$ and the partial derivatives of H at $\boldsymbol{\mu}$.

As an example, we give the expression of the term of order $O(n^{-1/2})$ as follows: The coefficient of $n^{-1/2}\varphi(x)$ is

$$-(2\sigma)^{-1}\{l_{11}EZ_{11}^2 + 2l_{12}EZ_{11}Z_{21} + l_{22}EZ_{21}^2\} \tag{1.5}$$

and the coefficient of $n^{-1/2}H_2(x)\varphi(x)$ is

$$\begin{aligned} -\tfrac{1}{6}\sigma^{-3}\Big\{&6l_{11}E^2(l_1Z_{11}^2 + l_2Z_{11}Z_{21}) + 6l_{22}E^2(l_1Z_{11}Z_{21} + l_2Z_{21}^2) \\ &+12l_{12}E(l_1Z_{11}^2 + l_2Z_{11}Z_{21})E(l_1Z_{11}Z_{21} + l_2Z_{21}^2) \\ &+E(l_1Z_{11} + l_2Z_{21})^3\Big\}. \end{aligned} \tag{1.6}$$

REMARK 1.2. If the Assumption 4 is not true, but the conditional characteristic function of some linear combination $\mathbf{aZ}'_1$ with $\mathbf{al}' \neq 0$ satisfies a similar condition as the Assumption 4, then Theorem 1 is still true. The proof of this easily follows from that of Theorem 1.

REMARK 1.3. It may be noted that the structure of the expansion obtained by us is the same as that of Bhattacharya (1985), but under a weaker condition to cover a wider set of practical situations, as in survival analysis [see, for instance, the recent paper by Babu, Rao and Rao (1989)]. We hope to consider an application of Theorem 1, in a forthcoming paper, to obtain the asymptotic expansion of the distribution of a statistic connected with the Kaplan–Meier estimator in survival analysis. It is also hoped to develop some formal rules for obtaining the expansion as was done by Bhattacharya (1985) and Chibisov (1972, 1980c).

Some earlier contributions which are related to our discussions on $H(\overline{Z}_n)$ are the results on stochastic expansions given in review papers by Pfanzagl (1981) and Chibisov (1984) and those of Chibisov (1980b) where one-term Edgeworth expansion is obtained under only the nonlattice condition on the first component. Some other relevant references are Babu (1990) and Bai and Rao (1989).

2. Lemmas. To begin with, we present some notations for the formal Edgeworth expansion. Let $X_1, X_2, \dots, X_n$ be independent random variables with mean zero and variances σ_j^2, $j = 1, 2, \dots, n$. Write $B_n^2 = \sum_{j=1}^n \sigma_j^2$ and denote by F_n the distribution function of $B_n^{-1}\sum_{j=1}^n X_j$. Assume that the mth moments of X_j's are finite and denote by $\gamma_{j\nu}$ the νth cumulant of X_j, $\nu = 3, \dots, m$; $j = 1, 2, \dots, n$. Write

$$\lambda_{\nu n} = n^{(\nu-2)/2} B_n^{-\nu} \sum_{j=1}^{n} \gamma_{j\nu}, \tag{2.1}$$

$$Q_{\nu n}(x) = \sum_{s=1}^{\nu} (-1)^{\nu+2s} \Phi^{(\nu+2s)}(x) h_{\nu s}, \tag{2.2}$$

$$= -\varphi(x) \sum_{s=1}^{\nu} H_{\nu+2s-1}(x) h_{\nu s}, \tag{2.3}$$

where

$$h_{\nu s} = h_{\nu s}(\lambda_{3n}, \dots, \lambda_{\nu+2,n}) = \sum{}' \prod_{i=1}^{\nu} (k_i!)^{-1} \left(\frac{\lambda_{i+2,n}}{(i+2)!} \right)^{k_i},$$

the summation Σ' running over all possible nonnegative integers with the restrictions

$$k_1 + k_2 + \cdots + k_\nu = s,$$
$$k_1 + 2k_2 + \cdots + \nu k_\nu = \nu,$$

$\Phi(x)$ and $\varphi(x)$ are distribution and density functions of the standard normal variable and $H_\nu(x)$ is the Chebyshev–Hermitian polynomial of degree ν. Then the formal Edgeworth expansion of F_n is defined by

$$U_{mn}(x) = \Phi(x) + \sum_{\nu=1}^{m-2} n^{-\nu/2} Q_{\nu n}(x).$$

Then we have the following lemma.

LEMMA 1. *With the above notation, assume that there are positive constants* $a_1 < a_2$ *such that*

$$n a_1 \le B_n^2 \le n a_2.$$

Then, we have

$$\|F_n - U_{mn}\| = \sup_x |F_n(x) - U_{mn}(x)|$$
$$\le C(m) \left\{ \Lambda_{mn} + B_{m+1,n} + \int_{\Delta_1 \le |t| \le \Delta_2} |t|^{-1} \prod_{j=1}^{n} |v_{nj}(t)|\, dt \right\},$$

where

$$\Lambda_{mn} = B_n^{-m} \sum_{j=1}^{n} E|X_j|^m I\left[|X_j| > \sqrt{an}\right],$$

$$B_{m+1,n} = B_n^{-m-1} \sum_{j=1}^{n} E|X_j|^{m+1} I\left[|X_j| \le \sqrt{an}\right],$$

Δ_1 is a constant such that $\sum_{j=1}^{n} E|X_j|^2 I[|X_j| \geq 1/12\Delta_1] \leq B_n^2/8$ and

$$\Delta_2 = (B_n B_{m+1,n})^{-1},$$

$I[A]$ denotes the indicator function of the set A, $v_{nj}(t)$ is the characteristic function of $X_j I[|X_j| \leq \sqrt{an}\,]$, a is a positive constant and $C(m)$ is a constant which may depend on the distribution through a, a_1 and s_2 only.

PROOF. This lemma can be proved by following the same lines as in Bai and Zhao (1986). One only needs to note that when $|t| \leq \Delta_1 B_n$,

$$\prod_{j=1}^{n} |v_{nj}(t/B_n)| \leq \exp\{-\tfrac{5}{24}t^2\},$$

which has been proved in page 42 of Hall [(1982), page 42]. Therefore, the quantity of $(B_n B_{3,n})^{-1}$ used in Bai and Zhao [(1986), page 19] can be replaced by Δ_1. □

LEMMA 2. *Suppose that $\psi(x) = V(x/b)$, where V is the standard normal distribution function or its derivatives and $b \neq 0$ is a constant. Then we have*

$$|\psi * (F_n - U_{mn})(x)| \leq C(m)\left\{\frac{\Lambda_{mn}(x)}{(1+|x|)^m} + \left(|b|^{-2m} + |b|^{m+1}\right)\frac{B_{m+1,n}(x)}{(1+|x|)^{m+1}}\right\},$$

where $C(m)$ is a constant depending on m only and

$$\Lambda_{mn}(x) = B_n^{-m} \sum_{j=1}^{n} E|X_j|^m I\left[|X_j| > B_n(1+|x|)\right],$$

$$B_{m+1,n}(x) = B_n^{-m-1} \sum_{j=1}^{n} E|X_j|^{m+1} I\left[|X_j| \leq B_n(1+|x|)\right].$$

where $\psi * F$ denotes the convolution of the functions ψ and F.

The proof is given in Bai (1989).

