EFFECTS OF THE MINIMAL LENGTH ON THE THERMAL PROPERTIES OF A TWO-DIMENSIONAL DIRAC OSCILLATOR

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The effect of the minimal length on the thermal properties of a Dirac oscillator is considered. The canonical partition function is well-determined by using the method based on the Epstein zeta function. Through this function, all thermodynamics properties, such as the free energy, the total energy, the entropy, and the specific heat, have been determined. Moreover, this study leads to a minimal length in the interval of $10^{-16} < \Delta x < 10^{-14}$ m with the following physically acceptable condition $\beta > \beta_0 = \frac{1}{m_0^2 c^2}$. We show that this condition is obtained directly through the properties of the Epstein zeta function, and the minimal length Δx coincide with the reduced Compton wavelength $\bar{\lambda} = \frac{\hbar}{m_0 c}$.

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1. Introduction

The Dirac relativistic oscillator is an important potential both for theory and application. It was for the first time studied by Itô *et al.* [1]. They considered a Dirac equation in which the momentum $\vec{p}$ is replaced by $\vec{p} - im\beta\omega\vec{r}$, with $\vec{r}$ being the position vector, m the mass of particle, and ω the frequency of the oscillator. The interest in the problem was revived by Moshinsky and Szczepaniak [2], who gave it the name of Dirac oscillator (DO) because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Physically, it can be shown that the (DO) interaction is a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field [3, 4]. The electromagnetic potential associated with the DO has been found by Benitez *et al.* [5]. The Dirac oscillator has attracted a lot of interest both because it provides one of the examples of the Dirac's equation exact solvability and because of its numerous physical applications (see [6] and references therein). Recently, Franco-Villafane *et al.* [7] exposed the proposal of the first experimental microwave realization of the one-dimensional DO. Quimbay *et al.* [8, 9] show that the Dirac oscillator can describe a naturally occurring physical system. Specifically, the case of a two-dimensional Dirac oscillator can be used to describe the dynamics of the charge carriers in graphene, and hence its electronic properties. This idea has been also proved in the calculations of the thermal properties of graphene using a method base on zeta function [10]: this method allowed Adra *et al.* [11] to determine the thermodynamics functions for the Dirac equation with a Lorentz scalar and inverse-linear potential in the range of all temperatures.

The unification between the general theory of relativity and the quantum mechanics is one of the most important problems in theoretical physics. This unification predicts the existence of a minimal measurable length of the order of the Planck length. All approaches of quantum gravity show the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be reformulated. The minimal length uncertainty relation has appeared in the context of the string theory, where it is a consequence of the fact that the string cannot probe distances smaller than the string scale $\hbar\sqrt{\beta}$, where β is a small positive parameter called the deformation parameter. This minimal length can be introduced as an additional uncertainty in position measurement, so that the usual canonical commutation relation between position and momentum operators becomes $[\hat{x},\hat{p}] = i\hbar(1+\beta p^2)$. This commutation relation leads to the standard Heisenberg uncertainty relation $\Delta\hat{x}\Delta\hat{p} \geq i\hbar(1+\beta(\Delta p)^2)$, which clearly implies the existence of a non-zero minimal length $\Delta x_{\mathrm{min}} = \hbar\sqrt{\beta}$. This modification of the uncertainty relation is usually termed the generalized uncertainty principle (GUP) or the minimal length uncertainty principle [12–15]. Investigating the influence of the minimal length assumption on the energy spectrum of quantum systems has become an interesting issue primarily for two reasons. First, this may help to set some upper bounds on the value of the minimal length. In this context, we can cite some studies of the hydrogen atom and a two-dimensional

Dirac equation in an external magnetic field. Moreover, the classical limit has also provided some interesting insights into some cosmological problems. Second, it has been argued that quantum mechanics with a minimal length may also be useful to describe non-point-like particles, such as quasi-particles and various collective excitations in solids, or composite particles (see Ref. [16] and references therein). Nowadays, the reconsideration of the relativistic quantum mechanics in the presence of a minimal measurable length has been studied extensively. In this context, many papers were published where different quantum systems in space with Heisenberg algebra were studied. They are: the Abelian Higgs model [17], the thermostatics with minimal length [18], the one-dimensional hydrogen atom [19], the Casimir effect in minimal length theories [20], the effect of minimal lengths on electron magnetism [21], the Dirac oscillator in one and three dimensions [22–26], the solutions of a two-dimensional Dirac equation in presence of an external magnetic field [27], the non-commutative phase space Schrödinger equation [28], Schrödinger equation with harmonic potential in the presence of a magnetic field [29], and finally, the two-dimensional Dirac oscillator in both commutative and non-commutative phase space [30, 31].

