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EFFECT OF FINITE SPAN ON THE AIRLOAD DISTRIBUTIONS FOR OSCILLATING WINGS

I - AERODYNAMIC THEORY OF OSCILLATING WINGS OF FINITE SPAN

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SUMMARY

A formula is derived for the pressure distribution on an oscillating lifting surface of finite span under the assumption of a ratio of span to average chord (aspect ratio) that is not too small. The range of validity of this formula, so far as aspect-ratio limitations are concerned, is not less than the range of validity of lifting-line theory for the non-oscillating wing.

It is found that the effect of three-dimensionality of the flow may be incorporated in the results of the two-dimensional theory by adding a correction factor σ to the basic function $C(k)$ of the two-dimensional theory.

The correction term σ is a function that depends on wing plan form, wing deflection function, and reduced frequency k. Its determination requires the solution of an integral equation which is similar to the integral equation of lifting-line theory.

The present report concludes with an explicit statement of the form which the results of the theory assume for the spanwise variations of lift, total moment, and hinge moments on a wing which is oscillating in bending, torsion, and aileron and tab deflection.

Methods for the numerical evaluation of the results obtained as well as numerical applications to specific problems are given in part II of this report.

INTRODUCTION

The present report deals with the linear aerodynamic theory of oscillating airfoils of finite span. It contains the outcome of attempts to obtain simplifications and extensions of earlier results given in references 1 to 3. The guiding thought in the developments is to deduce, from a rigorous formulation of the problem of nearly plane lifting surfaces, simplified results in applicable form which depend on the assumption of sufficiently large aspect ratio. It is found that such results can be obtained and that their range of applicability, so far as aspect-ratio limitations are concerned, is at least as inclusive as that of lifting-line theory for the airfoil in uniform motion.

In reference 1 wings of rectangular plan form were considered and expressions were obtained for the spanwise distribution of lift and moment on the airfoil for arbitrary, small periodic displacements of the points of the wing surface. In reference 2 these results were extended to wings of arbitrary plan form. In reference 3 expressions were obtained, on the basis of results in references 1 and 2, for the spanwise variation of aileron hinge moment and tab hinge moment. Moreover, it was established that the effect of finite span could be incorporated into the results of the two-dimensional theory, as given in references 4 and 5, by a modification of the basic function $C = F + iG$ in an explicitly specified manner. The term to be added to the function C in order to account for the three-dimensionality of the problem was found to be the same for lift, moment, and hinge moments but to depend on the nature of the motion being dealt with. Determination of these three-dimensional correction terms requires the solution of an integral equation for the variation of the circulation along the span. The derivation of this integral equation is an important part of the work.

In this report the foregoing results are obtained in what appears at present to be the simplest possible way. In particular, a considerable reduction in the necessary analysis is accomplised by showing that the effect of finite span manifests itself in the expression for the chordwise pressure distribution solely by a modification of the function C. With this result, use may be made of the known formulas for lift and moments of the two-dimensional theory in order to establish the final results of the present theory without further integrations.

SYMBOLS

U	velocity of flight
x, y, z	Cartesian coordinates
$u+U, v, w$	components of fluid velocity
t	time

p	pressure
ρ	density
ϕ	velocity potential
γ, δ	discontinuities of u and v in the xy-plane
$x_l(y)$	coordinate of leading edge
$x_t(y)$	coordinate of trailing edge
b	semichord
b_o	semichord of midspan
s	ratio of span to chord at midspan
w_a	velocity component w at the airfoil
Z_a	instantaneous deflection of points of wing surface measured normal to xy-plane
R_a, R_w, R_r	regions in xy-plane (airfoil, wake, remaining)
ω	circular frequency
—	barring of terms, defined by equation (12)
Γ	circulation per unit of span
Ω	circulation function defined by equation (15)
K	function defined by equations (39a, b) and occurring as kernel of the integral equation (62)
y^*, η^*	dimensionless spanwise coordinates; $y^* = y/b_o$, $\eta^* = \eta/b_o$
z, ζ	dimensionless chordwise coordinates defined by equation (40b)
z_m	dimensionless coordinate of midchord line; $z_m=(x_l + x_t)/2b_o$
k	reduced frequency; $k = \omega b/U$
k_o	reduced frequency at midspan

k_m	$k_o\ z_m$
Q	function defined by equation (47)
Λ_1, Λ_2	functions defined by equations (52b) and (53b)
μ	function defined by equation (59)
$\overline{\Omega}^{(2)}$	two-dimensional circulation function given by equation (60)
λ	variable of integration
S	function defined by equation (72)
C(k)	function defined by Theodorsen
σ	correction term defined by equation (77)
a	location of elastic axis in units of semichord b
c	location of aileron leading edge in units of b
e	location of aileron hinge line in units of b
d	location of tab leading edge in units of b
f	location of tab hinge line in units of b
l, m	aileron and tab overhang in units of b; $l = e - c$, $m = f - d$
h	bending deflection of wings
α	angle of attack of wings
β, γ	aileron and tab deflection angles
L	lift per unit of span
M_a	moment about elastic axis per unit of span
M_β, M_γ	aileron and tab hinge moments per unit of span
F	function defined by equation (86)

FORMULATION OF THE PROBLEM

The wing considered is of arbitrarily given plan form and is placed in the path of a uniform incompressible air stream of velocity U. Small camber, thickness, angle of attack, and deformations are assumed for the wing which, for this reason, may be represented by a plane surface of discontinuity of pressure and tangential velocity in the fluid, parallel to the direction of U. In addition to the surface of discontinuity representing the wing itself, there is admitted the possibility of a surface of discontinuity of tangential velocity (but not of pressure) which extends downstream from the trailing edge of the wing and which is also taken to be parallel to U.

Let x and y be the coordinates in the plane of the surfaces of discontinuity, the direction of U determining the x-axis, and let z be the coordinate perpendicular to x and y. Let u + U, v, and w be the components of fluid velocity in the x, y, and z directions, respectively. Let p designate the pressure in the fluid and ρ the density. With the higher order terms in the velocities neglected, the equations of fluid motion and of continuity become

$$\left.\begin{aligned} &(a) \quad \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ &(b) \quad \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ &(c) \quad \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2)$$

Exterior to the surfaces of discontinuity the flow is without vorticity so that in terms of a velocity potential ϕ,

$$\begin{aligned}
&(a) \quad u = \frac{\partial \phi}{\partial x} \\
&(b) \quad v = \frac{\partial \phi}{\partial y} \\
&(c) \quad w = \frac{\partial \phi}{\partial z}
\end{aligned} \qquad (3)$$

From equations (3) and (1) an appropriate form of the Bernoulli equation follows for the pressure p,

$$-\frac{p}{\rho} = \frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} \qquad (4)$$

Let

$$\begin{aligned}
&(a) \quad \gamma = (\Delta u)_{z=0} = \left(\Delta \frac{\partial \phi}{\partial x}\right)_{z=0} \\
&(b) \quad \delta = (\Delta v)_{z=0} = \left(\Delta \frac{\partial \phi}{\partial y}\right)_{z=0}
\end{aligned} \qquad (5)$$

stand for the discontinuity distributions of the velocity components u and v over the surfaces of discontinuity in the xy-plane. According to the Biot-Savart theorem the velocity component w in the interior of the fluid is then given by the integral

$$w = \frac{-1}{4\pi}\iint \frac{\gamma(\xi, \eta, t)(x-\xi) + \delta(\xi, \eta, t)(y-\eta)}{\left\{(x-\xi)^2 + (y-\eta)^2 + z^2\right\}^{3/2}}\, d\xi\, d\eta \qquad (6)$$

From equations (5) it follows that γ and δ are related to each other in the form

$$\frac{\partial \gamma}{\partial y} = \frac{\partial \delta}{\partial x} \qquad (7)$$

From equations (4) and (5) it follows that the discontinuity of pressure Δp is given by

$$-\frac{\Delta p}{\rho} = \frac{\partial}{\partial t}\left(\int_{-\infty}^{x} \gamma \, dx'\right) + U\gamma \tag{8}$$

The problem of the oscillating lifting surface may now be formulated. Designate the coordinates of the leading and trailing edge of the wing by $x_l(y)$ and $x_t(y)$, respectively, with $x_l(0) = -b_o$ and $x_t(0) = b_o$, so that b_o represents the semichord at midspan. Let sb_o stand for the length of the semispan. Distinguish the following three regions in the xy-plane; (1) the airfoil region R_a, (2) the wake region R_w which is the strip of width $2sb_o$ extending downstream from the trailing edge, (3) the remaining region R_r. Then there is the condition of continuous flow outside the airfoil and wake regions, in R_r,

$$\gamma = \delta = 0 \tag{9}$$

and the condition of continuous pressure in the wake region R_w,

$$\Delta p = 0 \tag{10}$$

At the airfoil the condition of tangential flow is to be satisfied. If $Z_a(x, y, t)$ stands for the instantaneous deflection of points of the wing surface measured normal to the xy-plane, this condition in the region R_a is

$$w_a = w(x, y, o, t) = \frac{\partial Z_a}{\partial t} + U\frac{\partial Z_a}{\partial x} \tag{11}$$

By introducing equations (9) to (11) in the integral representation for w, equation (6), and taking account of equations (7) and (8), there may be obtained the general integral equation of lifting-surface theory for the distribution of the velocity discontinuities γ_a and δ_a over the airfoil region. With the solution of this integral equation,

equation (8) furnishes the chordwise pressure distribution Δp_a. Appropriate integration of Δp_a results in expressions for spanwise lift, moment, and aileron and tab hinge-moment distributions.

