# EDGEWORTH EXPANSIONS FOR INTEGER VALUED ADDITIVE FUNCTIONALS OF UNIFORMLY ELLIPTIC MARKOV CHAINS 

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#### Abstract

We obtain asymptotic expansions for probabilities $\mathbb{P}\left(S_{N}=k\right)$ of partial sums of uniformly bounded integer-valued functionals $S_{N}=\sum_{n=1}^{N} f_{n}\left(X_{n}\right)$ of uniformly elliptic inhomogeneous Markov chains. The expansions involve products of polynomials and trigonometric polynomials, and they hold without additional assumptions. As an application of the explicit formulas of the trigonometric polynomials, we show that for every $r \geq 1, S_{N}$ obeys the standard Edgeworth expansions of order $r$ in a conditionally stable way if and only if for every $m$, and every $\ell$ the conditional distribution of $S_{N}$ given $X_{j_{1}}, \ldots, X_{j_{\ell}} \bmod m$ is $o_{\ell}\left(\sigma_{N}^{1-r}\right)$ close to uniform, uniformly in the choice of $j_{1}, \ldots, j_{\ell}$, where $\sigma_{N}=\sqrt{\operatorname{Var}\left(S_{N}\right)}$.


## 1. Introduction

Let $Y_{1}, Y_{2}, \ldots$ be a sequence of integer-valued random variables. Let $S_{N}=Y_{1}+\cdots+$ $Y_{N}$ and suppose that $V_{N}=V\left(S_{N}\right)=\operatorname{Var}\left(S_{N}\right) \rightarrow \infty$. Recall that the local central limit theorem (LLT) states that, uniformly in $k$ we have

$$
\mathbb{P}\left(S_{N}=k\right)=\frac{1}{\sqrt{2 \pi} \sigma_{N}} e^{-\left(k-\mathbb{E}\left(S_{N}\right)\right)^{2} / 2 V_{N}}+o\left(\sigma_{N}^{-1}\right) .
$$

where $\sigma_{N}=\sqrt{V_{N}}$. For independent random variables, the stable local central limit theorem (SLLT) states that the LLT holds true for any integer-valued square integrable independent sequence $Y_{1}^{\prime}, Y_{2}^{\prime}, \ldots$ which differs from $Y_{1}, Y_{2}, \ldots$ by a finite number of elements. We recall a classical result due to Prokhorov 1

1. Theorem. [19] If $Y_{n}$ are independent and bounded then the SLLT holds true iff for each integer $h>1$,

$$
\begin{equation*}
\sum_{n} \mathbb{P}\left(Y_{n} \neq m_{n} \bmod h\right)=\infty \tag{1}
\end{equation*}
$$

where $m_{n}=m_{n}(h)$ is the most likely residue of $X_{n}$ modulo $h$.
We refer the readers to [20, 25] for extensions of this result to the case when $Y_{n}$ 's are not necessarily bounded (for instance, the result holds true when $\sup _{n}\left\|Y_{n}\right\|_{L^{3}}<\infty$ ). Related results for local convergence to more general limit laws are discussed in [2, 17].

[^0]The central limit theorem (CLT) for inhomogeneous Markov chains was obtained for the first time in [3], and we refer to [23, 18] for two modern approaches. In the past decades local limit theorems were extended to stationary homogeneous Markov chains. The first result in this direction was obtained in [24], and we refer to [11] for a general approach. Despite the fact that results in the homogeneous case where already known in the 50 's ([24]), only very recently [4] the general case of an inhomogeneous (uniformly elliptic) Markov chains was solved (see also [16]). It turns out that the theory of the the local limit theorem in the inhomogeneous case is far richer than the homogeneous case. In particular, one of the main problems in the inhomogeneous setting arises from a possibility of non-linear growth of $V_{N}$. We refer to [4] for a detailed discussion about the obstructions for the LLT (in both lattice and non-lattice cases).

The local limit theorem deals with approximation of $P\left(S_{N}=k\right)$ up to an error term of order $o\left(\sigma_{N}^{-1}\right)$. In this paper, for uniformly bounded integer-valued additive functional $Y_{n}=f_{n}\left(X_{n}\right)$ of uniformly elliptic inhomogeneous Markov chains $\left\{X_{n}\right\}$ we will characterize a more refined type of approximations of the probabilities $\mathbb{P}\left(S_{N}=k\right)$. Given $r \geq 1$, the Edgeworth expansion of order $r$ holds true if there are polynomials $P_{b, N}$, whose coefficients are uniformly bounded in $N$ and their degrees do not depend on $N$, so that uniformly in $k \in \mathbb{Z}$ we have that

$$
\begin{equation*}
\mathbb{P}\left(S_{N}=k\right)=\sum_{b=1}^{r} \frac{P_{b, N}\left(k_{N}\right)}{\sigma_{N}^{b}} \mathfrak{g}\left(k_{N}\right)+o\left(\sigma_{N}^{-r}\right) \tag{2}
\end{equation*}
$$

where $k_{N}=\left(k-\mathbb{E}\left(S_{N}\right)\right) / \sigma_{N}$ and $\mathfrak{g}(u)=\frac{1}{\sqrt{2 \pi}} e^{-u^{2} / 2}$.
During the 20th century, the work of many authors led to the development of asymptotic expansions in both the CLT and the LLT, see [12, 10] and references therein for more details. Recently in [6], for bounded independent random variables $Y_{n}$ we gave a complete characterization for expansions of an arbitrary order $r$ by means of the rate of decay of the characteristic function of $S_{N}-\mathbb{E}\left[S_{N}\right]$ and their first $r-1$ derivatives at nonzero "resonant points" of the form $t=\frac{2 \pi l}{m}$ with $0<m \leq 2 K$ and $0 \leq l<m$, where $K=\sup _{n}\left\|Y_{n}\right\|_{L^{\infty}}$. The main probabilistic interpretation of these results was a characterization of "super stable" Edgeworth expansions of an arbitrary order $r$ using only the decay rates of the distance of the distribution of $S_{N}-\sum_{j=1}^{\ell} Y_{k_{j, N}}$ modulo $m$ from the uniform distribution, for all $m$, for an arbitrary choice of indexes $k_{1, N}, \ldots, k_{\ell, N}$. In this paper we will characterize a certain type of super stable Edgeworth expansions of an arbitrary order $r$ (see the precise definition below). That is, we will generalize [6, Theorem 1.8] to integer valued additive functionals of uniformly elliptic Markov chains.

We say that the Edgeworth expansion of order $r$ holds true in a conditionally stable ${ }^{2}$ way if the usual Edgeworth expansion of order $r$ holds true under conditioning by finite

[^1]elements $X_{j_{1}}, \ldots, X_{j_{\ell}}$, with error terms which depend only on $\ell$ and polynomials whose coefficients are bounded by some constants which depend only on $\ell$.

We begin from a quantitative version of Prokhorov theorem which is a direct consequence of the general asymptotic expansion which will be described in Section 2.
2. Theorem. Let $\varepsilon_{0}>0$ be so that for every measurable set $A$,

$$
\varepsilon_{0} \mathbb{P}\left(X_{n+1} \in A\right) \leq P\left(X_{n+1} \in A \mid X_{n}=x\right) \leq \varepsilon_{0}^{-1} \mathbb{P}\left(X_{n+1} \in A\right)
$$

for each $n$ and a.e. $x$. Let $Y_{n}=f_{n}\left(X_{n}\right)$ with $\sup _{n}\left\|Y_{n}\right\|_{L^{\infty}} \leq K$. For each $r \in \mathbb{N}$ there is a constant $R=R\left(r, K, \varepsilon_{0}\right)$ such that the conditionally stable Edgeworth expansion of order $r$ holds if for all $N$ we have

$$
M_{N}:=\min _{2 \leq h \leq 2 K} \sum_{n=1}^{N} \mathbb{P}\left(Y_{n} \neq m_{n}(h) \bmod h\right) \geq R \ln V_{N} .
$$

In particular, $S_{N}$ obeys Edgeworth expansions of all orders if

$$
\lim _{N \rightarrow \infty} \frac{M_{N}}{\ln V_{N}}=\infty
$$

This theorem is a quantitative version of Prokhorov's Theorem 1. We observe that logarithmic in $V_{N}$ growth of various non-periodicity characteristics of individual summands are often used in the theory of local limit theorems (see e.g. [13, 14, 16]). However, to justify the optimality we need to understand the conditions necessary for the validity of the Edgeworth expansion.

To this end we obtain an expansion for the probabilities $\mathbb{P}\left(S_{N}=k\right)$ which holds true without additional assumptions $\sqrt[3]{ }$. In order to not to overload the exposition we will formulate the general trigonometric expansion later (see Theorem 7). Our generalized expansion is a key step in proving a complete characterization of the conditionally stable expansions of an arbitrary order which extends [6, Theorem 1.8] (which dealt with independent summands).
3. Definition. Call $t$ resonant if $t=\frac{2 \pi l}{m}$ with $0<m \leq 2 K$ and $0 \leq l<m$.
4. Theorem. For arbitrary $r \geq 1$ the following conditions are equivalent:
(a) $S_{N}=\sum_{j=1}^{N} f_{n}\left(X_{n}\right)$ obeys the conditionally stable Edgeworth expansions of order $r$.
(b) For each $\ell, \sup _{1 \leq j_{1}, \ldots, j_{\ell} \leq N}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{j_{1}}, . ., X_{j_{\ell}}\right]\right\|_{L^{1}}=o_{\ell}\left(\sigma_{N}^{-(r-1)}\right)$.
(c) For each $1 \leq j_{1}, \ldots, j_{\ell} \leq N$, and each $h \leq 2 K$ the conditional distribution of $S_{N}$ given $X_{j_{1}}, \ldots, X_{j_{\ell}} \bmod h$ is $o_{\ell}\left(\sigma_{N}^{1-r}\right)$ close to uniform.
5. Remark. It will follow from our proofs in the case $r=1$ that the conditionally stable local limit theorem (namely the LLT after conditioning on a finite number number of

[^2]elements) is equivalent to the conditionally stable Edgeworth expansion of order 1. Thus we see that the conditionally stable local limit theorem holds true iff for $\ell$,
$$
\sup _{1 \leq j_{1}, \ldots, j_{\ell} \leq N}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{j_{1}}, . ., X_{j_{\ell}}\right]\right\|_{L^{1}}=o_{\ell}(1)
$$
that is iff (c) holds with $r=1$, namely, for each integers $h$ and $L$ if we are given a sequence $\left(j_{1, N}, \ldots, j_{\ell_{N}, N}\right)$ of tuples with $\ell_{N} \leq L$ then the conditional distributions of $\left(S_{N} \mid X_{j_{1}}, \ldots, X_{j_{\ell_{N}}}\right) \bmod h$ converge to uniform as $N \rightarrow \infty$. Equivalently for each $m \in \mathbb{Z}$
$$
\lim _{N \rightarrow \infty} \mathbb{E}\left[e^{i m S_{N} / h} \mid X_{j_{1}}, . ., X_{j_{\ell}}\right]=0
$$
6. Remark. Note that if
$$
M_{N}(h)=\sum_{n=1}^{N} \mathbb{P}\left(Y_{n} \neq m_{n}(h)\right) \leq R\left(r, K, \varepsilon_{0}\right) \ln \sigma_{N}
$$
then for most $n$, the distributions of $Y_{n}$ are sufficiently close to being concentrated on a single point modulo $h$. Our arguments also show that if this closeness holds for all $n$, then the Edgeworth expansions of order $r$ is valid iff $\mathbb{E}\left[e^{i t S_{N}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)$ for every nonzero resonant point $t$. This is a particular case of Theorem 9 formulated in Section 2, which shows that the condition $\mathbb{E}\left[e^{i t S_{N}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)$ is always necessary for the usual expansions to hold, and that a certain weaker version of condition (b) is sufficient.

We also note that if the distribution of $Y_{n} \bmod m$ is not approximately concentrated on a single point the condition $\mathbb{E}\left[e^{i t S_{N}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)$ does not imply that the Edgeworth expansion holds even in the independent case, see [6, Example 10.2]. In fact, in the independent case if one of the $Y_{n}$ 's is uniformly distributed modulo $m$ then $\mathbb{E}\left[e^{i t S_{N}}\right]=0$ for all $N$ large enough and for every nonzero resonant point of the form $t=\frac{2 \pi l}{m}$. However, this does not imply that the derivatives of the characteristic function of $S_{N}{ }^{m}$ $\mathbb{E}\left[S_{N}\right]$ vanish at $t$, and hence by [6, Theorem 1.5] expansions of an arbitrary order $r$ might not hold.

## 2. Main Results

Let $\left\{X_{j}\right\}$ be a Markov chain and assume that each $X_{j}$ takes values on some countably generated measurable space. Denote by $\mu_{n}$ the law of $X_{n}$. We assume that there is a constant $C>1$ so that for $\mu_{n}$-a.e. $x$ and all measure subsets $A$ on the state space of $X_{n+1}$ we have

$$
\begin{equation*}
C^{-1} \mathbb{P}\left(X_{n+1} \in A\right) \leq P\left(X_{n+1} \in A \mid X_{n}=x\right) \leq C \mathbb{P}\left(X_{n+1} \in A\right) \tag{3}
\end{equation*}
$$

The latter condition is equivalent to the following representation of the transition probabilities:

$$
\mathbb{P}\left(X_{n+1} \in A \mid X_{n}=x\right)=\int_{A} p_{n}(x, y) d \mu_{n+1}(y)
$$

where the transition densities $p_{n}(x, y)$ take values in the interval $\left[C^{-1}, C\right]$. Then $\left\{X_{n}\right\}$ is exponentially fast $\psi$-mixing (see e.g. [4]), which means that there are constants
$C_{1}>0$ and $\delta \in(0,1)$ so that if $\bar{X}$ is a function of $X_{1}, \ldots, X_{m}$ and $\bar{Y}$ is a function of $X_{m+n}, X_{m+n+1}, \ldots$ for some $m$ and $n$ then for every relevant measurable sets $A, B$

$$
\begin{equation*}
|\mathbb{P}(\bar{X} \in A, \bar{Y} \in B)-P(\bar{X} \in A) P(\bar{Y} \in B)| \leq C_{1} P(\bar{X} \in A) P(\bar{Y} \in B) \delta^{n} \tag{4}
\end{equation*}
$$

Next, for each $n$, let $f_{n}$ be a of measurable integer valued-function on the state space of $X_{n}$ and set $Y_{n}=f_{n}\left(X_{n}\right)$. We assume that $K:=\sup \left\|Y_{n}\right\|_{L^{\infty}}<\infty$.

Let $q_{n}(m)$ denote the second largest value among $P\left(Y_{n} \equiv j \bmod m\right)$, $j=0,1, \ldots, m-1$. Set

$$
M_{N}=\min _{2 \leq m \leq 2 K} \sum_{n=1}^{N} q_{n}(m) .
$$

7. Theorem. Let $S_{N}=Y_{1}+Y_{2}+\ldots+Y_{N}$ and $\sigma_{N}=\sqrt{V\left(S_{N}\right)}$. There is $J=J(K)<\infty$ and polynomials $P_{a, b, N}$ with degrees depending only on a and $b$, whose coefficients are uniformly bounded in $N$ such that, for any $r \geq 1$ uniformly in $k \in \mathbb{Z}$ we have

$$
\mathbb{P}\left(S_{N}=k\right)-\sum_{a=0}^{J-1} \sum_{b=1}^{r} \frac{P_{a, b, N}\left(\left(k-a_{N}\right) / \sigma_{N}\right)}{\sigma_{N}^{b}} \mathfrak{g}\left(\left(k-a_{N}\right) / \sigma_{N}\right) e^{2 \pi i a k / J}=o\left(\sigma_{N}^{-r}\right)
$$

where $a_{N}=\mathbb{E}\left(S_{N}\right)$ and $\mathfrak{g}(u)=\frac{1}{\sqrt{2 \pi}} e^{-u^{2} / 2}$.
Moreover, given $K, r$, there exists $R=R(K, r)$ such that if $M_{N} \geq R \ln V_{N}$ then we can choose $P_{a, b, N}=0$ for $a \neq 0$.