LEMMA 3. *Suppose a and b are nonzero constants. Then we have*

$$\int_{-\infty}^{\infty} \Phi^{(k)}\left(\frac{(x-by)}{a}\right) d\Phi^{(l)}(y) = \frac{a^k b^l}{(a^2+b^2)^{(k+l)/2}} \Phi^{(k+l)}\left(\frac{x}{\sqrt{a^2+b^2}}\right).$$

PROOF. Noting that $\Phi^{(k)}((x-by)/a) = a^k (d^k/dx^k)\Phi((x-by)/a)$, the lemma follows by direct computation and the details are omitted. □

LEMMA 4. *Suppose X_j, $j = 1, 2, \ldots, n$, are independent random variables with mean zero and variances σ_j^2. Write $B_n^2 = \sum_{j=1}^{n} \sigma_j^2$ and define $Y_j = X_j I[|X_j| \leq aB_n]$, where $a > 0$ is a constant. Then for each integer $m > 0$, there* *is an absolute constant* $C(m)$ (*which may depend on* a) *such that*

$$E\left|\left(B_n^{-1}\sum_{j=1}^{n} Y_j\right)^m\right| \le C(m).$$

PROOF. The proof of this lemma can be found in Lemma 1 of Bai and Zhao (1986) or in Corollary 14.4 of Bhattacharya and Rao (1976). □

LEMMA 5. *Suppose that* $\{X_j,\ j = 1,2,\dots\}$ *is a sequence of* iid *random variables with mean zero and finite* mth *moments and* $m \ge 3$ *is an integer. Then for any* $\frac{1}{2} < b < a < 1$, *we have*

$$P\left\{\left|\sum_{j=1}^{n} X_j\right| \ge \varepsilon n^a\right\} = o(n^{-b(m-2)}).$$

The proof is routine using the truncation technique and applying the Bennett inequality to the sum of the truncated variables. The result also follows from Corollary 17.12 of Bhattacharya and Rao (1976) or the arguments contained in Nagaev (1979) and Chibisov (1980a).

LEMMA 6. *Suppose that* $\{(X_{j1},\dots,X_{jp}),\ j = 1,2,\dots\}$ *is a sequence of* iid *random* p-*vectors with mean zero. Suppose that* $X_{11},\dots,X_{1s}$ *have finite second moment and that* $X_{1,s+1},\dots,X_{1p}$ *have finite* $m/2$ *moment,* $m \ge 3$. *Then for any* $\beta_i > 0$ *and for any integers* α_i, $i = 1,\dots,p$, *such that* $\gamma = \alpha_{s+1} + \cdots + \alpha_p$ *is even or is not equal to* $(m-2)$ *if odd, we have*

$$Q_n = E\prod_{i=1}^{p}\left|\frac{1}{n}\sum_{j=1}^{n} X_{ji}\right|^{\alpha_i} I[A] \le Cn^{-\alpha/2}\left(n^{-\gamma/2} + o(n^{-(m-2)/2})\right), \tag{2.4}$$

where $A = \{|X_{ji}| \le \sqrt{n\beta_i},\ i = 1,\dots,s,\ |X_{ju}| \le n\beta_u,\ u = s+1,\dots,p,\ j = 1,\dots,n\}$, $\alpha = \alpha_1 + \cdots + \alpha_s$, *and* C *is a positive constant.*

PROOF. As an illustration, we only give the proof for the case where $s = 1$, $p = 2$, $\beta_j = 1$, $j = 1,2$. For ease of notation, redenote $\alpha = \alpha_1$, $\gamma = \alpha_2$, $X_j = X_{j1}$ and $Y_j = X_{j2}$. Then for any even integers α and γ, we have

$$\begin{aligned} &E\left|\sum_{j=1}^{n} X_j\right|^{\alpha}\left|\sum_{j=1}^{n} Y_j\right|^{\gamma} I\left[|X_j| \le \sqrt{n},\ |Y_j| \le n,\ j = 1,2,\dots,n\right] \\ &\quad \le \sum{}^{*} n^{T+S+K}\prod_{t=1}^{T}\left|EX_1^{i_t} I\left[|X_1| \le \sqrt{n}\right]\right|\left|\prod_{s=1}^{S}\left|EY_1^{j_s} I\left[|Y_1| \le n\right]\right|\right| \\ &\qquad \times \prod_{k=1}^{K}\left|EX_1^{h_k}Y_1^{g_k}\right| I\left[|X_1| \le \sqrt{n},\ |Y_1| \le n\right], \end{aligned} \tag{2.5}$$

where the summation Σ^* run over all possible integers such that i_t, j_s, h_k,

$g_k \geq 1$ and $\Sigma i_t + \Sigma h_k = \alpha$ and $\Sigma j_s + \Sigma g_k = \gamma$. Now we have the following:

$$\text{(i)} \qquad \left|EX_1^{i_t} I\left[|X_1| \leq \sqrt{n}\right]\right| \leq C n^{(i_t-2)/2} \quad \text{for all } t = 1, \dots, T,$$

$$\text{(ii)} \qquad \left|EX_1^{h_k} Y_1^{g_k} |I\left[|X_1| \leq \sqrt{n}, |Y_1| \leq n\right]\right| \leq \begin{cases} Cn^{h_k/2+g_k-3/2} & \text{if } g_k \leq m/2, \\ o(n^{h_k/2+g_k-m/2}) & \text{if } g_k > m/2, \end{cases}$$

$$\text{(iii)} \qquad \left|EY_1^{j_s} I[|Y_1| \leq n]\right| \leq \begin{cases} Cn^{j_s-2} & \text{if } 2 \leq j_s \leq m/2, \\ o(n^{j_s-m/2}) & \text{if } j_s = 1 \text{ or } > m/2. \end{cases}$$

Thus, if in a term on the right-hand side of (2.5) there is at least one g_k or j_s greater than $m/2$ or one j_s equal to 1, then this terms is dominated by $o(n^{-\alpha/2-(m-2)/2})$. If all g_k and j_s are less than or equal to $m/2$ and all j_s greater than 1, this term is dominated by $Cn^{-\alpha/2-S/2-K} \leq Cn^{-(\alpha+\gamma)/2}$. And we proved the lemma for the case where α and γ are even.

Now we assume $\alpha = 2\theta + 1$ and γ is even. Then by what we have proved we obtain

$$\begin{aligned} &E\left|\frac{1}{n}\sum_{j=1}^{n} X_j\right|^{2\theta+1}\left|\frac{1}{n}\sum_{j=1}^{n} Y_j\right|^{\gamma} I[A] \\ &\leq E^{1/2}\left|\frac{1}{n}\sum_{j=1}^{n} X_j\right|^{2\theta}\left|\frac{1}{n}\sum_{j=1}^{n} Y_j\right|^{\gamma} I[A] E^{1/2}\left|\frac{1}{n}\sum_{j=1}^{n} X_j\right|^{\alpha}\left|\frac{1}{n}\sum_{j=1}^{n} Y_j\right|^{\gamma} I[A] \\ &\leq C\left\{n^{-\theta}\left(o(n^{-(m-2)/2}) + n^{-\gamma/2}\right)n^{-(\theta+1)}\left(o(n^{-(m-2)/2}) + n^{-\gamma/2}\right)\right\}^{1/2} \\ &\leq Cn^{-\alpha/2}\left(o(n^{-(m-2)/2}) + n^{-\gamma/2}\right). \end{aligned}$$

This proves the lemma for any integer α and even γ. Using the similar approach, one can prove the lemma is true for any integers α and γ. □