Further development is here based on the assumption of simply harmonic motion,

$$Z_a(x, y, t) = \bar{Z}_a(x, y)e^{i\omega t} \tag{12}$$

As a consequence of the linearity of the theory, it follows also that velocities and pressure become products of $e^{i\omega t}$ and of amplitude functions independent of time which are designated by bars.

The first step in the deduction of the final form of the integral equation of the problem consists in the determination of the values of γ and δ in the wake region, by means of equations (8), (7), (9), and (10). Equation (8) takes on the form

$$0 = i\omega \left(\int_{x_l}^{x_t} \bar{\gamma}_a \, dx' + \int_{x_t}^{x} \bar{\gamma}_w \, dx' \right) + U\bar{\gamma}_w \tag{13}$$

If the circulation Γ is defined by

$$\bar{\Gamma} = \int_{x_l}^{x_t} \bar{\gamma}_a \, dx' \tag{14}$$

and the circulation function Ω by

$$\bar{\Omega} = \frac{1}{b_o} \bar{\Gamma} e^{i \frac{\omega}{U} x_t} \tag{15}$$

it follows from equation (8) and its differentiated form that γ_w is given by

$$\bar{\gamma}_w = -i b_o \frac{\omega}{U} \bar{\Omega} e^{-i \frac{\omega}{U} x} \tag{16}$$

When γ_w of equation (16) is substituted in equation (7), there is obtained for δ_w,

$$\bar{\delta}_w = \int_{-\infty}^{x} \frac{\partial \bar{\gamma}}{\partial y} dx' = b_o \frac{d\bar{\Omega}}{dy} e^{-i \frac{\omega}{U} x} \tag{17}$$

Equations (17), (16), and (11) are now introduced in the integral representation for w, equation (6), in which the coordinate z is made to approach zero. As was shown in reference 1, the two limit processes of integration and of letting z approach zero may be interchanged if the Cauchy principal value of the integrals is taken at the points $\xi = x$, $\eta = y$ for which the integrand becomes infinite. There results

$$\bar{w}_a(x, y) = \frac{-1}{4\pi} \iint_{R_a} \frac{\bar{\gamma}_a(\xi, \eta)(x - \xi) + \bar{\delta}_a(\xi, \eta)(y - \eta)}{\left[(x - \xi)^2 + (y - \eta)^2\right]^{3/2}} d\xi\, d\eta$$

$$- \frac{b_o}{4\pi} \iint_{R_w} e^{-i \frac{\omega \xi}{U}} \frac{-i \frac{\omega}{U} \bar{\Omega}(\eta)(x - \xi) + \frac{d\bar{\Omega}}{d\eta}(y - \eta)}{\left[(x - \xi)^2 + (y - \eta)^2\right]^{3/2}} d\xi\, d\eta \tag{18}$$

Equation (18) holds for all points x, y inside R_a. Its left-hand side is the given function

$$\bar{w}_a = i\omega \bar{Z}_a + U \frac{\partial \bar{Z}_a}{\partial x} \tag{19}$$

and, according to equations (7) and (9), $\bar{\delta}_a$ is expressed in terms of $\bar{\gamma}_a$ by means of the formula,

$$\bar{\delta}_a = \int_{x_l}^{x} \frac{\partial \bar{\gamma}_a}{\partial y} \, dx' \tag{20}$$

In addition, there is imposed the Kutta-Joukowsky condition stating that $\bar{\gamma}$ and $\bar{\delta}$ are finite along the trailing edge of the wing. The amplitude of the chordwise pressure distribution $\Delta \bar{p}_a$ in terms of the function $\bar{\gamma}_a$, for which equation (18) is to be solved, is given by

$$-\frac{\Delta \bar{p}_a}{\rho} = i\omega \int_{x_l}^{x} \bar{\gamma}_a \, dx' + U \bar{\gamma}_a \tag{21}$$

The general problem is the solution of equations (18) to (20) for arbitrary shape (plan form) of the airfoil region R_a. Results equivalent to this solution are known for the following two special cases: (1) the two-dimensional problem to which the general problem reduces when the region R_a is the strip $|x| \leq b = b_o$ and w_a of equation (19) is independent of y (see for instance reference 5), and (2) the problem of the wing with circular plan form (references 6 and 7).

The purpose of the remainder of this report is to obtain approximate solutions of the problem which are applicable subject only to the restriction that the airfoil region R_a is of sufficiently large aspect ratio, a restriction that is roughly equivalent to requiring that the ratio s of span to chord at midspan is sufficiently large. The nature of these results is, presumably, that of the dominant terms of an asymptotic development of the exact solution in terms of the parameter s. However, no investigation of this aspect of the problem is made in the present report.

Concerning the range of validity of the results as far as aspect-ratio limitations are concerned the statement may be made, in view of the nature of the analysis, that the results obtained in this report for oscillating wings are certainly applicable for all wings for which lifting-line theory is considered applicable in the case of uniform motion. There is some reason to believe that the results for oscillating wings may have a somewhat wider range of validity than lifting-line theory, as the chordwise waves which occur in the vibration problem may be responsible for a reduction of the effective chord length while at the same time having no influence on the effective span length.

DERIVATION OF SIMPLIFIED INTEGRAL EQUATION OF LIFTING-SURFACE THEORY

Equation (18) may be written in the form

$$x_l \leq x \leq x_t, \quad -sb_o \leq y \leq sb_o; \quad \bar{w}_a = I_1 + I_2 \tag{22}$$

where

$$\text{(a)} \quad I_1 = \frac{-1}{4\pi} \int_{-sb_o}^{sb_o} \int_{x_l(\eta)}^{x_t(\eta)} \frac{\bar{\gamma}_a(\xi, \eta)(x - \xi) + \bar{\delta}_a(\xi, \eta)(y - \eta)}{\left[(x - \xi)^2 + (y - \eta)^2\right]^{3/2}} d\xi \, d\eta$$

$$\text{(b)} \quad I_2 = \frac{-b_o}{4\pi} \int_{-sb_o}^{sb_o} \int_{x_t(\eta)}^{\infty} e^{-i\frac{\omega}{U}\xi} \frac{-i\frac{\omega}{U}\bar{\Omega}(\eta)(x - \xi) + \bar{\Omega}'(\eta)(y - \eta)}{\left[(x - \xi)^2 + (y - \eta)^2\right]^{3/2}} d\xi \, d\eta \tag{23}$$

and where, from equation (20),

$$\bar{\delta}_a(\xi, \eta) = \int_{x_l(\eta)}^{\xi} \frac{\partial \bar{\gamma}_a(\xi', \eta)}{\partial \eta} d\xi'$$

The decisive step in the present solution of the problem as expressed by equations (22) and (23) is the reduction of the two-dimensional integral equation to 2 one-dimensional integral equations which can be solved successively. For this reduction it is assumed that the airfoil region is sufficiently elongated in span direction, that the rate of taper is moderate, and that the velocities do not vary appreciably along the span over distances which are a fraction of the mean chord. With these assumptions it may be postulated that at every spanwise section the contribution to the normal velocity w_a induced by the tangential velocities <u>at the airfoil</u> is approximately as if every spanwise section were part of a wing without taper and as if the flow were two-dimensional. Equation (23a) then becomes

$$I_1 \approx \frac{-1}{4\pi} \oint_{-\infty}^{\infty} \oint_{x_l(y)}^{x_t(y)} \frac{\bar{\gamma}_a(\xi, y)(x-\xi)\,d\xi\,d\eta}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}$$

$$= \frac{-1}{2\pi} \oint_{x_l(y)}^{x_t(y)} \frac{\bar{\gamma}_a(\xi, y)}{x-\xi}\,d\xi = \tilde{I}_1 \qquad (24)$$

