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Effect of Hard Cores on the Binding Energies of H^3 and He^3
——*Corrected Values*——

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There was a slight error in the kinetic energy formula of the paper I by Ohmura (formely Kikuta), Morita, and Yamada. The binding energy of H^3 as well as the Coulomb energy of He^3 are recalculated as described in I by using the correct value of the kinetic energy. The results obtained are, however, almost the same as in I and II.

In the previous papers,[1,2] (referred to as I, II respectivly hereafter), the binding energy of H^3 has been calculated by assuming two-body central forces with hard cores of various sizes. It has been shown that the binding energy is reduced considerably with the increase of the hard core radius, and that the binding energy is not very sensitive to the shape of the potential if the hard core radius is quite large (for example, 0.6×10^{-13}cm). The observed energy difference between He^3 and H^3 ($\sim$0.76 Mev) is much smaller than the Coulomb energies calculated by many authors[3], which are about 1 Mev or more. However, if the two-body nuclear force has a strong repulsive core in it, the two protons of He^3 may be pushed away from one another so that the Coulomb energy can be expected to be reduced. In I and II we found this to be the case. Such an effect of the hard core on the binding energy of H^3 and on the Coulomb energy of He^3 are assumed to be valid even when the two-body potential has the tensor force included in it.

Meanwhile Dr. G. Derrick of the University of Sydney used some numerical values of I as a check on his coding for SILLIAC, and found a small difference ($\sim$1 Mev) between his kinetic energy values and ours. Then he checked the cumbersome formulae for kinetic energy given in I, and has kindly informed me in the fall of 1957 that the term $\cdots+(D^2-16D/x-8/x^2)C_1(x)$ in the expression on the second line of page 232 of I should be replaced by $\cdots+(D^2-14D/x-8/x^2)C_1(x)$. Thus we decided to recalculate the binding energy of H^3 and the Coulomb energy of He^3 by using the correct values of kinetic energy, though the corrections were expected to be rather small. The same assumptions, the same notation and the same method of calculations as those in I and II are used. We use the same potentials as in I and II; namely, (1), (2), Table I in I and (1), Table I in II. The

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trial function is the same as (4) in I. The binding energy and the values of parameters μ and ν determined with variational method in this way are given in Tables I and II.

Table I. Variationally computed binding energy of H^3.

Hard core radius D in 10^{-13} cm	B. E. (H^3) in Mev			
	$r_{0s} \doteqdot 2.7\times10^{-13}$ cm		$r_{0s} \doteqdot 2.4\times10^{-13}$ cm	
	Yukawa pot.	Exponential pot.	Yukawa pot.	Exponential pot.
0.0	12.49	10.26	15.97	11.38
0.2	9.15	8.81	10.0	9.98
0.4		7.52		8.85
0.6	6.43	5.79	7.41	7.38
Experimental Value	8.49			

Table II. Adjusted values of μ and ν in $10^{13}\,cm^{-1}$

D in 10^{-13} cm	$r_{0s} \doteqdot 2.7\times10^{-13}$ cm				$r_{0s} \doteqdot 2.4\times10^{-13}$ cm			
	μ		ν		μ		ν	
	Yukawa	Expon.	Yukawa	Expon.	Yukawa	Expon.	Yukawa	Expon.
0.0	0.605	0.479	∞	∞	0.659	0.503	∞	∞
0.2	0.447	0.462	5.93	5.03	0.489	0.479	6.1	5.20
0.4		0.457		4.09		0.500		4.27
0.6	0.452	0.450	4.28	4.20	0.498	0.494	4.47	4.51

Since the adjustment of the potential parameters is somewhat inaccurate in the case of the Yukawa type potential, all the values for the Yukawa well are considered to be about one order of magnitude less precise than those for the exponential well. But if we assume the validity of potential parameters, all the tabulated figures in Table I～X have validity. The exceptions are Tables II and III, where the tabulated values contain errors of the order 1%, or even up to several per cent.

If charge independence of nuclear forces is assumed, the Coulomb energy of He^3 can be calculated by the perturbation method with reasonable precision. The results are given in Table III. The detailed numerical results which appear in the course of our calculation of kinetic energy, potential energy, etc., are given in Table IV～X in the Appendix. By comparing with Tables in references I and II, we see that: 1) The kinetic energies are reduced by an amount of the order of magnitude of 1 Mev in case of $D\neq0$. Consequently, the binding energies increase by a similar amount. 2) Adjusted values of μ increase somewhat, but ν practically remains unchanged. 3) The results for $D=0$ remain correct, because the corrected term contains a factor D. 4) Some qualitaive conclusions stated in I and II are not altered, namely, a) Hard cores push out the wave functions of H^3 and He^3,