3. Proof of the main theorem. Without loss of generality, we can assume that $\boldsymbol{\mu} = \mathbf{0}$ and $H(\mathbf{0}) = 0$. For simplicity of writing, we give the proof only for the case of $k = 2$. It is easy to see that the proof for the case $k = 2$ can be extended to the general case although it may involve complicated notations. Since H is $m - 1$ times continuously differentiable in a neighborhood of $\mathbf{0}$ and $l_1 = \partial H/\partial z_1|_{\mathbf{z}=\mathbf{0}} > 0$, by inverse function theorem there is a constant $\delta > 0$ and an open interval $\Theta = (a, b)$ containing zero as an inner point, such that for each $\mathbf{z} \in D = \{|z_1| \leq \delta, |z_2| \leq \delta\}$ and $x \in \Theta$, the equation $x = H(\mathbf{z})$ has a unique solution $z_1 = g(\mathbf{x})$ [here $\mathbf{x} = (x, z_2) \in D^* = \{x \in \Theta, |z_2| \leq \delta\}$] satisfying the following:

1. g is $m - 1$ times continuously differentiable on D^* and its partial derivatives at $\mathbf{0}$ can be expressed as functions of those of H at $\mathbf{0}$. For some

examples, we have

$$\left.\frac{\partial g}{\partial x}\right|_{\mathbf{x}=\mathbf{0}} = l_1^{-1},$$

$$\left.\frac{\partial g}{\partial z_2}\right|_{\mathbf{x}=\mathbf{0}} = -\frac{l_2}{l_1},$$

$$\left.\frac{\partial^2 g}{\partial x^2}\right|_{\mathbf{x}=\mathbf{0}} = -\frac{l_{11}}{l_1^3},$$

$$\left.\frac{\partial^2 g}{\partial x\,\partial z_2}\right|_{\mathbf{x}=\mathbf{0}} = -l_1^{-3}(l_1 l_{12} - l_2 l_{11}),$$

$$\left.\frac{\partial^2 g}{\partial z_2^2}\right|_{\mathbf{x}=\mathbf{0}} = -l_1^{-3}\left(l_1^2 l_{22} - 2 l_1 l_2 l_{12} + l_2^2 l_{11}\right),$$

where

$$l_i = \left.\frac{\partial H}{\partial z_i}\right|_{\mathbf{z}=\mathbf{0}}, \qquad l_{ij} = \left.\frac{\partial^2 H}{\partial z_i\,\partial z_j}\right|_{\mathbf{z}=\mathbf{0}}, \qquad i, j = 1, 2.$$

2. $g(\mathbf{0}) = 0$,

$$H(\mathbf{z}) \le x \text{ is equivalent to } z_1 \le g(\mathbf{x}). \tag{3.1}$$

Note that

$$P\left\{|Z_{ij}| \ge \sqrt{n\beta} \text{ for some } i = 1, 2,\ j = 1, 2, \dots, n\right\} = o\left(n^{-(m-2)/2}\right).$$

We can replace Z_{ij} by $Z_{ij} I[|Z_{ij}| < \sqrt{n\beta}\,]$ in $\overline{Z}_i$. For simplifying our notations, we still use Z_{ij} for denoting the replacement variables $Z_{ij} I[|Z_{ij}| < \sqrt{n\beta}\,]$. The reader should remember this replacement in the proof.

Define the following events

$$\xi_1 = \left\{|\overline{Z}_1| \le \delta n^{-3/7}\right\}, \tag{3.2}$$

$$\xi_2 = \left\{|\overline{Z}_2| \le \delta n^{-3/7}, |\bar{\alpha}| \le \delta n^{-3/7}\right\}, \tag{3.3}$$

where $\bar{\alpha} = (1/n)(\alpha_1^* + \cdots + \alpha_n^*)$ and $\alpha_j^* = E(Z_{1j} | Z_{2j})$.

Now we assume first that $x\sigma n^{-1/14} \in \Theta$. Write

$$\begin{aligned} F_{n1}(x) &= P\left\{\overline{Z}_1 \le g\left(x\sigma/\sqrt{n}, \overline{Z}_2\right); \xi_1, \xi_2\right\}, \\ F_{n2}(x) &= P\left\{\overline{Z}_1 \le g\left(x\sigma/\sqrt{n}, \overline{Z}_2\right); \xi_2\right\}. \end{aligned} \tag{3.4}$$

Then by Lemma 5, we have

$$
\begin{aligned}
&\sup_{x\sigma n^{-1/14}\in\Theta} |F_n(x) - F_{n2}(x)| \\
(3.5)\qquad &\le \sup_{x\sigma n^{-1/14}\in\Theta} \{|F_n(x) - F_{n1}(x)| + |F_{n1}(x) - F_{n2}(x)|\} \\
&\le 2P(\xi_1^c) + P(\xi_2^c) = o(n^{-(m-2)/2}),
\end{aligned}
$$

where ξ_l^c denotes the complement of the event ξ_l, $l = 1, 2$.

Let $D_n = \{Z_{2j},\ j = 1, 2, \dots, n\}$, P^*, E^* denote the conditional probability and expectation given D_n. Denote the conditional variance, νth cumulant and the c.f. (characteristic function) of Z_{1j} given D_n by β_j^*, $\gamma_{\nu j}^*$ and $v_j^*(t)$, respectively, and write

$$
(3.6)\qquad B_n^2 = \sum_{j=1}^{n} \beta_j^*,
$$

$$
(3.7)\qquad F_n^*(y) = P^*\left(\sum_{j=1}^{n} (Z_{1j} - \alpha_j^*) \le B_n y\right) = P^*\left(\bar{Z}_1 \le \bar{\alpha} + \frac{1}{n} B_n y\right).
$$

Define the events

$$
(3.8)\qquad \xi_3 = \left\{\left|B_n^2 - n\beta\right| < \tfrac{1}{2} n\beta\right\},
$$

$$
(3.9)\qquad \xi_4 = \{\Delta_1 \ge \Delta_3\},
$$

where Δ_1 is such that $\sum_{j=1}^{n} E^*|Z_{1j} - \alpha_j^*|^2 I[|Z_{1j} - \alpha_j^*| \ge 1/12\Delta_1] \le B_n^2/8$, Δ_3 is some suitably chosen positive constant and $\beta = E\beta_1^*$, which is positive by assumption. We have

$$
(3.10)\qquad P(\xi_l^c) = o(n^{-(m-2)/2}), \qquad l = 3, 4.
$$

Here the proof for $l = 3$ is trivial and for $l = 4$ follows from the same lines in Hall (1987).

When $\prod_{i=2}^{4} \xi_i$ occurs, using B_n, $\gamma_{\nu j}^*$, $\nu = 3, \dots, m$; $j = 1, \dots, n$, we can construct the conditional formal Edgeworth expansion of F_n^*, denoted by $U_{mn}^*(x)$. Denote the conditional c.f. of $Z_{1j} I[|Z_{1j} - \alpha_j^*| \le \sqrt{n\beta}\,]$ given D_n by $v_{nj}^*(t)$. Now we proceed to prove

$$
(3.11)\qquad \sup_{x\sigma n^{-1/14}\in\Theta} \left| F_n(x) - EU_{mn}^*(y(x)) I\left[\prod_{i=2}^{4} \xi_i\right]\right| = o(n^{-(m-2)/2}),
$$

where $y = y(x) = (n/B_n) g(x\sigma/\sqrt{n}, \bar{Z}_2) - (n/B_n)\bar{\alpha}$. Then by Lemma 1, we have

$$
\begin{aligned}
&\|F_n^* - U_{mn}^*\| = \sup_y |F_n^*(y) - U_{mn}^*(y)| \\
(3.12)\qquad &\le C(m)\left\{\Lambda_{mn}^* + B_{m+1,n}^* + \int_{\Delta_1 \le |t| \le \Delta_2} |t|^{-1} \prod_{j=1}^{n} \left|v_{nj}^*(t)\right| dt\right\},
\end{aligned}
$$