In particular, $S_{N}$ obeys the Edgeworth expansion of all orders if

$$
\lim _{N \rightarrow \infty} \frac{M_{N}}{\ln \sigma_{N}}=\infty
$$

Next, we say that the Edgeworth expansions of order $r$ hold true in a conditionally stable way if they hold true under conditioning by finite elements $X_{j_{1}}, \ldots, X_{j_{\ell}}$, with error terms $o_{\ell}\left(\sigma_{N}^{-(r-1)}\right)$ which depend only on $\ell, r$ and $\sigma_{N}$ (and not on the indexes $j_{1}, \ldots, j_{\ell}$ ).
8. Theorem. $S_{n}$ obeys the Edgeworth expansions of order $r$ in a conditionally stable way if and only if for every nonzero resonant point $t_{j}$ and every $\ell$,

$$
\sup _{1 \leq j_{1}, \ldots, j_{\ell} \leq N}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{j_{1}}, \ldots, X_{j_{\ell}}\right]\right\|_{L^{1}}=o_{\ell}\left(\sigma_{N}^{-(r-1)}\right)
$$

In the course of the proof of Theorem 8 we obtain the following result.
9. Theorem. (i) The condition $\mathbb{E}\left[e^{i t_{j} S_{N}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)$ is necessary for the usual Edgeworth expansions of order $r$ to hold true.
(ii) There is a natural number $\ell_{r}$ which depends only on $r$ so that the condition

$$
\max _{\ell \leq \ell_{r}} \sup _{j_{1}, \ldots, j_{\ell} \in \mathcal{B}}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{j_{1}}, . ., X_{j_{\ell}}\right]\right\|_{L^{1}}=o_{\ell}\left(\sigma_{N}^{-(r-1)}\right)
$$

is sufficient for the usual Edgeworth expansions of order r to hold true.

[^3]The number $\ell_{r}$ in part (ii) can be recovered from the proof of the theorem (for instance, we have $\ell_{2}=6$ ).

The reason that Theorem 8 follows from Theorem 9 is that we can apply it to the conditional law of $S_{N}$ given a finite number of $X_{j}$ 's, and that in part (i) the term $o\left(\sigma_{N}^{-(r-1)}\right)$ depends only on the error term of the Edgeworth expansions.

## 3. Background and some preparations

3.1. A generalized sequential Perron-Frobenius theorem. Let $B_{j}$ denote the space of bounded functions of $X_{j}$, and let $\|\cdot\|_{\infty}$ be the supremum norms. Let $B_{j}^{*}$ denote the dual space of $B_{j}$.

Let us take uniformly bounded real-valued functions $U_{n}=u_{n}\left(X_{n}\right)$ and for every complex number $z$ consider the operator $R_{z}^{(j)}: B_{j} \mapsto B_{j+1}$ defined by

$$
R_{z}^{(j)} g(x)=\mathbb{E}\left[e^{i U_{j+1}+z Y_{j+1}} g\left(X_{j+1}\right) \mid X_{j}=x\right] .
$$

For each $j$ and $n$ in $\mathbb{N}$ let

$$
R_{z}^{j, n}=R_{z}^{(j)} \cdots R_{z}^{(j+n-1)}
$$

The next result serves as one of our key technical tools.
10. Theorem. There exist a number $\delta_{0}>0$ which depends only on the uniform bound $K$ of $Y_{n}$ and on the ellipticity constant of $X_{n}$ so that the following holds. If sup $\left\|U_{n}\right\|_{L^{1}}+$ $|z|<\delta_{0}$ then for every $j \in \mathbb{Z}$ there exists a triplet $\lambda_{j}(z), h_{j}^{(z)}$ and $\nu_{j}^{(z)}$ consisting of a nonzero complex number $\lambda_{j}(z)$, a complex function $h_{j}^{(z)} \in B_{j}$ and a continuous linear functional $\nu_{j}^{(z)} \in B_{j}^{*}$ satisfying $\nu_{j}^{(z)}(\mathbf{1})=1, \nu_{j}^{(z)}\left(h_{j}^{(z)}\right)=1$,

$$
R_{z}^{(j)} h_{j+1}^{(z)}=\lambda_{j}(z) h_{j}^{(z)}, \quad \text { and }\left(R_{z}^{(j)}\right)^{*} \nu_{j}^{(z)}=\lambda_{j}(z) \nu_{j+1}^{(z)}
$$

where $\left(R_{z}^{(j)}\right)^{*}: B_{j}^{*} \rightarrow B_{j+1}^{*}$ is the dual operator of $R_{j}^{(z)}$ and $B_{j}^{*}$ is the dual space of $B_{j}$. When $z=t \in \mathbb{R}$ and $U_{n} \equiv 0$ then $h_{j}^{(t)}$ is strictly positive, $\nu_{j}^{(t)}$ is a probability measure and there are constants $a, b>0$, so that $\lambda_{j}^{(t)} \in[a, b]$ and $h_{j}^{(t)} \geq a$. When $t=0$ we have $\lambda_{j}(0)=1$ and $h_{j}^{(0)}=1$.

Moreover, this triplet is analytic and uniformly bounded. Namely, the maps

$$
\lambda_{j}(\cdot): \mathbb{U} \rightarrow \mathbb{C}, h_{j}^{(\cdot)}: \mathbb{U} \rightarrow B_{j} \text { and } \nu_{j}^{(\cdot)}: \mathbb{U} \rightarrow B_{j}^{*}
$$

where $\mathbb{U}=\left\{z \in \mathbb{C}:|z|<\delta_{0}\right\}$ are analytic, and there exists a constant $C>0$ so that

$$
\begin{equation*}
\max \left(\sup _{z \in \mathbb{U}}\left|\lambda_{j}(z)\right|, \sup _{z \in \mathbb{U}}\left\|h_{j}^{(z)}\right\|_{\infty}, \sup _{z \in \mathbb{U}}\left\|\nu_{j}^{(z)}\right\|_{\infty}\right) \leq C \tag{5}
\end{equation*}
$$

where $\|\nu\|_{\infty}$ is the operator norm of a linear functional $\nu: B_{j} \rightarrow \mathbb{C}$. In addition, $\lambda_{j}(z), h_{j}(z)$ and $\nu_{j}(z)$ depend continuously on $U_{j}$ in the sense that they converge uniformly to the triplets corresponding to the choice $U_{j}=0$ as $\sup \left\|U_{n}\right\|_{L^{1}} \rightarrow 0$.

Furthermore, there exist constants $C>0$ and $\delta \in(0,1)$ such that for any $n \geq 1$, $j \in \mathbb{Z}, z \in U$ and $q \in B_{j+n}$,

$$
\begin{equation*}
\left\|\frac{R_{z}^{j, n} q}{\lambda_{j, n}(z)}-\left(\nu_{j+n}^{(z)}(q)\right) h_{j}^{(z)}\right\|_{\infty} \leq C\|q\|_{\infty} \cdot \delta^{n} \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\frac{\left(R_{z}^{j, n}\right)^{*} \mu}{\lambda_{j, n}(z)}-\left(\mu h_{j}^{(z)}\right) \nu_{j+n}(z)\right\|_{\infty} \leq C\|\mu\|_{\infty} \cdot \delta^{n} \tag{7}
\end{equation*}
$$

where $\lambda_{j, n}(z)=\prod_{k=0}^{n-1} \lambda_{j+k}(z)$. Here $\|\cdot\|_{\infty}$ are the appropriate operator norms corresponding the the norms in the spaces $B_{j}$.

Proof. This theorem is proved similarly to [9, Ch. 6] (which makes a stronger assumption sup $\left\|U_{n}\right\|_{L^{\infty}}<\delta_{0}$ ). In the course of the proof we will use several definitions and properties of real and complex cones. In order not to overload the paper we will not present them here, and instead we refer to the Appendix of [9] for a summary of all the necessary background.

Let $Q_{j}$ be the Markov operator given by

$$
Q_{j} g(x)=\mathbb{E}\left[g\left(X_{j+1}\right) \mid X_{j}=x\right]=\int p_{j}(x, y) g(y) d \mu_{j+1}(y)
$$

Then $Q_{j}$ maps $B_{j+1}$ to $B_{j}$ and the corresponding operator norm equals 1. Let $\mathcal{K}_{j, L}$, $L>0$ be the real Birkhoff cone which consists of the positive function $g_{j}$ on the range of $X_{j}$ so that $g>0$ and $g\left(x_{1}\right) \leq L g\left(x_{2}\right)$ for all $x_{1}, x_{2}$. Then, since $C^{-1} \leq p_{j}(x, y) \leq C$ for all $x$ and $y$, we see that for every nonnegative bounded function $g$ on the state space of $X_{j+1}$ we have $Q_{j} g \in \mathcal{K}_{C^{2}, j}$. Let $L=2 C^{2}$. Then by [9, Lemma 6.5.1] the projective diameter of $\mathcal{K}_{C^{2}, j}$ inside $\mathcal{K}_{L, j}$ (with respect to the real Hilbert metric associated with the cone $\mathcal{K}_{j, L}$ ) does not exceed $d_{0}=d_{0}(C)=2 \ln \left(2 C^{2}\right)$. We conclude that $Q_{j} \mathcal{K}_{j+1, L} \subset \mathcal{K}_{j, L}$, and the projective diameter of the image is bounded above by $d_{0}$.

Next, let us explain in what sense $R_{z}^{(j)}$ is a small perturbation of $Q_{j}$ with respect to the dual of the cones $\mathcal{K}_{j, L}$. Since $U_{j}$ and $f_{j}$ are uniformly bounded and because of the uniform ellipticity we get that for every point $x$ and a function $g \in \mathcal{K}_{j+1, L}$ we have

$$
\begin{equation*}
\left|R_{z}^{(j)} g(x)-Q g(x)\right| \leq C\|g\|_{\infty} \mathbb{E}\left[\left|U_{j+1}+|z|\right| Y_{j+1} \mid\right] \leq C^{\prime} L\left(\left\|U_{j+1}\right\|_{L^{1}}+|z| K\right) Q g(x) \tag{8}
\end{equation*}
$$

for some constant $C^{\prime}$. Next, let us recall that the dual of the cone $\mathcal{K}_{j, L}$ is generated by the the linear functional $h \rightarrow h\left(x_{0}\right)$ and $h \rightarrow g\left(x_{1}\right)-C^{-2} h\left(x_{2}\right)$ where $x_{0}, x_{1}, x_{2}$ are arbitrary points in the state space of $X_{j}$. Now, using (8), by repeating the arguments in [9, Proposition 6.6.1] we see that if $s$ is one of the latter linear functionals then for every $g \in \mathcal{K}_{K, j+1}$ we have

$$
\left|s\left(R_{z}^{(j)} g\right)-s\left(Q^{(j)} g\right)\right| \leq A C^{4}\left(\left\|U_{j+1}\right\|_{L^{1}}+|z| K\right)
$$

where $A$ is an absolute constant. By [9, Theorem A.2.4] (taking into account Theorem 6.2.1 and Lemma 6.4 .1 of [9]), there is a constant $\delta_{0}$ so that if $\left\|U_{j+1}\right\|_{L^{1}}+|z| K<\delta_{0}$
then $R_{z}^{(j)} g$ maps the canonical complexification $\mathcal{K}_{L, j+1, \mathbb{C}}$ of $\mathcal{K}_{K, j}$ to the canonical complexification $\mathcal{K}_{L, j, \mathbb{C}}$ and the projective diameter of the image (with respect to the complex Hilbert metric associated with the complex cone $\mathcal{K}_{j, L, \mathrm{C}}$ ) does not exceed $2 d_{0}$. Once this is established, the rest of the proof of Theorem 10 proceeds as in [9, Ch. 6] by a repeated application of a conic perturbation theorem due to H.H. Rugh [21] and the explicit limiting expressions for $\lambda_{j}(z), h_{j}^{(z)}$ and $\nu_{j}^{(z)}$.
3.2. Behavior around 0. Let

$$
\Lambda_{N}(h)=\ln \mathbb{E}\left[e^{i h\left(S_{N}-\mathbb{E}\left[S_{N}\right]\right) / \sigma_{N}}\right]+h^{2} / 2
$$

By [5. Section 5] for every $m$ there exist constants $\delta_{m}, C_{m}>0$ so that for all $j \geq 3$

$$
\sup _{h \in\left[-\delta_{m} \sigma_{N}, \delta_{m} \sigma_{N}\right]}\left|\Lambda_{N}^{(j)}(h)\right| \leq C_{m} \sigma_{N}^{-(j-2)} .
$$

Set

$$
Q_{r, N}(t)=\sum_{\bar{k}} \frac{1}{k_{1}!\cdots k_{r}!}\left(\frac{\Lambda_{N}^{(3)}(0)}{3!}\right)^{k_{1}} \cdots\left(\frac{\Lambda_{N}^{(r+2)}(0)}{(r+2)!}\right)^{k_{r}}(i t)^{3 k_{1}+\ldots+(r+2) k_{r}}
$$

where the summation ranges over the collection of $r$ tuples of nonnegative integers $\left(k_{1}, \ldots, k_{r}\right)$ that are not all 0 so that $\sum_{j} j k_{j} \leq r$. Then

$$
\begin{equation*}
Q_{r, N}(t)=\sum_{j=1}^{r} \sigma_{N}^{-j} P_{j, N}(t) \tag{9}
\end{equation*}
$$

with

$$
\begin{equation*}
P_{j, N}(x)=\sum_{\bar{k} \in A_{j}} C_{\bar{k}} \prod_{j=1}^{s}\left(\sigma_{N}^{-2} \Lambda_{N}^{(j+2)}(0)\right)^{k_{j}}(i x)^{3 k_{1}+\ldots+(s+2) k_{s}} \tag{10}
\end{equation*}
$$

where $A_{j}$ is the set of all tuples of nonnegative integers $\bar{k}=\left(k_{1}, \ldots, k_{s}\right), k_{s} \neq 0$ for some $s=s(\bar{k}) \geq 1$ so that $\sum_{s} s k_{s}=j$ (note that when $j \leq r$ then $s \leq r$ since $k_{s} \geq 1$ ). Moreover

$$
C_{\bar{k}}=\prod_{j=1}^{s} \frac{1}{k_{j}!(j+2)^{k_{j}}}
$$

11. Lemma. ([5, Section 4.3]). Let $W_{N}=\left(S_{N}-\mathbb{E}\left[S_{N}\right]\right) / \sigma_{N}$. For every $r \geq 1$ there are constants $\delta_{r}, C_{r}>0$ so that for every $t \in\left[-\delta_{r} \sigma_{N}, \delta_{r} \sigma_{N}\right]$ we have

$$
\left|\mathbb{E}\left[e^{i t W_{N}}\right]-e^{-t^{2} / 2}\left(1+Q_{r, N}(t)\right)\right| \leq C e^{-c t^{2}} \sigma_{N}^{-(r+1)} \max \left(|t|,|t|^{(r+3)(r+2)}\right)
$$

where $c>0$ is a constant independent of $r$ (and, by decreasing $\delta_{r}$, it can be made arbitrarily close to $1 / 2$ ).