Table III. Coulomb energy of He^3

Hard core radius D in 10^{-13} cm	C. E. of He^3 in Mev			
	$r_{0s} \doteqdot 2.7\times10^{-13}$ cm		$r_{0s} \doteqdot 2.4\times10^{-13}$ cm	
	Yukawa pot.	Exponential pot.	Yukawa pot.	Exponential pot.
0.0	1.246	0.986	1.358	1.037
0.2	0.830	0.836	0.896	0.861
0.4		0.746		0.798
0.6	0.688	0.686	0.736	0.733
Experimental Value	0.764			

so that the Coulomb energy of He^3, which is too large without the hard cores, is brought into agreement with the experimental value of the difference between the binding energies of H^3 and He^3, b) Hard cores reduce the binding energies of H^3 and He^3 considerably, c) The hard core effects are slightly more enhanced for the Yukawa potential than for the exponential, especially near $D=0$, possibly because of the rapid change of the potential form in this region, d) Discussions in I and II about the approximations can be applied also to the present case, and so the values of B.E. (H^3) are expected to be about 1 Mev larger than the values in Table I, while the Coulomb energies may be several per cent smaller than the values in Table III.

If we take the hard core radius D as $0.3\sim0.4\times10^{-13}$cm for $r_{os}=2.7\times10^{-13}$cm or $D=0.4\sim0.6\times10^{-13}$cm for $r_{os}=2.4\times10^{-13}$cm, reasonable fit to the experimental values is obtained.

Since the binding energy thus calculated is rather sensitive to unestablished details of the nuclear force, and since the tensor potential must be taken into account, the absolute value of the binding energies of H^3 should not be considered literally. The inclusion of the tensor force reduces the binding energy[3)] of H^3 to some extent. On the contrary, we can clearly see from Table III the general trend of the effect of hard cores on the Coulomb energy of He^3. Because, in the case of the exponential potential, it is inferred from Table II that the diminution of the Coulomb energy for the exponential potential is not due to the behavior of the wave function at large distances, which is mainly specified by the parameter μ, but due to the vanishing of the wave function at short distances.

The inclusion of the tensor potential in the two-body force may decrease the Coulomb energy. This fact will be understood by comparing the calculated values of Pease and Feshbach[3)] with Table III of the present paper. Using the Yukawa well for both central and tensor parts, Pease and Feshbach obtained $1.01\sim1.04$ Mev for the Coulomb energy, while Table III gives 1.245 Mev (assuming $r_{os}\doteqdot2.7\times10^{-13}$cm) and 1.358 Mev (assuming $r_{as}\doteqdot2.4\times10^{-13}$cm) for the Yukawa well. The reason may be as follows: The tensor potential mixes D-states (and P-states) in H^3

state. These states are pushed away outside by the centrifugal forces, so that the Coulomb energy arising from these mixed parts of the wave function decreases considerably. There is also another reason that the inclusion of tensor force reduces the binding energy so that the nucleus has a larger extension.

Summary. This note reports the corrected values of our results in the previous papers I and II. The correction is done by replacing $(D^2-16D/x-8/x^2)C_1(x)$, the last part of the second line of p. 232 in I, by $(D^2-14D/x-8/x^2)C_1(x)$. Consequently, the kinetic energies decrease about 1 Mev for $D\neq 0$. There is no change for $D=0$. Otherwise the results obtained are almost the same as in I and II. The absolute value of binding energy is relatively sensitive to unestablished details of the nuclear force, but we can clearly see from Table III the general trend of the effect of hard cores on the Coulomb energy of He^3. Because it is inferred from Table II that the diminution of the Coulomb energy for the exponential potential is not due to the behavior of the wave function at large distances, which is mainly specified by the parameter μ, but due to the vanishing of the wave function at short distances. The Coulomb energy of He^3 is being calculated by assuming an extended proton. The Coulomb energy may be reduced further.

Table IV. No hard core. Exponential potential. ($\nu=\infty$).

