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Effect of Inertia on Losses from a Plasma in Toroidal Equilibrium

H.P. Zehrfeld, B.J. Green

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B.J.Green

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Abstract

We investigate theoretically the MHD equilibrium of a resistive, low-density plasma in a model stellarator field. The effect of inertia on plasma motion is treated exactly, and its influence on plasma loss determined. The results are valid for arbitrary aspect ratio. For the existence of a stationary equilibrium we show that there are two unconnected regions of solution, described in terms of the mass fluxes the long and the short way within a magnetic surface. Both these regions contain subregions corresponding to subsonic and supersonic fluid flow. For the region containing the case of the wellknown classical resistive diffusion we argue that the increase in plasma loss due to inertia is strongly limited and does not appreciably exceed the classical diffusion rate.

I. Introduction

In recent years several attempts have been made to investigate the effects of inertia on plasma diffusion in configurations of the stellarator type. A first step in this direction was taken by Knorr [1] . He ordered the plasma parameters so that both finite conductivity and inertia of the ions were first order effects. With such an ordering, the influence of inertia on plasma diffusion is necessarily a second order effect. Discussing the results in zeroth and first orders only, Knorr could not display the influence of inertia on plasma losses. However, this search for a complete equilibrium solution, order by order, raised interesting questions concerning the definiteness of such an equilibrium.

It is a typical feature of such problems that so called "surface quantities" occur in each order. Depending on the particular structure of the equations and the expansion procedure used, these surface quantities are determined either by the relationship between different orders, or they remain arbitrary within reasonable physical limits. In the latter case their presence indicates that the equilibrium is not described uniquely by the equations and boundary conditions used. In Knorr's paper there was such an arbitrariness of the equilibrium solution which the author removed by additional assumptions. Later, in a revised treatment of the same problem [2] , Karlson pointed out that such assumptions are superfluous and that solving consistently, order by order, should give an unique solution. In the genuine stationary equilibrium problem investigated here, we come to the same conclusion.

In the present work we employ the usual hydromagnetic equations for the stationary equilibrium of a low-density plasma with high but finite electric conductivity. In the equation of motion we include the complete inertia term. We solve these equations in toroidal axisymmetric geometry. For the confining magnetic field we take the simple model configuration that was first introduced by Pfirsch and Schlüter [3] . We make a perturbation expansion in resistive effects which means that the solution found in zeroth order describes ideal hydromagnetic flows. Further, it means that already in first order we obtain effects of inertia on diffusion. The structure of our zeroth order equations has been investigated rather generally by Greene and Karlson [4] . They treated the question of the uniqueness of certain toroidal equilibria and gave plausible arguments concerning the minimum information which must be given to obtain a well-posed problem for ideal hydromagnetic flows. Their results indicate that five suitably chosen functions of a variable labeling the magnetic surfaces, and the location of a rigid conducting boundary define a unique stationary flow. Apart from a surface quantity related to the low-β expansion this is in agreement with what we find in zeroth order. For our case we can further specify limitations for the range within which some of the surface quantities, the mass fluxes, can be freely chosen. Such limitations come from the condition for an equilibrium solution to exist.

Though we indicate how to get an equilibrium solution in each order, we focus our attention only on the zeroth order, and the determination of plasma losses in first order. The derived plasma loss expression relates for given temperature distribution, the average particle flux in outward direction, the mass distribution on the magnetic surfaces, the magnetic field and the zeroth order mass fluxes the "long" and the "short way". If it were true that these fluxes could be chosen independently from the magnetic field and from each other, plasma rotation would lead to considerably enhanced diffusion for small rotational transform. Formally, this is because the term containing the mass flux the short way, increases as ι^{-4} for $\iota \to 0$ whereas the classical resistive diffusion grows as ι^{-2}. This seems to be related to the result of Stringer [5] who investigated the effect of plasma rotation on diffusion using a mixture of guiding center and fluid equations. However, the detailed analysis we give shows that such a conclusion on greatly enhanced diffusion due to mass flow is not correct. A numerical study of this problem [8] appears to support this conclusion.

Firstly, as we have indicated above there are already in zeroth order limitations for the range within which the mass fluxes can be chosen and this range is a narrow one for usual values of the aspect ratio and for small rotational transform.

Secondly, the condition for a single-valued solution in first order give differential equations for the maxx fluxes in zeroth order. Together with boundary conditions, these equations determine these fluxes completely. Although we have not solved the ordinary differential equations for the mass fluxes, which are nonlinear and extremely complicated, it is clear that any restrictions they impose on the solution are additional to those arising from the existence of a solution for the density in zeroth order.

On the basis of these latter restrictions we estimate the plasma losses and argue that there is no appreciable contribution due to inertia. A quantitative determination of the plasma losses requires further investigation.

II. Basic Equations

The equations we use are

$$\rho\left(\tfrac{1}{2}\nabla v^2 - \underline{v}\times \mathrm{rot}\,\underline{v}\right) = \underline{J}\times\underline{B} - \nabla p \tag{1}$$

$$\underline{v}\times\underline{B} = \eta\underline{J} + \nabla\phi \tag{2}$$

$$\mathrm{div}\,\rho\underline{v} = Q \tag{3}$$

$$\mathrm{div}\,\underline{J} = 0 \tag{4}$$

where the symbols have their usual meaning. We consider a low β plasma, so that $\underline{B}$ is a given external magnetic field. Q, the plasma source distribution, is also assumed given as a function in space. We assume that the plasma is injected with the local plasma velocity $\underline{v}$. In Ohm's law (2), we have neglected those parts of the Hall term which are unimportant if, at each point in the plasma, the frequency of material rotation is small compared to the ion gyrofrequency. The remaining contributions to the Hall term are included in the generalized electric potential ϕ .