A corresponding approximation $\tilde{I}_2$ for the integral I_2 is obtained by a less direct procedure. Write

$$I_2 = \frac{-b_o}{4\pi}\left(I_3 - i\frac{\omega}{U} I_4\right) \qquad (25)$$

where

$$I_3 = \int_{-sb_o}^{sb_o} \int_{x_t(\eta)}^{\infty} \frac{e^{-i\frac{\omega}{U}\xi}\,\bar{\Omega}'(\eta)(y-\eta)}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}\,d\xi\,d\eta \qquad (26)$$

$$I_4 = \int_{-sb_o}^{sb_o} \int_{x_t(\eta)}^{\infty} \frac{e^{-i\frac{\omega}{U}\xi}\,\bar{\Omega}(\eta)(x-\xi)}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}\,d\xi\,d\eta \qquad (27)$$

Write next

$$I_3 = \oint_{-sb_o}^{sb_o} \oint_{x}^{\infty} - \oint_{-sb_o}^{sb_o} \oint_{x}^{x_t(\eta)} \qquad (28)$$

In the integral extended from x to $x_t(\eta)$, that is, over part of the airfoil region, the same approximation can be made as in the expression for I_1. If this is done, its contribution to I_3 vanishes and

$$I_3 \approx \int_{-sb_o}^{sb_o} \int_{x}^{\infty} \frac{e^{-i\frac{\omega}{U}\xi}\, \bar{\Omega}'(\eta)(y-\eta)}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{3/2}} d\xi\, d\eta$$

With a new variable of integration $\lambda = \xi - x$

$$I_3 \approx e^{-i\frac{\omega}{U}x} \int_{-sb_o}^{sb_o} \frac{d\bar{\Omega}}{d\eta} \left\{ \int_{0}^{\infty} \frac{e^{-i\frac{\omega}{U}\lambda}(y-\eta)d\lambda}{\left[\lambda^2 + (y-\eta)^2\right]^{3/2}} \right\} d\eta \tag{29}$$

In order to approximate the remaining integral I_4 this integral is separated into two parts, one part which equals the value of I_4 in the two-dimensional theory and the other part being the difference between the two-dimensional and the three-dimensional value of I_4,

$$I_4 = I_4^{(2)} + \Delta I_4 \tag{30}$$

$$I_4^{(2)} = \bar{\Omega}(y) \int_{-\infty}^{\infty} \int_{x_t(y)}^{\infty} \frac{e^{-i\frac{\omega}{U}\xi}(x-\xi)d\xi\, d\eta}{\left[(x-\xi)^2 + (y-\eta)^2\right]^{3/2}}$$

$$= 2\,\bar{\Omega}(y) \int_{x_t(y)}^{\infty} \frac{e^{-i\frac{\omega}{U}\xi}}{(x-\xi)} d\xi \tag{31}$$

The factor $\bar{\Omega}(y)$ may be written in the alternate form

$$\bar{\Omega}(y) = \frac{1}{2} \int_{-sb_o}^{sb_o} \bar{\Omega}'(\eta) \frac{|y - \eta|}{y - \eta} d\eta \tag{32}$$

if account is taken of the fact that $\bar{\Omega}(\pm sb_o) = 0$.

If equations (31) and (32) are combined, the value of $I_4^{(2)}$ may be written

$$I_4^{(2)} = \int_{-sb_o}^{sb_o} \int_{x_t(y)}^{\infty} e^{-i \frac{\omega}{U} \xi} \frac{\bar{\Omega}'(\eta) \, |y - \eta|}{(x - \xi)(y - \eta)} d\xi \, d\eta \tag{33}$$

The next step is transformation of I_4 by integration by parts with respect to η. Before this step is carried out it is noted that it is consistent with the preceding approximations to replace in equation (27) the limit of integration $x_t(\eta)$ by $x_t(y)$. Then

$$I_4 \approx \int_{x_t(y)}^{\infty} e^{-i \frac{\omega}{U} \xi} (x - \xi) \left\{ \int_{-sb_o}^{sb_o} \frac{\bar{\Omega}(\eta) \, d\eta}{\left[(x - \xi)^2 + (y - \eta)^2\right]^{3/2}} \right\} d\xi$$

$$= \int_{x_t(y)}^{\infty} e^{-i \frac{\omega}{U} \xi} (x - \xi)$$

$$\times \left[\int_{-sb_o}^{sb_o} \frac{\bar{\Omega}'(\eta)(y - \eta) \, d\eta}{(x - \xi)^2 \sqrt{(x - \xi)^2 + (y - \eta)^2}} \right] d\xi \tag{34}$$

as the integrated portion of the inner integral vanishes because

$$\bar{\Omega}\ (\pm\ sb_o) = 0$$

By combining equations (34) and (33) there follows

$$I_4 - I_4^{(2)} = \Delta I_4 \approx \int_{-sb_o}^{sb_o} \int_{x_t(y)}^{\infty} \frac{\bar{\Omega}'(\eta)\ e^{-i\frac{\omega}{U}\xi}}{x - \xi}$$

$$\times \left[\frac{y - \eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} - \frac{|y - \eta|}{y - \eta} \right] d\xi\, d\eta \qquad (35)$$

Equation (35) for ΔI_4 may again be separated into two integrals, in the same manner as I_3 of equation (28). If this separation is made, the integral between the limits x and $x_t(y)$ may again be neglected with the result that

$$\Delta I_4 \approx \int_{-sb_o}^{sb_o} \int_{x}^{\infty} e^{-i\frac{\omega}{U}\xi} \frac{\bar{\Omega}'(\eta)}{x - \xi}$$

$$\times \left[\frac{y - \eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} - \frac{|y - \eta|}{y - \eta} \right] d\xi\, d\eta$$

and with $\lambda = \xi - x$

$$\Delta I_4 \approx e^{-i\frac{\omega}{U}x} \int_{-sb_o}^{sb_o} \bar{\Omega}'(\eta) \left[- \int_0^{\infty} \frac{1}{\lambda} \left(\frac{y - \eta}{\sqrt{\lambda^2 + (y - \eta)^2}} - \frac{|y - \eta|}{y - \eta} \right) e^{-i\frac{\omega}{U}\lambda} d\lambda \right] d\eta \qquad (36)$$

Now, by combining equations (36) and (31), there follows:

$$I_4 \approx 2\bar{\Omega}(y)\int_{x_t(y)}^{\infty} \frac{e^{-i\frac{\omega}{U}\xi}}{x-\xi}\,d\xi - e^{-i\frac{\omega}{U}x}\oint_{-sb_o}^{sb_o} \bar{\Omega}'(\eta)$$

$$\times\left[\int_0^{\infty} \frac{1}{\lambda}\left(\frac{y-\eta}{\sqrt{\lambda^2+(y-\eta)^2}} - \frac{|y-\eta|}{y-\eta}\right) e^{-i\frac{\omega}{U}\lambda}\,d\lambda\right] d\eta$$

$$= \tilde{I}_4 \qquad (37)$$

From combination of equations (37), (29), and (25) there follows finally as an approximate value of the wake integral I_2,

$$\tilde{I}_2 = i\,\frac{\omega b_o}{U}\,\frac{\bar{\Omega}(y)}{2\pi}\int_{x_t(y)}^{\infty} \frac{e^{-i\frac{\omega}{U}\xi}}{(x-\xi)}\,d\xi$$

$$-\frac{e^{-i\frac{\omega}{U}x}}{4\pi}\oint_{-sb_o}^{sb_o} \frac{d\bar{\Omega}}{d\eta}\,K\left[\frac{\omega}{U}(y-\eta)\right] d\eta \qquad (38)$$

where

$$K(x) = \frac{\omega b_o}{U}\int_0^{\infty} \frac{e^{-i\lambda}\,x}{[\lambda^2+x^2]^{3/2}}\,d\lambda$$

$$+ i\,\frac{\omega b_o}{U}\int_0^{\infty} \frac{e^{-i\lambda}}{\lambda}\left(\frac{x}{\sqrt{\lambda^2+x^2}} - \frac{|x|}{x}\right) d\lambda \qquad (39a)$$