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev	
			$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm		$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm
0.5	∞	33.42	21.19	22.39	44.42	*10.18	*11.38
0.4835	∞	31.67	20.12	21.19	41.54	*10.25	*11.33
0.45	∞	28.15	17.96	18.79	35.98	*10.12	*10.96

Table V. Hard core radius $D=0.2\times10^{-13}$ cm. Exponential potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E (H^3) in Mev		N in $(2\times10^{-13}\text{cm})^6$
			$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm		$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm	
0.55	4.5	41.92	28.12	29.79	62.81	*7.23	*8.90	0.005945
0.5	6.5	40.51	27.04	28.79	60.14	7.41	9.16	0.011327
0.5	5.5	38.77	26.02	27.55	56.51	*8.27	*9.81	0.010975
0.5	5	37.63	25.35	26.74	54.49	*8.49	*9.89	0.010723
0.5	4.5	36.26	24.53	25.77	52.28	*8.51	*9.75	0.010393
0.5	3.5	32.53	22.27	23.11	47.06	7.74	8.58	0.009344
0.45	5.5	32.61	22.09	23.17	45.97	*8.73	*9.81	0.020033
0.45	5	31.70	21.55	22.53	44.49	8.76	9.74	0.019672
0.45	4.5	30.60	20.89	21.74	42.87	*8.62	*9.47	0.019194
0.45	4	29.25	20.07	20.78	41.07	8.26	8.97	0.018546
0.4	4.5	25.01	17.11	17.77	34.19	7.93	8.59	0.037981

Table VI. Hard core radius $D=0.4\times10^{-13}$ cm. Exponential Potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in $(2\times10^{-13}\text{cm})^6$
			$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm		$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm	
0.5	5.5	48.04	34.06	36.46	77.36	4.74	7.14	0.017722
0.5	4.5	43.73	31.42	33.21	68.14	*7.01	*8.80	0.016515
0.5	3.5	37.64	27.70	28.75	58.07	*7.26	*8.32	0.014507
0.5	3	33.77	25.25	25.89	52.55	6.46	7.10	0.012975
0.45	3.5	31.65	23.58	24.24	48.07	*7.16	*7.82	0.026619
0.4	4.5	29.72	21.93	22.70	44.39	*7.25	*8.02	0.056483
0.4	4	27.92	20.82	21.38	41.63	*7.10	*7.66	0.05462
0.4	3.5	25.73	19.47	19.80	38.69	*6.52	*6.84	0.05208
0.35	4.5	23.09	17.30	17.70	34.03	6.37	6.77	0.11764

Table VII. Hard core radius $D=0.6\times10^{-13}$ cm. Exponential Potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in $(2\times10^{-13}\text{cm})^6$
			$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm		$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm	
0.5	4.5	51.18	39.70	41.89	85.70	*5.18	*7.38	0.025025
0.5	4	46.75	36.98	38.53	78.16	*5.56	*7.12	0.023497
0.5	3.5	42.03	34.03	34.93	70.88	5.17	6.07	0.021423
0.5	3	36.42	30.32	30.57	62.79	3.95	4.21	0.018909
0.45	4.5	42.60	33.56	35.04	70.46	*5.70	*7.18	0.04371
0.4	5.5	39.17	30.43	32.04	64.93	4.68	6.29	0.08576
0.4	5	36.88	29.07	30.33	60.63	*5.32	*6.58	0.08382
0.4	4.5	34.32	27.05	28.38	56.25	*5.57	*6.45	0.08132
0.4	4	31.42	25.67	26.16	51.81	*5.29	*5.77	0.07815
0.35	4.5	26.49	21.62	22.03	43.23	4.87	5.29	0.16429
0.3	5.5	21.88	17.65	18.11	35.76	3.78	4.24	0.38152
0.3	5	20.67	16.92	17.20	33.70	3.88	4.17	0.37685
0.3	4.5	19.30	16.06	16.15	31.62	3.74	3.83	0.37085
0.3	3.5	15.97	13.88	13.58	27.17	2.69	2.38	0.35203

Table VIII. No hard core. Yukawa potential. ($\nu=\infty$)

μ in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev	
		$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm		$r_{0s}=2.7\times10^{-13}$ cm	$r_{0s}=2.4\times10^{-13}$ cm
0.55	38.92	27.04	29.56	53.75	*12.22	14.73
0.6	45.08	31.38	34.48	63.97	*12.49	*15.58
0.65	51.44	35.89	39.58	75.07	*12.26	*15.96
0.7	57.99	40.52	44.86	87.07	11.45	*15.79

Table IX. Hard core radius $D=0.2\times10^{-13}$ cm. Yukawa potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in $(2\times10^{-13}\text{cm})^6$
			$r_{0s}=2.7$ $\times10^{-13}$ cm	$r_{0s}=2.4$ $\times10^{-13}$ cm		$r_{0s}=2.7$ $\times10^{-13}$ cm	$r_{0s}=2.4$ $\times10^{-13}$ cm	
0.55	4.5	41.26	28.83	29.90	62.81	7.29	8.36	0.005945
0.5	6.5	40.47	28.25	29.44	60.14	*8.58	**9.77	0.011327
0.5	5.5	38.50	26.87	27.88	56.51	*8.86	**●9.87	0.010975
0.5	5	37.17	25.99	26.90	54.49	8.68	●9.59	0.010723
0.5	4.5	35.67	24.96	25.75	52.28	*8.34	**9.13	0.010393
0.45	6	33.19	23.21	23.98	47.31	9.09	●9.86	0.020312
0.45	5.5	32.33	22.72	23.32	45.97	*9.08	**●9.68	0.020033
0.45	5	31.32	21.93	22.54	44.49	8.76	●9.37	0.019672
0.45	4.5	30.12	21.11	21.63	42.87	*8.35	**8.87	0.019194
0.45	4	28.68	20.12	20.53	41.07	7.73	8.14	0.018546
0.4	4.5	24.67	17.32	17.61	34.19	*7.80	●**8.09	0.037981