where

$$\Lambda^*_{mn} = B_n^{-m} \sum_{j=1}^{n} E^*|Z_{1j} - \alpha_j^*|^m I\left[|Z_{1j} - \alpha_j^*| > \sqrt{n\beta}\right], \tag{3.13}$$

$$B^*_{m+1,n} = B_n^{-m-1} \sum_{j=1}^{n} E^*|Z_{1j} - \alpha_j^*|^{m+1} I\left[|Z_{1j} - \alpha_j^*| \le \sqrt{n\beta}\right], \tag{3.14}$$

$$\Delta_2 = (B_n B^*_{m+1,n})^{-1} \le \Delta_4 \triangleq (M/2) n^{(m-2)/2} \tag{3.15}$$

and $C(m)$ is a constant depending on m only. By the existence of the mth moment of $\mathbf{Z}_1$, one can easily prove that

$$E\Lambda^*_{mn} I\left[\prod_{i=2}^{4} \xi_i\right] \le (2\beta)^{m/2} n^{-(m-2)/2} E|Z_1 f_1 - \alpha_1^*|^m I\left[|Z_{11} - \alpha_1^*| > \sqrt{n\beta}\right]$$
$$= o(n^{-(m-2)/2})$$

and

$$EB^*_{m+1,n} I\left[\prod_{i=2}^{4} \xi_i\right] \le (2\beta)^{(m+1)/2} n^{-(m-1)/2}$$
$$\times E|Z_{11} - \alpha_1^*|^{m+1} I\left[|Z_{11} - \alpha_1^*| \le \sqrt{n\beta}\right]$$
$$= o(n^{-(m-2)}).$$

By Assumption 4, for $|t| \ge \Delta_3$, there exists a constant $\kappa < 1$ such that for all large n,

$$E|v^*_{n1}(t)| \le E|v^*_1(t)| + P\left(|Z_{11} - \alpha_1^*| > \sqrt{n\beta}\right)$$
$$\le E|v_1(t)| + P\left(|Z_{11}| \ge \sqrt{n\beta}\right) + P\left(|Z_{11} - \alpha_1^*| > \sqrt{n\beta}\right) \le \kappa.$$

Hence by Assumption 4 and (3.10), we obtain

$$E \sup_y |F^*_n(y) - U^*_{mn}(y)| I\left\{\prod_{i=2}^{4} \xi_i\right\}$$
$$\le o(n^{-(m-2)/2}) + C(m) \int_{\Delta_3 \le |t| \le \Delta_4} |t|^{-1} \prod_{j=1}^{n} E|v^*_{nj}(t)|\, dt \tag{3.16}$$
$$\le o(n^{-(m-2)/2}) + C(m)(\log n) \sup_{|t| \ge \Delta_3} \left(E|v^*_{n1}(t)|\right)^n$$
$$= o(n^{-(m-2)/2}).$$

Substituting $y = y(x)$, we obtain

$$E \sup_{x\sigma n^{-1/14} \in \Theta} |F^*_n(y(x)) - U^*_{mn}(y(x))| I\left[\prod_{i=2}^{4} \xi_i\right] = o(n^{-(m-2)/2}). \tag{3.17}$$

Note that $F_{n2}(x) = EF^*_n(y(x))I[\xi_2] = EF^*_n(y(x))I[\prod_{i=2}^{4}\xi_i] + o(n^{-(m-2)/2})$, so that from (3.5) and (3.17) we obtain (3.11).

By Lemma 4, there is an absolute constant $C(m)$ such that

$$|U_{mn}^*(y(x))|I[\xi_3] \le C(m).$$

Thus $E|U_{mn}^*(y(x))|I[\xi_3\xi_4^c] \le C(m)P(\xi_4^c) = o(n^{-(m-2)/2})$. Therefore, we can remove ξ_4 from (3.11). Namely, we have

$$\sup_{x\sigma n^{-1/14}\in\Theta}\left|F_n(x) - EU_{mn}^*(y(x))I\left[\prod_{i=2}^{3}\xi_i\right]\right| = o(n^{-(m-2)/2}). \tag{3.18}$$

Write $x_n = x\sigma/\sqrt{n}$, $B(n) = (n\beta - B_n^2)/(n\beta)$. Assume $\prod_{i=2}^3\xi_i$ occurs. Since $y(x) = \sqrt{n/\beta}\,(g(x_n, \bar{Z}_2) - \bar{\alpha})(1 - B(n))^{-1/2}$, by (3.1) and Taylor expansion for a multivariate function, we can write

$$y(x) = \sum_{w=1}^{m-1} y_w(x) + R_1(x), \tag{3.19}$$

where

$$y_1(x) = \sqrt{\frac{n}{\beta}}\,(\eta_1 x_n + \eta_2\bar{Z}_2 - \bar{\alpha}) \quad \text{or}$$

$$x_n = l_1\sqrt{\frac{\beta}{n}}\,y_1(x) + l_1\bar{\alpha} + l_2\bar{Z}_2, \tag{3.20}$$

$$y_w(x) = 2^{-2(w-1)}\binom{2(w-1)}{w-1}y_1(x)B(n)^{w-1} + \sqrt{\frac{n}{\beta}}\sum_{u=2}^{w}\sum_{l=0}^{u}\frac{\eta_{l,u-l}}{l!\,(u-l)!}2^{-2(w-u)}\binom{2w-2u}{w-u}x_n^l\bar{Z}_2^{u-l}B(n)^{w-u}, \tag{3.21}$$

$$w=2,\dots,m-1,$$

$$\begin{aligned}|R_1(x)| &\le \varepsilon_1(n)\sqrt{n}\left\{|x_n|^{m-1} + |\bar{Z}_2^{m-1}| + \left(|x_n| + |\bar{Z}_2|\right)|B(n)|^{m-1}\right\}\\ &\le C\sqrt{n}\left\{\varepsilon_1(n)\left[\left|\frac{1}{\sqrt{n}}y_1(x)\right|^{m-1} + |\bar{Z}_2^{m-1}| + |\bar{\alpha}|^{m-1}\right]\right.\\ &\qquad\left. + \left[\frac{1}{\sqrt{n}}|y_1(x)| + |\bar{\alpha}| + |\bar{Z}_2|\right]|B(n)|^{m-1}\right\},\end{aligned} \tag{3.22}$$

$$\eta = \frac{1}{l_1},\ \eta_2 = -\frac{l_2}{l_1},$$

and

$$\eta_{l,u-l} = \left.\frac{\partial^u g}{\partial x^l\,\partial z_2^{u-l}}\right|_{\mathbf{x}=\mathbf{0}}, \qquad l = 0, 1, \dots, u,$$

which can be determined by the partial derivatives of H at $\mathbf{0}$, as mentioned in (3.1), and by Assumption 1, $\varepsilon_1(n) \to 0$ as $n \to \infty$, depending on n and H only.