Lemma 11 will be crucial to determine the contribution of the resonant point 0 . To determine the contribution of other non-resonant points we will also need the following more general result, whose proof proceeds exactly as the proof of [5, Proposition 23].
12. Proposition. Fix some integer $r \geq 1$. Let $\mathcal{L}_{N}: \mathbb{R} \rightarrow \mathbb{C}$ be an $r+2$ times differentiable function so that

$$
\mathcal{L}_{N}(0)=\mathcal{L}_{N}^{\prime}(0)=\mathcal{L}_{N}^{\prime \prime}(0)=0
$$

and that for each $3 \leq j \leq r+2$ for every $t \in\left[-\delta_{r}, \delta_{r}\right]$ we have

$$
\left|\mathcal{L}_{N}^{(j)}(t)\right| \leq A_{r} \sigma_{N}^{2}
$$

where $\delta_{r}$ and $A_{r}$ are constants which do not depend on $t$ and $N$. Set $\overline{\mathcal{L}}(t)=\mathcal{L}\left(t / \sigma_{N}\right)$. Then there are constants $0<c<\frac{1}{2}$ and $B_{r}, \varepsilon_{r}>0$ depending only on $\delta_{r}$ and $A_{r}$ so that for every $t \in\left[-\varepsilon_{r} \sigma_{N}, \varepsilon_{r} \sigma_{N}\right]$ we have

$$
\left|e^{\overline{\mathcal{L}}_{N}(t)}-\mathcal{H}_{N, r}(t)\right| \leq B_{r}\left(\sigma_{N}\right)^{-(r+1)} \mid \max \left(|t|,|t|^{(r+2)(r+3)}\right)
$$

where

$$
\begin{equation*}
\mathcal{H}_{N, r}(t)=1+\sum_{\bar{k}} \frac{1}{k_{1}!\cdots k_{r}!}\left(\frac{\overline{\mathcal{L}}_{N}^{(3)}(0)}{3!}\right)^{k_{1}} \cdots\left(\frac{\overline{\mathcal{L}}_{N}^{(r+2)}(0)}{(r+2)!}\right)^{k_{r}}(i t)^{3 k_{1}+\ldots+(r+2) k_{r}} \tag{11}
\end{equation*}
$$

and the summation runs over the collection of $r$ tuples of nonnegative integers $\left(k_{1}, \ldots, k_{r}\right)$ that are not all 0 so that $\sum_{j} j k_{j} \leq r$.

### 3.3. Mixing properties and moment estimates.

13. Lemma. For each $j$, let $G_{j}=g_{j}\left(X_{j}\right)$ be a real valued function of $X_{j}$ so that $\|G\|_{\infty}:=\sup _{j}\left\|G_{j}\right\|_{L^{\infty}}<\infty$. For every $k, n \in \mathbb{N}$ such that $k \leq n$ let $G_{k, n}=\sum_{j=k}^{n} G_{j}$. Then
(i) There are constants $C_{1}, C_{2}$ which depend only on the ellipticity constant $C$ so that

$$
C_{1} \sum_{j=k}^{n} \operatorname{Var}\left(G_{j}\right) \leq \operatorname{Var}\left(G_{k, n}\right) \leq C_{2} \sum_{j=k}^{n} \operatorname{Var}\left(G_{j}\right)
$$

(ii) For every $p>2$ there is a constant $R_{p}$ depending only on $p, C$ and $\|G\|_{\infty}$ so that

$$
\left\|G_{k, n}-\mathbb{E}\left[G_{k, n}\right]\right\|_{L^{p}} \leq R_{p}\left(1+\sqrt{\operatorname{Var}\left(G_{k, n}\right)}\right)
$$

The first part follow: $5^{5}$ from [18, Proposition 13], while the harder lower bound in the first estimate was obtained in [23, Proposition 3.2 ]. We also refer to [4, Theorem 2.1], which in the case one step elliptic Markov chains and functionals of the form $G_{j}$ reduces to these variance estimates of part (i).

The second result was essentially obtained in [4, Lemma 2.16] and it also follows from [15, Theorem 6.17].

[^4]14. Lemma (Proposition 1.11 (2), 4). Let $G_{j}=g_{j}\left(X_{j}\right)$ be a real valued function of $X_{j}$ so that $\sup \left\|G_{j}\right\|_{L^{\infty}}<\infty$. Then there exist $\delta \in(0,1)$ and $A>0$ which depend only on the ellipticity constant $C$ and on $\|G\|_{\infty}$ so that for all $n, k \in \mathbb{N}$ we have
$$
\left|\operatorname{Cov}\left(g_{n}\left(X_{n}\right), g_{n+k}\left(X_{n+k}\right)\right)\right| \leq A \delta^{k}
$$
15. Remark. We note that by (4) and (12) we can get that
$$
\left|\operatorname{Cov}\left(g_{n}\left(X_{n}\right), g_{n+k}\left(X_{n+k}\right)\right)\right| \leq C_{1}\left\|g_{n}\left(X_{n}\right)\right\|_{L^{1}}\left\|g_{n+k}\left(X_{n+k}\right)\right\|_{L^{1}} \delta^{k} .
$$

However, this stronger estimate will not be used in this paper.

### 3.4. Mixing and transition densities.

16. Lemma. We have

$$
\mathbb{P}\left(X_{n+k} \in A \mid X_{n}=x\right)=\int p_{n}^{(k)}(x, y) d \mu_{n+k}(y)
$$

and the transition densities $p_{n}^{(k)}(x, y)$ take values in $\left[C^{-1}, C\right]$.
Proof. We have

$$
\mathbb{P}\left(X_{n+k} \in A \mid X_{n}\right)=\mathbb{E}\left[\mathbb{P}\left(X_{n+k} \in A \mid X_{n+k-1}, X_{n}\right) \mid X_{n}\right]=\mathbb{E}\left[\mathbb{P}\left(X_{n+k} \in A \mid X_{n+k-1}\right) \mid X_{n}\right] .
$$

To complete the proof, note that $\mathbb{P}\left(X_{n+k} \in A \mid X_{n+k-1}\right) \in\left[C^{-1}, C\right]$, a.s.
In the course of the proofs will also need the following result.
17. Lemma. $\operatorname{ess-sup}_{x, y}\left|p_{n}^{(k)}(x, y)-1\right| \leq C_{1} \delta^{k}$.

Proof. Let $x$ be fixed. Let $\Gamma(A)$ be the singed measure given by

$$
\Gamma(A)=\mathbb{P}\left(X_{n+k} \in A \mid X_{n}=x\right)-\mathbb{P}\left(X_{n+k} \in A\right)
$$

Then $y \rightarrow p_{n}^{(k)}(x, y)-1$ is the Radon-Nikodym derivative $d \Gamma / d \mu_{n+k}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Recall that by [1, Ch.4], for every two sub- $\sigma$-algebras $\mathcal{G}, \mathcal{H}$ of $\mathcal{F}$,

$$
\begin{gather*}
\psi(\mathcal{G}, \mathcal{H}):=\sup \left\{\left|\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A) \mathbb{P}(B)}-1\right|: A \in \mathcal{G}, B \in \mathcal{H}, \mathbb{P}(A) \mathbb{P}(B)>0\right\}  \tag{12}\\
=\sup \left\{\|\mathbb{E}[h \mid \mathcal{G}]-\mathbb{E}[h]\|_{L^{\infty}}: h \in L^{1}(\Omega, \mathcal{H}, \mathbb{P}),\|h\|_{L^{1}} \leq 1\right\}
\end{gather*}
$$

Let $\mathcal{G}=\sigma\left\{X_{n}\right\}$ and $\mathcal{H}=\sigma\left\{X_{n+k}\right\}$. Then by condition (4) we have $\psi(\mathcal{G}, \mathcal{H}) \leq C_{1} \delta^{k}$. Hence, by applying (12) with the function $h=\mathbb{I}\left(X_{n+k} \in A\right) / \mathbb{P}\left(X_{n+k} \in A\right)$ we see that

$$
\left\|\mathbb{P}\left(X_{n+k} \in A \mid X_{n}\right)-\mathbb{P}\left(X_{n+k} \in A\right)\right\|_{L^{\infty}} \leq C_{1} \mathbb{P}\left(X_{n+k} \in A\right) \delta^{k}
$$

Since the state space $\left(\mathcal{X}_{n+k}, \mathcal{F}_{n+k}\right)$ of $X_{n+k}$ is countably generated we conclude that

$$
\sup _{A \in \mathcal{F}_{n+k}}\left(\mathbb{P}\left(X_{n+k} \in A\right)\right)^{-1}\left|\mathbb{P}\left(X_{n+k} \in A \mid X_{n}=x\right)-\mathbb{P}\left(X_{n+k} \in A\right)\right| \leq C_{1} \delta^{k}, \quad \mu_{n}-\text { a.s. }
$$

Hence $d \Gamma / d \mu_{n+k}$ is bounded by $C_{1} \delta^{k}$.
3.5. Conditioning. Let $\mathcal{E}=\mathcal{E}_{N}$ be the $\sigma$-algebra generated by $\left\{X_{n}: n \in \mathcal{B}\right\}$ where $\mathcal{B}$ is a subset of $\{1, \ldots, N\}$.
18. Lemma. (i) There is a constant $C_{2} \geq 1$ so that for any $n \notin \mathcal{B}$ and all $\omega$ in the sample space we have

$$
\begin{equation*}
C_{2}^{-1} V\left(Y_{n}\right) \leq V\left(Y_{n} \mid \mathcal{E}\right) \leq C_{2} V\left(Y_{n}\right) \tag{13}
\end{equation*}
$$

(ii) For any $n$, let $q_{n}(m \mid \mathcal{E})$ be the second largest among $P\left(Y_{n} \equiv j \bmod m \mid \mathcal{E}\right)$,
$j=0,1, \ldots, m-1$. Then there exists a constant $A \geq 1$ so that for any $n \in \mathbb{N}$ and all $\omega$ in the sample space we have

$$
\begin{equation*}
A^{-1} q_{n}(m) \leq q_{n}(m \mid \mathcal{E}) \leq A q_{n}(m) \tag{14}
\end{equation*}
$$

Proof. We first note that by considering iid copies of $X_{1}$ (which are also independent of $\left\{X_{n}\right\}$ ) we can always extend $\left\{X_{n}\right\}$ to a two sided uniformly elliptic Markov chain with the same ellipticity constant $C$.

Let us prove the first item. It is clearly enough to prove it in the case when $\mathbb{E}\left(Y_{n}\right)=0$. Now, for every positive integers $n, k$ and $l$ we have

$$
\begin{equation*}
\mathbb{P}\left(X_{n} \in A \mid X_{n-l}=a, X_{n+k}=b\right)=\int_{A} p_{n}^{(n-l, n+k)}(y \mid a, b) d \mu_{n}(y) \tag{15}
\end{equation*}
$$

with

$$
p_{n}^{(n-l, n+k)}(y \mid a, b)=\frac{p_{n-l}^{(l)}(a, y) p_{n}^{(k)}(y, b)}{p_{n-l}^{(l+k)}(a, b)}
$$

and $p_{m}^{(s)}$ is the transition density of $X_{m+s}$ given $X_{m}$. Hence by Lemma 16, for every possible value of $Y_{n}=f_{n}\left(X_{n}\right)$ we have

$$
\begin{equation*}
C^{-3} \mathbb{P}\left(Y_{n}=x\right) \leq \mathbb{P}\left(f_{n}\left(X_{n}\right)=x \mid X_{n-l}, X_{n+k}\right) \leq C^{3} \mathbb{P}\left(Y_{n}=x\right) \tag{16}
\end{equation*}
$$

We note that when $n$ is smaller than the first index $n_{1}$ in $\mathcal{B}$ then we only condition on the latter, but in this case we still get (16) from (15) by further conditioning on $X_{0}$. We conclude that

$$
\begin{gathered}
V\left(Y_{n} \mid \mathcal{E}\right)=\sum_{x} \mathbb{P}\left(Y_{n}=x \mid \mathcal{E}\right)\left(x-\mathbb{E}\left[Y_{n} \mid \mathcal{E}\right]\right)^{2} \\
\geq C^{-3} \sum_{x} \mathbb{P}\left(Y_{n}=x\right)\left(x-\mathbb{E}\left[Y_{n} \mid \mathcal{E}\right]\right)^{2} \geq C^{-3} V\left(Y_{n}\right)
\end{gathered}
$$

since $\mathbb{E}\left(Y_{n}-a\right)^{2} \geq V\left(Y_{n}\right)$ for any $a \in \mathbb{R}$. On the other hand, using again (16) we see that

$$
V\left(Y_{n} \mid \mathcal{E}\right) \leq \mathbb{E}\left[Y_{n}^{2} \mid \mathcal{E}\right]=\sum_{x} \mathbb{P}\left(Y_{n}=x \mid \mathcal{E}\right) x^{2} \leq C^{3} \sum_{x} \mathbb{P}\left(Y_{n}=x\right) x^{2}=C^{3} V\left(Y_{n}\right)
$$

and (13) follows.

To prove the second item we use the fact that if $Z$ is an integer valued random variable with $\|Z\|_{L^{\infty}} \leq K$ then

$$
\begin{equation*}
\frac{q(Z)}{4} \leq V(Z) \leq 8 K^{3} q(Z) \tag{17}
\end{equation*}
$$

where $q(Z)$ is the probability that $Z$ takes its second most likely value. Indeed let $Z^{\prime}$ and $Z^{\prime \prime}$ be independent copies of $Z$, and let $\bar{i}$ and $\hat{i}$ be the most likely and the second most likely values of $Z$. Then

$$
\begin{aligned}
V(Z) & =\frac{1}{2} \mathbb{E}\left[\left(Z^{\prime}-Z^{\prime \prime}\right)^{2}\right]= \\
\frac{1}{2} \sum_{|i|,|j| \leq K} \mathbb{P}(Z=i) \mathbb{P}(Z=j)(i-j)^{2} & \geq \frac{1}{2} \mathbb{P}(Z=\hat{i}) \mathbb{P}(Z \neq \hat{i})=\frac{q(Z)(1-q(Z))}{2} \geq \frac{q(Z)}{4}
\end{aligned}
$$

since $q(Z) \leq \frac{1}{2}$. On the other hand, using the above formula for $V(Z)$ we get

$$
V(Z) \leq \frac{1}{2} \times(2 K)^{2} \times \mathbb{P}\left(Z^{\prime} \neq \bar{i} \text { or } Z^{\prime \prime} \neq \bar{i}\right) \leq(2 K)^{2} \times \mathbb{P}(Z \neq \bar{i}) \leq(2 K)^{2} \times 2 K q(Z)
$$

Applying (17) with $Z=Y_{n} \bmod m$ and using item (i) proves item (ii).
19. Lemma (Conditional chains). After conditioning on $\mathcal{E}$, for almost every realization of $\mathcal{E}$ the sequence $\left\{X_{n}: n \notin \mathcal{E}\right\}$ forms a uniformly elliptic Markov chain. More precisely, let us write $\mathcal{B}=\left\{n_{1}<n_{2}<\ldots .<n_{d}\right\}$, where both $n_{i}$ and $d$ might also depend on $N$. For the sake of convenience, let us also set $n_{0}=0$ and $n_{d+1}=\infty$. Then, for almost every realization of $\mathcal{E}$ we have the following:
(i) The random variables $\left\{X_{n}: n_{s}<n<n_{s+1}\right\}$ are conditionally independent, namely they are independents with respect to $\mathbb{P}_{\mathcal{E}}$, where $\mathbb{P}_{\mathcal{E}}(\cdot)$ denotes $\mathbb{P}(\cdot \mid \mathcal{E})$.
(ii) If $n$ and $n+1$ belong to the same block $\left(n_{s}, n_{s+1}\right)$ then

$$
\mathbb{P}_{\mathcal{E}}\left(X_{n+1} \in A \mid X_{n}=x\right)=\int p_{n, \mathcal{E}}(x, y) d \mu_{n+1, \mathcal{E}}(y)
$$

with $C^{-6} \leq p_{\mathcal{E}, n}(x, y) \leq C^{6}$, where $d \mu_{n+1, \mathcal{E}}$ denotes the law of $X_{n+1}$ given $\mathcal{E}$.
Proof. The first part follows because $\left\{X_{n}\right\}$ is a Markov chain, and it does not require ellipticity.