The principal aim of this paper is to study the effect of the presence of a non-zero minimal length on the thermal properties of the Dirac oscillator in one and two dimensions. For this, we use the formalism based on the Epstein zeta function to calculate the canonical partition function in both cases. We expect that the introduction of a minimal length has important consequences on these properties.

This paper is organized as follows: in Sec. 2, we propose a method based on Epstein zeta function to calculate the canonical partition function of the Dirac oscillator in one and two dimensions. Section 3 is devoted to present the different results concerning the thermodynamics quantities of this oscillator. Finally, Sec. 4 will be a conclusion.

2. Zeta thermal partition function of a Dirac oscillator in one and two dimensions

2.1. Framework theory

The two-dimensional Epstein zeta function $\mathcal{Z}$ is defined for $\operatorname{Re} s > 1$, by [32–35]

$$\mathcal{Z}(s) = \sum_{n,m=-\infty}^{\infty} \frac{1}{\left(am^2 + bmn + cn^2\right)^s}\,, \tag{1}$$

where a, b, c are real numbers with $a > 0$ and $D = b^2 - 4ac$. By defining that $D = 4ac - b^2 > 0$, then the following quantity

$$Q(m,n) = am^2 + bmn + cn^2 \tag{2}$$

is a positive-definite binary quadratic form of discriminant D. In this case, we have

$$\mathcal{Z}(s) = \sum_{n,m=-\infty}^{\infty} \frac{1}{Q(m,n)^s} \,. \tag{3}$$

With substitutions

$$x = \frac{b}{2a}\,, \qquad y = \frac{\sqrt{D}}{2a}\,, \qquad \tau = x + iy\,, \tag{4}$$

equation (3) becomes

$$\mathcal{Z}(s) = \sum_{n,m=-\infty}^{\infty} \frac{1}{a^s \mid m + n\tau \mid^{2s}} \,. \tag{5}$$

Following the procedure used in [32], the final form of two-dimensional Epstein zeta function is

$$\mathcal{Z}(s) = 2a^{-s}\zeta(2s) + 2a^{-s}y^{1-2s}\sqrt{\pi}\,\frac{\zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \frac{2a^{-s}y^{\frac{1}{2}-s}\pi^s}{\Gamma(s)}H(s) \tag{6}$$

with [32]

$$H(s) = 4\sum_{k=1}^{\infty} \sigma_{1-2s}(k)k^{s-\frac{1}{2}}\cos(2k\pi x)K_{s-\frac{1}{2}}(2k\pi y)\,, \tag{7}$$

where $\sigma_\nu(k)$ denotes the sum of the v^{th} powers of the divisors of k, that is,

$$\sigma_\nu(k) = \sum_{d/k} d^\nu = \sum_{d/k} \left(\frac{k}{d}\right)^\nu \,. \tag{8}$$

2.2. The zeta thermal function

We start with the following eigenvalues of a one-dimensional Dirac oscillator in the presence of minimal length β [22]

$$\epsilon_n = m_0c^2\sqrt{1 + 2\frac{\hbar\omega}{m_0c^2}n + \beta\frac{\hbar^2\omega^2}{c^2}n^2}\,. \tag{9}$$