By (3.20) and the relationship between η's and l's, we can rewrite (3.21) as

$$(3.21') \quad y_w(x) = \sqrt{\frac{n}{\beta}} \sum{}^{**} \eta^*_{i_1 i_2 i_3 i_4 i_5} \left(\sqrt{\frac{\beta}{n}}\, y_1(x) \right)^{i_1} \bar{\alpha}^{i_2} \bar{Z}_2^{i_3+i_4} B(n)^{i_5},$$

$$w = 2, \dots, m-1,$$

where the summation Σ^{**} is taken subject to $i_1 + i_2 + i_3 + i_4 + i_5 = w$, $i_5 < w$, and

$$\eta^*_{i_1 i_2 i_3 i_4 i_5} = (i_1!\, i_2!\, i_3!\, i_4!)^{-1} l_1^{i_1+i_2} l_2^{i_3} \eta_{i_1+i_2+i_3,\, i_4} 2^{-2i_5} \binom{2i_5}{i_5}.$$

Now we begin to eliminate $R_1(x)$ from $y(x)$. We shall prove

$$(3.23) \quad \sup_{x\sigma n^{-1/14} \in \Theta} E|U^*_{mn}(y(x)) - U^*_{mn}(y'(x))| I\left[\prod_{i=2}^{3} \xi_i\right] = o(n^{-(m-2)/2}),$$

where $y'(x) = y(x) - R_1(x)$. Consider the term

$$(3.24) \quad J_{\nu s} = n^{-\nu/2} E|\Phi^{(\nu+2s)}(y(x)) - \Phi^{(\nu+2s)}(y'(x))| |h^*_{\nu s}| I\left[\prod_{i=2}^{3} \xi_i\right],$$

where $h^*_{\nu s} = h_{\nu s}(\lambda^*_{3n}, \dots, \lambda^*_{\nu+2,n})$, $h_{\nu s}$ is defined in (2.3), while λ^*_{ln}'s are similarly defined as in (2.1). Now assume that $\prod_{i=2}^{3}\xi_i$ occurs and $x\sigma n^{-1/14} \in \Theta$. Then we have the following:

$$(3.25) \quad y_w(x) \to 0, \qquad R_1(x) \to 0; \qquad w = 2, 3, \dots, m-1, \text{ uniformly},$$

$$(3.26) \qquad |h^*_{\nu s}| \le C n^{\nu/2}, \quad \text{by Lemma 4},$$

where C is a constant which may depend on ν, s and β. (Hereafter we shall use the same symbol C to denote constants with the similar properties, but may take different values at different appearances). If $|y_1(x)| \le 1$, then by (3.25) for all large n, $|y(x)| \le 2$ and $|y'(x)| \le 2$. Hence by the mean value theorem

$$(3.27) \quad \begin{aligned} &|\Phi^{(\nu+2s)}(y(x)) - \Phi^{(\nu+2s)}(y'(x))| \\ &= |\Phi^{(\nu+2s+1)}(\xi(x))| |R_1(x)| \\ &\le C\Big\{\varepsilon_1(n)\Big[n^{-(m-2)/2} + \sqrt{n}\left(|\bar{Z}_2^{m-1}| + |\bar{\alpha}|^{m-1}\right)\Big] \\ &\qquad + \left(1 + \sqrt{n}|\bar{\alpha}| + \sqrt{n}|\bar{Z}_2|\right)|B(n)|^{m-1}\Big]\Big\}, \end{aligned}$$

where $\xi(x)$ lies between $y(x)$ and $y'(x)$. Hereafter $\Phi^{(\nu+2s)}$ with $\nu = s = 0$ is used to denote Φ. If $|y_1(x)| > 1$, then by (3.22), when n large, $|y_1(x)/\xi(x)| \ge 2^{-1}$ for any $\xi(x)$ that lies between $y(x)$ and $y'(x)$. Hence by the mean value theorem,

$$
\begin{aligned}
&\left|\Phi^{(\nu+2s)}(y(x)) - \Phi^{(\nu+2s)}(y'(x))\right| \\
&\quad = \left|\Phi^{(\nu+2s+1)}(\xi(x))\right|\left|R_1(x)\right| \\
&\quad \le C\varepsilon_1(n)\Big\{n^{-(m-2)/2}\left|\xi(x)^{m-1}\Phi^{(\nu+2s+1)}(\xi(x))\right|\left|y_1(x)/\xi(x)\right|^{m-1} \\
&\qquad + \sqrt{n}\Big[\left|\bar{Z}_2^{m-1}\right| + |\bar{\alpha}|^{m-1} + \big(n^{-1/2}\left|\xi(x)\Phi^{(\nu+2s+1)}(\xi(x))\right| \\
&\qquad\qquad \times\left|y_1(x)/\xi(x)\right| + |\alpha| + \left|\bar{Z}_2\right|\big)\left|B(n)\right|^{m-1}\Big]\Big\}.
\end{aligned}
$$

Note that the same bound as in (3.27) can be obtained here. Thus (3.24)–(3.27) imply that

$$
\begin{aligned}
J_{\nu s} &\le C\varepsilon_1(n)\Big\{n^{-(m-2)/2} + \sqrt{n}\,E\big(\left|\bar{Z}_2^{m-1}\right| + |\bar{\alpha}|^{m-1}\big)\Big\} \\
&\quad + CE\Big\{\big[1 + \sqrt{n}\,|\bar{\alpha}| + \sqrt{n}\left|\bar{Z}_2\right|\big]\left|B(n)^{m-1}\right|\Big\} \qquad (3.28)\\
&\le o(n^{-(m-2)/2}),
\end{aligned}
$$

where the last step follows from the following facts:

1. $E|\bar{Z}_2^{m-1}| = O(n^{-(m-1)/2})$,
2. $E|\bar{\alpha}^{m-1}| = O(n^{-(m-1)/2})$,
3. $E|B(n)^{m-1}| = o(n^{-(m-2)/2})$,
4. $E|\bar{\alpha}B(n)^{m-1}| = o(n^{-(m-1)/2})$,
5. $E|\bar{Z}_2B(n)^{m-1}| = o(n^{-(m-1)/2})$.

Recalling the definition of U_{mn}^*, from (3.24)–(3.28), we obtain (3.23).

By Taylor expansion, we have

$$
\begin{aligned}
\Phi^{(\nu+2s)}(y'(x)) &= \Phi^{(\nu+2s)}(y_1(x)) \\
&\quad + \sum_{w=1}^{m-2}\sum_{u=0}^{w}\Phi^{(\nu+2s+u)}(y_1(x))\sum^{(w,u)}\prod_{l=2}^{m-1}\frac{1}{i_l!}y_l^{i_l}(x) + R_2(x) \qquad (3.29)\\
&\triangleq \Phi^{(\nu+2s)}(y_1(x)) + \Psi_{\nu,s}(x) + R_2(x),
\end{aligned}
$$

where the summation $\Sigma^{(w,u)}$ is taken subject to

$$
\begin{gathered}
i_2 + i_3 + \cdots + i_{m-1} = u \le w, \\
i_2 + 2i_3 + \cdots + (m-2)i_{m-1} = w \le m-2
\end{gathered}
$$

and the remainder term satisfies

$$
\begin{aligned}
\left|R_2(x)\right| &\le C\Big\{n^{-(m-1)/2} + \sum_{u=1}^{m-1} n^{u/2}\Big\{\left|\bar{Z}_2^{m+u-1}\right| + |\bar{\alpha}|^{m+u-1} \qquad (3.30)\\
&\qquad + \big(\left|\bar{Z}_2\right|^{u} + |\bar{\alpha}|^{u} + n^{-u/2}\big)\left|B(n)\right|^{m-1}\Big\}\Big\}.
\end{aligned}
$$

Now we proceed to eliminate $R_2(x)$ from $\Phi^{(\nu+2s)}(y'(x))$. By Lemma 6, we have

$$n^{(u-\nu)/2}E|h^*_{\nu,s}||\bar{Z}_2^{m+u-1}|I[\xi_3] \le Cn^{u/2}E|\bar{Z}_2^{m+u-1}| = O(n^{-(m-1)/2}) = o(n^{-(m-2)/2}). \tag{3.31}$$

Similarly we have

$$n^{(u-\nu)/2}E|h^*_{\nu,s}||\bar{\alpha}|^{m+u-1}I[\xi_3] = o(n^{(m-2)/2}). \tag{3.32}$$

By Lemma 6, we have

$$n^{(u-\nu)/2}E|h^*_{\nu,s}||\bar{Z}_2^u||B(n)|^{m-1}I[\xi_3] \le Cn^{u/2}E|\bar{Z}_2^u||B(n)|^{m-1} = o(n^{-(m-2)/2}). \tag{3.33}$$

Similarly we have

$$n^{(u-\nu)/2}E|h^*_{\nu,s}||\bar{\alpha}|^u|B(n)|^{m-1}I[\xi_3] = o(n^{-(m-2)/2}), \tag{3.34}$$

$$n^{-\nu/2}E|h^*_{\nu,s}||B(n)|^{m-1}I[\xi_3] = o(n^{-(m-2)/2}). \tag{3.35}$$

By (3.30)–(3.35), we obtain

$$E|R_2(x)n^{-\nu/2}h^*_{\nu,s}|I[\xi_3] = o(n^{-(m-2)/2}), \tag{3.36}$$

which shows that $R_2(x)$ in $\Phi^{(\nu+2s)}(y'(x))$ is negligible.