For the applications it is convenient to transform K(x) by an integration by parts of the first integral as follows:

$$\int_0^\infty \frac{e^{-i\lambda}\,d\lambda}{(\lambda^2 + x^2)^{3/2}} = e^{-i\lambda}\left[\frac{\lambda}{x^2\sqrt{\lambda^2 + x^2}} + c(x)\right]_0^\infty$$

$$+ i\int_0^\infty e^{-i\lambda}\left(\frac{\lambda}{x^2\sqrt{\lambda^2 + x^2}} + c\right)d\lambda$$

The right side assumes a definite value only if

$$c(x) = -x^{-2}$$

and then

$$\int_0^\infty \frac{e^{-i\lambda}\,d\lambda}{(\lambda^2 + x^2)^{3/2}} = \frac{1}{x^2} + i\int_0^\infty \frac{e^{-i\lambda}}{x^2}\left(\frac{\lambda}{\sqrt{\lambda^2 + x^2}} - 1\right)d\lambda$$

By introducing this formula in equation (39a) there follows:

$$K(x) = \frac{\omega b_o}{U}\frac{1}{x} - i\,\frac{\omega b_o}{U}\frac{1}{x}\int_0^\infty e^{-i\lambda}\left(1 - \frac{\sqrt{\lambda^2 + x^2} - |x|}{\lambda}\right)d\lambda \qquad (39b)$$

Combination of equations (38), (24), and (22) results in the approximate integral equation of lifting-surface theory, which is the basic result of the present work. Because the establishment of this equation depends on the condition of a ratio of wing span to average wing chord that is not too small (so that the wing plan form has the appearance of a strip), it might be considered to call this equation the integral equation of lifting-strip theory to distinguish it from the general equation of lifting-surface theory on the one hand, and from lifting-line theory for the stationary sufficiently elongated wing on the other hand. That it is not possible to speak of a lifting-line theory for the oscillating wing follows from the fact that according to equation (38) and (22) the effect of spanwise variation of circulation is not uniform across the chord but varies in a manner depending on the frequency ω.

In order to obtain a normalized form of the equations of the problem, the following dimensionless variables may be introduced:

$$\text{(a)} \qquad y^* = y/b_o$$

$$\text{(b)} \qquad z = \frac{x - \frac{1}{2}(x_t + x_l)}{\frac{1}{2}(x_t - x_l)} \tag{40}$$

As the space coordinate z does no longer occur, from equation (18) on, introduction of a dimensionless variable z at this stage will give no rise to doubts as to its meaning in the subsequent developments. Write as abbreviations

$$\text{(a)} \qquad \frac{1}{2}(x_t - x_l) = b$$

$$\text{(b)} \qquad \frac{1}{2b_o}(x_t + x_l) = z_m \tag{41}$$

The quantity z_m is a measure of the local sweep of the midchord line. The quantity b stands for the magnitude of the local chord so that $b(o) = b_o$. With these symbols equation (40b) becomes

$$x = b_o z_m + bz \tag{42}$$

Also let

$$k = \frac{\omega b}{U} \qquad k_o = \frac{\omega b_o}{U} \qquad k_m = k_o z_m \tag{43}$$

where k is the local reduced frequency and k_o the reduced frequency at midspan.

Upon introducing equations (40) to (43) in equations (38), (24), and (22) there follows as the normalized form of the integral equation to be solved

$$-\bar{w}_a(z, y^*) = \frac{1}{2\pi} \oint_{-1}^{1} \frac{\bar{\gamma}_a(\zeta, y^*)}{z - \zeta} d\zeta$$

$$- \frac{ik_o}{2\pi} e^{-ik_m} \bar{\Omega}(y^*) \int_1^{\infty} \frac{e^{-ik\zeta}}{z - \zeta} d\zeta$$

$$+ \frac{e^{-ikz}}{4\pi} e^{-ik_m} \oint_{-s}^{s} \frac{d\bar{\Omega}}{d\eta^*} K\left[k_o(y^* - \eta^*) \right] d\eta^* \qquad (44)$$

Equation (44) has first been established in reference 2. For the wing of rectangular plan form $(k = k_o, \; k_m = 0)$ it reduces to an equation given in reference 1.

The problem from here on is to solve equation (44) for the function $\bar{\gamma}_a$, to obtain an equation for the spanwise variation of Ω, and to express the chordwise pressure distribution in terms of $\bar{w}_a$.

In terms of the variable z as defined by equation (42) the function $\bar{\Omega}$ of equations (14) and (15) is to be written as

$$\bar{\Omega}(y^*) = \frac{b}{b_o} e^{i(k+k_m)} \int_{-1}^{1} \bar{\gamma}_a(\zeta, y^*) d\zeta \qquad (45)$$

and equation (21) for $\Delta\bar{p}_a$ becomes

$$-\frac{\Delta\bar{p}_a}{\rho U} = ik \int_{-1}^{z} \bar{\gamma}_a \, dz' + \bar{\gamma}_a(z, y^*) \qquad (46)$$

To be established is the manner in which the presence of the term containing $d\bar{\Omega}/d\eta^*$ in equation (44) modifies the result of the two-dimensional theory for $\Delta\bar{p}_a$.

CALCULATION OF PRESSURE DISTRIBUTION

The next step in the analysis is the determination of $\bar{\gamma}_a$ from equation (44) in which the following abbreviation is inserted:

$$\bar{Q} = \frac{1}{2} \oint_{-s}^{s} \frac{d\bar{\Omega}}{d\eta^*} K \left[k_o(y^* - \eta^*)\right] d\eta^* \tag{47}$$

Use is made of the following pair of formulas of integral-equation theory:

$$\begin{aligned} g(z) &= \frac{1}{2\pi} \oint_{-1}^{1} \frac{f(\zeta)}{z - \zeta} \, d\zeta, \quad f(1) \text{ finite} \\ f(z) &= -\frac{2}{\pi} \sqrt{\frac{1 - z}{1 + z}} \oint_{-1}^{1} \sqrt{\frac{1 + \zeta}{1 - \zeta}} \frac{g(\zeta)}{z - \zeta} \, d\zeta \end{aligned} \tag{48}$$

It may be noted that a derivation of the results of the two-dimensional theory in this manner has been given in reference 8.

By applying equations (48) to equation (44) there follows,

$$\bar{\gamma}_a = -\frac{2}{\pi}\sqrt{\frac{1-z}{1+z}}\left[-\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\,\frac{\bar{w}_a(\zeta, y^*)}{z-\zeta}\,d\zeta\right.$$

$$+\frac{ik_o}{2\pi}\,\bar{\Omega}\,e^{-ik_m}\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\left(\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\zeta-\lambda}\,d\lambda\right)\frac{d\zeta}{z-\zeta}$$

$$\left.-\frac{\bar{Q}}{2\pi}\,e^{-ik_m}\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\,\frac{e^{-ik_\zeta}}{z-\zeta}\,d\zeta\right] \tag{49}$$

The double integral in equation (49) is reduced to a single integral as follows:

$$\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\left(\int_{1}^{\infty}\frac{e^{-ik\lambda}}{\zeta-\lambda}\,d\lambda\right)\frac{d\zeta}{z-\zeta}$$

$$=\int_{1}^{\infty}e^{-ik\lambda}\left[\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\,\frac{d\zeta}{(\zeta-\lambda)(z-\zeta)}\right]d\lambda$$

$$=\int_{1}^{\infty}e^{-ik\lambda}\left[\frac{1}{z-\lambda}\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\left(\frac{1}{\zeta-\lambda}+\frac{1}{z-\zeta}\right)d\zeta\right]d\lambda$$

$$=\int_{1}^{\infty}\frac{e^{-ik\lambda}}{z-\lambda}\left[\pi\left(1-\sqrt{\frac{\lambda+1}{\lambda-1}}\right)-\pi\right]d\lambda \tag{50}$$

By introducing equation (50) in equation (49),

$$\bar{\gamma}_a = -\frac{2}{\pi}\sqrt{\frac{1-z}{1+z}}\left(-\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{\bar{w}_a}{z-\zeta}\,d\zeta\right.$$

$$-\frac{1}{2}ik_o\bar{\Omega}e^{-ik_m}\int_{1}^{\infty}\sqrt{\frac{\lambda+1}{\lambda-1}}\frac{e^{-ik\lambda}}{z-\lambda}\,d\lambda$$

$$\left.-\frac{\bar{Q}}{2\pi}e^{-ik_m}\oint_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{e^{-ik\zeta}}{z-\zeta}\,d\zeta\right) \tag{51}$$