To the above equations we must add an equation of state, which we assume to be of the form

$$p = p(\rho, F) \tag{5}$$

where F is the magnetic flux the long way and is used to parameterise the magnetic surfaces. Such a relation (5) results from the specification of a thermodynamic potential, together with the specification of either the entropy S or the temperature T as functions of F [1] . Later, for reasons of simplicity, we refer to an isothermal plasma.

As co-ordinate system we choose the orthogonal system r, θ, ξ as indicated in fig.1. For our calculations in such a co-ordinate system we need the metric $(ds)^2$, and the volume element $d^3\tau$, which are

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + R^2N^2(d\xi)^2 \tag{6}$$

$$d^3\tau = \sqrt{g}\, dr\, d\theta\, d\xi \quad ; \quad \sqrt{g} = rRN \tag{7}$$

We wish to solve the equations (1) to (5) in the case of axial symmetry and as a particularly simple choice of magnetic field [2] we take

$$\underline{B} = B_o R \left(\frac{f\varkappa}{N} \nabla\theta + \nabla\xi \right) \tag{8}$$

where $$N := 1 + \varkappa \cos\theta \quad , \quad \varkappa := \frac{r}{R} \tag{9}$$

and $$f := \frac{\iota(r)\varkappa}{(1-\varkappa^2)^{1/2}} \tag{10}$$

Here ι is the normalised rotational transform (the usual rotational transform divided by 2π) and B_o is the toroidal component of the magnetic field on the magnetic axis (r = 0).

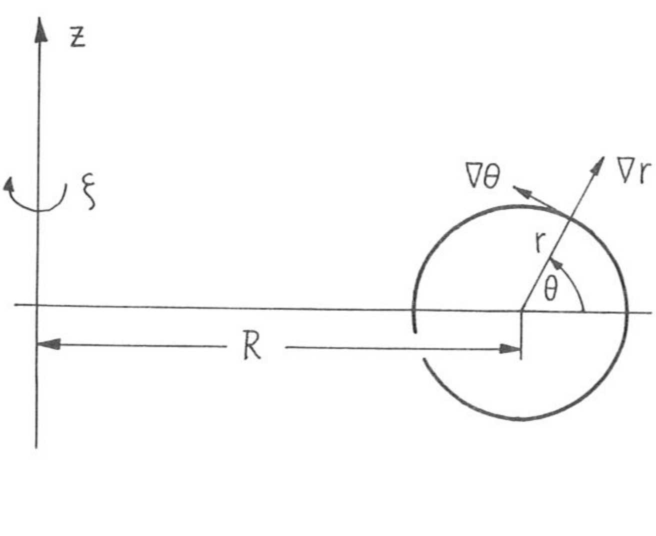

Fig. 1

III. Solution Procedure

Equations (1) to (5) can be displayed in dimensionless form by writing all magnitudes as hatched dimensionless variables and scale magnitudes with subscript o, e.g. $A=A_o\hat{A}$. We then make an ordering of the terms in the equations by relating the scale magnitudes. In particular we wish to solve the system of equations, exactly with respect to the effect of inertia but approximately with respect to resistivity effects. To do this we order the dimensionless parameter

$$\varepsilon \sim \frac{\eta J_o}{V_o B_o} \sim \frac{Q_o L_o}{\rho_o V_o} \ll 1$$

where L_o is the scale length of variation in the plasma flow. Then we make a perturbation expansion in ε, so that in zeroth order the effects of resistivity are absent. In this order no source term Q is required to maintain a steady state, because without resistive effects there are no losses. However, in higher orders of ε, these effects are considered and we show below how a solution is obtained in each order. Formally this expansion of the dimensionless equations in orders of ε is equivalent to an expansion of the dimensional equations in terms of the η and Q terms, so that we need consider only this latter expansion in what follows. We emphasise the important fact that this procedure is exact with respect to inertial effects while treating the effect of restivity in a well defined, approximate manner.

The equations (1) to (5) can be written in the following form

$$\underline{B}.\nabla\Phi = -\eta J_{\parallel} B \tag{11}$$

$$\underline{V}_{\perp} = \frac{1}{B^2}\underline{B}\times\nabla\Phi + \frac{\eta}{B^2}\underline{B}\times\underline{J} \tag{12}$$

$$\underline{B}.\nabla\left(\frac{1}{2}V_{\parallel}^2 - \frac{1}{2}V_{\perp}^2 + \frac{V_{\parallel}}{fB}\frac{\partial\Phi}{\partial r} + \tilde{c}^2 \ln\rho\right) = -\underline{V}.\mathrm{rot}\,\eta\underline{J} - \mathrm{div}\,\eta\left\{\frac{V_{\parallel}J_{\parallel}}{f}\nabla r + \frac{V_{\parallel}}{B}\underline{B}\times\underline{J}\right\} \tag{13}$$

$$\underline{B}.\nabla\left(\frac{\rho V_{\parallel}}{B} + \frac{\rho}{fB^2}\frac{\partial\Phi}{\partial r}\right) = Q - \mathrm{div}\,\eta\left\{\frac{\rho J_{\parallel}}{fB}\nabla r + \frac{\rho}{B^2}\underline{B}\times\underline{J}\right\} \tag{14}$$

$$\underline{J}_\perp = \frac{1}{B^2}\underline{B}\times(\nabla p + \tfrac{1}{2}\rho\nabla v^2) - \frac{\rho}{B^2}\underline{B}\times(\underline{v}\times \mathrm{rot}\,\underline{v}) \tag{15}$$

$$\underline{B}.\nabla\frac{1}{B}J_\parallel = -\,\mathrm{div}\,\underline{J}_\perp \tag{16}$$

$$\tilde{c}^2 := \frac{1}{\ln\rho}\left\{\int_{\rho^*}^{\rho}\frac{\partial p}{\partial\rho'}\frac{d\rho'}{\rho'} + \ln\rho^*\left.\frac{\partial p}{\partial\rho}\right|_{x=x^*}\right\} \tag{17}$$