To prove the second part, notice that by (15) together with Lemma 16 the conditional law of $X_{n}$ is equivalent to the law of $X_{n}$, and the Radon-Nikodym derivative is bounded above by $C^{3}$ and below by $C^{-3}$. Note that if $n<n_{1}$ then we can still use (15) by taking $n-l=0$ and setting $X_{0}$ to be an independent copy of $X_{1}$ which is independent of $\left\{X_{n}\right\}$.

Next, since $\left\{X_{n}: n_{s}<n<n_{s+1}\right\}$ are conditionally independent it is enough to show that each $\left\{X_{n}: n_{s}<n<n_{s+1}\right\}$ forms uniformly elliptic Markov chain after conditioning by $\mathcal{E}$. To show that, let $n$ satisfy that $n_{s}<n<n+1<n_{s+1}$ for some $s$. If $0<n_{s}$ and $n_{s+1}<\infty$ then

$$
\mathbb{P}_{\mathcal{E}}\left(X_{n+1} \in A \mid X_{n}\right)=\mathbb{P}\left(X_{n+1} \in A \mid X_{n}, X_{n_{s+1}}\right)
$$

and so it follows from (15) that

$$
\mathbb{P}_{\mathcal{E}}\left(X_{n+1} \in A \mid X_{n}\right)=\int_{A} p_{n}^{\left(n_{s}, n_{s+1}\right)}\left(y \mid X_{n}, X_{n_{s+1}}\right) d \mu_{n+1}(y)
$$

where the densities are bounded above by $C^{3}$ and below by $C^{-3}$. Now the result follows since $\mu_{n+1}$ and $\mu_{n+1, \mathcal{E}}$ are equivalent with Radon-Nikodym derivatives bounded between
$C^{-3}$ and and $C^{3}$. The proof when $n_{s}=1$ is similar, and the case $n_{s+1}=\infty$ reduces to the unconditioned chain.

## 4. Classical estimates.

Recall Definition 3, Let $\mathcal{R}=\left\{t_{j}\right\}$ be the set of all resonant points. Divide $\mathbb{T}$ into intervals $I_{j}$ of small size $\delta$ such that each interval contains at most one resonant point and this point is strictly inside $I_{j}$. We call an interval resonant if it contains a resonant point inside. Then

$$
\begin{equation*}
2 \pi \mathbb{P}\left(S_{N}=k\right)=\sum_{j} \int_{I_{j}} e^{-i t k} \mathbb{E}\left(e^{i t S_{N}}\right) d t \tag{18}
\end{equation*}
$$

In the case of sums of independent identically distributed integer valued random variables the Edgeworth expansion comes from the expansion of the characteristic function near zero while the other intervals give negligible contributions. In this section we obtain a similar estimates for the integer valued additive functionals of uniformly elliptic Markov chains. However, in contrast to the iid case, in order to be able to disregard the contribution of an interval $I_{j}$ we need to assume that this interval is either nonresonant, or it is resonant but the value of $M_{N}(m)$ is large (where $m$ is the denominator of the corresponding resonant point).
4.1. The contribution a neighborhood of 0 . In this section we will estimate the integral $\int_{I_{j}} e^{-i t k} \mathbb{E}\left(e^{i t S_{N}}\right) d t$ when $t_{j}=0$. Namely, we will expand the integral

$$
\int_{-\delta}^{\delta} \mathbb{E}\left(e^{i t S_{N}}\right) d t=\sigma_{N}^{-1} \int_{-\delta \sigma_{N}}^{\delta \sigma_{N}} e^{i t \mathbb{E}\left[S_{N}\right] / \sigma_{N}} \mathbb{E}\left(e^{i t\left(S_{N}-\mathbb{E}\left[S_{N}\right]\right) / \sigma_{N}}\right) d t
$$

for a sufficiently small $\delta=\delta_{r}$. First, by Lemma 11, if $\delta$ is small enough then

$$
\int_{-\delta \sigma_{N}}^{\delta \sigma_{N}} e^{i t \mathbb{E}\left[S_{N}\right]} \mathbb{E}\left(e^{i t\left(S_{N}-\mathbb{E}\left[S_{N}\right]\right) / \sigma_{N}}\right) d t=\int_{-\infty}^{\infty} e^{i t \mathbb{E}\left[S_{N}\right] / \sigma_{N}} e^{-t^{2} / 2}\left(1+Q_{r, N}(t)\right) d t+o\left(\sigma_{N}^{-r-1}\right)
$$

Second, recall that for every real $\alpha$ we have

$$
\int_{-\infty}^{\infty} e^{-i \alpha h} e^{-h^{2} / 2} h^{k} d h=(-1)^{k} H_{k}(\alpha) \varphi(\alpha)
$$

where $H_{k}$ is the $k$-th Hermite polynomial. Applying this formula with $\alpha=-\mathbb{E}\left[S_{N}\right] / \sigma_{N}$ and taking into account (9) and (10) we get the following result.
20. Proposition. There are polynomials $P_{0, b, N}$ with uniformly bounded coefficients so that for $t_{j}=0$, for every $r \geq 1$ we have the following: if the length of the resonant interval $I_{j}$ around 0 is small enough then

$$
\int_{I_{j}} e^{-i t k} \mathbb{E}\left(e^{i t S_{N}}\right) d t=\sum_{b=1}^{r} \frac{P_{a, b, N}\left(\left(k-a_{N}\right) / \sigma_{N}\right)}{\sigma_{N}^{b}} \mathfrak{g}\left(\left(k-a_{N}\right) / \sigma_{N}\right)+o\left(\sigma_{N}^{-r}\right) .
$$

The above Proposition shows that the contribution of the neighborhood of 0 corresponds to the polynomials $P_{0, b, N}$ and the choice $a=0$ (in the notations of Theorem (7).
4.2. The negligible contribution: non-resonant intervals and resonant points with $M_{N}(m) \geq R \ln \sigma_{N}$ and the proof of Theorem 2. For the sake of simplicity, assume that $\sum_{n=1}^{[N / 2]} q_{2 n}(m) \geq \frac{R \ln \sigma_{N}}{2}$ (otherwise we will work with odd indexes instead). Let us condition on $X_{1}, X_{3}, X_{5}, \ldots$ Then $X_{2}, X_{4}, X_{6}, \ldots$ are independent after such a conditioning. Moreover, by Lemma 18 the ratio between $q_{2 n}(m)$ and their conditioned versions is uniformly bounded and bounded away from 0 . Therefore, we can assume that, after the conditioning, we still have $\sum_{n=1}^{[n / 2]} q_{2 n}(m) \geq R_{0} \ln \sigma_{N}$ with $R_{0}$ large enough. This reduces the problem to the case of independent variables which was considered in [6, Lemma 3.4] and it shows that the contribution of the integrals over such resonant intervals is $o\left(\sigma_{N}^{-r}\right)$. Note also that similar arguments show that the contribution coming from non-resonant intervals is $O\left(e^{-c V_{N}}\right)$ for some $c>0$. Indeed, we assume that the sums of the variances of $X_{2 n}, n \leq N / 2$ is lager than the sum of corresponding sum along the odd indexes, and then condition on the odd indexes. Now we can apply [6, Lemma 3.3].

We note that it is immediate from the formulation of Theorem 2 that it is enough to prove that the usual Edgeworth expansions hold (not in a conditionally stable way) since after conditioning by a finite number of elements $q_{n}(m)$ can only change by a multiplicative constant, see Lemma 18(ii). Hence, if $M_{N}(m) \geq R \ln \sigma_{N}$ for large enough $R$ and all denominators $m$ of nonzero resonant points, then the conditionally stable Edgeworth expansions of order $r$ hold true, and the proof of Theorem 2 is complete.
5. Contribution of nonzero resonant intervals with $M_{N}(m) \leq R \ln \sigma_{N}$

In this Section we prove Theorem 7 .
Let $t_{j}=2 \pi l / m$ be a nonzero resonant point so that $M_{N}(m)=\sum_{n=1}^{N} q_{n}(m) \leq R \ln V_{N}$.
Let us fix some $\bar{\varepsilon}>0$, and let $N_{0}=N_{0}\left(t_{j}, N\right)$ be the number of indexes $n$ between 1 to $N$ so that $q_{n}(m) \geq \bar{\epsilon}$. Then $N_{0} \leq \frac{R \ln V_{N}}{\bar{\epsilon}}$. Let us denote the latter indexes by $n_{1}<n_{2}<\cdots<n_{N_{0}}$ and set $\mathcal{B}=\mathcal{B}_{N}=\left\{n_{1}, \ldots, n_{N_{0}}\right\}$. Let $\mathcal{E}=\mathcal{E}_{N}$ be the $\sigma$-algebra generated by $\left\{X_{n}: n \in \mathcal{B}\right\}$.

### 5.1. More on conditioning.

21. Lemma. There is a constant $C>0$ so that

$$
\begin{equation*}
\left|\mathbb{E}\left(S_{N} \mid \mathcal{E}\right)-\mathbb{E}\left(S_{N}\right)\right| \leq C N_{0} \leq \frac{C R \ln V_{N}}{\bar{\varepsilon}} \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\operatorname{Var}\left(S_{N} \mid \mathcal{E}\right)-V_{N}\right| \leq C \ln ^{2} V_{N} \tag{20}
\end{equation*}
$$

Proof. In order to prove (19), let us take $n$ so that $n_{i}<n<n_{i+1}$ for some $i$. Then

$$
\begin{equation*}
\mathbb{E}\left(Y_{n} \mid \mathcal{E}\right)-\mathbb{E}\left(Y_{n}\right)=\int\left(p_{n}^{\left(n_{i}, n_{i+1}\right)}\left(x \mid X_{n_{i}}, X_{n_{i+1}}\right)-1\right) f_{n}(x) d \mu_{n}(x) \tag{21}
\end{equation*}
$$

where $p_{n}^{\left(n_{i}, n_{i+1}\right)}\left(x \mid X_{n_{i}}, X_{n_{i+1}}\right)$ is defined by

$$
p_{n}^{\left(n_{i}, n_{i+1}\right)}\left(x \mid X_{n_{i}}, X_{n_{i+1}}\right)=\frac{p_{n_{i}}^{\left(n-n_{i}\right)}\left(X_{n_{i}}, x\right) p_{n}^{\left(n_{i+1}-n_{i}\right)}\left(x, X_{n_{i+1}}\right)}{p_{n_{i-1}}^{\left(n_{i+1}-n_{i}\right)}\left(X_{n_{i}}, X_{n_{i+1}}\right)} .
$$

Now, by Lemmas 16 and 17 we have

$$
\begin{gathered}
\left|p_{n_{i}}^{\left(n-n_{i}\right)}\left(X_{n_{i}}, x\right)-1\right| \leq C_{1} \delta^{n-n_{i}}, \quad\left|p_{n_{i}}^{\left(n_{i+1}-n\right)}\left(x, X_{n_{i+1}}\right)-1\right| \leq C_{1} \delta^{n_{i+1}-n}, \\
C^{-1} \leq p_{n_{i-1}}^{\left(n_{i+1}-n_{i}\right)}\left(X_{n_{i}}, X_{n_{i+1}}\right) \leq C,\left|p_{n_{i-1}}^{\left(n_{i+1}-n_{i}\right)}\left(X_{n_{i}}, X_{n_{i+1}}\right)-1\right| \leq C_{1} \delta^{n_{i+1}-n_{i}} .
\end{gathered}
$$

We thus conclude that

$$
\left|p_{n}^{\left(n_{i}, n_{i+1}\right)}\left(x \mid X_{n_{i}}, X_{n_{i+1}}\right)-1\right| \leq C_{2} \delta^{\min \left(n-n_{i}, n_{i+1}-n\right)}
$$

for some constant $C_{2}$. Therefore,

$$
\begin{equation*}
\left|\mathbb{E}\left(Y_{n} \mid \mathcal{E}\right)-\mathbb{E}\left(Y_{n}\right)\right| \leq C_{2} \mathbb{E}\left[\left|Y_{n}\right|\right] \delta^{\min \left(n-n_{i}, n_{i+1}-n\right)} \tag{22}
\end{equation*}
$$

Similarly when $n<n_{1}$ or $n>n_{N_{0}}$ we have

$$
\left|\mathbb{E}\left(Y_{n} \mid \mathcal{E}\right)-\mathbb{E}\left(Y_{n}\right)\right| \leq C \mathbb{E}\left[\left|X_{n}\right|\right] \delta^{d\left(n, \mathcal{B}_{N}\right)}
$$

where $\mathcal{B}_{N}=\left\{n_{1}, n_{2}, \ldots, n_{N_{0}}\right\}$ and $d(n, B)=\min \{|n-b|: b \in B\}$ for any $n$ and a finite set $B$, and we set $n_{0}=0$ and $n_{N_{0}+1}=N$. It follows that there exists a constant $C>0$ so that for any $i$,

$$
\begin{equation*}
\sum_{n_{i}<n \leq n_{i+1}}\left|\mathbb{E}\left(Y_{n} \mid \mathcal{E}\right)-\mathbb{E}\left(Y_{n}\right)\right| \leq C \tag{23}
\end{equation*}
$$

Therefore,

$$
\left|\mathbb{E}\left(S_{N} \mid \mathcal{E}\right)-\mathbb{E}\left(S_{N}\right)\right| \leq \sum_{i=0}^{N_{0}} \sum_{n_{i}<n \leq n_{i+1}}\left|\mathbb{E}\left(Y_{n} \mid \mathcal{E}\right)-\mathbb{E}\left(Y_{n}\right)\right| \leq C\left(N_{0}+1\right)
$$

Now we prove (20). First, let $A_{1}<A_{2}$ be two sufficiently large numbers. Using (13) and that $\left\{X_{n}\right\}$ also satisfies (3) given $\mathcal{E}$ with a deterministic constant (see Lemma (19), we can divide $\{1, \ldots, N\}$ into blocks $B_{1}, B_{2}, \ldots, B_{a_{N}}, a_{N} \asymp V_{N}$ so that for any $k$ both $V\left(S_{B_{k}}\right)$ and $V\left(S_{B_{k}} \mid \mathcal{E}\right)$ lie between $A_{1}$ and $A_{2}$, where $S_{B}=\sum_{n \in B} Y_{n}$ for any finite set $B \subset \mathbb{N}$. Let us put $Z_{k}=S_{B_{k}}$.

Next, let us define $\tilde{S}_{N}=\sum_{k \in \mathcal{N}_{N}} Z_{k}$ where $\mathcal{N}_{N}$ is the set of indexes $1 \leq k \leq a_{N}$ so that the distance between $B_{k}$ and $\mathcal{B}_{N}$ is at least $A \ln a_{N}$, where $A$ is a constant so large that $a_{N}^{2} \delta^{A \ln a_{N}} \leq 1$. We claim first that

$$
\begin{equation*}
V_{N}=V\left(\tilde{S}_{N}\right)+O\left(\ln ^{2} a_{N}\right) \text { and }\left\|V\left(S_{N} \mid \mathcal{E}\right)-V\left(\tilde{S}_{N} \mid \mathcal{E}\right)\right\|_{L^{\infty}}=O\left(\ln ^{2} a_{N}\right) \tag{24}
\end{equation*}
$$

Indeed, by (4) we have $\left|\operatorname{Cov}\left(Z_{n+k}, Z_{n}\right)\right| \leq C \delta^{k}$ and also $\left|\operatorname{Cov}\left(Z_{n+k}, Z_{n} \mid \mathcal{E}\right)\right| \leq C \delta^{k}$, where $C>0$ is some constant. As a consequence, for any $k$ we have

$$
\begin{equation*}
\left|V\left(S_{N}\right)-V\left(S_{N}-Z_{k}\right)\right| \leq C \text { and }\left|V\left(S_{N} \mid \mathcal{E}\right)-V\left(S_{N}-Z_{k} \mid \mathcal{E}\right)\right| \leq C \tag{25}
\end{equation*}
$$

Now (24) follows by a repeated application of (25) taking into account that $\tilde{S}_{N}$ was obtained by removing at most $O\left(\ln ^{2} a_{N}\right)$ individual summands.

