long wave side it appears that there may be some additional bands. The magnitude of this spacing indicates that we have to do with one of the transverse or "deformation" oscillations of the molecule, and the complexity of the bands is evidently due to the existence of several vibrational levels having nearly the same energy. In the case of the ideal linear molecule with Hooke's law binding the energy levels corresponding to the transverse vibrations are degenerate, but in the actual molecule this degeneracy is of course at least partly removed and there will be several transitions which will give rise to bands in nearly the same position in the spectrum. A somewhat similar complexity of structure has been observed in the ultraviolet bands of cyanogen.[3]

[1] Hurd and Pilgrim, *J. Am. Chem. Soc.*, **55**, 757 (1933).
[2] L. O. Brockway and L. Pauling, *Proc. Nat. Acad. Sci.*, **19**, 860 (1933).
[3] Sho-Chow Woo and Richard M. Badger, *Phys. Rev.*, **39** (1932).

EFFECT OF INHOMOGENEITY ON COSMOLOGICAL MODELS

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1. *Introduction.*—In the application of relativistic mechanics and relativistic thermodynamics to cosmology, it has been usual to consider homogeneous models of the universe, filled with an idealized fluid, which at any given time has the same properties throughout the whole of its spatial extent. This procedure has a certain heuristic justification on account of the greater mathematical simplicity of homogeneous as compared with non-homogeneous models, and has a measure of observational justification on account of the approximate uniformity in the large scale distribution of extra-galactic nebulae, which is found out to the some 10^8 light-years which the Mount Wilson 100-inch telescope has been able to penetrate. Nevertheless, it is evident that some preponderating tendency for inhomogeneities to disappear with time would have to be demonstrated, before such models could be used with confidence to obtain extrapolated conclusions as to the behavior of the universe in very distant regions or over exceedingly long periods of time.

It is the object of the present note to contribute to our knowledge of the effects of inhomogeneity on the theoretical behavior of cosmological models. For the immediate purposes of this investigation we shall confine our attention to very simple models composed of dust particles (nebulae) which exert negligible pressure and which are distributed non-uniformly

but nevertheless with spherical symmetry around some particular origin. This will permit us to employ expressions for the line element and its consequences which are equivalent to those recently developed by Lemaître[1] for investigating the formation of nebulae. The result of the investigation will be to emphasize the possible dangers of drawing conclusions as to the actual universe from long range extrapolations made on the basis of a homogeneous model.

2. *The Energy-Momentum Tensor.*—For the purposes of the investigation it will be simplest to use a set of co-moving coördinates such that the spatial components are determined by a network of meshes drawn so as to connect neighboring particles and allowed to move therewith. Making use of the postulated spherical symmetry and the absence of pressure and hence also of pressure gradients, it can then readily be shown that the line element for the model can be reduced to the form

$$ds^2 = -e^\lambda \, dr^2 - e^\omega(d\theta^2 + \sin^2\theta d\phi^2) + dt^2 \tag{1}$$

where λ and ω are functions of r and t. We may hence consider the energy-momentum tensor corresponding to our model and to this line element.

On the one hand, since the material filling the model is by hypothesis dust, exerting no pressure, we can use

$$T^{\alpha\beta} = \rho \, \frac{dx^\alpha}{ds} \, \frac{dx^\beta}{ds} \tag{2}$$

as an expression for this tensor, where ρ is the density of the dust as measured by a local observer moving therewith, and the quantities (dx^α/ds) and (dx^β/ds) are components of the velocity of the dust with respect to the coördinates in use. And in co-moving coördinates with the line element (1), this reduces to give a single surviving component

$$T_4^4 = \rho \qquad T_\beta^\alpha = 0 \ (\alpha \text{ or } \beta \neq 4). \tag{3}$$

On the other hand, the components of the energy-momentum tensor corresponding to the line element (1) can be computed from the formulae given for this general form by Dingle,[2] and by combining the results thus provided with the information given by (3), we easily obtain as a set of equations, connecting the metrical variables λ and ω with the density ρ,

$$8\pi T_1^1 = e^{-\omega} - e^{-\lambda}\frac{\omega'^2}{4} + \ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \Lambda = 0 \tag{4}$$

$$8\pi T_2^2 = 8\pi T_3^3 = -e^{-\lambda}\left(\frac{\omega''}{2} + \frac{\omega'^2}{4} - \frac{\lambda'\omega'}{4}\right) + \frac{\ddot{\lambda}}{4} + \frac{\dot{\lambda}^2}{4} + \frac{\ddot{\omega}}{2} +$$
$$\frac{\dot{\omega}^2}{4} + \frac{\dot{\lambda}\dot{\omega}}{4} - \Lambda = 0 \tag{5}$$

$$8\pi T_4^4 = e^{-\omega} - e^{-\lambda}\left(\omega'' + \frac{3}{4}\omega'^2 - \frac{\lambda'\omega'}{2}\right) + \frac{\dot{\omega}^2}{4} + \frac{\dot{\lambda}\dot{\omega}}{2} - \Lambda = 8\pi\rho \qquad (6)$$

$$8\pi e^{\lambda} T_4^1 = -8\pi T_1^4 = \frac{\omega'\dot{\omega}}{2} - \frac{\dot{\lambda}\omega'}{2} + \dot{\omega}' = 0, \qquad (7)$$

where the accents denote differentiation with respect to r and the dots with respect to t, and Λ is the cosmological constant.