According to equations (46) and (45) it is necessary to integrate equation (51) between the limits -1 and z, with z' as the variable of integration. By interchanging the order of integration with respect to z', ζ, and λ, respectively, the following integrals occur (see reference 8):

$$\oint_{-1}^{z}\sqrt{\frac{1-z'}{1+z'}}\frac{dz'}{\zeta-z'} = \frac{\pi}{2} + \sin^{-1} z + \sqrt{\frac{1-\zeta}{1+\zeta}}\Lambda_1(z, \zeta) \tag{52a}$$

$$\int_{-1}^{z}\sqrt{\frac{1-z'}{1+z'}}\frac{dz'}{\lambda-z'} = \frac{\pi}{2} + \sin^{-1} z + \sqrt{\frac{\lambda-1}{\lambda+1}}\Lambda_2(z, \lambda) \tag{53a}$$

The functions Λ_n are defined by

$$\Lambda_1(z, \zeta) = \frac{1}{2}\ln\frac{1-z\zeta+\sqrt{1-\zeta^2}\sqrt{1-z^2}}{1-z\zeta-\sqrt{1-\zeta^2}\sqrt{1-z^2}} \tag{52b}$$

$$\Lambda_2(z,\lambda) = 2\tan^{-1}\left(\sqrt{\frac{(1-z)(\lambda+1)}{(1+z)(\lambda-1)}}\right) - \pi \tag{53b}$$

The following formulas will also be needed:

$$\frac{\partial \Lambda_1}{\partial \zeta} = \frac{\sqrt{1 - z^2}}{\sqrt{1 - \zeta^2}} \frac{1}{z - \zeta} = \sqrt{\frac{1 - z}{1 + z}} \left(\frac{1}{\sqrt{1 - \zeta^2}} + \sqrt{\frac{1 + \zeta}{1 - \zeta}} \frac{1}{z - \zeta} \right) \tag{52c}$$

$$\frac{\partial \Lambda_2}{\partial \lambda} = \sqrt{\frac{1 - z}{1 + z}} \left(\frac{1}{\sqrt{\lambda^2 - 1}} + \sqrt{\frac{\lambda + 1}{\lambda - 1}} \frac{1}{z - \lambda} \right) \tag{53c}$$

With equations (52a) and (53a) there follows from equation (51),

$$\int_{-1}^{z} \bar{\gamma}_a \, dz' = \frac{-2}{\pi} \int_{-1}^{1} \left[\sqrt{\frac{1 + \zeta}{1 - \zeta}} \left(\frac{\pi}{2} + \sin^{-1} z \right) + \Lambda_1 \right] \bar{w}_a \, d\zeta$$

$$- \frac{ik_o}{\pi} \bar{\Omega} \, e^{-ik_m} \int_{1}^{\infty} \left[\sqrt{\frac{\lambda + 1}{\lambda - 1}} \left(\frac{\pi}{2} + \sin^{-1} z \right) + \Lambda_2 \right] e^{-ik\lambda} \, d\lambda$$

$$- \frac{\bar{Q}}{\pi^2} e^{-ik_m} \int_{-1}^{1} \left[\sqrt{\frac{1 + \zeta}{1 - \zeta}} \left(\frac{\pi}{2} + \sin^{-1} z \right) + \Lambda_1 \right] e^{-ik\zeta} \, d\zeta \tag{54}$$

Equation (54) leads immediately to the integral equation for the spanwise variation of $\bar{\Omega}$. Let $z = 1$ and there follows, in view of equation (45) and in view of the fact that $\Lambda_1(1, \zeta) = 0$ and $\Lambda_2(1, \lambda) = -\pi$,

$$\frac{b_o}{b} e^{-i(k+k_m)} \bar{\Omega} (y^*) = -2 \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}} \bar{w}_a \, d\zeta$$

$$-ik_o \bar{\Omega} e^{-ik_m} \int_{1}^{\infty} \left(\sqrt{\frac{\lambda+1}{\lambda-1}} - 1 \right) e^{-ik\lambda} \, d\lambda$$

$$- \frac{\bar{Q}}{\pi} e^{-ik_m} \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}} e^{-ik\zeta} \, d\zeta \tag{55}$$

Equation (55) can be simplified by means of the following relations of the theory of Bessel functions:

(a)

$$\int_{1}^{\infty} \left(\sqrt{\frac{\lambda+1}{\lambda-1}} - 1 \right) e^{-ik\lambda} \, d\lambda$$

$$= -\frac{\pi}{2} \left(H_1^{(2)} (k) + iH_o^{(2)} (k) \right) - \frac{e^{-ik}}{ik} \tag{56}$$

(b)

$$\int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}} e^{-ik\zeta} \, d\zeta = \pi \left(J_o(k) - iJ_1 (k) \right)$$

Hence

$$\bar{\Omega} = -2 \frac{b}{b_o} e^{i(k+k_m)} \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}} \bar{w}_a \, d\zeta$$

$$+ ik\bar{\Omega} e^{ik} \left\{ \frac{\pi}{2} \left[H_1^{(2)} + i \, H_o^{(2)} \right] + \frac{e^{-ik}}{ik} \right\} - \frac{b}{b_o} \bar{Q} e^{ik} \left(J_o - iJ_1 \right) \tag{57}$$

and by canceling two of the $\bar{\Omega}$-terms against each other

$$\bar{\Omega} - \frac{J_0 - iJ_1}{\frac{1}{2}\pi ik\left[H_1^{(2)} + iH_0^{(2)}\right]} \frac{b}{b_0} \bar{Q} = \frac{2b}{b_0} \frac{e^{ik_m}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\bar{w}_a d\zeta}{\frac{1}{2}\pi ik\left[H_1^{(2)} + iH_0^{(2)}\right]} \tag{58}$$

Define now a function μ by the relation

$$\mu = -\frac{J_0 - iJ_1}{\pi ik\left[H_1^{(2)} + iH_0^{(2)}\right]} = \frac{J_0 - iJ_1}{\pi k\left[(J_0 - Y_1) - i\,(J_1 + Y_0)\right]} \tag{59}$$

and write as an abbreviation

$$\bar{\Omega}^{(2)} = 4\frac{b}{b_0} e^{ik_m} \frac{\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\bar{w}_a\, d\zeta}{\pi ik\left[H_1^{(2)} + iH_0^{(2)}\right]} \tag{60}$$

Equation (58) then assumes the form

$$\bar{\Omega}(y^*) + \mu(k)\frac{b}{b_0} 2\bar{Q}(y^*) = \bar{\Omega}^{(2)}(y^*) \tag{61}$$

Equation (61) is the integral equation for the spanwise variation of circulation. The function $\bar{\Omega}^{(2)}$ is the distribution of $\bar{\Omega}$ according to the two-dimensional theory. The $\bar{Q}$-term represents the influence of finite span. With equation (47) for $\bar{Q}$ there may be written

$$\bar{\Omega}\,(y^*) + \mu(k)\,\frac{b}{b_o}\oint_{-s}^{s}\frac{d\bar{\Omega}}{d\eta^*}\,K[\,k_o(y^* - \eta^*)]\;d\eta^* = \bar{\Omega}^{(a)}\,(y^*) \qquad (62)$$

The next step in the analysis is to subtract equation (55) multiplied by $\frac{1}{\pi}\left(\frac{\pi}{2} + \sin^{-1} z\right)$ from equation (54):

$$\int_{-1}^{z}\bar{\gamma}_a\,dz' - \frac{1}{\pi}\left(\frac{\pi}{2} + \sin^{-1} z\right)\frac{b}{b_o}\,e^{-i(k+k_m)}\,\bar{\Omega}$$

$$= \frac{2}{\pi}\int_{-1}^{1}\Lambda_1\,(z,\ \zeta\,)\bar{w}_a\,d\,\zeta$$

$$- \frac{ik_o}{\pi}\,\bar{\Omega}\,e^{-ik_m}\int_{1}^{\infty}\left(\frac{\pi}{2} + \sin^{-1}z + \Lambda_2\right)\,e^{-ik\lambda}\,d\lambda$$

$$- \frac{\bar{Q}}{\pi^2}\,e^{-ik_m}\int_{-1}^{1}\Lambda_1\,(z,\,\zeta\,)e^{-ik\zeta}\,d\,\zeta \qquad (63)$$