where we denote vector components parallel and perpendicular to $\underline{B}$ with the subscripts $\parallel$ and $\perp$ respectively. Briefly these equations are derived from (1) to (5) as follows: (11) is the parallel component of (2); (12) and (15) are the perpendicular components of (2) and (1) respectively; (16) comes directly from (4) where we have used $\mathrm{div}\underline{B} = 0$. (13) is essentially the parallel component of (1) where Ohm's law (2), and (12) have been used to introduce η terms. The definition (17) is required to transform the term $1/\rho\,\underline{B}.\nabla p$ to $\underline{B}.\nabla\,\tilde{c}^2\ln\rho$. (14) is the continuity equation (3) where, as in the derivation of (13), (12) has been used to eliminate $\underline{v}_\perp$. In both (13) and (14) use has been made of the relation

$$\mathrm{div}\,\gamma(\underline{B}\times\nabla\phi) = \underline{B}.\nabla\left(\frac{\gamma}{f}\frac{\partial\phi}{\partial r}\right) + \mathrm{div}\left(\frac{\eta\gamma}{f}J_\parallel B\nabla r\right) \tag{18}$$

where γ is an arbitrary scalar and where a $\underline{B}.\nabla\phi$ has been replaced by means of equation (11).

We now discuss the general structure of the system of equations (11) to (16) with a view to formulating a procedure for solution in each order of the η expansion. (12) and (16) are algebraic equations for the components of $\underline{J}_\perp$ and $\underline{v}_\perp$ in terms of the other quantities in the same order of expansion. In a given order (11), (13) and (14) are written in the form of magnetic differential equations [6]. These equations give solutions in terms of the right hand sides, which involve only quantities of lower order than those constituting the operands of $\underline{B}.\nabla$. In each solution a surface quantity is included, which remains undetermined for the present. (15) is also a magnetic differential equation which gives $J_\parallel$ in terms of $\underline{J}_\perp$ in the same order, and a surface quantity.

To illustrate the systematic procedure for solution order by order, we assume that all quantities up to order n-1 are known, and show the nth order quantities are determined. First of all the generalised electric potential $\Phi^{(n)}$ is given by (11) which allows us to calculate $\underline{v}_{\perp}^{(n)}$ from (12). (13) and (14) must be solved simultaneously for $\rho^{(n)}$ and $v_{\parallel}^{(n)}$, then $\underline{J}_{\perp}^{(n)}$ comes from (16) and finally $J_{\parallel}^{(n)}$ is obtained from (15). We can make the procedure somewhat clearer (although in principle, it is unaltered) if instead of (15) we write (4) in terms of the r, θ, ξ components. We obtain J_r from the ξ component of (1) together with the use of (3) and the ξ component of (2), in the form

$$rNJ_r = \frac{1}{fB_0}\left\{\frac{\partial}{\partial r}\left(\frac{1}{fB_0}\eta\rho r v_\xi J_\xi N^3\right) - rN^2 Q v_\xi\right\} + \frac{\partial}{\partial\theta}\left(\rho N^2 v_\theta v_\xi\right) \tag{20}$$

and so instead of (15) we have for div$\underline{J}$ = 0 the equation

$$B.\nabla\left(NJ_\theta + \frac{\partial}{\partial r}\left(\frac{1}{fB_0}v_\theta v_\xi\right)\right) = -\frac{f}{rN}\frac{\partial}{\partial r}\left\{\frac{1}{f}\left(\frac{\partial}{\partial r}\left(\frac{1}{fB_0}\eta\rho r v_\xi J_\xi N^3\right) - rN^2 Q v_\xi\right)\right\} \tag{21}$$

which has the general form of the other magnetic differential equations discussed above. This emphasises that the solution for quantities of order n is determined by quantities of lower order. The details of the solution procedure remain the same for $\underline{v}^{(n)}$, $\rho^{(n)}$, $\Phi^{(n)}$, but instead of using (15) and (16) we now use (20) for $J_r^{(n)}$, (21) for $J_\theta^{(n)}$ and the ξ component of equation (2) for $J_\xi^{(n)}$.

Clearly we are interested only in solutions periodic in the angle variable θ. This requirement places restrictions upon the surface functions in the solution as we will now show. The magnetic differential equations in order n are of the form

$$\frac{\partial G^{(n)}}{\partial\theta} = R^{(n)}(r,\theta) \tag{22}$$

where $G^{(n)}$ is the column of functions $G_j^{(n)}$ (j = 1,..,4) which are the operands of $\partial/\partial\theta$ appearing in (11), (13) and (21) respectively. $R^{(n)}$ is defined as the column of the correspon-ding right hand sides of these equations. The formal solution of (22) is

$$G^{(n)}(r,\theta) = \int_0^\theta R^{(n)}(r,\theta')\,d\theta' + S^{(n)}(r) \tag{23}$$

where $S^{(n)}(r)$ is the column of surface functions. Thus the n-th order quantities depend on the $S_j^{(n)}$ and their derivatives. Closer examination reveals that the $R_j^{(n+1)}$, which are made up of quantities of order n, depend on r, θ, the $S_j^{(n)}$ and the first three derivatives of the $S_j^{(n)}$. Because the generalised electric potential is determined up to a constant it is possible to redefine these surface functions so that only second deriva-tives appear. In what follows these are the surface functions discussed. Formally we can write in order n+1

$$\frac{\partial G^{(n+1)}}{\partial\theta} = R^{(n+1)}\left(r,\theta,S^{(n)},\frac{dS^{(n)}}{dr},\frac{d^2S^{(n)}}{dr^2}\right) \tag{24}$$

The periodicity requirement on the solution $G^{(n+1)}$ gives the relations

$$\langle R^{(n+1)}\rangle := \frac{1}{2\pi}\int_0^{2\pi} R^{(n+1)}\,d\theta = 0 \tag{25}$$

These four relations are differential equations, which when solved with boundary conditions give the $S_j^{(n)}$. Hence the solution of order n is completed by solving in terms of the solution of order n-1 and obtaining these surface functions appearing in that solution from the relations imposed by the periodicity requirement on the order n+1 solution. This systematic procedure gives the stationary equilibrium solution, order by order.