Next, let us show that there exists $C_{1}>0$ so that

$$
\begin{equation*}
\left|\operatorname{Var}\left(\tilde{S}_{N} \mid \mathcal{E}\right)-V\left(\tilde{S}_{N}\right)\right| \leq C_{1} \tag{26}
\end{equation*}
$$

Indeed, let $1 \leq k_{1} \leq k_{2} \leq a_{N}$ be so that $d\left(B_{k_{i}}, \mathcal{B}_{N}\right) \geq A \ln a_{N}$ for $i=1,2$, where $d(A, B)$ denotes the distance between two finite sets. Then there are $s_{1}, s_{2}$ so that $B_{k_{i}}$ is contained in $\left(n_{s_{i}}, n_{s_{i}+1}\right)$, for $i=1,2$. Let us first assume that $s_{1} \neq s_{2}$. Then $Z_{k_{1}}$ and $Z_{k_{2}}$ are independent given $\mathcal{E}$ and

$$
\left|\operatorname{Cov}\left(Z_{k_{1}}, Z_{k_{2}}\right)\right| \leq C \delta^{A a_{N}}
$$

Therefore, the contribution of such pairs to the left hand side of (26) is $O(1)$.
Next, let us assume that $B_{k_{1}}, B_{k_{2}} \subset\left(n_{s}, n_{s+1}\right)$ for some $s$. We will first show that, with $W_{k_{i}}=\left\{f_{n}\left(X_{n}\right): n \in B_{k_{i}}\right\} \quad i=1,2$, for any possible values $u$ and $v$ of $W_{k_{1}}$ and $W_{k_{2}}$, respectively, for all possible values $x$ and $y$ of $X_{n_{s}}$ and $X_{n_{s+1}}$ respectively, we have

$$
\begin{array}{r}
P\left(W_{k_{1}}=v, W_{k_{2}}=u \mid X_{n_{s}}=x, X_{n_{s+1}}=y\right)-P\left(W_{k_{1}}=v, W_{k_{2}}=u\right)  \tag{27}\\
=P\left(W_{k_{1}}=v, W_{k_{2}}=u\right) O\left(\delta^{A \ln a_{N}}\right) .
\end{array}
$$

Assuming that (27) holds we get ${ }^{67}$ that

$$
\begin{equation*}
\left|\operatorname{Cov}\left(Y_{k_{1}}, Y_{k_{2}}\right)-\operatorname{Cov}\left(Y_{k_{1}}, Y_{k_{2}} \mid \mathcal{E}\right)\right| \leq C \sqrt{V\left(Y_{k_{1}}\right) V\left(Y_{k_{2}}\right)} \delta^{A \ln a_{N}}=O\left(a_{N}^{-2}\right) \tag{28}
\end{equation*}
$$

which yields that the contribution of such pairs $k_{1}$ and $k_{2}$ to the left hand side of (26) is also $O(1)$.

Now let us prove (27). Let $X\left(B_{k_{i}}\right)=\left\{X_{n}: n \in B_{k_{i}}\right\}$ and let

$$
\Gamma_{1}=\left\{X\left(B_{k_{1}}\right): f_{j}\left(X_{j}\right)=v_{j}\right\}, \quad \Gamma_{2}=\left\{X\left(B_{k_{2}}\right): f_{j}\left(X_{j}\right)=u_{j}\right\}
$$

For any $n, k$ and $x=\left(x_{0}, \ldots, x_{k}\right)$ let us write

$$
p^{n, n+k}(x)=\prod_{j=1}^{k} p_{n+j-1}\left(x_{j-1}, x_{j}\right)
$$

where we recall that $y \rightarrow p_{n}(x, y)$ is the transition density of $X_{n+1}$ given $X_{n}=x$. Denote

$$
\bar{x}=\left(x_{j}\right)_{j=n_{s}+1}^{n_{s+1}}, \quad \bar{x}^{(i)}=\left(x_{j}\right)_{j \in B_{k_{i}}}, \quad i=1,2
$$

and let

$$
D=\left\{\bar{x}: \bar{x}^{(i)} \in \Gamma_{i}, i=1,2\right\} .
$$

Let us assume that $k_{1} \leq k_{2}$ and write $B_{k_{i}}=\left[\alpha_{i}, \beta_{i}\right] \cap \mathbb{N}$. Then the difference on the left hand side of (27) can be written as

$$
\int_{D} p^{\alpha_{1}, \beta_{1}}\left(\bar{x}^{(1)}\right) p_{\beta_{1}}^{\left(\alpha_{2}-\beta_{1}\right)}\left(x_{\alpha_{2}}\right) p^{\alpha_{2}, \beta_{2}}\left(\bar{x}^{(2)}\right)\left(\frac{p_{n_{s}}^{\left(\alpha_{1}-n_{s}\right)}\left(x, x_{\alpha_{1}}\right) p_{\beta_{2}}^{\left(n_{s+1}-\beta_{2}\right)}\left(x_{\beta_{2}}, y\right)}{p_{n_{s}}^{\left(n_{s+1}-n_{s}\right)}(x, y)}-1\right) \prod_{t=\alpha_{1}}^{\beta_{2}} d \mu_{t}\left(x_{t}\right)
$$

where we recall that $y \rightarrow p_{j}^{(k)}(x, y)$ is the transition density of $X_{j+k}$ given $X_{j}=x$. The term in the parenthesis is bounded exactly as the corresponding term from (21). Since $d\left(B_{k_{i}}, \mathcal{B}\right) \geq A \ln a_{N}$ and $B_{k_{i}} \subset\left(n_{s}, n_{s+1}\right)$, it follows that $n_{s+1}-n_{s} \geq A \ln a_{N}$. Therefore, by Lemma 17 the term in the parenthesis is $O\left(\delta^{A \ln a_{N}}\right)$, and (27) follows.

[^5]5.2. The conditional cumulant generating function around nonzero resonant points and its additive behavior. By Lemma [4, Lemma 3.4] we can find a bounded sequences of integers $\left(c_{n}\right)$ so that $\mathbb{E}\left(S_{N}\right)-\sum_{n=1}^{N} c_{n}$ is bounded in $N$. Therefore, by replacing $Y_{n}$ with $Y_{n}-c_{n}$ we can assume that $\sup _{N}\left|\mathbb{E}\left(S_{N}\right)\right|<\infty$.

Now, let us fix some nonzero resonant point $t_{j}=2 \pi l / m$. Let $j\left(Y_{n}, m \mid \mathcal{E}\right)$ be the most likely residue of $Y_{n} \bmod m$ given $\mathcal{E}$, and set

$$
\begin{equation*}
Z_{n}=Y_{n} \bmod \mathrm{~m}-j\left(Y_{n}, m \mid \mathcal{E}\right), \quad \bar{Z}_{n}=Z_{n}-\mathbb{E}\left[Z_{n} \mid \mathcal{E}\right] \tag{29}
\end{equation*}
$$

and for any $n<k$,

$$
S_{n, k} \bar{Z}=\sum_{s=n}^{k-1} \bar{Z}_{s}
$$

Note that for every $p \in[1,2]$ we have

$$
\begin{equation*}
E\left[\left|S_{n, k} \bar{Z}\right|^{p} \mid \mathcal{E}\right] \leq b \sum_{s=n}^{k-1} q_{s}(m \mid \mathcal{E}) \tag{30}
\end{equation*}
$$

for some positive constant $b$. Indeed, since $|x|^{p} \leq|x|+|x|^{2}$ for all real $x$ and $p \in(1,2)$ it is enough to prove (30) for $p=1$ and 2 . For $p=1$ it follows from the triangle inequality that

$$
\mathbb{E}\left[\left|\bar{Z}_{s}\right| \mid \mathcal{E}\right] \leq 2 \mathbb{E}\left[\left|Z_{s}\right| \mid \mathcal{E}\right] \leq 2 K^{2} \mathbb{P}\left(Z_{n} \neq 0 \mid \mathcal{E}\right) \leq 2 K^{2} q_{s}(m \mid \mathcal{E})
$$

For $p=2$ the result follows since by Lemmata 18 and 19, there is a constant $c>0$ so that

$$
\operatorname{Var}\left(S_{n, k} Z \mid \mathcal{E}\right) \leq c \sum_{s=n}^{k-1} \operatorname{Var}\left(Z_{s} \mid \mathcal{E}\right) \leq c \sum_{s=n}^{k-1} \mathbb{E}\left(Z_{s}^{2} \mid \mathcal{E}\right) \leq c(4 K)^{3} \sum_{s=n}^{k-1} q_{s}(m \mid \mathcal{E})
$$

where in the last inequality we have used that

$$
\mathbb{E}\left(Z_{s}^{2} \mid \mathcal{E}\right) \leq \sum_{0<|j| \leq 4 K} j^{2} \mathbb{P}\left(Z_{s}=j \mid \mathcal{E}\right) \leq(4 K)^{3} q_{s}(m \mid \mathcal{E})
$$

Recall that by Lemma 19 conditioned on $\mathcal{E}$, for each $s$ the Markov chains $\left\{X_{n}: n_{s}<\right.$ $\left.n<n_{s+1}\right\}$ are uniformly elliptic with constants not depending on $s$ and $\mathcal{E}$. Moreover, the processes $\left\{X_{n}: n_{s}<n<n_{s+1}\right\}$ are independent, where we set $n_{0}=0$ and $n_{N_{0}+1}=\infty$. Consider the family of functions $g_{n}^{(z)}=z Y_{n}+i t_{j} Z_{n}=$ for $z$ small enough (recall that both $Y_{n}$ and $Z_{n}$ are functions of $X_{n}$ ). Then, since the $L^{2}$-norm of $Z_{n}$ is $O(\sqrt{\bar{\varepsilon}})$ (both unconditionally and conditionally on $\mathcal{E}$ ), we can apply Theorem 10 after conditioning on $\mathcal{E}$ provided that $\bar{\varepsilon}$ is small enough. This means that there is a constant $r_{0}>0$ so that for any complex number $z$ with $|z| \leq r_{0}$ there are complex numbers $\lambda_{n}(z)=\lambda_{n, \mathcal{E}}(z)$, uniformly bounded functions $h_{n}^{(z)}=h_{n, \mathcal{E}}^{(z)}$ and uniformly bounded measures $\nu_{n}^{(z)}=\nu_{n, \mathcal{E}}^{(z)}$ (which depend on the realizations of $\mathcal{E}$ ) so that $\nu_{n}^{(z)}(\mathbf{1})=1$,
$\left|\lambda_{n}(z)-1\right|<\frac{1}{2},\left\|h_{n}^{(z)}-1\right\|_{\infty} \leq \frac{1}{2}$ and for any bounded function $g, n \geq 1$ and $p>0$ we have

$$
\begin{equation*}
\left\|\mathbb{E}\left[e^{i t_{j} S_{n, p} Z+z S_{n, p}} g\left(X_{n+p}\right) \mid \mathcal{E}\right] / \lambda_{n, p}(z)-\nu_{n+p}^{(z)}(g) \int h_{n}^{(z)} d \mu_{n, \mathcal{E}}\right\|_{\infty} \leq A_{0}\|g\|_{\infty} \gamma^{p} \tag{31}
\end{equation*}
$$

where $\gamma \in(0,1)$ and $A_{0}>0$ are some constants which do not depend on $\mathcal{E}$ and $N$ and $\mu_{n, \mathcal{E}}$ is the law of $X_{n}$ given $\mathcal{E}$. Set $\Pi_{n}(z)=\Pi_{n, \mathcal{E}}(z)=\ln \lambda_{n, \mathcal{E}}(z)$ and

$$
\Pi_{n, p}=\Pi_{n, p, \mathcal{E}}=\sum_{s=n}^{n+p-1} \Pi_{s}(z)
$$

Taking the logarithms in (31) we get that

$$
\begin{equation*}
\left|\Pi_{n, p, \mathcal{E}}(z)-\Gamma_{n, p}(z)\right| \leq C \tag{32}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma_{n, p}(z)=\Gamma_{t_{j}, n, p, \mathcal{E}}(z)=\ln \mathbb{E}\left[e^{i t_{j} S_{n, p} Z+z S_{n, p}} \mid \mathcal{E}\right] . \tag{33}
\end{equation*}
$$

Let use also set $\Gamma_{N}(z)=\Gamma_{1, N}(z)=\Gamma_{t_{j}, N, \mathcal{E}}(z)$. Using the analyticity in $z$ we get that for any $u$ there is a constant $C_{u}>0$ so that for any complex number $z$ with $|z| \leq r_{0} / 2$ the derivatives of $\Gamma_{n, p}$ satisfy

$$
\begin{equation*}
\left|\Pi_{n, p}^{(u)}(z)-\Gamma_{n, p}^{(u)}(z)\right| \leq C_{u} . \tag{34}
\end{equation*}
$$

### 5.3. Estimates on the derivatives of the conditional cumulant generating

 function. We need the following result.22. Lemma. For every $p \in(1,2)$ we have the following:
(i) $\left.\left.\Pi_{1, N, \mathcal{E}}^{\prime}(0)=\mathbb{E}\left(S_{N}\right)+O\left(\left(\ln \sigma_{N}\right)^{\frac{2}{p+2}}\left(\sigma_{N}\right)^{\frac{2 p}{p+2}}\right)\right)=O\left(\left(\ln \sigma_{N}\right)^{\frac{2}{p+2}}\left(\sigma_{N}\right)^{\frac{2 p}{p+2}}\right)\right)$ and so

$$
\left.\left.\Gamma_{t_{j}, N, \mathcal{E}}^{\prime}(0)=\mathbb{E}\left(S_{N}\right)+O\left(\left(\ln \sigma_{N}\right)^{\frac{2}{p+2}}\left(\sigma_{N}\right)^{\frac{2 p}{p+2}}\right)\right)=O\left(\left(\ln \sigma_{N}\right)^{\frac{2}{p+2}}\left(\sigma_{N}\right)^{\frac{2 p}{p+2}}\right)\right)
$$

(ii) We have

$$
\Pi_{1, N, \mathcal{E}}^{\prime \prime}(0)=V_{N}+O\left(\left(\sigma_{N}\right)^{\frac{2 p}{p+1}}\left(\ln \sigma_{N}\right)^{\frac{1}{p+1}}\right)
$$

and so

$$
\Gamma_{t_{j}, N, \mathcal{E}}^{\prime \prime}(0)=V_{N}+O\left(\left(\sigma_{N}\right)^{\frac{2 p}{p+1}}\left(\ln \sigma_{N}\right)^{\frac{1}{p+1}}\right) .
$$

(iii) For any $u \geq 3$ there exist constants $D_{u}>0$ and $\delta_{u}>0$ so that for any $N$ and all $h \in\left[-\delta_{u}, \delta_{u}\right]$ we have

$$
\left|\Pi_{1, N, \mathcal{E}}^{(u)}(i h)\right| \leq D_{u} V_{N}
$$

and therefore there is a constant $D_{u}^{\prime}$ such that

$$
\left|\Gamma_{t_{j}, N, \mathcal{E}}^{(u)}(i h)\right| \leq D_{u}^{\prime} V_{N}
$$