The second and third integral in the right-hand side of equation (63) are now integrated by parts,

$$\int_{-1}^{z} \bar{\gamma}_a \, dz' - \frac{1}{\pi}\left(\frac{\pi}{2} + \sin^{-1} z\right) \frac{b_o}{b} e^{-i(k+k_m)} \bar{\Omega}$$

$$= \frac{2}{\pi} \int_{-1}^{1} \Lambda_1 \bar{w}_a \, d\zeta - \frac{ik_o}{\pi} \bar{\Omega} e^{-ik_m}$$

$$\times \left\{ \left[\frac{\pi}{2} + \sin^{-1} z + \Lambda_2 (z, \lambda) \right] \frac{e^{-ik\lambda}}{-ik} \right\}_1^{\infty}$$

$$- \frac{ik_o}{\pi} \bar{\Omega} e^{-ik_m} \int_1^{\infty} \frac{\partial \Lambda_2}{\partial \lambda} \frac{e^{-ik\lambda}}{-ik} d\lambda$$

$$- \frac{\bar{Q}}{\pi^2} e^{-ik_m} \left(\Lambda_1 \frac{e^{-ik\zeta}}{-ik} \Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{\partial \Lambda_1}{\partial \zeta} \frac{e^{-ik\zeta}}{-ik} d\zeta \right) \tag{64}$$

According to equations (52b) and (53b),

$$\left. \begin{aligned} \Lambda_1 (z, \pm 1) = 0 \qquad & \Lambda_2 (z, 1) = 0 \\ \Lambda_2 (z, \infty) = - & \left(\frac{\pi}{2} + \sin^{-1} z \right) \end{aligned} \right\} \tag{65}$$

The third of these relations is verified by means of the identity

$$2 \tan^{-1} \sqrt{\frac{1 - z}{1 + z}} = \frac{\pi}{2} - \sin^{-1} z$$

By means of equation (65), and by noting that $b_o/b = k_o/k$, it is seen that the integrated terms in equation (64) cancel and

$$\int_{-1}^{z} \bar{\gamma}_a \, dz' = \frac{2}{\pi} \int_{-1}^{1} \Lambda_1 \bar{w}_a \, d\zeta - \frac{k_o}{k} \frac{\bar{\Omega}}{\pi} e^{-ik_m} \int_{1}^{\infty} \frac{\partial \Lambda_2}{\partial \lambda} e^{-ik\lambda} \, d\lambda$$

$$- \frac{1}{ik} \frac{\bar{Q}}{\pi^2} e^{-ik_m} \int_{-1}^{1} \frac{\partial \Lambda_1}{\partial \zeta} e^{-ik\zeta} \, d\zeta \tag{66}$$

Now by combining equations (66) and (51), as in equation (46), there follows for the pressure distributions on the airfoil

$$- \frac{\Delta \bar{p}_a}{\rho U} = - \frac{2}{\pi} \oint_{-1}^{1} \left(\sqrt{\frac{1 - z}{1 + z}} \sqrt{\frac{1 + \zeta}{1 - \zeta}} \frac{1}{z - \zeta} - ik\Lambda_1 \right) \bar{w}_a \, d\zeta$$

$$+ \frac{ik_o}{\pi} \bar{\Omega} e^{-ik_m} \int_{1}^{\infty} \left(\sqrt{\frac{1 - z}{1 + z}} \sqrt{\frac{\lambda + 1}{\lambda - 1}} \frac{1}{z - \lambda} - \frac{\partial \Lambda_2}{\partial \lambda} \right) e^{-ik\lambda} \, d\lambda$$

$$+ \frac{1}{\pi^2} \bar{Q} e^{-ik_m} \oint_{-1}^{1} \left(\sqrt{\frac{1 - z}{1 + z}} \sqrt{\frac{1 + \zeta}{1 - \zeta}} \frac{1}{z - \zeta} - \frac{\partial \Lambda_1}{\partial \zeta} \right) e^{-ik\zeta} \, d\zeta$$

$$\tag{67}$$

Equation (67) simplifies considerably when equations (52c) and (53c) are taken into account. There follows

$$\frac{\Delta \bar{p}_a}{\rho U} = \frac{2}{\pi} \int_{-1}^{1} \left(\sqrt{\frac{1 - z}{1 + z}} \sqrt{\frac{1 + \zeta}{1 - \zeta}} \frac{1}{z - \zeta} - ik\Lambda_1 \right) \bar{w}_a \, d\zeta$$

$$+ \frac{ik_o}{\pi} \bar{\Omega} e^{-ik_m} \sqrt{\frac{1 - z}{1 + z}} \int_{1}^{\infty} \frac{e^{-ik\lambda}}{\sqrt{\lambda^2 - 1}} d\lambda$$

$$+ \frac{1}{\pi^2} \bar{Q} e^{-ik_m} \sqrt{\frac{1 - z}{1 + z}} \int_{-1}^{1} \frac{e^{-ik\zeta}}{\sqrt{1 - \zeta^2}} d\zeta \qquad (68)$$

Equation (68) contains the noteworthy result that the term with $\bar{\Omega}$ and also the term with $\bar{Q}$ lead to chordwise pressure distributions which are both proportional to $\sqrt{(1 - z)/(1 + z)}$. Note that according to equation (44) these two pressure terms are caused by downward velocities across the wing of the form $\int_{1}^{\infty} e^{-ik\xi} (z - \zeta)^{-1} d\zeta$ and e^{-ikz}. The fact that these two different velocity distributions lead to the same simple pressure distribution is here arrived at by an analysis which does not reveal the inner reason for this occurrence. A modification of the analysis so as to clarify this point appears to be worth while, particularly as the fact itself accounts for the relatively simple form in which the aerodynamic span effect modifies the expressions for the air forces and moments of the two-dimensional theory.

In order to obtain the final form of the expression for $\Delta \bar{p}_a$, use the following known formulas of the theory of Bessel functions:

$$\text{(a)} \quad \int_{1}^{\infty} \frac{e^{-ik\lambda}}{\sqrt{\lambda^2 - 1}} d\lambda = -i \frac{\pi}{2} H_o^{(2)}(k)$$

$$\text{(b)} \quad \int_{-1}^{1} \frac{e^{-ik\zeta}}{\sqrt{1 - \zeta^2}} d\zeta = \pi J_o(k) \qquad (69)$$

and express the quantity $\bar{Q}$ in terms of $\bar{\Omega}$ by means of equation (61),

$$\bar{Q} = \frac{\bar{\Omega}^{(2)} - \bar{\Omega}}{2\mu(k)\ k/k_o} \tag{70}$$

Then write

$$\frac{\Delta\bar{p}_a}{\rho U} = \frac{2}{\pi} \oint_{-1}^{1} \left(\sqrt{\frac{1 - z}{1 + z}} \sqrt{\frac{1 + \zeta}{1 - \zeta}} \frac{1}{z - \zeta} - ik\Lambda_1 \right) \bar{w}_a \, d\zeta + \sqrt{\frac{1 - z}{1 + z}} S \tag{71}$$

where

$$S = e^{-ik_m} \left(\frac{k_o}{2} H_o^{(2)} \bar{\Omega} + \frac{1}{\pi} J_o \bar{Q} \right)$$

$$= \frac{k_o}{2} e^{-ik_m} \left[H_o^{(2)} \bar{\Omega}^{(2)} + \left(\bar{\Omega} - \bar{\Omega}^{(2)} \right) \left(H_o^{(2)} - \frac{J_o}{\pi k \mu} \right) \right]$$

$$= \frac{k_o}{2} e^{-ik_m} \bar{\Omega}^{(2)} \left[H_o^{(2)} + \left(\frac{\bar{\Omega}}{\bar{\Omega}^{(2)}} - 1 \right) \left(H_o^{(2)} - \frac{J_o}{\pi k \mu} \right) \right] \tag{72}$$

Introduce the value of $\bar{\Omega}^{(2)}$ from equation (60) and the value of μ from equation (59). Then

$$S = \frac{2 \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}}\, \bar{w}_a \, d\zeta}{\pi i}$$

$$\times \frac{H_o^{(2)} + \left(\frac{\bar{\Omega}}{\bar{\Omega}^{(2)}} - 1 \right) \left[H_o^{(2)} + J_o \frac{i\left(iH_o^{(2)} + H_1^{(2)} \right)}{J_o - iJ_1} \right]}{H_1^{(2)} + i\, H_o^{(2)}}$$

$$= \frac{2}{\pi i} \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}}\, \bar{w}_a \, d\zeta$$

$$\times \left[\frac{H_o^{(2)}}{H_1^{(2)} + iH_o^{(2)}} + \left(\frac{\bar{\Omega}}{\bar{\Omega}^{(2)}} - 1 \right) \right.$$

$$\left. \times \frac{H_o + i\left(iH_o + H_1 \right) \left(1 + \frac{i\, J_1}{J_o - iJ_1} \right)}{H_1 + i\, H_o} \right] = \frac{2}{\pi} \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}}\, \bar{w}_a \, d\zeta$$

$$\times \left[\frac{H_1}{H_1 + iH_o} - 1 + \left(\frac{\bar{\Omega}}{\bar{\Omega}^{(2)}} - 1 \right) \left(\frac{H_1}{H_1 + iH_o} + \frac{i\, J_1}{J_o - iJ_1} \right) \right] \tag{73}$$