The four conditions (25) are now written explicitly (neglecting the order superscript) as

$$\langle \eta J_{\parallel} B N \rangle = 0 \tag{26}$$

$$\langle N \underline{v} . \mathrm{rot}\, \eta \underline{J} \rangle + \langle N\, div\left(\frac{\eta v_{\parallel}}{B} \underline{B} \times \underline{J}\right)\rangle + \frac{1}{r}\frac{\partial}{\partial r}\langle \eta v_{\parallel} J_{\parallel} \frac{r}{f} N\rangle = 0 \tag{27}$$

$$\langle N Q \rangle = \langle N\, div\left(\frac{\eta \rho}{B^2} \underline{B} \times \underline{J}\right)\rangle + \frac{1}{r}\frac{\partial}{\partial r}\left\langle \frac{\eta \rho}{B f} J_{\parallel} r N\right\rangle \tag{28}$$

$$\frac{1}{f B_0^2}\frac{\partial}{\partial r}\left\langle \frac{1}{f} \eta \rho r v_{\xi} J_{\xi} N^3 \right\rangle - \frac{r}{f B_0}\langle N^2 Q v_{\xi}\rangle = \frac{i}{4\pi^2 R} = \mathrm{const.} \tag{29}$$

where i is the net current flowing through the magnetic surfaces, i.e.

$$i = 4\pi^2 R r \langle N J_r \rangle$$

(29), which is the integrated form on the equation div$\underline{J}$ = 0, states that i is a constant, determined only by boundary conditions.

IV. The Zeroth Order

So long as we are not interested in effects which come from finite conductivity the appropriate equations are (1) to (4) or (11) to (16) without the η , Q terms. These describe ideal hydromagnetic flows in the magnetic field configuration we use. As made plausible by Greene and Karlson [4] , such flows as solutions of the equations under discussion should be determined uniquely, if we specify the fluxes of magnetic field $\underline{B}$ and mass flux density $\rho \underline{v}$ both the long and the short way, and also specify temperature or entropy as a function of Φ. We treat here the case of low β so it is clear that we get a sixth surface quantity, because a divergencefree $\underline{J}_{\parallel}$ can always be added to the solution of the zeroth order equations.

To show that the result of Greene and Karlson is indeed correct in our case, we write down the zeroth order solutions. From now on we drop the superscripts denoting quantities of zeroth order.

$$\Phi = \int_0^r S_1(r')\,dr' \tag{30}$$

$$\underline{V}_\perp = \frac{1}{B^2}\underline{B}\times\nabla\Phi = \frac{S_1}{B^2}\underline{B}\times\nabla r \tag{31}$$

$$\frac{1}{2}(V_\parallel^2 - V_\perp^2) + \frac{S_1}{fB}V_\parallel + \tilde{C}^2 \ln\rho = S_2 \tag{32}$$

$$\frac{1}{B}\rho V_\parallel + \frac{\rho}{B^2 f}S_1 = S_3 \tag{33}$$

An expression for $\underline{J}$ we give later. Assume now all fluxes and temperature are given in the sense of Greene and Karlson. Apart from the surface quantity for $\underline{J}$ we have here S_1, S_2, S_3, and T (or S) as unspecified functions of r. A prescription of the toroidal magnetic flux $F(\Phi)$ determines $\Phi(r)$, because by means of (8) we can calculate F(r) so that we obtain S_1. We can then use the magnetic flux $\chi(\Phi)$ the short way to calculate

$$\iota(\Phi) = \frac{d\chi}{dF} = \frac{\chi'(\Phi)}{F'(\Phi)} \tag{34}$$

which together with $\Phi(r)$ gives $\iota(r)$. The temperature is given by $T(\Phi(r))$. The remaining surface quantities S_2 and S_3 are related to the fluxes of $\rho\underline{v}$ the long and the short way, which we denote by $2\pi\Psi$ and $2\pi\Gamma$ respectively. These are given by the equations (32) and (33) and by the definitions of Ψ and Γ

$$\Psi = \int_0^r \langle\sqrt{g}\,\rho\underline{V}\cdot\nabla\xi\rangle\,dr' \tag{35}$$

$$\Gamma = \int_0^r \sqrt{g}\,\rho\underline{V}\cdot\nabla\theta\,dr' \tag{36}$$

Thus we have shown how to derive our surface functions from those of the work [4] .

For our particular considerations it is useful to work with the following surface quantities as functions of r: Ψ' and Γ' , the derivatives of Ψ and Γ with respect to r, and $\rho_\circ$, which is defined by

$$\rho_\circ(r) := \rho(r, \tfrac{\pi}{2}) \tag{37}$$

and is undetermined by the zeroth order equations alone. In what follows we refer to an isothermal plasma and put $\tilde{c} = c =$ const. Using the functions introduced by (35), (36) and (37), and the abbreviations

$$u := \rho^2/\rho_\circ^2 \tag{38}$$

$$\gamma^2 := (1+f^2)^{-1} \tag{39}$$

$$M := \frac{\Gamma'(1-\varkappa^2)^{1/2}}{\iota\gamma\rho_\circ c r} \tag{40}$$

$$E := \frac{\Gamma' - \iota\Psi'}{\iota\rho_\circ c r\langle N\sqrt{u}\rangle} \tag{41}$$

we get the following two expressions for $\underline{v}$ and u

$$\underline{V} = \frac{1}{\rho_\circ}\left\{ \frac{\varkappa\Gamma'}{N\sqrt{u}}\nabla\theta + \frac{1}{\iota\varkappa}\left(\frac{\Gamma'(1-\varkappa^2)^{1/2}}{\sqrt{u}} - \frac{N^2}{\langle N\sqrt{u}\rangle}(\Gamma' - \iota\Psi')\right)\nabla\xi \right\} \tag{42}$$

$$u \ln u = \left(M^2 + E^2(N^2-1)\right)u - \frac{M^2}{N^2} \tag{43}$$

To illustrate the structure and the range of solvability of the transcendental equation (43) for u we refer to fig.2. This figure corresponds to $M < 1$, however the following arguments apply equally well to the case $M > 1$. We have plotted the function u.ln u and the straight line which represents the right hand side of (43) for a general value of Θ. These lines are bounded by those given by $\Theta = 0$ and $\Theta = \pi$.