With substitutions

$$b = 2r\,, \qquad a = r^2\frac{\beta}{\beta_0}\,, \qquad \left(r = \frac{\hbar\omega}{m_0c^2}, \beta_0 = \frac{1}{m_0^2c^2}\right)\,, \tag{10}$$

we get

$$\epsilon_n = m_0c^2\sqrt{an^2 + bn + 1}\,. \tag{11}$$

In what follows, we choose $r = 1$. Given the energy spectrum, we can define the partition function via

$$Z_{\mathrm{1D}} = \sum_n e^{-\tilde{\beta}\epsilon_n}\,, \tag{12}$$

where $\tilde{\beta} = \frac{1}{k_{\mathrm{B}}T}$ with k_{B} is the Boltzmann constant. In our case, Z reads

$$Z_{\mathrm{1D}} = \sum_n e^{-\frac{1}{\tau}\sqrt{an^2+bn+1}} \tag{13}$$

with $\tau = \frac{k_{\mathrm{B}}T}{m_0c^2}$. Now, when we put that

$$\chi = \frac{1}{\tau}\sqrt{an^2 + bn + 1}\,, \tag{14}$$

and according the following relation [36]

$$e^{-\chi} = \frac{1}{2\pi i}\int_C \mathrm{d}s\chi^{-s}\Gamma(s)\,, \tag{15}$$

the sum is transformed into

$$\begin{aligned}\sum_n e^{-\frac{1}{\tau}\sqrt{an^2+bn+1}} &= \frac{1}{2\pi i}\int_C \mathrm{d}s\left(\frac{1}{\tau}\right)^{-s}\sum_n\left\{an^2 + bn + 1\right\}^{-\frac{s}{2}}\Gamma(s)\\ &= \frac{1}{2\pi i}\int_C \mathrm{d}s\left(\frac{1}{\tau}\right)^{-s}\mathcal{Z}(s)\Gamma(s)\,,\end{aligned} \tag{16}$$

where $\Gamma(s)$ and $\mathcal{Z}(s)$ are respectively the Euler and one-dimensional Epstein zeta function [33], and with

$$\mathcal{Z}(s) = \sum_n \frac{1}{Q(1,n)^{\frac{s}{2}}}\,, \tag{17}$$

where

$$Q(1,n) = an^2 + bn + 1\,. \tag{18}$$

Setting that

$$x = \frac{b}{2}\,, \qquad y = \frac{\sqrt{D}}{2}\,, \qquad \text{with} \qquad D = 4a - b^2 > 0\,, \tag{19}$$

we find the restriction on the deformation parameter β

$$\beta > \beta_0 = \frac{1}{m_0^2 c^2}\,. \tag{20}$$

Consequently, (17) is transformed into

$$\mathcal{Z}(s) = 2a^{-\frac{s}{2}}\zeta(s) + \frac{2a^{-\frac{s}{2}}y^{1-s}\sqrt{\pi}}{\Gamma\left(\frac{s}{2}\right)}\zeta(s-1)\Gamma\left(\frac{s}{2}-\frac{1}{2}\right) + \frac{2a^{-\frac{s}{2}}y^{\frac{1}{2}-\frac{s}{2}}\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}H\left(\frac{s}{2}\right). \tag{21}$$

Thus, the final partition function is

$$Z_{1\mathrm{D}} = \frac{1}{2\pi i}\int\limits_C \mathrm{d}s \left(\frac{1}{\tau}\right)^{-s} \mathcal{Z}(s)\Gamma(s) \tag{22}$$

or

$$\begin{aligned} Z_{1\mathrm{D}} &= \frac{1}{2\pi i}\int\limits_C \mathrm{d}s \left(\frac{1}{\tau}\right)^{-s} 2a^{-\frac{s}{2}}\zeta(s)\Gamma(s) \\ &\quad + \frac{1}{2\pi i}\int\limits_C \mathrm{d}s \left(\frac{1}{\tau}\right)^{-s} \frac{2a^{-\frac{s}{2}}y^{1-s}\sqrt{\pi}}{\Gamma\left(\frac{s}{2}\right)}\zeta(s-1)\Gamma\left(\frac{s}{2}-\frac{1}{2}\right)\Gamma(s) \\ &\quad + \frac{1}{2\pi i}\int\limits_C \mathrm{d}s \left(\frac{1}{\tau}\right)^{-s} \frac{2a^{-\frac{s}{2}}y^{\frac{1}{2}-\frac{s}{2}}\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}H\left(\frac{s}{2}\right)\Gamma(s)\,. \end{aligned} \tag{23}$$