Proof. Denote $\Pi_{n}=\Pi_{t_{j}, n, \mathcal{E}}$. For any $I$ we set $\Pi_{I}(z)=\sum_{n \in I} \Pi_{n}(z)$.
First we prove (ii). Let $\varepsilon_{N}<\bar{\epsilon}$. Then the number of $n$ 's between 1 and $N$ so that $q_{n}(m \mid \mathcal{E}) \geq \varepsilon_{N}$ is $O\left(\ln \sigma_{N} / \varepsilon_{N}\right)$. We subdivide the set of $n$ 's between 1 to $N$ so that $q_{n}(m)<\varepsilon_{N}$ into blocks $B_{1}, \ldots, B_{l_{N}}$ so that for each $k$ we have $\varepsilon_{N} \leq \sum_{n \in B_{k}} q_{n}(m \mid \mathcal{E}) \leq 2 \varepsilon_{N}$. Then

$$
\begin{equation*}
l_{N}=O\left(\ln \sigma_{N} / \varepsilon_{N}\right) \tag{35}
\end{equation*}
$$

and each $B_{k}$ is contained in one of the blocks $\left(n_{s}, n_{s+1}\right)$. By (32) for any $k$ we have

$$
\begin{gathered}
\Pi_{B_{k}}^{\prime \prime}(0)=\frac{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} S_{B_{k}}^{2} \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} \mid \mathcal{E}\right]}-\left(\frac{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} S_{B_{k}} \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} \mid \mathcal{E}\right]}\right)^{2}+O(1) . \\
\quad=\frac{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} S_{B_{k}}^{2} \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} \mid \mathcal{E}\right]}-\left(\frac{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} S_{B_{k}} \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} \mid \mathcal{E}\right]}\right)^{2}+O(1) .
\end{gathered}
$$

To estimate the first term on the right hand side, we first write

$$
\frac{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} S_{B_{k}}^{2} \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} \mid \mathcal{E}\right]}=\frac{\mathbb{E}\left[\left(e^{i t_{j} S_{B_{k}} \bar{Z}}-1\right)\left(S_{B_{k}}^{2}-\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]\right) \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} \mid \mathcal{E}\right]}+\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]
$$

To estimate the first summand on the right hand side, by (30) we have

$$
\mathbb{E}\left[\left|S_{B_{k}} \bar{Z}\right| \mid \mathcal{E}\right] \leq C \sum_{n \in B_{k}} q_{n}(m \mid \mathcal{E}) \leq 2 C \varepsilon_{N}
$$

and so when $\varepsilon_{N}$ is smaller than some sufficiently small constant $c_{0}>0$ we have

$$
\left|\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} \mid \mathcal{E}\right]-1\right| \leq \frac{1}{2}
$$

which implies that

$$
\left|\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{z}} \mid \mathcal{E}\right]\right| \geq \frac{1}{2}
$$

Next, let us estimate the numerator. Since

$$
\begin{equation*}
\left|e^{i t_{j} S_{B_{k}} \bar{Z}}-1\right| \leq\left|t_{j}\right|\left|S_{B_{k}} \bar{Z}\right| \tag{36}
\end{equation*}
$$

for every $p \in(1,2)$ we have

$$
\left|\mathbb{E}\left[\left(e^{i t_{j} S_{B_{k}} \bar{Z}}-1\right)\left(S_{B_{k}}^{2}-\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]\right) \mid \mathcal{E}\right]\right| \leq C\left\|S_{B_{k}} \bar{Z}\right\|_{L^{p}(\mathcal{E})}\left(\left\|S_{B_{k}}^{2}\right\|_{L^{q}(\mathcal{E})}+\left|\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]\right|\right)
$$

where $L^{q}(\mathcal{E})$ denotes the $L^{q}$ norm with respect to the conditional measure and $q$ is the conjugate exponent of $p$. To estimate $\left\|S_{B_{k}}^{2}\right\|_{L^{q}(\mathcal{E})}$, let $q_{0}=[q]+1$. Then

$$
\mathbb{E}\left[S_{B_{k}}^{2 q_{0}} \mathcal{E}\right]=\mathbb{E}\left[\left(\bar{S}_{B_{k}}+\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]\right)^{2 q_{0}} \mid \mathcal{E}\right]=\sum_{j=0}^{2 q_{0}}\binom{2 q_{0}}{j} \mathbb{E}\left[\bar{S}_{B_{k}}^{j} \mid \mathcal{E}\right]\left(\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]\right)^{2 q_{0}-j}
$$

Since $\sup _{a<b}\left|\mathbb{E}\left[S_{b}\right]-\mathbb{E}\left[S_{a}\right]\right|<\infty$ due to (23), we have that $\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]$ is uniformly bounded in $k$. Applying the moment estimates of Lemma 13 to the conditioned Markov chain we see that for every integer $w \geq 1$ there is a constant $C_{w} \geq 1$ so that

$$
\begin{equation*}
\left\|\bar{S}_{B_{k}}\right\|_{L^{w}(\mathcal{E})} \leq C_{w}\left(1+\left\|\bar{S}_{B_{k}}\right\|_{L^{2}(\mathcal{E})}\right) \tag{37}
\end{equation*}
$$

We thus conclude that

$$
\left\|S_{B_{k}}^{2}\right\|_{L^{q}(\mathcal{E})} \leq\left\|S_{B_{k}}^{2}\right\|_{L^{q_{0}}(\mathcal{E})}=\left\|S_{B_{k}}\right\|_{L^{2 q_{0}}(\mathcal{E})}^{2}=O\left(1+\left\|\bar{S}_{B_{k}}\right\|_{L^{2}(\mathcal{E})}^{2}\right)
$$

To estimate the term $\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]$ we have

$$
\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]=V\left(S_{B_{k}} \mid \mathcal{E}\right)+\left(\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]\right)^{2}
$$

The second term is $O(1)$ because of (23). Combining the above estimates and using (30) we see that

$$
\left|\frac{\mathbb{E}\left[\left(e^{i t_{j} S_{B_{k}} \bar{Z}}-1\right)\left(S_{B_{k}}^{2}-\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]\right) \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} \mid \mathcal{E}\right]}\right| \leq C\left(\varepsilon_{N}\right)^{1 / p}\left(1+V\left(S_{B_{k}} \mid \mathcal{E}\right)\right)
$$

We conclude that for every $p \in(1,2)$ we have

$$
\frac{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} S_{B_{k}}^{2} \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} \mid \mathcal{E}\right]}=\mathbb{E}\left[S_{B_{k}}^{2} \mid \mathcal{E}\right]+O\left(\left(\varepsilon_{N}\right)^{1 / p}\right) V\left(S\left(B_{k}\right) \mid \mathcal{E}\right)+O\left(\left(\varepsilon_{N}\right)^{1 / p}\right)
$$

Next, similar arguments show that for every $p \in(1,2)$ we have

$$
\begin{gather*}
\left|\frac{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} S_{B_{k}} \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} Z} \mid \mathcal{E}\right]}-\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]\right|=\left|\frac{\mathbb{E}\left[\left(e^{i t_{j} S_{B_{k}} \bar{Z}}-1\right)\left(S_{B_{k}}-\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]\right) \mid \mathcal{E}\right]}{\mathbb{E}\left[e^{i t_{j} S_{B_{k}} \bar{Z}} \mid \mathcal{E}\right]}\right|  \tag{38}\\
\left.=O\left(\varepsilon_{N}\right)^{1 / p}\right) \sqrt{V\left(S\left(B_{k}\right) \mid \mathcal{E}\right)}
\end{gather*}
$$

Now, by (23) and since each $B_{k}$ is contained in one of the block $\left(n_{s}, n_{s+1}\right)$ we have

$$
\left|\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]-\mathbb{E}\left(S_{B_{k}}\right)\right| \leq C
$$

and therefore, since $\left|\mathbb{E}\left(S_{B_{k}}\right)\right|$ is also bounded in $k$, we get that

$$
\left|\mathbb{E}\left[S_{B_{k}} \mid \mathcal{E}\right]\right|=O(1)
$$

Combining the above estimates we derive that for every $p \in(1,2)$ we have

$$
\left.\Pi_{B_{k}}^{\prime \prime}(0)=V\left(S_{B_{k}} \mid \mathcal{E}\right)\left(1+O\left(\varepsilon_{N}\right)^{1 / p}\right)\right)+O\left(\left(\varepsilon_{N}\right)^{1 / p}\right) \sqrt{V\left(S_{B_{k}} \mid \mathcal{E}\right)}+O(1)
$$

Next, set $\mathfrak{s}_{k}=V\left(S\left(B_{k}\right) \mid \mathcal{E}\right)$. Then

$$
\begin{equation*}
\sum_{k=1}^{l_{N}} \sqrt{\mathfrak{s}_{k}} \leq \sqrt{l_{N}} \sqrt{\sum_{k=1}^{l_{N}} \mathfrak{s}_{k}} \leq A\left(\frac{\ln \sigma_{N}}{\varepsilon_{N}}\right)^{1 / 2} \sigma_{N} \tag{39}
\end{equation*}
$$

for some constant $A$. Therefore, the contribution to $\Pi_{1, N}^{\prime \prime}(0)$ coming from the terms $O\left(\left(\varepsilon_{N}\right)^{1 / p}\right) \sqrt{V\left(S_{B_{k}} \mid \mathcal{E}\right)}$ is $O\left(\sigma_{N}\left(\varepsilon_{N}\right)^{1 / p-1 / 2} \sqrt{\ln \sigma_{N}}\right)$. Now, because of the Lemma 19 and exponential decay of correlation (Lemma 14) we have

$$
V\left(S_{N} \mid \mathcal{E}\right)=\sum_{k=1}^{l_{N}} V\left(S_{B_{k}} \mid \mathcal{E}\right)+O\left(l_{N}\right)
$$

Since $l_{N}=O\left(\ln \sigma_{N} / \varepsilon_{N}\right)$ we conclude that

$$
\begin{equation*}
\sum_{n=1}^{N} \Pi_{n}^{\prime \prime}(0)= \tag{40}
\end{equation*}
$$

$$
O\left(\ln \sigma_{N} / \varepsilon_{N}\right)+\left(1+O\left(\left(\varepsilon_{N}\right)^{1 / p}\right)\right)\left(V\left(S_{N} \mid \mathcal{E}\right)+O\left(\ln \sigma_{N} / \varepsilon_{N}\right)\right)+O\left(\sigma_{N}\left(\varepsilon_{N}\right)^{1 / p-1 / 2} \sqrt{\ln \sigma_{N}}\right)
$$

$$
=V\left(S_{N} \mid \mathcal{E}\right)+O\left(\left(\varepsilon_{N}\right)^{1 / p}\right) V\left(S_{N} \mid \mathcal{E}\right)+O\left(\ln \sigma_{N} / \varepsilon_{N}\right)+O\left(\sigma_{N}\left(\varepsilon_{N}\right)^{1 / p-1 / 2} \sqrt{\ln \sigma_{N}}\right)
$$

This together with (200) and the choice $\varepsilon_{N}=\sigma_{N}^{-\frac{2 p}{p+1}}\left(\ln \sigma_{N}\right)^{\frac{p}{p+1}}$ yields (ii), where we have used that, by (13), $V\left(S_{N} \mid \mathcal{E}\right) / V_{N}$ is uniformly bounded and bounded away from 0 .

Now we derive (i) using the estimates obtained in the proof of (ii). By (38) and (19)

$$
\Pi_{1, N}^{\prime}(0)=\mathbb{E}\left(S_{N}\right)+O\left(\ln \sigma_{N} / \varepsilon_{N}\right)+O\left(\left(\varepsilon_{N}\right)^{1 / p}\right) \sum_{k=1}^{l_{N}} \sqrt{V\left(S\left(B_{k}\right) \mid \mathcal{E}\right)}
$$

By (39)

$$
\Pi_{1, N}^{\prime}(0)=\mathbb{E}\left(S_{N}\right)+O\left(\ln \sigma_{N} / \varepsilon_{N}\right)+O\left(\left(\varepsilon_{N}\right)^{1 / p}\left(\frac{\ln \sigma_{N}}{\varepsilon_{N}}\right)^{1 / 2} \sigma_{N}\right)
$$

Taking $\varepsilon_{N}=\left(\frac{\ln \sigma_{N}}{\sigma_{N}^{2}}\right)^{\frac{p}{p+2}}$ we get (i).
In order to prove (iii), let $c_{0}>0$ be such that for any $n<k$ with $\sum_{s=n}^{k-1} q_{s}(m \mid \mathcal{E}) \leq c_{0}$ we have

$$
\begin{equation*}
\left|\mathbb{E}\left(e^{i t_{j} S_{n, k}} \mid \mathcal{E}\right)\right|=\left|\mathbb{E}\left(e^{i t_{j} S_{n, k} \bar{Z}} \mid \mathcal{E}\right)\right| \geq \frac{1}{2} \tag{41}
\end{equation*}
$$

Fix some $s$, and decompose $\left\{n_{s}<n<n_{s+1}\right\}$ into blocks $B_{1}, B_{2}, \ldots, B_{L_{s}}, L_{s} \leq R^{\prime} \ln \sigma_{N}$, so that for each $k$ we have $\frac{c_{0}}{2}<\sum_{n \in B_{k}} q_{n}(m \mid \mathcal{E})<c_{0}$ (this is possible if $\bar{\epsilon}$ is small enough).

Next, let us fix some large constant $A$. If $V\left(S_{B_{j}} \mid \mathcal{E}\right)$ is larger than $2 A$ then we subdivide the block $S_{B_{j}}$ into smaller blocks so that the variance along each blocks is between $A$
and $2 A$. We conclude that there is a partition of $\{1,2, \ldots, N\}$ into blocks $\tilde{B}_{1}, \ldots, \tilde{B}_{\tilde{L}}$ so that $\tilde{L} \asymp \sigma_{N}^{2}$, the conditional variances along the blocks are uniformly bounded and

$$
\begin{equation*}
\left|\mathbb{E}\left(e^{i t_{j} S_{\tilde{B}_{s}}} \mid \mathcal{E}\right)\right| \geq \frac{1}{2} \tag{42}
\end{equation*}
$$

for each block $\tilde{B}_{s}$. Since $V\left(S_{\tilde{B}_{s}} \mid \mathcal{E}\right) \leq 2 A$ we have

$$
\left|\mathbb{E}\left(e^{i t_{j} S_{\tilde{B}_{s}}+i h S_{\tilde{B}_{s}}} \mid \mathcal{E}\right)-\mathbb{E}\left(e^{i t_{j} S_{\tilde{B}_{l}}} \mid \mathcal{E}\right)\right| \leq 4 A\left|t_{j}\right||h|
$$

and thus there exists $0<h_{0}<r_{0} / 2$ so that for every $h \in\left[-h_{0}, h_{0}\right]$ we have

$$
\begin{equation*}
\left|\mathbb{E}\left(e^{i t_{j} S_{\tilde{B}_{l}}+i h S_{\tilde{B}_{l}}} \mid \mathcal{E}\right)\right| \geq \frac{1}{4} . \tag{43}
\end{equation*}
$$

Next, let us decompose $\Pi_{1, N}$ according to the blocks $\tilde{B}_{s}$ :

$$
\Pi_{1, N}=\sum_{s=1}^{\tilde{L}} \Pi_{\tilde{B}_{s}} .
$$

Differentiating both $\Pi_{\tilde{B}_{s}}$ and $\Gamma_{t_{j}, \tilde{B}_{s}, \mathcal{E}}=\Gamma_{\tilde{B}_{s}} u$-times and using the Cauchy integral formula together with (32) we see that if $|h|<h_{0}$ then

$$
\begin{equation*}
\left|\Pi_{1, \tilde{B}_{s}, \mathcal{E}}^{(u)}(i h)\right| \leq C_{u}+\left|\Gamma_{\tilde{B}_{s}}^{(u)}(i h)\right| \tag{44}
\end{equation*}
$$

where $C_{u}$ is a constant which depends on $u$ but not on $\mathcal{E}, N$ or $h$. Next, let us bound

$$
\psi(h)=\psi_{s}(h):=\Gamma_{\tilde{B}_{j}}^{(u)}(i h)=\ln \mathbb{E}\left[e^{i t_{j} S_{\tilde{B}_{s}}} e^{i h S_{\tilde{B}_{s}}} \mid \mathcal{E}\right] .
$$

To ease the notation, let us abbreviate $S_{\tilde{B}_{j} s}=S$ and $W=t_{j} S_{\tilde{B}_{s}}$. Then by Faá di Bruno's formula, for every $h \in\left[-r_{0}, r_{0}\right]$ we have

$$
\left|\psi^{(u)}(h)\right|=\left|\sum_{\left(m_{1}, \ldots, m_{u}\right)} \frac{u!}{\prod_{q=1}^{u}\left(m_{q}!(q!)^{m_{q}}\right)} \cdot \frac{1}{\psi(h)^{\sum_{l=1}^{u} m_{q}}} \prod_{w=1}^{u}\left((i)^{w} \mathbb{E}\left[S^{w} e^{i W+i h S} \mid \mathcal{E}\right]\right)^{m_{w}}\right|
$$

where $\left(m_{1}, \ldots, m_{u}\right)$ range over all the $u$-tuples of nonnegative integers such that $\sum_{q} q m_{q}=u$. By applying (43) (which provides lower bounds on the denominators)
together with Lemma 13 (taking into account Lemma 19) and the Hölder inequality (to bound the numerators) we see that if $|h|<h_{0}$ then

$$
\left|\psi^{(u)}(h)\right| \leq C(u, A)
$$

for some constant $C(u, A)$ which depends only on $u$ and $A$. Thus by (44)

$$
\left|\Pi_{1, \tilde{B}_{s}, \mathcal{E}}^{(u)}(i h)\right| \leq C^{\prime}(u, A)
$$

[^6]and since the number of blocks is $O\left(\sigma_{N}^{2}\right)$ we conclude that if $|h|<h_{0}$ then
$$
\left|\Pi_{1, N, \mathcal{E}}^{(u)}(h)\right| \leq C^{\prime \prime} \sigma_{N}^{2}
$$
for some other constant $C^{\prime \prime}$.