Substituting (3.21′) into the expression of $\Psi_{\nu,s}(x)$ in (3.29), we rewrite $\Psi_{\nu,s}(x)$ as a linear combination of terms of the form

$$n^{-w/2}(y_1(x))^{i_1}\Phi^{(\nu+2s+u)}(y_1(x))(\sqrt{n}|\bar{\alpha}|)^{i_2}(\sqrt{n}\bar{Z}_2)^{i_3}((n\beta - B_n^2)/\sqrt{n}\beta)^{i_4},$$

$i_1 + i_2 + i_3 + i_4 = w + u$, $1 \le u \le w$, $i_4 \le w \le m - \nu - 2$. Note that $x^i\Phi^{(j)}(x)$, $j \ge 1$, can be written as a linear combination of $\Phi^{(u)}(x)$, $u = 1 \vee (j - i), \dots, i + j$, where $a \vee b = \max(a, b)$. Thus $\Psi_{\nu,s}(x)$ can be rewritten as a linear combination of terms of the following form:

$$n^{-w/2}\Phi^{(v)}(y_1(x))(\sqrt{n}|\bar{\alpha}|)^{i_1}(\sqrt{n}\bar{Z}_2)^{i_2}((n\beta - B_n^2)/\sqrt{n})^{i_3}, \tag{3.37}$$

$m - \nu - 2 \ge w \ge u \ge 1$, $i_1 + i_2 + i_3 \le w + u$, $\nu + 2s - w + i_1 + i_2 + i_3 \le v \le \nu + 2s + 2u + w - i_1 - i_2 - i_3$, $i_3 \le w$. The coefficients of terms of (3.37) may depend on the partial derivatives and β.

For given $\nu \le m - 2$, write

$$\begin{aligned} B_n^{-\nu} &= (n\beta)^{-\nu/2}(1 - B(n))^{-\nu/2} \\ &= (n\beta)^{-\nu/2}\left\{\sum_{i=0}^{m-2} c(i,\nu)B(n)^i + R_{3\nu}\right\} \\ &= B(n,\nu,w) + (n\beta)^{-\nu/2}R_{3\nu}, \end{aligned} \tag{3.38}$$

where

$$c(j,\nu) = \frac{(2j-2+\nu)!!}{2^j j!\,(\nu-2)!!}$$

and

$$|R_{3\nu}| \le C|B(n)|^{m-1}, \quad \text{when } \xi_3 \text{ occurs}.$$

Now we proceed to eliminate $R_{3\nu}$ from $B_n^{-\nu}$. Consider a term of form (3.37), denoted by $T(\nu, w)$. Then following the steps in the proof of (3.33), we have, by Lemma 6,

$$\begin{aligned} n^{-\nu/2}E|T(\nu,w)h_{\nu,s}^{**}R_{3\nu}|I[\xi_3] &\le CE\left|\bar{Z}_2^{i_1}\bar{\alpha}^{i_2}B(n)^{m-1+i_3}\right| \\ &= o(n^{-(m+i_1+i_2-2)/2}), \end{aligned} \tag{3.39}$$

where $h_{\nu,s}^{**} = B_n^{\nu}h_{\nu,s}^{*}$.

Then by (3.11), (3.23), (3.33) and (3.39), we have

$$\sup_{x\sigma n^{-1/14}\in\Theta}\left|F_n(x) - EU_{mn}^{**}(y_1(x))I\left[\prod_{i=2}^{3}\xi_i\right]\right| = o(n^{-(m-2)/2}), \tag{3.40}$$

where $U_{mn}^{**}(y_1(x))$ is obtained from $U_{mn}^{*}(y(x))$ by substituting $y(x)$ with $y'(x)$, $\Phi^{(\nu+2s)}(y(x))$ with $\Phi^{(\nu+2s)}(y_1(x)) + \Psi_{\nu,s}(y_1(x))$ and $B_n^{-\nu}$ in the terms associated with w and $h_{\nu s}^{*}$ with $(B(n,\nu,w)$. Note that the leading term of $U_{mn}^{**}(y_1(x))$ is $\Phi(y_1(x))$. The other general terms in $U_{mn}^{**}(y_1(x))$ with suitable coefficients, depending on the partial derivatives of H and β, have the following form:

$$\begin{aligned} &n^{-(\nu+w)/2}\Phi^{(v)}(y_1(x))\left(\sqrt{n}\,\bar{Z}_2\right)^{i_1}\left(\sqrt{n}\,\bar{\alpha}\right)^{i_2} \\ &\quad\times\left((n\beta - B_n^2)/\sqrt{n}\right)^{i_3}\prod_{i=1}^{\nu}\left(\frac{1}{n}\sum_{j=1}^{n}\gamma_{j,i+2}^{*}\right)^{k_i}, \end{aligned} \tag{3.41}$$

where

$$\begin{gathered} k_1 + k_2 + \cdots + k_\nu = s, \\ k_1 + 2k_2 + \cdots + \nu k_\nu = \nu \le m-2, \\ m-\nu-2 \ge w \ge u \ge 1, \qquad i_1+i_2+i_3 \le w+u, \qquad i_3 \le u, \\ 1 \vee (\nu+2s-w+i_1+i_2+i_3) \le v \le \nu+2s+2u+w-i_1-i_2-i_3. \end{gathered}$$

Now we proceed to eliminate ξ_2 and ξ_3 from (3.40). Denote the general term of (3.41) by $W(x)$. B Lemmas 4, 5 and 6,

$$E|W(x)|I[\xi_2^c\xi_3] \le C(m)E\left|\sqrt{n}\,\bar{Z}_2\right|^{i_1}\left|\sqrt{n}\,\bar{\alpha}\right|^{i_2}\left(\left|\bar{Z}_2^m\right| + |\bar{\alpha}|^m\right) = o(n^{-(m-2)/2}).$$

Hence ξ_2 can be removed from (3.40). By Lemma 6, we have

$$\begin{aligned} E|W(x)|I[\xi_3^c] &\le CE\left|\sqrt{n}\,\bar{Z}_2\right|^{i_1}\left|\sqrt{n}\,\bar{\alpha}\right|^{i_2}|B(n)|^{i_3}\left(B_n^2/n\right)^{s}I[\xi_3^c] \\ &\le CE\left|\bar{Z}\right|^{i_1}|\bar{\alpha}|^{i_2}|B(n)|^{m-1}\left(B_n^2/n\right)^{s} = o(n^{-(m-2)/2}). \end{aligned}$$

This proves ξ_3 is removable from (3.40). Therefore, we proved

$$\sup_{x\sigma n^{-1/14}\in\Theta} |F_n(x) - EU_{mn}^{**}(y_1(x))| = o(n^{-(m-2)/2}). \tag{3.42}$$