Introduce in conformity with reference 4 the function

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_o^{(2)}(k)} \tag{74}$$

Equation (73) then becomes

$$S = \frac{2}{\pi} \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}}\, \bar{w}_a d\zeta \left[C - 1 + \left(\frac{\bar{\Omega}}{\bar{\Omega}^{(2)}} - 1 \right) \left(C + \frac{i\, J_1}{J_o - iJ_1} \right) \right] \tag{75}$$

By introducing equation (75) in equation (71) there follows:

$$\frac{\Delta\bar{p}_a}{\rho U} = \frac{2}{\pi} \oint_{-1}^{1} \left(\sqrt{\frac{1-z}{1+z}} \sqrt{\frac{1+\zeta}{1-\zeta}} \frac{1}{z-\zeta} - ik\Lambda_1 \right) \bar{w}_a \, d\zeta$$

$$- \frac{2}{\pi} \sqrt{\frac{1-z}{1+z}} \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}} \bar{w}_a \, d\zeta$$

$$+ \frac{2}{\pi} \left[C + \left(\frac{\bar{\Omega}}{\bar{\Omega}^{(2)}} - 1 \right) \left(C + \frac{i J_1}{J_o - iJ_1} \right) \right]$$

$$\times \sqrt{\frac{1-z}{1+z}} \int_{-1}^{1} \sqrt{\frac{1+\zeta}{1-\zeta}} \bar{w}_a \, d\zeta \tag{76}$$

Equation (76) is the general result to be established. A corresponding formula for the two-dimensional theory is given in reference 8. Equation (76) shows that the aerodynamic span effect in the present theory manifests itself solely by a modification of the basic function C(k). An additive correction term occurs, which is given in the form

$$\sigma = \left(\frac{\bar{\Omega}}{\bar{\Omega}^{(2)}} - 1 \right) \left[C(k) + \frac{iJ_1(k)}{J_o(k) - iJ_1(k)} \right] \tag{77}$$

and which is seen to depend on the ratio of three-dimensional to two-dimensional circulation function $\bar{\Omega}$. In order to evaluate the correction term for a given wing deflection function Z_a, it is then necessary only to solve the integral equation (62) for the spanwise variation of circulation.

LIFT AND MOMENTS FOR BENDING, TORSIONAL, AILERON, AND TAB DEFLECTION

For the applications (see reference 10) the following deflection functions are of particular interest:

Bending deflection: $\bar{Z}_a = \bar{h}(y) = \bar{h}_r f_h(y^*)$

Torsional deflection: $\bar{Z}_a = \bar{\alpha}(y)(x - ab) = \bar{\alpha}_r(x - ab)f_\alpha(y^*)$

Aileron deflection: $\bar{Z}_a = \bar{\beta}(y)(x - eb)$ for $cb \leq x \leq b$

Tab deflection: $\bar{Z}_a = \bar{\gamma}(y)(x - fb)$ for $db \leq x \leq b$

Lift and moment functions according to the two-dimensional theory may be found in reference 5. Modified so as to include the aerodynamic span effect these functions can be written in the following form, as was shown in reference 3:

$$\frac{\bar{L}}{2\rho U^2 b} = \pi\left\{-\frac{k^2}{2} + ik\left[C + \sigma_h\right]\right\}\frac{\bar{h}}{b}$$

$$+\pi\left\{\frac{ik}{2} + \frac{ak^2}{2} + \left[1 + \left(\frac{1}{2} - a\right)ik\right]\left[C + \sigma_\alpha\right]\right\}\bar{\alpha}$$

$$+\left\{\frac{\pi}{2}\left[ikC_{\beta_2} - k^2C_{\beta_1}\right] + \left[E_1 + \frac{ik}{2}E_2\right]\left[C + \sigma_\beta\right]\right\}\bar{\beta}$$

$$+\left\{\frac{\pi}{2}\left[ikC_{\gamma_2} - k^2C_{\gamma_1}\right] + \left[E_1(d) + \frac{ik}{2}E_2(d)\right]\left[C + \sigma_\gamma\right]\right\}\bar{\gamma} \qquad (78)$$

$$\frac{\overline{M}_a}{2\rho U^2 b^2} = \pi \left\{ \frac{ak^2}{2} - \left(\frac{1}{2} + a\right) ik \left[C + \sigma_h\right] \right\} \frac{\overline{h}}{b}$$

$$+ \pi \left\{ \frac{ik}{2} A_{\alpha_2} - \frac{k^2}{2} A_{\alpha_1} - \left(\frac{1}{2} + a\right) \left[1 + \left(\frac{1}{2} - a\right) ik\right] \left[C + \sigma_\alpha\right] \right\} \overline{\alpha}$$

$$+ \left\{ \frac{\pi}{2} \left[A_{\beta_3} + ikA_{\beta_2} - k^2 A_{\beta_1}\right] - \left(\frac{1}{2} + a\right) \left[E_1 + \frac{ik}{2} E_2\right] \left[C + \sigma_\beta\right] \right\} \overline{\beta}$$

$$+ \left\{ \frac{\pi}{2} \left[A_{\gamma_3} + ikA_{\gamma_2} - k^2 A_{\gamma_1}\right] \right.$$

$$\left. - \left(\frac{1}{2} + a\right) \left[E_1(d) + \frac{ik}{2} E_2(d)\right] \left[C + \sigma_\gamma\right] \right\} \overline{\gamma} \qquad (79)$$

$$\frac{\overline{M}_\beta}{2\rho U^2 b^2} = \left\{ -\frac{\pi k^2}{2} B_{h_1} + \frac{1}{2} E_3 ik \left[C + \sigma_h\right] \right\} \frac{\overline{h}}{b}$$

$$+ \left\{ \frac{\pi}{2} \left[ikB_{\alpha_2} - k^2 B_{\alpha_1}\right] + \frac{1}{2} E_3 \left[1 + \left(\frac{1}{2} - a\right) ik\right] \left[C + \sigma_\alpha\right] \right\} \overline{\alpha}$$

$$+ \left\{ \frac{\pi}{2} \left[B_{\beta_3} + ikB_{\beta_2} - k^2 B_{\beta_1}\right] + \frac{1}{2\pi} E_3 \left[E_1 + \frac{ik}{2} E_2\right] \left[C + \sigma_\beta\right] \right\} \overline{\beta}$$

$$+ \left\{ \frac{\pi}{2} \left[B_{\gamma_3} + ikB_{\gamma_2} - k^2 B_{\gamma_1}\right] \right.$$

$$\left. + \frac{1}{2\pi} E_3 \left[E_1(d) + \frac{ik}{2} E_2(d)\right] \left[C + \sigma_\gamma\right] \right\} \overline{\gamma} \qquad (80)$$

$$\frac{\overline{M}_\gamma}{2\rho U^2 b^2} = \left\{ -\frac{\pi k^2}{2} D_{h_1} + \frac{1}{2} E_3(d) ik \left[C + \sigma_h \right] \right\} \frac{\overline{h}}{b}$$

$$+ \left\{ \frac{\pi}{2} \left[ikD_{\alpha_2} - k^2 D_{\alpha_1} \right] + \frac{1}{2} E_3(d) \left[1 + \left(\frac{1}{2} - a \right) ik \right] \left[C + \sigma_\alpha \right] \right\} \overline{\alpha}$$

$$+ \left\{ \frac{\pi}{2} \left[D_{\beta_3} + ikD_{\beta_2} - k^2 D_{\beta_1} \right] + \frac{1}{2\pi} E_3(d) \left[E_1 + \frac{ik}{2} E_2 \right] \left[C + \sigma_\beta \right] \right\} \overline{\beta}$$

$$+ \left\{ \frac{\pi}{2} \left[D_{\gamma_3} + ikD_{\gamma_2} - k^2 D_{\gamma_1} \right] + \frac{1}{2\pi} E_3(d) \left[E_1(d) + \frac{ik}{2} E_2(d) \right] \right.$$

$$\left. \times \left[C + \sigma_\gamma \right] \right\} \overline{\gamma} \qquad (81)$$

The terms A, B, and D are defined in reference 5. The terms E are of the form

$$\left.\begin{aligned} E_1 &= T_{10} - lT_{21} & E_1(d) &= T_{10}(d) - mT_{21}(d) \\ E_2 &= T_{11} - 2lT_{10} & E_2(d) &= T_{11}(d) - 2mT_{10}(d) \\ E_3 &= T_{12} - 2lT_{20} & E_3(d) &= T_{12}(d) - 2mT_{20}(d) \end{aligned}\right\} \qquad (82)$$

with the terms T also defined in reference 5.