### 5.4. The canonical form of the generalized Edgeworth polynomials at reso-

 nant points. Let$$
\begin{equation*}
\Lambda_{t_{j}, N, \mathcal{E}}(h)=\Gamma_{t_{j}, N, \mathcal{E}}\left(i h / \sigma_{N}\right)=\ln \mathbb{E}\left[e^{i t_{j} S_{N} Y} e^{i h S_{N} / \sigma_{N}} \mid \mathcal{E}\right] \tag{45}
\end{equation*}
$$

and set

$$
\begin{gather*}
H_{t_{j}, N, \mathcal{E}, r}(t)=  \tag{46}\\
1+\sum_{\bar{k}} \frac{1}{k_{1}!\cdots k_{r}!}\left(\frac{\Lambda_{t_{j}, N, \mathcal{E}}^{(3)}(0)}{3!}\right)^{k_{1}} \cdots\left(\frac{\Lambda_{t_{j}, N, \mathcal{E}}^{(r+2)}(0)}{(r+2)!}\right)^{k_{r}}(i t)^{3 k_{1}+\ldots+(r+2) k_{r}}
\end{gather*}
$$

where the summation runs over the collection of $r$ tuples of nonnegative integers $\left(k_{1}, \ldots, k_{r}\right)$ that are not all 0 so that $\sum_{j} j k_{j} \leq r$. Then we can also write

$$
\begin{equation*}
H_{t_{j}, N, \mathcal{E}, r}(t)=1+\sum_{q=1}^{r} \sigma_{N}^{-j} \tilde{P}_{t_{j}, \mathcal{E}, q}(t) \tag{47}
\end{equation*}
$$

with

$$
\tilde{P}_{t_{j}, N, \mathcal{E}, q}(x)=\sum_{\bar{k} \in A_{q}} C_{\bar{k}} \prod_{j=1}^{s}\left(\sigma_{N}^{-2} \Gamma_{t_{j}, N, \mathcal{E}}^{(j+2)}(0)\right)^{k_{j}}(i x)^{3 k_{1}+\ldots+(s+2) k_{s}}
$$

where $A_{q}$ is the set of all tuples of nonnegative integers $\bar{k}=\left(k_{1}, \ldots, k_{s}\right)$, for some $s=s(\bar{k}) \geq 1$ such that $\sum_{s} s k_{s}=q\left(\right.$ note that when $j \leq r$ then $s \leq r$ since $k_{s} \geq 1$ ).
Moreover $C_{\bar{k}}=\prod_{j=1}^{s} \frac{1}{k_{j}!(j+2)^{k_{j}}}$.
By Lemma 22, the $L^{\infty}$ norm of the coefficients of each $\tilde{P}_{t_{j}, N, \mathcal{E}, q}$ are uniformly bounded. Next, set

$$
\tilde{\Lambda}_{t_{j}, N, \mathcal{E}}(h)=\Lambda_{t_{j}, N, \mathcal{E}}(h)-\left(\Lambda_{t_{j}, N, \mathcal{E}}(0)+h \Lambda_{t_{j}, N, \mathcal{E}}^{\prime}(0)+\left(h^{2} / 2\right) \Lambda_{t_{j}, N, \mathcal{E}}^{\prime \prime}(0)\right) .
$$

Set also

$$
d_{N}=\Lambda_{t_{j}, N, \mathcal{E}}^{\prime}(0)-\frac{i \mathbb{E}\left[S_{N}\right]}{\sigma_{N}} \quad \text { and } \quad u_{N}=\Lambda_{t_{j}, N, \mathcal{E}}^{\prime \prime}(0)-1
$$

Then by Lemma 22, for every $p \in(1,2)$ we have

$$
\begin{equation*}
\left\|d_{N}\right\|_{L^{\infty}}=O\left(\sigma_{N}^{-\frac{2-p}{p+2}} \ln \frac{2}{p+2} \sigma_{N}\right), \quad\left\|u_{N}\right\|_{L^{\infty}}=O\left(\sigma_{N}^{-\frac{2}{p+1}}\left(\ln \sigma_{N}\right)^{\frac{1}{p+1}}\right) \tag{48}
\end{equation*}
$$

23. Proposition. For every $r$ there are constants $\delta_{r}, C_{r}>0$ so that for every realization of $\mathcal{E}$ and every real $h$ with $|h| \leq \delta_{r} \sigma_{N}$ we have
$\mathbb{E}\left(e^{i\left(t_{j}+h / \sigma_{N}\right) S_{N}} \mid \mathcal{E}\right)=\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) e^{-h \Lambda_{t_{j}, N, \mathcal{E}}^{\prime}(0)+\left(h^{2} / 2\right) \Lambda_{t_{j}, N, \mathcal{E}}^{\prime \prime}(0)} H_{t_{j}, N, \mathcal{E}, r}(h)+\theta_{N, r, \mathcal{E}} \sigma_{N}^{-r-1} e^{-h^{2} / 4}$
where $\theta_{N, r, \mathcal{E}}$ is a random variable so that $\sup _{N}\left\|\theta_{N, r, \mathcal{E}}\right\|_{L^{\infty}}<\infty$. As a consequence,

$$
\begin{gather*}
\mathbb{E}\left(e^{i\left(t_{j}+h / \sigma_{N}\right) S_{N}} \mid \mathcal{E}\right)=\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) e^{-i h \mathbb{E}\left[S_{N}\right] / \sigma_{N}} e^{-h^{2} / 2} \times  \tag{49}\\
\left(e^{h d_{N}+h^{2} u_{N} / 2} e^{-h \Lambda_{t_{j}, N, \mathcal{E}}^{\prime}(0)+h^{2} / 2 \Lambda_{t_{j}, N, \mathcal{E}}^{\prime \prime}(0)} H_{t_{j}, \mathcal{E}, r}(h)\right)+\theta_{N, r, \mathcal{E}} \sigma_{N}^{-r-1} e^{-h^{2} / 4}
\end{gather*}
$$

Proof. We have

$$
\begin{gathered}
\mathbb{E}\left(e^{i\left(t_{j}+h / \sigma_{N}\right) S_{N}} \mid \mathcal{E}\right)=\exp \left(\Lambda_{N, \mathcal{E}}(h)+i t_{j} \sum_{n=1}^{N} j\left(Y_{n}, m \mid \mathcal{E}\right)\right)= \\
\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) e^{-h \Lambda_{N, \mathcal{E}}^{\prime}(0)+\left(h^{2} / 2\right) \Lambda_{N, \mathcal{E}}^{\prime \prime}(0)} \exp \left(\tilde{\Lambda}_{N, \mathcal{E}}(h)\right) .
\end{gathered}
$$

Notice now that $\tilde{\Lambda}_{N, \mathcal{E}}^{(q)}(0)=0, q=0,1,2$ and that for $j \geq 3$,

$$
\left\|\tilde{\Lambda}_{N, \mathcal{E}}^{(j)}(h)\right\|_{L^{\infty}}=\left\|\Lambda_{N, \mathcal{E}}^{(j)}(h)\right\|_{L^{\infty}}=O\left(\sigma_{N}^{-(j-2)}\right)
$$

and $\sigma_{N}=\sigma_{N, \mathcal{E}}(1+o(1))$. Now the proof of the proposition is completed using Proposition 12, applied for every realization of $\mathcal{E}$.
5.5. Proof of Theorem 7. Recall the decomposition (18) of $\mathbb{P}\left(S_{N}=k\right)$. In this section we will expand the integrals $\sum_{j} \int_{I_{j}} e^{-i t k} \mathbb{E}\left(e^{i t S_{N}}\right) d t$ for resonant points $t_{j}=\frac{2 \pi l}{m}$ so that $M_{N}(m) \leq R \sigma_{N}$ for some constant $R$.

Recall first that

$$
\int_{-\infty}^{\infty} e^{-i \alpha h} e^{-h^{2} / 2} h^{k} d h=(-1)^{k} H_{k}(\alpha) \varphi(\alpha)
$$

where $H_{k}$ is the $k$-th Hermite polynomial. Now, let us write

$$
\int_{I_{j}} e^{-i k t} \mathbb{E}\left(e^{i t S_{N}}\right) d t=\mathbb{E}\left[e^{-i t_{j} k} \sigma_{N}^{-1} \int_{-\delta \sigma_{N}}^{\delta \sigma_{N}} e^{-i k h} \mathbb{E}\left(e^{i\left(t_{j}+h / \sigma_{N}\right) S_{N}} \mid \mathcal{E}\right) d h\right]
$$

Expanding the terms $e^{h d_{N}}$ and $e^{-h^{2} u_{N} / 2}$ in (49) and using (48) yields that the contribution of $t_{j}$ up to $o\left(\sigma_{N}^{-r}\right)$ equals to the expectation of

$$
\sigma_{N}^{-1} \mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) \int_{-\infty}^{\infty} e^{-i\left(t_{j} k+h k / \sigma_{N}\right)}\left(1+\sum_{j=1}^{3 r-2} \frac{h^{j} d_{N}^{j}}{j!}\right)\left(1+\sum_{j=1}^{r} \frac{h^{2 j} u_{N}^{j}}{j!2^{j}}\right) H_{t_{j}, N, \mathcal{E}, r}(h) e^{-h^{2} / 2} d h
$$

Next,

$$
\begin{equation*}
\left(1+\sum_{j=1}^{3 r-2} \frac{h^{j} d_{N}^{j}}{j!}\right)\left(1+\sum_{j=1}^{r} \frac{h^{2 j} u_{N}^{j}}{j!2^{j}}\right) H_{t_{j}, N, \mathcal{E}, r}(h)=1+\sum_{s=1}^{w_{r}} A_{t_{j}, s, N, \mathcal{E}} h^{s}+g_{N, r}(h) \tag{50}
\end{equation*}
$$

where $w_{r}=5 r-2$ and $g_{N, r}(h)$ is a polynomials whose coefficients are $o\left(\sigma_{N}^{-r-1}\right)$ in the $L^{\infty}$ norm. Then with $\hat{k}_{N}=\frac{k-\mathbb{E}\left[S_{N}\right]}{\sigma_{N}}$ the contribution is the expectation of

$$
\begin{equation*}
\sigma_{N}^{-1} \mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) \varphi\left(\hat{k}_{N}\right)\left(1+\sum_{s=1}^{w_{r}} A_{t_{j}, s, N, \mathcal{E}}(-1)^{s} H_{s}\left(\hat{k}_{N}\right)\right) \tag{51}
\end{equation*}
$$

Proof of Theorem 7. The theorem follows from (47) and (51) and the results in Section 4.2 (showing that the contribution of nonzero resonant points is negligible when $M_{N}(m) \geq R \ln \sigma_{N}$ with $R$ large enough).

## 6. Classical Edgeworth expansions.

24. Proposition. The condition $\mathbb{E}\left[e^{i t_{j} S_{N}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)$ is necessary for the usual expansions of order r to hold.

Proof. Let us write

$$
\begin{equation*}
1+\sum_{s=1}^{w_{r}} A_{t_{j}, s, N, \mathcal{E}}(-1)^{s} H_{s}\left(\hat{k}_{N}\right)=\sum_{u=0}^{w_{r}} B_{t_{j}, u, N, \mathcal{E}} \hat{\mathcal{k}}_{N}^{u} . \tag{52}
\end{equation*}
$$

Using (51) and [6, Lemma 5.1] we conclude that the expansions hold iff

$$
\mathbb{E}\left[\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) B_{t_{j}, u, N, \mathcal{E}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)
$$

for all $u$. However, since $H_{k}$ is of degree $k$ we conclude that the expansions hold iff

$$
\mathbb{E}\left[\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) A_{t_{j}, s, N, \mathcal{E}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)
$$

Indeed, the leading coefficient on the right hand side of (52) is $A_{t_{j}, w_{r}, N, \mathcal{E}}$, which yields that

$$
\mathbb{E}\left[\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) A_{t_{j}, w_{r}, N, \mathcal{E}}\right]=o\left(\sigma_{N}^{-(r-1)}\right) .
$$

Now we can proceed by induction on s, using [6, Lemma 5.1]. This means that the contribution is reduced to

$$
\sigma_{N}^{-1} e^{i t_{j} k} \mathbb{E}\left[\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right)\right]=\sigma_{N}^{-1} e^{i t_{j} k} \mathbb{E}\left[e^{i t_{j} S_{N}}\right]
$$

and thus $\mathbb{E}\left[e^{i t_{j} S_{N}}\right]=o\left(\sigma_{N}^{-(r-1)}\right)$.
25. Remark. The proof shows that if the conditionally stable expansions of order $r$ hold then for all $\ell$,

$$
\sup _{j_{1}, \ldots, j_{\ell}}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{j_{1}}, \ldots, X_{j_{\ell}}\right]\right\|_{L^{\infty}}=o_{\ell}\left(\sigma_{N}^{-(r-1)}\right)
$$

Indeed we can just replace the chain by the chain conditioned on $X_{j_{1}}, \ldots, X_{j_{\ell}}$ and use that the error term in the definition of the conditionally stable expansion depends only on $r, \sigma_{N}$ and the number of conditioned variables $\ell$.
26. Proposition. The condition

$$
\begin{equation*}
\max _{k \leq 8 r-4} \sup _{j_{1}, \ldots, j_{k} \in \mathcal{B}}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{j_{1}}, \ldots, X_{j_{k}}\right]\right\|_{L^{1}}=o\left(\sigma_{N}^{-(r-1)}\right) \tag{53}
\end{equation*}
$$

is sufficient for the usual expansions of order $r$ to hold. Similarly, the condition that for each $\ell$

$$
\sup _{j_{1}, \ldots, j_{\ell}}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{j_{1}}, \ldots, X_{j_{\ell}}\right]\right\|_{L^{1}}=o_{\ell}\left(\sigma_{N}^{-(r-1)}\right)
$$

is is sufficient for the conditionally stable expansions of order $r$ to hold.