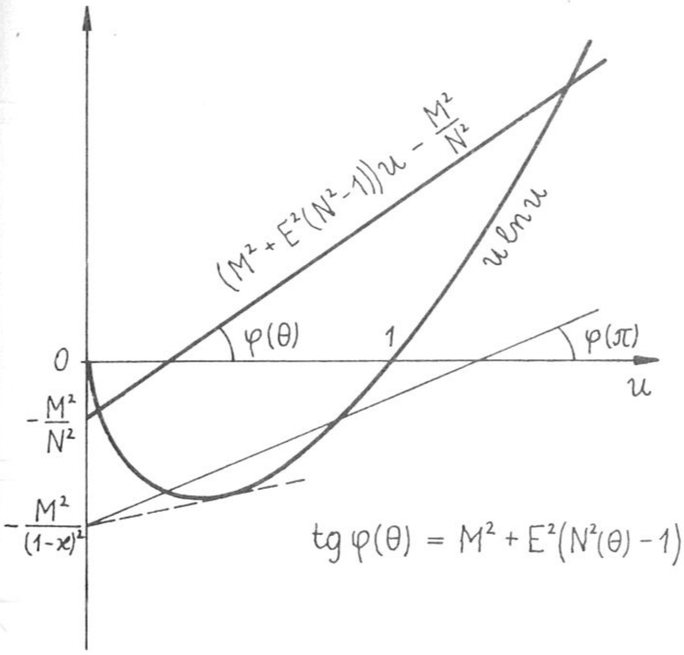

Fig.2

We have also drawn the tangent to u.lnu which has the same intercept on the vertical axis as the $\theta=\pi$ line. If for all θ there exist solutions for the mass density ρ, the procedure to obtain ρ as function of θ and r is as follows: From (43) we get u as $u(\theta, \varkappa, M^2, E^2)$. We obtain M from (40) and E from (41) by specifying Ψ, Γ and ρ_0. So if we substitute for M and E in u, by means of (38), we then have ρ.

However, we can see that solutions exist only for a restricted class of mass fluxes. Assume we fix a certain amount of mass flux Γ the short way. Then for all θ we get a real solution for u only, when we restrict Ψ so that

$$E^2 \leqslant \frac{\ln\left\{\frac{e^{M^2-1}}{M^2}(1-\varkappa)^2\right\}}{1-(1-\varkappa)^2} =: F(M^2,\varkappa) \tag{44}$$

This inequality comes from the fact that to intersect the curve u.ln u the straight line represented by the right hand side of (43) for $\theta=\pi$, must have a sufficiently large slope, i.e. larger than the slope of the tangent line with the same intercept on the vertical axis. In fig.3 the function F is plotted in a M^2-E^2 plane for two different values of $\varkappa$. Since by (40) and (41) M and E are real quantities, these curves, each specified by a definite aspect ratio, together with the lines M=0 and E=0, bound the regions corresponding to a stationary solution. For any nonzero $\varkappa$ it is impossible to reach one region from the other by a continuous change of the mass fluxes.

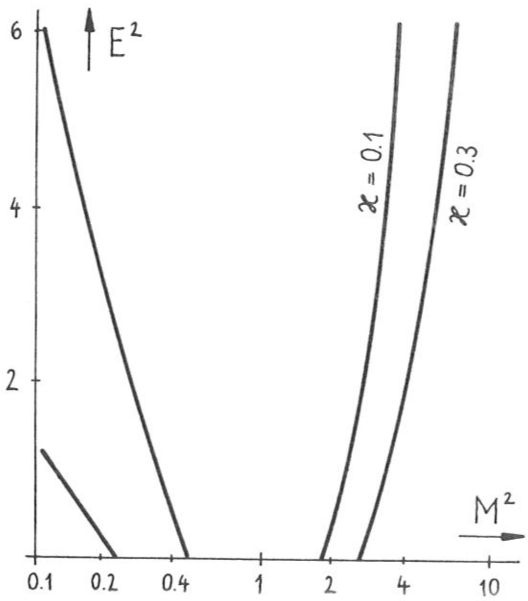

Fig.3

Mapping now the indicated region into a $\Gamma' - \Psi'$ plane we can illustrate the corresponding restrictions for the derivatives of the mass fluxes. By (40) and (41) it is clear that, apart from geometrical factors and the sound speed c, the rotational transform ι, and the mass density ρ_0 defined by equation (37) will enter in the map. If we denote the total mass contained in a torus with small radius r by $4\pi^2 \mathcal{M}$, then for given mass fluxes, ρ_0 is related to $\mathcal{M}' = d\mathcal{M}/dr$

$$\mathcal{M}' = \rho_0 r R \langle N\sqrt{u}\rangle (M^2, E^2) \tag{45}$$

Here we have written out the functional dependence of the average $\langle N \sqrt{u}\rangle$. In what follows we take the liberty to indicate for functions only those arguments which are important in the situation under consideration. Filling each magnetic surface with a certain amount of mass which is in zeroth order at our disposal, obviously corresponds for given mass fluxes to a definite function $\rho_0(r)$ and vice versa. We take this arbitrariness in the mass distribution into account by plotting in fig.4 dimensionless mass fluxes. Analytically the boundary curves of the allowed regions are obtained as follows: The quantities to be plotted we calculate from equation (40) and (41) in terms of M and E. This is possible after expressing $\langle N \sqrt{u}\rangle$, by means of the solution of (43), as a function of M and E. The resulting expression taken on the boundary is