The first integral has two poles in $s = 0$ and $s = 1$, the second has three poles in $s = 0$, $s = 1$ and $s = 2$, and finally, the third has a pole at $s = 0$. By applying the residues theorem, we get

$$Z_{1\mathrm{D}} = 2\zeta(0) + \frac{2}{\sqrt{a}}\left\{\zeta(1) + \zeta(0)\right\}\tau + \frac{2\pi}{ay}\tau^2\,. \tag{24}$$

The last integral goes to zero because of the following relation

$$\frac{1}{\Gamma(s)} = s e^{\gamma s}\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right\}, \tag{25}$$

where γ is Euler's constant given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right) . \tag{26}$$

In this stage, Elizalde [33, 34] mentioned that this formula is very useful and its practical application quite simple: in fact, the first two terms are just nice, while the last one is quickly convergent and thus absolutely useless in practice. Finally, the partition function for the one-dimensional Dirac oscillator becomes

$$Z_{1\mathrm{D}}(\tau,\alpha) = \frac{2\pi}{\alpha\sqrt{\alpha-1}}\tau^2 + \frac{1}{\sqrt{\alpha}}\tau - 1 \tag{27}$$

with $\alpha = \frac{\beta}{\beta_0}$, $a = \alpha$ and $y = \sqrt{\alpha - 1}$.

The two-dimensional case can be treated in the same way as one-dimensional: starting with the following form of the spectrum of energy (see [30])

$$\bar{\epsilon}_n = m_0c^2\sqrt{1 + 4\frac{\hbar\omega}{m_0c^2}n + 4\beta\frac{\hbar^2\omega^2}{c^2}n^2} \tag{28}$$

and by the same approach as used above, the wanted partition function of a two-dimensional Dirac oscillator can be written as

$$Z_{2\mathrm{D}}\,(\tau,\alpha) = \frac{\pi}{4\alpha\sqrt{\alpha-1}}\tau^2 + \frac{1}{2\sqrt{\alpha}}\tau - 1\,. \tag{29}$$

All thermal properties for both cases can be obtained using the following relations

$$\mathcal{F} \equiv \frac{F}{mc^2} = -\tau\ln(Z)\,, \qquad \mathcal{U} \equiv \frac{U}{mc^2} = \tau^2\frac{\partial\ln(Z)}{\partial\tau}\,,$$

$$\mathcal{S} \equiv \frac{S}{k_{\mathrm{B}}} = \ln(Z) + \tau\frac{\partial\ln(Z)}{\partial\tau}\,, \qquad \mathcal{C} \equiv \frac{C}{k_{\mathrm{B}}} = 2\tau\frac{\partial\ln(Z)}{\partial\tau} + \tau^2\frac{\partial^2\ln(Z)}{\partial\tau^2}\,. \tag{30}$$

3. Numerical results and discussions

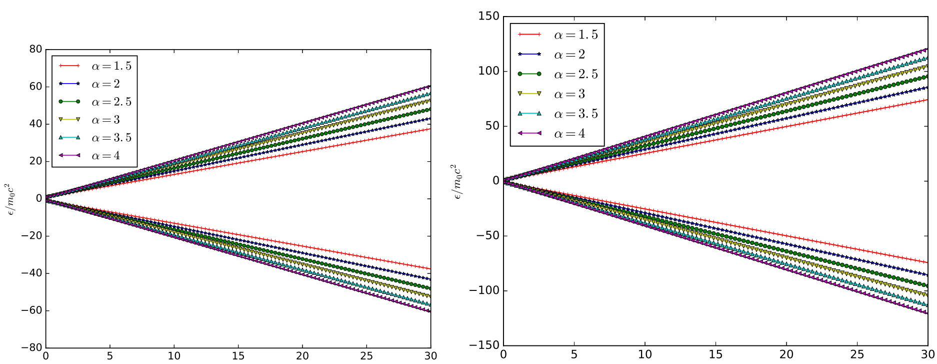
Before presenting our results concerning the thermal quantities of one- and two-dimensional Dirac oscillator, two remarks can be made: *(i)* in Table I, we show some values of β_0 together with the minimal length Δx for some fermionic particles.

TABLE I

Some values of both β_0 and minimal length $\Delta x = \sqrt{\beta_0}$.