Proof. Since the variables $X_{j}$ coming from different blocks $\left(n_{k}, n_{k+1}\right)$ are conditionally independent (under $\mathcal{E}$ ) we have ${ }^{8}$

$$
\Gamma_{t_{j}, N, \mathcal{E}}(z)=\sum_{k=0}^{N_{0}} \Gamma_{t_{j}, n_{k}, n_{k+1}, \mathcal{E}}(z)
$$

Thus recalling (45) we have

$$
\begin{equation*}
(i)^{-s} \sigma_{N}^{s} \Lambda_{t_{j}, N, \mathcal{E}}^{(s)}(0)=\sum_{k=0}^{N_{0}} \Gamma_{t_{j}, n_{k}, n_{k+1}, \mathcal{E}}^{(s)}(0) . \tag{54}
\end{equation*}
$$

For $s \geq 3$, let

$$
G_{k, s}=G_{t_{j}, k, s, \mathcal{E}}=(i)^{s} \Gamma_{t_{j}, n_{k}, n_{k+1}, \mathcal{E}}^{(s)}(0) .
$$

Then $G_{k, s}$ are functions of $\left(X_{n_{k}}, X_{n_{k+1}}\right)$. Arguing similarly to the proof of Lemma 22(iii) we get

$$
\left\|G_{k, s}\right\|_{L^{\infty}} \leq C \sigma_{n_{k}, n_{k+1}}^{2}, s \geq 3
$$

For $j=1,2$ we need to estimate $d_{N}$ and $u_{N}$ and not only the cumulants. To estimate $d_{N}$, note that

$$
\begin{equation*}
\sigma_{N} d_{N}=i \sum_{k}\left(\Gamma_{t_{j}, n_{k}, n_{k+1}, \mathcal{E}}^{\prime}(0)-i \mathbb{E}\left[S_{n_{k}, n_{k+1}}\right]\right):=\sum_{k=0}^{N_{0}} G_{k, 1} . \tag{55}
\end{equation*}
$$

Notice also that $G_{k, 1}$ depend only on $\left(X_{n_{k}}, X_{n_{k+1}}\right)$. Arguing as in the proof of Lemma 22 we see that, for every $p \in(1,2)$

$$
\begin{equation*}
\left\|G_{k, 1}\right\|_{L^{\infty}}=O\left(\left(\sigma_{n_{k}, n_{k+1}}\right)^{\frac{2 p}{p+2}}\left(\ln \sigma_{n_{k}, n_{k+1}}\right)^{\frac{2}{p+2}}\right)+O(1) \tag{56}
\end{equation*}
$$

where the $O(1)$ term is only needed when $\sigma_{n_{k}, n_{k+1}}=\sqrt{\operatorname{Var}\left(S_{n_{k}, n_{k+1}}\right)}$ is small.
To estimate $u_{N}$, by applying Lemma 21 we see that

$$
-\sigma_{N}^{2} u_{N}=\Gamma_{N}^{\prime \prime}(0)-\sigma_{N}^{2}=\Gamma_{t_{j}, N, \mathcal{E}}^{\prime \prime}(0)-V\left(S_{N} \mid \mathcal{E}\right)+O\left(\ln ^{2} \sigma_{N}\right)
$$

in $L^{\infty}$. Observe now that

$$
V\left(S_{N} \mid \mathcal{E}\right)=\sum_{k} V\left(S_{n_{k}, n_{k+1}} \mid \mathcal{E}\right)
$$

because the blocks between two bad times are conditionally independent. Thus,

$$
\begin{equation*}
-\sigma_{N}^{2} u_{N}=\Gamma_{N}^{\prime \prime}(0)-\sigma_{N}^{2}=\sum_{k}\left(\Gamma_{t_{j}, n_{k}, n_{k+1}, \mathcal{E}}^{\prime \prime}(0)-V\left(S_{n_{k}, n_{k+1}} \mid \mathcal{E}\right)\right)+O\left(\ln ^{2} \sigma_{N}\right) \tag{57}
\end{equation*}
$$

Let

$$
\begin{equation*}
G_{2, k}=\Gamma_{t_{j}, n_{k}, n_{k+1}, \mathcal{E}}^{\prime \prime}(0)-V\left(S_{n_{k}, n_{k+1}}\right) \tag{58}
\end{equation*}
$$

[^7]Then arguing 9 as in the proof of Lemmata 21 and 22 we see that for every $p \in(1,2)$ we have

$$
\left\|G_{2, k}\right\|_{L^{\infty}}=O(1)+O\left(\left(\sigma_{n_{k}, n_{k+1}}\right)^{\frac{2 p}{p+1}}\left(\ln \sigma_{n_{k}, n_{k+1}}\right)^{\frac{1}{p+1}}\right)
$$

where the $O(1)$ term is needed to cover the case when $\sigma_{n_{k}, n_{k+1}}$ is small.
Next, by using the explicit formula (51) of the generalized Edgeworth polynomials and the above formulas we see that their coefficients are linear combinations of expressions of the form

$$
\begin{equation*}
A_{N} \mathbb{E}\left[e^{i t_{j} S_{N}}\right]+\sum_{1 \leq k_{1}, \ldots, k_{\ell_{r}} \leq N_{0}} c_{k_{1}, \ldots, k_{\ell}} \mathbb{E}\left[\mathbb{E}\left[e^{i t_{j} S_{N}} \mid \bar{X}_{k_{1}}, \ldots, \bar{X}_{k_{\ell_{r}}}\right] G_{k_{1}, \ldots k_{\ell}, N}\right] \tag{59}
\end{equation*}
$$

where $\ell_{r}=4 r-2$, with $\bar{X}_{k}=\left(X_{n_{k}}, X_{n_{k+1}}\right)$, and $A_{N}$ is either 0 or $1, c_{k_{1}, \ldots, k_{3 r}}$ are combinatorial coefficients bounded by some constant $C=C_{r}$, and

$$
\begin{equation*}
G_{k_{1}, \ldots, k_{\ell_{r}}, N}=\prod_{s=1}^{3 r-2} \sigma_{N}^{-j_{k_{s}}} G_{j_{k_{s}}, k_{s}} \tag{60}
\end{equation*}
$$

for appropriate $0 \leq j_{k_{s}} \leq m_{r}$ (for some $m_{r}$ which depend only on $r$ ), where for $j=0$ we set $G_{0, k}=0$. Before we proceed with the proof let us give more detailed explanation of (60). First, the coefficient of the polynomials defined on the right hand side of (52) are linear combinations of $A_{t_{j}, s, N, \mathcal{E}}, s \leq w_{r}$. Next, by (50) and (46) each $A_{t_{j}, s, N, \mathcal{E}}$ has the form

$$
A_{t_{j}, s, N, \mathcal{E}}=\mathcal{P}_{s}\left(d_{N}, u_{N}, \Lambda_{t_{j}, N, \mathcal{E}}^{(3)}(0), \ldots, \Lambda_{t_{j}, N, \mathcal{E}}^{\left(v_{s}\right)}(0)\right)
$$

for some polynomial $\mathcal{P}_{s}$ whose degree does not exceed $4 s-2$, where $v_{s}$ is some positive integer. Indeed, the term of the smallest order in the brackets on the left hand side of (50) is $d_{N}$, and the largest relevant power of $d_{N}$ is $3 s-2$. On the other hand, the term $H_{t_{j}, N, \mathcal{E}, r}(t)$ contributes at most $s$ variables among $\Lambda_{t_{j}, N, 0}^{(u)}$ to $A_{t_{j}, s, N, \mathcal{E}}$, where there is an actual contribution only if $u \leq s+2$ since in the computation of $A_{t_{j}, s, N, \mathcal{E}}$ we need only to take into account the partial term $H_{t_{j}, N, \mathcal{E}, s}(t)$.

Overall we get at most $4 s-2$ appearances of variables of the form $\left(X_{n_{k}}, X_{n_{k+1}}\right)$ which amounts in at most $2(4 r-2)=8 r-4$ appearances of variables of the form $X_{n_{j}}$, which is the maximal number of conditioned variables in (53). Now we arrive at (60) by taking expectation of the expression in (51), using (54) and (55), and the fact that for every function $Q=Q\left(X_{m_{1}}, \ldots, X_{m_{s}}\right)$ with $m_{\ell} \in \mathcal{B}$ we have
$\mathbb{E}\left[\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) Q\right]=\mathbb{E}\left[Q \cdot \mathbb{E}\left[\mathbb{E}\left(e^{i t_{j} S_{N}} \mid \mathcal{E}\right) \mid X_{m_{1}}, \ldots, X_{m_{s}}\right]\right]=\mathbb{E}\left[Q \cdot \mathbb{E}\left(e^{i t_{j} S_{N}} \mid X_{m_{1}}, \ldots, X_{m_{s}}\right)\right]$.
To prove that the contribution coming from the nonzero resonant point is negligible it is enough to show that the above coefficients are $o\left(\sigma_{N}^{-(r-1)}\right)$.

We claim that

$$
\begin{equation*}
\|G\|:=\sum_{k_{1}, \ldots, k_{\ell_{r}}}\left\|G_{k_{1}, \ldots k_{\ell_{r}}, N}\right\|_{L^{\infty}} \leq C \tag{61}
\end{equation*}
$$

[^8]for some $C$ which depends only on $r$. Note that (61) implies that
$$
\left|\sum_{k_{1}, \ldots, k_{\ell_{r}}} \mathbb{E}\left[\mathbb{E}\left[e^{i t_{j} S_{N}} \mid \bar{X}_{k_{1}}, \ldots, \bar{X}_{k_{\ell}}\right] G_{k_{1}, \ldots, k_{\ell_{r}, N}}\right]\right| \leq C \sup _{a_{1}, \ldots, a_{2 \ell_{r}} \in \mathcal{B}}\left\|\mathbb{E}\left[e^{i t_{j} S_{N}} \mid X_{a_{1}}, \ldots, X_{a_{2 \ell_{r}}}\right]\right\|_{L^{1}}
$$
and so the first condition is indeed sufficient.
In order to prove (61), let us first consider the case where one of $j_{k_{s}}=j$ is larger than 2. In this case we have
$$
\sigma_{N}^{-j} G_{j_{k_{s}}, k_{s}}=\sigma_{N}^{-j}\left[O\left(\sigma_{n_{k}, n_{k+1}}^{2}\right)+O(1)\right], k=k_{s}
$$
while the other terms are bounded. Thus the contribution to $\|G\|$ of such terms is $O\left(N_{0}^{\ell_{r}}\right) \sigma_{N}^{-1}=o(1)$. Otherwise, $j_{k_{s}}$ is either 1 or 2 . If one of them is 1 , then, since the other terms in the product are bounded we see that the contribution to $\|G\|$ of such terms is dominated by
$$
N_{0}^{\ell_{r}} \sum_{k=0}^{N_{0}} \sigma_{N}^{-1}\left\|G_{1, k}\right\|_{L^{\infty}}
$$

However, by (56), if $p$ is close enough to 1 then $\left\|G_{k, 1}\right\|_{L^{\infty}}=O\left(\sigma_{N}^{3 / 4}\right)$ and so

$$
N_{0}^{k_{\ell}} \sum_{k=0}^{N_{0}} \sigma_{N}^{-1}\left\|G_{1, k}\right\|_{L^{\infty}}=O\left(N_{0}^{k_{\ell}+1} \sigma_{N}^{-1 / 4}\right)=o(1)
$$

where we have used that $N_{0}=O\left(\ln \sigma_{N}\right)$.
It remains to consider $\left(k_{1}, \ldots, k_{\ell_{r}}\right)$ so that $j_{k_{s}}=2$ for all $s$. In this case the contribution to $\|G\|$ from such terms is at most

$$
N_{0}^{k_{\ell}} \sum_{k=0}^{N_{0}} \sigma_{N}^{-2}\left\|G_{2, k}\right\|_{L^{\infty}}
$$

Note that by (58), if $p$ is close enough to 1 then

$$
\left\|G_{2, k}\right\|_{L^{\infty}}=O\left(\ln \sigma_{N}\left(\sigma_{N}\right)^{3 / 2}\right)
$$

where we have again used that $\sigma_{n_{k}, n_{k+1}}=O\left(\sigma_{N}\right)$. Hence

$$
N_{0}^{\ell_{r}} \sum_{k} \sigma_{N}^{-2}\left\|G_{2, k}\right\|_{L^{\infty}} \leq C \sigma_{N}^{-1 / 2} N_{0}^{\ell_{r}+1} \ln \sigma_{N}=o(1)
$$

The proof that the second condition is sufficient for the stable expansions is similar, we first condition on a finite number of variables, and then repeat the arguments above.

Proof of Theorem 8. The theorem follows now by Remark 25) and Proposition 26 (note that the case when $M_{N}(m) \geq R \ln \sigma_{N}$ was already treated in §4.2).

Proof of Theorem 4. The theorem follows from Theorem 8 together with the standard fact that a sequence of probability measures on $\left\{\mu_{N}\right\}$ on $\mathbb{Z} / m \mathbb{Z}$ satisfies $\mu_{N}(a)=$ $\frac{1}{m}+O\left(\gamma_{N}\right)$ for some sequence $\gamma_{N}$ and all $a \in \mathbb{Z} / m \mathbb{Z}$ iff $\hat{\mu}_{N}(b)=O\left(\gamma_{N}\right)$ for all $b \in$ $(\mathbb{Z} / m \mathbb{Z}) \backslash\{0\}$ where $\hat{\mu}$ is the Fourier transform of $\mu$ (see e.g. [6, Lemma 6.2]).

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[^0]:    ${ }^{1}$ The local limit theorem has origins in the de Moivre-Laplace theorem, and Prokhorov's theorem can be viewed as a generalization.

[^1]:    ${ }^{2}$ Note that for independent summands $X_{n}$, in the case when $X_{n}=Y_{n}$, the conditionally stable expansions are equivalent to the super stable expansions defined in [6. However, in general these two types of expansions do not coincide, which is why we decided to introduce the notion of "conditionally stable expansions".

[^2]:    ${ }^{3}$ Expansions in the CLT for additive functionals of uniformly elliptic Markov chains were considered in [8 and [5].

[^3]:    ${ }^{4}$ Recall that $\ell$ is the number of indices we are allowed to fix.

[^4]:    ${ }^{5}$ Note that for one step uniformly elliptic chains the correlation coefficient of the chain as defined in [18] is strictly smaller than 1 .

[^5]:    ${ }^{6}$ It is enough to prove this when $\mathbb{E}\left(Y_{k_{i}}\right)=0, i=1,2$. In this case (28) is a direct consequence of (27).

[^6]:    ${ }^{7}$ Since $L_{s} \leq R^{\prime} \ln \sigma_{N}$ the blocks $B_{j}$ for which $V\left(S_{B_{j}} \mid \mathcal{E}\right) \leq 2 A$ only contribute $O\left(N_{0} \ln \sigma_{N}\right)=O\left(\ln ^{2} \sigma_{N}\right)$ to the total variance, and so in order to estimate $\tilde{L}$ we can disregard these blocks when one of the original blocks has small variance.

[^7]:    ${ }^{8}$ Recall that $\Gamma$ is defined by (33).

[^8]:    ${ }^{9}$ We first approximate the conditional variance by $\sigma_{n_{k}, n_{k+1}}^{2}$ as in Lemma 21, and then approximate the second derivative as in Lemma 22 .
    ${ }^{10} A_{N}$ is 1 only in the coefficients of the polynomial multiplied by $\sigma_{N}^{-1}$, but for the proof to work we actually only need $A_{N}$ to be bounded.