$$\frac{\Psi'}{\rho_0 c R} = \pm\left\{ \frac{\varkappa \Upsilon}{(1-\varkappa^2)^{1/2}} |M| \mp \varkappa \sqrt{F}(M^2) \langle N\sqrt{u}\rangle (M^2, F(M^2)) \right\} \tag{46}$$

$$\frac{\Gamma'}{\rho_o c R} = \pm \iota \frac{\varkappa \gamma}{(1-\varkappa^2)^{1/2}} |M| \tag{47}$$

where

$$0 \leqslant |M| \leqslant M_c \quad \text{or} \quad |M| \geqslant \hat{M}_c \tag{48}$$

M_c, $\hat{M}_c$ are the positive roots of the transcendental equation

$$F(M^2, \varkappa) = 0 \tag{49}$$

and depend on $\varkappa$. The bounding curves ((46), (47)) are shown in fig. 4 where a logarithmic scale is used for $\Gamma'/\rho_o c R \times 10^2$.

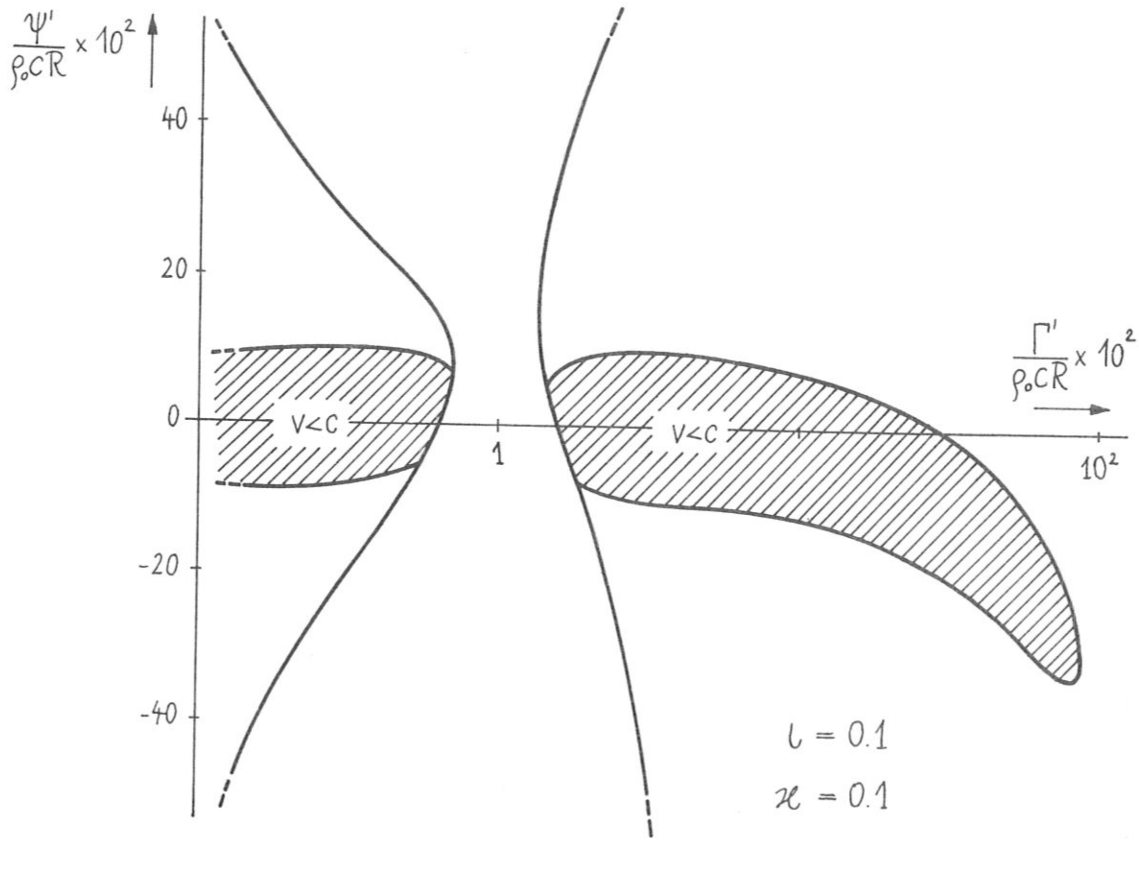

Fig.4

As can be seen from (46) and (47) these curves are invariant with respect to the transformation $(\Gamma', \Psi') \rightarrow (-\Gamma', -\Psi')$. Within the regions where a solution exists we have indicated the subregions for which the flow is subsonic. The region containing the map of the origin of the M^2-E^2 plane (fig.5) includes the quasistatic case [3] . For this reason and for reasons of comparison we exclude from the present work, consideration of the other region ($M>1$).

Fig.5 clearly demonstrates the stringent restrictions placed on the mass flux derivatives by the condition for a physical, stationary solution in zeroth order. In particular it shows that as $\iota \rightarrow 0$ the allowed range of Γ' is progressively reduced. From the definition of the mass flow rotational transform

$$\iota_M = \Gamma'/\Psi' \qquad (50)$$

it is clearly seen that for a given nonzero plasma rotation, $|\iota_M|$ cannot be chosen arbitrarily small, and the lower limit of $|\iota_M|$ increases sharply with increasing plasma rotation.