	Symbol	Mass $\left[\frac{\mathrm{MeV}}{c^2}\right]$	$\beta_0 = \frac{1}{m_0^2 c^2}$ $\left[\frac{\mathrm{s}^2}{\mathrm{kg}^2\mathrm{m}^2}\right]$	$\Delta x \simeq \hbar\sqrt{\beta_0}$ [m]
Electron/positron	e^-/e^+	0.511	$1.339109486 \times 10^{43}$	$38.615926800 \times 10^{-14}$
Proton/antiproton	$p/\bar{p}$	938.272	$3.971566887 \times 10^{36}$	$2.1030891047 \times 10^{-16}$
Muon	μ^-/μ^+	105.7	$3.143705046 \times 10^{38}$	$1.867594294 \times 10^{-15}$
Tauon	τ^-/τ^+	1777	$1.105704217 \times 10^{36}$	1.11056×10^{-16}

This parameter has been determined through the properties of Epstein zeta function, and this restriction leads to the minimal length $\Delta x \simeq \hbar\sqrt{\beta_0}$. According to Table I, the minimal length lies in the interval of $10^{-16} < \Delta x < 10^{-14}$ m. In addition, we can see that $\Delta x \simeq \hbar\sqrt{\beta_0} = \frac{\hbar}{m_0 c} = \bar{\lambda}$, where $\bar{\lambda}$ is the reduced Compton wavelength: the minimal length has the same order as the reduced Compton wavelength. *(ii)* In Fig. 1, we study the effect of the presence of minimal length on the spectrum of energy. In this context, the reduced spectrum of energy as a function of the quantum number n for different values of α are depicted in Fig. 1. This figure reveals that the effect deformation parameter β on the energy spectrum is significant.



(a) 1D Dirac oscillator (b) 2D Dirac oscillator

Fig. 1. Energy spectrum $\frac{\epsilon}{m_0 c^2}$ *versus* quantum number n for different values of $\alpha = \frac{\beta}{\beta_0}$.

We note here that when we have determined the thermodynamics functions of our oscillators in both one and two dimensions, we have only restricted ourselves to stationary states of positive energy. The reason for this is twofold [37]: *(i)* The Dirac oscillator possesses an exact Foldy–Wouthuysen transformation (FWT): so, the positive- and negative-energy solutions never mix. *(ii)* The solutions with infinite degeneracy do not correspond to physical states since there is not Lorentz finite representation for them. Thus, according to these arguments, we can assume that only particles with positive energy are available in order to determine the thermodynamic properties of our oscillator in question.

Now, we are ready to present our numerical results on the thermal properties of the Dirac oscillator in one and two dimensions: in Fig. 2, we show all thermal properties of the one-dimensional Dirac oscillator for different values of α. According to this figure, we can confirm that the parameter β plays a significant role in these properties, and the effect of this parameter is very important for the thermodynamic properties. In particular, the curves of the reduced specific heat, for different values of β, tend to the asymptotic limit at 2, and they separate in the range of the reduce temperature τ between 0 and ~ 10.