Recall that

$$y_1(x) = \frac{\sigma}{l_1\sqrt{\beta}}\left(x - \frac{\tilde{\sigma}}{\sigma}\Gamma_n\right), \tag{3.43}$$

where $\Gamma_n = (1/\tilde{\sigma}\sqrt{n})\Sigma_{j=1}^n(l_2 Z_{2j} + l_1\alpha_j^*)$ and $\tilde{\sigma}^2 = \boldsymbol{l}[\Sigma - \begin{pmatrix}\beta & 0\\ 0 & 0\end{pmatrix}]\boldsymbol{l}' = \sigma^2 - l_1^2\beta$. Since the summands in Γ_n has finite mth moments, we can construct the formal Edgeworth expansion of the distribution function of Γ_n, say, $V_{mn}(y)$. Then we have

$$V_{mn}(y) = \Phi(y) + \sum_{\nu=1}^{m-2} n^{-\nu/2} \sum_{s=1}^{\nu} \left(\frac{-1}{\tilde{\sigma}}\right)^{\nu+2s} \Phi^{(\nu+2s)}(y) \times h_{\nu s}(\lambda_3, \dots, \lambda_{\nu+2}), \tag{3.44}$$

where λ's are cumulants of $l_2 Z_{21} + l_1\alpha_1^*$, whereas the function $h_{\nu s}$ is defined in (2.3).

REMARK 3.1. Because of truncation, the expectation of Γ_n may not be zero and the variance and cumulants of the summands of Γ_n may depend on n. However, if we substitute those quantities by the corresponding values of the untruncated case, then following the steps of Bai and Zhao (1986) or Bhattacharya and Rao [(1976), Section 14], one can prove that there is only a difference of order $o(n^{-(m-2)/2})$ between the Edgeworth expansions when the summands of Γ_n are truncated or not.

First, we consider the leading term $\Phi(y_1(x))$ in $U_{mn}^{**}(x)$. By Lemmas 2 and 3, we obtain that

$$E\Phi(y_1(x)) = \Phi(x) + \sum_{\nu=1}^{m-2} n^{-\nu/2} \sum_{s=1}^{\nu} \left(\frac{-1}{\sigma}\right)^{\nu+2s} \Phi^{(\nu+2s)}(x) \times h_{\nu s}(\lambda_3, \dots, \lambda_{\nu+2}). \tag{3.45}$$

Now we consider the integral of the general terms given in (3.41). Expand the product

$$\left(\sum_{j=1}^n Z_{2j}\right)^{i_1}\left(\sum_{j=1}^n \alpha_j^*\right)^{i_2}\left(\sum_{j=1}^n (\beta - \beta_j^*)\right)^{i_3} \prod_{i-1}^{\nu}\left(\sum_{j=1}^n \gamma_{j,i+2}^*\right)^{k_i},$$

appearing in a term of (3.41). Then we make integration for each term after the expansion. Consider a term and to each index j, $j = 1, 2, \dots, n$, give a score d_j, which is defined to be the sum of the multiplicities of appearance in this term of Z_{2j}, α_j^*, two times of that of $(\beta - \beta_j^*)$ and $i + 2$ times of that of $\gamma_{j,i+2}^*$, $j = 1, 2, \dots, \nu$. We consider the sum of all those terms such that $d_1 \geq \cdots \geq d_\tau > 1$ and $\Sigma_{j=\tau+1}^n d_j = \mu - \tau \geq 0$, $d_j \leq 1$ for $j = \tau + 1, \dots, n$.

Denote such a sum by $T(\tau, \mu, d)$. Let r denote the number of d's which are greater than m [note that $0 \le r \le 2$, since $\Sigma d_j \le 3(m-2)$]. When $r \ge 1$, by the Schwarz inequality, we have

$$(3.46)\qquad \begin{aligned} E|T(\tau,\mu,d)| &\le o\big(n^{(d_1-m)/2+\cdots+(d_r-m)/2+(\mu-\tau)/2}\big)\\ &\le o\big(n^{\Sigma d_j/2-r(m/2-1)-\tau}\big), \end{aligned}$$

where and hereafter the symbol $o(\cdot)$ denotes a small quantity whose order is independent of x varying subject to $x\sigma n^{-1/14} \in \Theta$. Note that the power of n in the denominator of the term described by (3.41) is one half of

$$\nu + 2s + w + i_1 + i_2 + i_3 \ge \nu + 2s + i_1 + i_2 + 2i_3 = d_1 + d_2 + \cdots + d_n,$$

which with (3.46) imply that all those terms with some $d_j > m$ in the expansion of the term given by (3.41) can be omitted, in the sense that their sum is of the order $o(n^{-(m-2)/2})$. Now we consider the sum of all those terms such that $m \ge d_1 \ge \cdots \ge d_\tau > 1$, $\Sigma_{j=\tau+1}^n d_j$, $d_j \le 1$ for all $j > \tau$. We still denote such a sum by $T(\tau, \mu, d)$. Let

$$y_1^{(1)}(x) = y_1(x) - \zeta_1,$$

where $\zeta_1 = -(1/l_1\sqrt{n}\,\beta)(l_2 Z_{21} + l_1\alpha_1^*)$. Expand $\Phi^{(v)}(y_1(x))$ as

$$(3.47)\quad \Phi^{(v)}(y_1(x)) = \Phi^{(v)}\big(y_1^{(1)}(x)\big) + \sum_{i=1}^{m-d_1} \frac{\zeta_1^i}{i!}\Phi^{(v+i)}\big(y_1^{(1)}(x)\big) + R_4(x).$$

Here we have, when ξ_4 occurs,

$$|R_4(x)| \le C|\zeta_1|^{m-d_1+1}.$$

By a similar argument as before, one can prove that $R_4(x)$ is negligible.

Then we write

$$y_1^{(2)}(x) = y_1^{(1)}(x) - \zeta_2,$$

where $\zeta_2 = -(1/l_1\sqrt{n}\,\beta)(l_2 Z_{22} + l_1\alpha_2^*)$. Expand $\Phi^{(v+i)}(y_1^{(1)}(x))$ as

$$(3.48)\qquad \begin{aligned} &\Phi^{(v+i)}\big(y_1^{(1)}(x)\big)\\ &\quad= \Phi^{(v+i)}\big(y_1^{(2)}(x)\big) + \sum_{j=1}^{m-d_1} \frac{\zeta_1^j}{j!}\Phi^{(v+i+j)}\big(y_1^{(2)}(x)\big) + R_5(x). \end{aligned}$$

Here we have

$$|R_5(x)| \le C|\zeta_2|^{m-d_2+1}.$$

By repeating the same argument as before, one can prove that $R_5(x)$ is negligible.