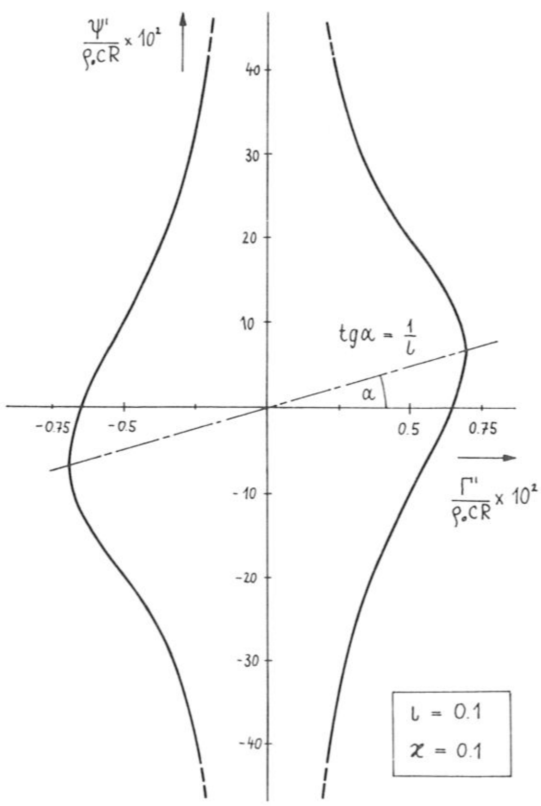

Fig.5

V. Lowest Order Plasma Diffusion

In our ordering of plasma parameters plasma diffusion, a resistive effect, appears in first order. If we denote the mass flux in the outward direction by $4\pi^2 . W^{(1)}$, then the defining expression for $W^{(1)}$ is

$$W^{(1)}(r) = \frac{1}{4\pi^2} \int_{r(x)=r} \rho \underline{v}^{(1)} . d\underline{S} \tag{60}$$

We recall that we are neglecting the order symbol of zeroth order quantities. From the expression (42) for $\underline{v}$ we see that $\rho^{(1)}\underline{v}$ gives no contribution to the surface integral (60). To simplify the consideration we assume a plasma source localised at $r = 0$. In this case $W^{(1)}$ is a constant. Then Ohm's law in first order yields for $W^{(1)}$

$$W^{(1)} = \langle \rho \underline{v}^{(1)} . \nabla r \rangle = \frac{\eta r R^2}{B_0 f} \langle \rho N^3 \underline{J} \cdot \nabla \xi \rangle \tag{61}$$

This is nothing other than the integrated version of the consistency relation (28) in first order for a localised source. For purposes of discussion and for the calculation of (61) we give the complete expressions for the components of the current density $\underline{J}$

$$J_\theta = -\frac{1}{B_0 N}\left\{\frac{\partial}{\partial r}\left(\frac{1}{f}\rho N^2 V_\theta V_\xi\right) - \langle N^{-1}\rangle^{-1}\left\langle\frac{1}{N}\frac{\partial}{\partial r}\left(\frac{1}{f}\rho N^2 V_\theta V_\xi\right)\right\rangle - \gamma^2 \langle N^{-1}\rangle^{-1}\left\langle N\left(\rho a_r + \frac{\partial p}{\partial r}\right)\right\rangle\right\} \tag{62}$$

$$J_r = \frac{1}{f B_0 N}\,\frac{1}{r}\,\frac{\partial}{\partial \theta}\left(\rho N^2 V_\theta V_\xi\right) = \frac{\rho V_\theta}{r B_0 f}\,\frac{\partial}{\partial \theta}\left(N V_\xi\right) \tag{63}$$

$$\begin{aligned} J_\xi = -\frac{N}{B_0 f}\Bigg\{ & \frac{1}{N^2}\left[\frac{\partial}{\partial r}\left(\frac{1}{f}\rho N^2 V_\theta V_\xi\right) - \langle N^{-1}\rangle^{-1}\left\langle\frac{1}{N}\frac{\partial}{\partial r}\left(\frac{1}{f}\rho N^2 V_\theta V_\xi\right)\right\rangle\right] \\ & + \left(\rho a_r - \gamma^2\langle N^{-1}\rangle^{-1} N^{-2}\langle \rho N a_r\rangle\right) + \left(\frac{\partial p}{\partial r} - \gamma^2 \langle N^{-1}\rangle^{-1} N^{-2}\left\langle N\frac{\partial p}{\partial r}\right\rangle\right)\Bigg\} \end{aligned} \tag{64}$$

where

$$a_r \equiv (\underline{v}.\nabla\underline{v}).\nabla r = -\frac{1}{r}\left(V_\theta^2 + \varkappa\,\frac{\cos\theta}{N}\,V_\xi^2\right) \tag{65}$$

is the r component of the local acceleration. We have made use of the consistency relation (26) to determine a zeroth order surface quantity which, without reference to first order, was unspecified. J_ζ , substituted in (61), gives for W

$$W^{(1)} = -\eta \frac{rRc^2}{B_0^2 f^2}\Big\{\langle \rho \frac{\partial \rho}{\partial r} N^3\rangle - \gamma^2 \langle N^{-1}\rangle^{-1}\langle \rho N\rangle\langle N \frac{\partial \rho}{\partial r}\rangle$$
$$- \frac{1}{r}\langle \rho^2 N^3 (\frac{V_\theta^2}{c^2} + \varkappa \frac{\cos\theta}{N}\frac{V_\zeta^2}{c^2})\rangle + \frac{1}{r}\gamma^2\langle N^{-1}\rangle^{-1}\langle \rho N\rangle\langle \rho N(\frac{V_\theta^2}{c^2} + \varkappa \frac{\cos\theta}{N}\frac{V_\zeta^2}{c^2})\rangle$$
$$+ \langle \rho N \frac{\partial}{\partial r}(\frac{1}{f}\rho N^2 \frac{V_\theta}{c}\frac{V_\zeta}{c})\rangle - \langle N^{-1}\rangle^{-1}\langle \rho N\rangle\langle \frac{1}{N}\frac{\partial}{\partial r}(\frac{1}{f}\rho N^2 \frac{V_\theta}{c}\frac{V_\zeta}{c})\rangle\Big\} \quad (66)$$

We now give a short explanation of the various terms appearing in the diffusion expression.