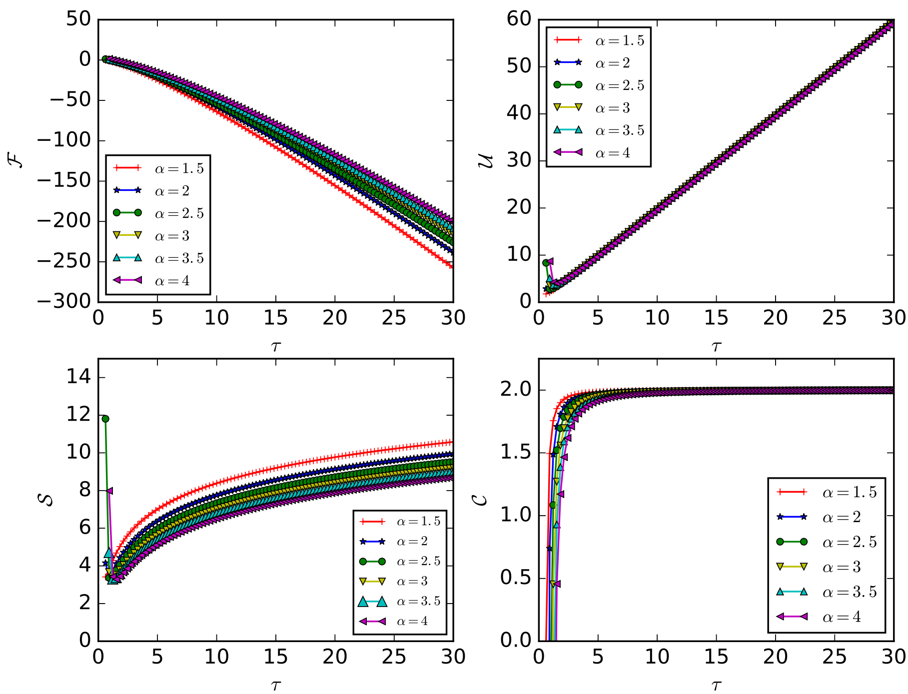


Fig. 2. Thermal properties of a one-dimensional Dirac oscillator for different values of $\alpha = \frac{\beta}{\beta_0}$.

For the case of a two-dimensional oscillator, we conclude that the method of determining the canonical partition function will be the same in both cases (see Eqs. (9) and (28)). As a consequence, all thermal properties can be found by the same manner as in the one-dimensional case. These properties are depicted in Fig. 3 and all phenomena observed can be argued in the same way as in the one-dimensional case.

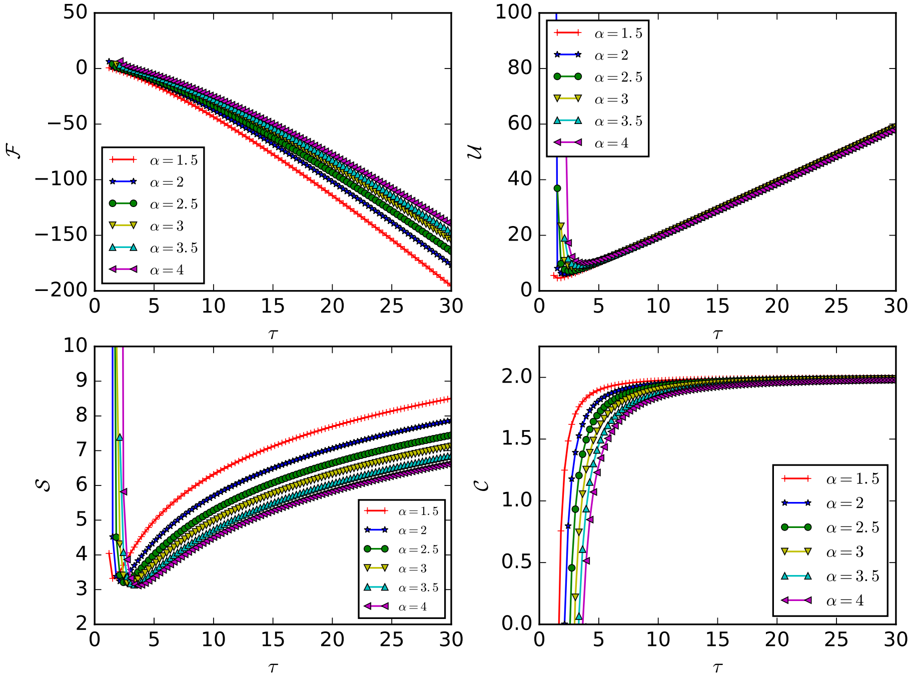


Fig. 3. Thermal properties of a two-dimensional Dirac oscillator for different values of $\alpha = \frac{\beta}{\beta_0}$.

4. Conclusion

In this work, we have study the influence of the minimal length on the thermal properties of the Dirac oscillator in one and two dimensions. The statistical quantities of both cases were investigated by employing the Epstein zeta function method. All this properties such as the free energy, the total energy, the entropy, and the specific heat, show the important effect of the presence of minimal length on the thermodynamics properties of the Dirac oscillator. Moreover, the formalism based on the properties of the zeta Epstein function allows us to calculate the values of minimal length $\Delta x = \hbar\sqrt{\beta}$ for some fermionic particles as shown in Table I. These values coincide well with the reduced Compton wavelength $\bar{\lambda}$ of these particles.

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