Repeat the same argument. After τ steps, we need only to consider the sum of those terms that $d_{\tau+1} = 1$, $\Sigma_{j=\tau+2}^n d_j = \mu - \tau - 1$, $d_j \le 1$ for all $j \ge \tau + 2$. Then repeat the same argument. Finally, we find that the integration on the set ξ_4 of terms of the form (3.41) can be written as a linear combination of terms of the following form, with an error term of order $o(n^{-(m-2)/2})$,

$$n^{-w/2}EX_1 \cdots X_\mu \Phi^{(v)}(\tilde{y}_1(x)), \tag{3.49}$$

where $\tilde{y}(x) = y_1(x) - \zeta$, $\zeta = -(1/l_1\sqrt{n\beta})\Sigma_{j=1}^{\mu}(l_2 Z_{2j} + l_1\alpha_j^*)$, X_j is a product of Z_{2j}, α_j^*, $\beta - \beta_j^*$ and/or $\gamma_{i+2,j}^*$, $i = 1, 2, \dots, \nu$, with suitable powers, μ is an integer bounded by a constant depending on m (and k in general) only. If we still use d_j to denote the score of X_j, then we can rewrite (3.49) in a more explicit form

$$\begin{aligned} &n^{-s/2}\left\{\prod_{j=1}^{\tau} EX_j n^{-d_j/2+1}\right\}\left\{E\sqrt{n}(\beta - \beta_1^*)\right\}^{i_1} \\ &\quad\times\left\{E\sqrt{n}Z_{21}\right\}^{i_2}\left\{E\sqrt{n}\alpha_i^*\right\}^{i_3} E\Phi^{(v)}(\tilde{y}_1(x)), \end{aligned} \tag{3.50}$$

$i_1 \le i_2 + i_3 + s$, $i_j \ge 2$, $3 \le d_j \le m$ for $j = 1, 2, \dots, \tau$. By applying Lemmas 2 and 3 and the expansion (3.44), we obtain

$$\begin{aligned} \Phi^{(v)}(\tilde{y}_1(x)) &= \Phi^{(v)}\left(\frac{x}{\sqrt{1-\tau\rho/n}}\right) \\ &\quad+ \sum_{\nu=1}^{m-2} n^{-\nu/2}\sum_{s=1}^{\nu}\left(\frac{-1}{\sigma\sqrt{1-\tau\rho/n}}\right)^{\nu+2s}\Phi^{(\nu+2s+v)}\left(\frac{x}{\sqrt{1-\tau\rho/n}}\right) \\ &\quad\times h_{\nu s}(\lambda_3, \dots, \lambda_{\nu+2}) + o(n^{-(m-2)/2}), \end{aligned} \tag{3.51}$$

where $\rho = \tilde{\sigma}^2/\sigma^2$. Finally, making ordinary Taylor expansion for $\sqrt{1-\tau\rho/n}$ on the right-hand side and omitting all terms with higher order, we conclude that

$$\sup_{x\sigma n^{-1/14}\in\Theta} |F_n(x) - U_{mn}(x)| = o(n^{-(m-2)/2}), \tag{3.52}$$

by recalling (3.40), (3.44), (3.50) and (3.51).

It is easy to see that

$$\sup_{x\le -an^{1/14}/(2\sigma)} |U_{mn}(x)| = o(n^{-(m-2)/2}). \tag{3.53}$$

Hence

$$\begin{aligned} \sup_{x\le -an^{1/14}/(2\sigma)} F_n(x) &\le F_n(-an^{1/14}/(2\sigma)) \\ &\le \sup_{x\le -an^{1/14}/(2\sigma)} |U_{mn}(x)| + \sup_{x\sigma n^{-1/14}\in\Theta} |F_n(x) - U_{mn}(x)| \\ &= o(n^{-(m-2)/2}). \end{aligned} \tag{3.54}$$

Similarly, we have

$$\sup_{x \geq bn^{1/14}/(2\sigma)} |1 - U_{mn}(x)| = o(n^{-(m-2)/2}), \tag{3.55}$$

$$\sup_{x \geq bn^{1/14}/(2\sigma)} (1 - F_n(x)) \leq 1 - F_n(bn^{1/14}/(2\sigma)) = o(n^{-(m-2)/2}). \tag{3.56}$$

(3.52)–(3.56) complete the proof of Theorem 1.

We shall compute the coefficients of the first term. The procedure gives a clue as to how the later terms can be computed. The term of order $1/\sqrt{n}$ has three sources:

(i) From the leading term of the first step expansion [see (3.45)], we have

$$-\frac{1}{6\sqrt{n}\,\sigma^3}\lambda_3 H_2(x)\varphi(x), \tag{3.57}$$

where $\lambda_3 = E(l_1\alpha_1^* + l_2 Z_{21})^3$, $\varphi(x)$ is the density of standard normal variable and $H_\nu(x)$ denotes the Chebyshev–Hermitian polynomial of the νth degree.

(ii) From the second term of the first step expansion, approximate B_n^2 by $n\beta$ and apply Lemma 3 to the integration to obtain

$$\begin{aligned} &-\frac{1}{6\sqrt{n}\,\beta^{3/2}}E(Z_{11} - \alpha_1^*)^3\Phi^{(3)}(\tilde{y}_1(x)) \\ &\quad = -\frac{1}{6\sqrt{n}\,\sigma^3}l_1^3 E(Z_{11} - \alpha_1^*)^3 H_2(x)\varphi(x) + O(n^{-1}). \end{aligned} \tag{3.58}$$

(iii) From Taylor expansion of the leading term, that is, $y_2(x)\Phi'(y_1(x))$. Recall that

$$\begin{aligned} y_2(x) &= \sqrt{n/\beta}\left\{\tfrac{1}{2}l_1^{-1}B(n)(x_n - l_2\bar{Z}_2) + \tfrac{1}{2}\eta_{20}x_n^2 + \eta_{11}x_n\bar{Z}_2 + \tfrac{1}{2}\bar{Z}_2^2\right\} \\ &= \tfrac{1}{2}\sqrt{n/\beta}\Big\{B(n)\big(\sqrt{\beta/n}\,y_1(x) + \bar{\alpha}\big) - l_1^{-1}\Big[l_{11}\big(\sqrt{\beta/n}\,y_1(x) + \bar{\alpha}\big)^2 \\ &\qquad + 2l_{12}\bar{Z}_2\big(\sqrt{\beta/n}\,y_1(x) + \bar{\alpha}\big) + l_{22}\bar{Z}_2^2\Big]\Big\}. \end{aligned}$$

After elementary computation, we obtain that the coefficient of $n^{-1/2}\varphi(x)$ is

$$-(2\sigma)^{-1}\{l_{11}EZ_{11}^2 + 2l_{12}EZ_{11}Z_{21} + l_{22}EZ_{21}^2\} \tag{3.59}$$

and the coefficients of $n^{-1/2}H_2(x)\varphi(x)$ is

$$\begin{aligned} &-\tfrac{1}{2}\sigma^{-3}\{l_{11}E^2(l_1Z_{11}^2 + l_2Z_{11}Z_{21}) + l_{22}E^2(l_1Z_{11}Z_{21} + l_2Z_{21}^2) \\ &\qquad + 2l_{12}E(l_1Z_{11}^2 + l_2Z_{11}Z_{21})E(l_1Z_{11}Z_{21} + l_2Z_{21}^2) \\ &\qquad\qquad + E\beta_1(l_1\alpha_1^* + l_2Z_{21})\}. \end{aligned} \tag{3.60}$$

Summing (3.57), (3.58) and (3.60) up, we obtain the coefficient of $n^{-1/2}H_2(x)\varphi(x)$ as follows:

$$
\begin{aligned}
(3.61)\qquad & -\tfrac{1}{6}\sigma^{-3}\Big\{6l_{11}E^2\big(l_1Z_{11}^2 + l_2Z_{11}Z_{21}\big) + 6l_{22}E^2\big(l_1Z_{11}Z_{21} + l_2Z_{21}^2\big) \\
& \quad + 12l_{12}E\big(l_1Z_{11}^2 + l_2Z_{11}Z_{21}\big)E\big(l_1Z_{11}Z_{21} + l_2Z_{21}^2\big) \\
& \quad + E\big(l_1Z_{11} + l_2Z_{21}\big)^3\Big\}.
\end{aligned}
$$

Now we get the first term in the expansion as the linear combination of $n^{-1/2}\varphi(x)$ with coefficient given in (3.59) and $n^{-1/2}H_2(x)\varphi(x)$ with coefficient in (3.61).

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