[**] gives B. E. $=9.94$ Mev with adjusted values of $\mu=0.492$, $\nu=5.89\times10^{13}$ cm^{-1}, while [●] gives B. E. $=10.07$ Mev with $\mu=0.487$, $\nu=6.36\times10^{+13}$ cm^{-1}. Since the author does not quite understand such a large difference between values of different two sets, the averaged values are tabulated in Table I and II.

Table X. Hard core radius $D=0.6\times10^{-13}$ cm. Yukawa potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in $(2\times10^{-13}\text{cm})^6$
			$r_{0s}=2.7$ $\times10^{-13}$ cm	$r_{0s}=2.4$ $\times10^{-13}$ cm		$r_{0s}=2.7$ $\times10^{-13}$ cm	$r_{0s}=2.4$ $\times10^{-13}$ cm	
0.5	4.5	51.13	40.50	41.97	85.70	*5.93	*7.41	0.025025
0.5	4	46.68	37.65	38.55	78.16	*6.17	*7.07	0.023497
0.5	3.5	41.95	34.58	34.89	70.88	*5.65	*5.96	0.021423
0.5	3	36.37	30.78	30.31	62.79	4.36	3.89	0.018909
0.45	4.5	42.58	34.24	35.08	70.46	*6.36	*7.20	0.04371
0.4	5.5	39.22	31.16	32.20	64.93	5.45	6.49	0.08576
0.4	5	36.91	29.72	30.44	60.63	6.00	6.72	0.08382
0.4	4.5	34.32	28.07	28.44	56.25	*6.14	*6.52	0.08132
0.4	4	31.42	26.16	26.31	51.81	*5.78	*5.92	0.07815
0.35	4.5	26.53	22.08	22.09	43.23	5.37	5.39	0.16429

Added note after completion of the manuscript.

We have come to know of a paper by J.M. Blatt and G. Derrick[4] entitled "Repulsive Core Forces in the Triton". They have concluded that the binding energy of H^3 increases with increasing core radius. They use a family of central and tensor potentials for the two-body nuclear force. The 4 adjustable parameters in the potential for the triplet even state can probably be determined so as to give the correct binding energy, the effective range and the electric quadrupole moment of the deuteron, and also the binding energy of the triton, if the core radius is not too large. Therefore, if we want to argue whether the binding energy of H^3 increases or decreases with increasing core radius, one plausible extra-restriction must be imposed on the 4 potential parameters in order to compare the binding energies of H^3 for the different values of the core radii. It seems that Blatt and Derrick assume that the well depth of the central force should be taken equal for the different core radii when comparing the B.E. of H^3. The author does not quite understand the reason why they choose the above mentioned criterion. Even if their criterion was most plausible, their conclusion could be doubtful, because the tensor force is completely omitted in their actual calculation of the binding energy of H^3. The numerical values obtained by them may of course be useful for some specialists as a guide.

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Appendix. Details of the results of calculation

In order to get the minimum of the stationary expression of energy by varying the adjustable parameters μ and ν, we have used the quadratic function approximation as in I and II. In the following tables we give the detailed values which have appeared in the course of our calculation. The values with asterisks are used to obtain Tables I, II and III. Tables IV～VII are revised versions of Table VI in reference I, and Tables XIII～X are revised versions of Table V in reference II. Potential parameters were given in Table I of reference I and Table I of reference II respectively. The values of normalization are given in $(2\times10^{-13}\text{cm})^6$.

References

1) T. Ohmura (Kikuta), M. Morita and M. Yamada, Prog. Theor. Phys. **15** (1956), 222.
2) T. Ohmura, M. Morita and M. Yamada, Prog. Theor. Phys. **17** (1957), 326.
3) See, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (1952), 205; R. L. Pease and H. Feshbach, Phys. Rev. **88** (1952), 945 and the papers cited in these references
4) J. M. Blatt and G. Derrick, Nuclear Physics **8** (1958), 602.