As eqn.(61) shows that the plasma loss is expressible in terms of the toroidal component of $\underline{J}$, we consider the role of this component of current density in the balance of forces. Indeed, we identify a correspondence between the terms of (66), and the forces balanced by those $\underline{J} \times \underline{B}$ forces involving J_ζ .

The component of the eqn. of of motion (1) normal to the magnetic surfaces describes how that part of the current density which is tangent to the magnetic surfaces supports the r components of the pressure gradient and the inertial force. As can be seen from (65) the latter is a centripetal force giving rise to an acceleration with respect to the axis of symmetry. Now the azimuthal current density component is related to J_r from div$\underline{J}$ = 0. The toroidal component of the eqn. of motion states that a fluid particle with nonzero V_θ and V_ζ suffers an acceleration in the ζ direction (a Coriolis acceleration), and the balancing force involves only this J_r. Thus it is clear that the part of $\underline{J}$, poloidal and tangent to the magnetic surfaces corresponds to a Coriolis force. In this way the r component of the pressure gradient, and the two types of accelerating force discussed above contribute to J_ζ and appear in the diffusion expression.

The first line of eqn.(66) is very similar to the usual resistive diffusion term, and in fact when inertia is neglected, i.e. when the density variation on magnetic surfaces vanishes, it is exactly that term. Clearly, it corresponds to the pressure gradient force. The second line corresponds to centrifugal forces, and the third line corresponds to the Coriolis force.

If $\underline{v}$, as prescribed by (42), is substituted in (66), we get the right hand side completely in terms of the magnetic field, ρ_0 and the mass fluxes Ψ and Γ. The point of view we have adopted is that $W^{(1)}$ is a specified quantity, so equation (66) can be visualised as a relation determining the profile $\rho_0(r)$ in terms of Ψ and Γ.

To determine the mass fluxes we must consider the remaining consistency relations (27) and (29). We have not solved these equations. However, whatever the boundary conditions and the corresponding solutions may be, the restrictions shown in fig.3 or fig.5 must hold for an equilibrium solution to exist. These restrictions prevent ι^{-4} like terms from playing a dominant role in the diffusion expression. Such terms formally arise when we replace v_ζ by (42), and their appearance is coupled with a nonzero Γ'.

We now examine more closely the contribution to plasma losses as a result of inertia.

For this investigation we use the dimensionless quantities M and E. We obtain for v_Θ and v_ζ from (42)

$$\frac{V_\theta}{C} = f\,\frac{\gamma M}{N\sqrt{u}} \quad , \qquad \frac{V_\zeta}{C} = \frac{\gamma M}{N\sqrt{u}} - NE \qquad (67)$$

The discussion of equation (43) for u reveals the following upper and lower limits for $\sqrt{u}$

$$\frac{|M|}{1-\varkappa} \leqslant \sqrt{u} < e^{\frac{1}{2}\{M^2+E^2((1+\varkappa)^2-1)\}} \quad \text{for} \quad M \neq 0 \tag{68}$$

$$e^{-\frac{1}{2}E^2(1-(1-\varkappa)^2)} \leqslant \sqrt{u} \leqslant e^{\frac{1}{2}E^2((1+\varkappa)^2-1)} \quad \text{for} \quad M = 0 \tag{69}$$

This always gives for v_Θ

$$\left|\frac{V_\theta}{c}\right| \leqslant f\gamma \tag{70}$$

whereas v_ξ does not seem to be limited since,

$$\left|\frac{V_\xi}{c}\right| \leqslant \gamma + (1+\varkappa)|E| \tag{71}$$

and since for M = 0 the value of E is not bounded.

We must bear in mind, however, that we lack the information contained in the consistency relations (27) and (29). Without taking first order into account, the arbitrariness of E for M = 0 is obviously due to the fact that the plasma may rotate as a whole about the axis of symmetry. It seems reasonable to eliminate the possibility of such a rotation by putting the zeroth order total angular momentum within each magnetic surface equal to zero. This will be done, but such an assumption requires further comment. Equation (29) can be written in a form which shows that this relation is nothing but the integrated form of the balance equation for the angular momentum in the respective order of expansion. Interpreted in this sense, (29) states that the change of the mechanical angular momentum is balanced by the torques produced by the injection of plasma and by the net outward-flowing current. The fulfillment of this balance in first order need not, but may, place restrictions on the zeroth order surface quantities such that the angular momentum in lowest order cannot be chosen exactly equal to zero.

For physical reasons, however, we expect that this can be done approximately.

If in zeroth order there is then no z component of the total angular momentum

$$D(r) \equiv \int_{r(x)=r} \underline{e}_z \cdot (\underline{x} \times \rho \underline{v})\, d^3\tau \tag{72}$$

about the axis of symmetry within each of the magnetic surfaces, this is equivalent to

$$\gamma M = E\langle N^3 \sqrt{u}\rangle \tag{73}$$

Using this in (67) to (69) we obtain the following upper limit for v_ξ

$$\left|\frac{v_\xi}{c}\right| \leq \gamma\left[1 - \frac{1-\varkappa}{\langle N^3\rangle}|M| \exp\left\{-\frac{1}{2}M^2\left(1+\frac{\gamma^2}{\langle N^3\sqrt{u}\rangle}\left((1+\varkappa)^2-1\right)\right)\right\} \leq 1 \tag{74}$$

The average $\langle N^3 \sqrt{u}\rangle$ is approximately unity for typical values of ι and $\varkappa$, for the allowed values of M and for those of E calculated from (73) in terms of M. Thus, v_ξ is always less than the sound speed.

On the basis of the allowed M values and of reasonable values for the scale length of ρ_0 and M, we have numerically calculated the plasma losses. These calculations show that the contribution due to inertia is, at most, of the same order of magnitude as classical resistive diffusion [3, 7] .

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