The terms $\sigma_j (j = h, \alpha, \beta, \gamma)$ as defined by equation (77) are given by

$$\sigma_j = \left[C + \frac{iJ_1(k)}{J_o(k) - iJ_1(k)}\right]\left(\frac{\bar{\Omega}_j}{\bar{\Omega}_j^{(2)}} - 1\right) \tag{83}$$

The functions $\bar{\Omega}_j^{(2)}$ of the two-dimensional theory as defined by equation (60) are given by

$$\left.\begin{aligned}
\bar{\Omega}_h^{(2)} &= \frac{b}{b_o}\frac{4iCe^{ik_m}}{kH_1^{(2)}(k)}\, ik\,\frac{\bar{h}}{b} \\
\bar{\Omega}_\alpha^{(2)} &= \frac{b}{b_o}\frac{4iCe^{ik_m}}{kH_1^{(2)}(k)}\left[1 + \left(\frac{1}{2} - a\right) ik\right]\bar{\alpha} \\
\bar{\Omega}_\beta^{(2)} &= \frac{b}{b_o}\frac{4iCe^{ik_m}}{kH_1^{(2)}(k)}\,\frac{1}{\pi}\left(E_1 + E_2\frac{ik}{2}\right)\bar{\beta} \\
\bar{\Omega}_\gamma^{(2)} &= \frac{b}{b_o}\frac{4iCe^{ik_m}}{kH_1^{(2)}(k)}\,\frac{1}{\pi}\left[E_1(d) + E_2(d)\frac{ik}{2}\right]\bar{\gamma}
\end{aligned}\right\} \tag{84}$$

The functions $\bar{\Omega}_j$ of the three-dimensional theory are, according to equations(62) and (39b), the solutions of the integral equation,

$$\bar{\Omega}_j(y^*) + \frac{b}{b_o}\mu(k)\left[\oint_{-s}^{s}\frac{d\bar{\Omega}_j}{y^*-\eta^*}\right.$$

$$\left.-ik_o\int_{-s}^{s}\frac{y^*-\eta^*}{|y^*-\eta^*|}F(k_o|y^*-\eta^*|)d\bar{\Omega}_j\right] = \bar{\Omega}_j^{(2)}(y^*) \qquad (85)$$

The function μ is defined by equation (59) as

$$\mu(k) = \frac{J_o(k) - i\,J_1(k)}{\pi k\left\{[J_o(k) - y_1(k)] - i\,[J_1(k) + y_o(k)]\right\}}$$

The function F is, according to equation (39b),

$$F(x) = \int_0^{\infty} e^{-i\lambda}\left(\frac{1}{x} + \frac{1}{\lambda} - \frac{\sqrt{x^2+\lambda^2}}{x\,\lambda}\right)d\lambda \qquad (86)$$

This function occurred previously in reference 9, where a different theory of the problem of the oscillating wing of finite span was put forward. A discussion of the theory of reference 9 is given in reference 1. Tables 1 and 2 contain values of the two functions μ and F for a significant range of the variables k and x.

It is apparent that the main task in obtaining three-dimensional corrections to the results of the two-dimensional theory consists in solving the integral equation for $\bar{\Omega}$. The second part of this report (reference 10), which deals with applications of the theory, contains a practical method for doing this.

Massachusetts Institute of Technology,
Cambridge, Mass., December 3, 1945.

REFERENCES

1. Reissner, Eric: On the General Theory of Thin Airfoils for Nonuniform Motion. NACA TN No. 946, 1944.

2. Reissner, Eric: The Equations of Lifting-Strip Theory for Tapered Wings. Report SB-76-S-1, Res. Lab., Curtiss-Wright Corp., 1943.

3. Hunter, Maxwell W., Jr.: Calculation of the Aerodynamic Span Effect in Flutter Analysis. M. S. Thesis, Massachusetts Inst. of Technology, 1944.

4. Theodorsen, Theodore: General Theory of Aerodynamic Instability and the Mechanism of Flutter. NACA Rep. No. 496, 1935.

5. Theodorsen, Theodore, and Garrick, I. E.: Nonstationary Flow about a Wing-Aileron-Tab Combination including Aerodynamic Balance. NACA Rep. No. 736, 1942.

6. Schade, Th.: Theory of the Oscillating Circular Arc Airfoil on the Basis of the Potential Theory. Pt. I. Luftfahrtforschung, vol. 17, nos. 11-12, Dec. 10, 1940, pp. 387-400. (Translation)[1]

7. Krienes, H., and Schade, T.: Theorie der schwingenden Kreisförmigen Tragfläche auf potentialtheoretischer Grundlage. Pt. II. Luftfahrtforschung, vol. 19, 1942, pp. 282-291.

8. Schwarz, L.: Berechnung der Druckverteilung einer harmonisch sich verformenden Trägflache in ebener Strömung. Luftfahrtforschung, vol. 17, Dec. 10, 1940, pp. 379-386.

9. Cicala, P.: Comparison of Theory with Experiment in the Phenomenon of Wing Flutter. NACA TM No. 887, 1939.

10. Reissner, Eric, and Stevens, John E.: Effect of Finite Span on the Airload Distributions for Oscillating Wings. II - Methods of Calculation and Examples of Application. NACA TN No. 1195, 1947.

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[1]Available for reference or loan in the Office of Aeronautical Intelligence, NACA, Washington, D. C. Also pts. I and II (references 6 and 7) will be published as NACA TM No. 1098.

TABLE I.– VALUES OF THE FUNCTION F OF EQUATION (86)

x	F(x)	x	F(x)
0.00	∞ – 1.571i	2.2	0.113 – 0.425i
.05	2.750 – 1.468i	2.4	.097 – .395i
.10	2.109 – 1.375i	2.6	.083 – .369i
.20	1.490 – 1.248i	2.8	.072 – .345i
.30	1.155 – 1.146i	3.0	.063 – .324i
.40	.935 – 1.060i	3.2	.055 – .305i
.50	.778 – .987i	3.4	.048 – .289i
.60	.658 – .922i	3.6	.043 – .274i
.70	.564 – .865i	3.8	.039 – .260i
.80	.489 – .813i	4.0	.035 – .248i
.90	.427 – .768i	4.2	.031 – .237i
1.00	.376 – .726i	4.4	.028 – .226i
1.10	.333 – .689i	4.6	.026 – .217i
1.20	.297 – .654i	4.8	.024 – .208i
1.30	.265 – .624i	5.0	.022 – .200i
1.40	.238 – .593i	5.2	.020 – .192i
1.50	.214 – .567i	5.4	.018 – .185i
1.60	.194 – .540i	5.6	.017 – .179i
1.70	.176 – .517i	5.8	.016 – .172i
1.80	.160 – .496i	6.0	.015 – .167i
1.90	.146 – .475i	∞	$1/(2x^2)$ $1/x$
2.00	.134 – .458i		

TABLE II.– VALUES OF THE FUNCTION μ OF EQUATION (59)

k	μ(k)	k	μ(k)
0.000	0.5000 – 0.0000i	0.650	0.2139 – 0.0665i
.020	.4810 – .0423i	.700	.2062 – .0610i
.040	.4607 – .0665i	.750	.1991 – .0557i
.060	.4410 – .0829i	.800	.1924 – .0507i
.080	.4226 – .0942i	.900	.1801 – .0413i
.100	.4051 – .1018i	1.000	.1688 – .0329i
.125	.3857 – .1082i	1.250	.1436 – .0159i
.150	.3690 – .1120i	1.500	.1218 – .0042i
.175	.3522 – .1130i	1.750	.1027 + .0031i
.200	.3393 – .1139i	2.000	.0864 + .0066i
.225	.3268 – .1132i	2.100	.0807 + .0070i
.250	.3154 – .1116i	2.150	.0780 + .0072i
.275	.3049 – .1099i	2.200	.0754 + .0072i
.300	.2955 – .1076i	2.250	.0730 + .0072i
.350	.2787 – .1023i	2.300	.0706 + .0070i
.400	.2644 – .0964i	2.350	.0684 + .0069i
.450	.2519 – .0903i	2.400	.0663 + .0066i
.500	.2408 – .0842i	2.480	.0632 + .0061i
.550	.2288 – .0775i	2.540	.0610 + .0057i
.600	.2220 – .0722i	2.600	.0591 + .0052i