3. *Solution of the Equations.*—To treat these equations it is convenient to begin by eliminating λ. As a first integral of (7) we can evidently write

$$e^{\lambda} = \frac{e^{\omega}\omega'^2/4}{f^2(r)} \qquad (8)$$

where $f^2(r)$ is an undetermined function of r having values which are necessarily positive.

Substituting (8) into (1), we may then rewrite the line element in the form (3)

$$ds^2 = -\frac{e^{\omega}\omega'^2/4}{f^2(r)} dr^2 - e^{\omega}(d\theta^2 + \sin^2\theta d\phi^2) + dt^2. \qquad (9)$$

Substituting (8) into (4), we obtain

$$e^{\omega}\left(\ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \Lambda\right) + \{1 - f^2(r)\} = 0. \qquad (10)$$

As the first integral of this equation we may evidently write

$$e^{3\omega/2}\left(\frac{\dot{\omega}^2}{2} - \frac{2}{3}\Lambda\right) + 2e^{\omega/2}\{1 - f^2(r)\} = F(r), \qquad (11)$$

where $F(r)$ is a second undetermined function of r. And as the integral of this equation we can write

$$\int \frac{de^{\omega/2}}{\sqrt{f^2(r) - 1 + \frac{1}{2}F(r)e^{-\omega/2} + \frac{\Lambda}{3}e^{\omega}}} = t + \mathbf{F}(r), \qquad (12)$$

where $\mathbf{F}(r)$ is a third undetermined function.

Substituting (8) into (5), it is readily found that the result is equivalent to (10), so that further consideration of (5) is not necessary.

Finally, substituting (8) into (6), we obtain for the density of the dust

$$8\pi\rho = e^{-\omega}\left\{1 - f^2(r) - \frac{4f(r)f'(r)}{\omega'}\right\} + \frac{3}{4}\dot{\omega}^2 + \frac{\dot{\omega}\omega'}{\omega'} - \Lambda. \qquad (13)$$

This result can be expressed in a variety of forms. Eliminating $f^2(r)$ with the help of (10), we obtain

$$8\pi\rho = -3\ddot{\omega} - 2\frac{\ddot{\omega}'}{\omega'} - \frac{3}{2}\dot{\omega}^2 - 2\frac{\dot{\omega}\dot{\omega}'}{\omega'} + 2\Lambda. \tag{14}$$

On the other hand, combining with (11), we obtain

$$8\pi\rho = \frac{e^{-3\omega/2}}{\omega'} \frac{\partial F(r)}{\partial r}. \tag{15}$$

Differentiating (15) we then obtain

$$\frac{\partial \log \rho}{\partial t} = -\frac{3}{2}\dot{\omega} - \frac{\dot{\omega}'}{\omega'}. \tag{16}$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = -\frac{3}{2}\ddot{\omega} - \frac{\ddot{\omega}'}{\omega'} + \left(\frac{\dot{\omega}'}{\omega'}\right)^2 \tag{17}$$

or by combining with (14) and (16)

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda + \frac{1}{3}\left(\frac{\partial \log \rho}{\partial t}\right)^2 + \frac{2}{3}\left(\frac{\dot{\omega}'}{\omega'}\right)^2. \tag{18}$$

4. *Applications.*—We are now ready to consider the application of these results to the behavior of cosmological models. From a mathematical point of view, it is evident from equations (10), (11) and (12) that we can choose the three undetermined functions $f^2(r)$, $F(r)$ and $\mathbf{F}(r)$ in such a way as to correspond to any initial values of ω, $\dot{\omega}$ and $\ddot{\omega}$ as functions of r at $t = 0$ that we wish to consider, and then at least in principle could compute the later behavior of ω as a function of r and t with the help of (12). From a more physical point of view, this means that we can start our model off at $t = 0$, with $[e^{\omega}\omega'^2/4f^2(r)]$ so chosen as to give us in accordance with the form of the line element (9) any desired initial relation between the radial coördinate r and distances as actually measured from the origin, with $\dot{\omega}$ so chosen as to give us in accordance with the co-moving character of the coördinates any desired initial distribution for the measured radial velocity of the dust in the model as a function of r, with $\ddot{\omega}$ then further so chosen as to give us in accordance with (14) any desired initial distribution for the density of dust as a function of r, and with the help of our equations could then follow the later behavior of the dust composing the model. This possibility of procedure may now be applied to some specific cases.

a. *Static Einstein Model.*—At $t = 0$, let us choose the distributions

$$e^{\omega} = r^2 \qquad \dot{\omega} = 0 \qquad \ddot{\omega} = 0. \tag{19}$$

In accordance with (9), (10), (11) and (12) this leads to Einstein's original static cosmological line element

$$ds^2 = -\frac{dr^2}{1 - \Lambda r^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2, \tag{20}$$

and in accordance with (14) we should have the well-known uniform density in the Einstein model

$$4\pi\rho = \Lambda \tag{21}$$

which would remain static in agreement with (16) and (17).

  b. *Distorted Einstein Model.*—At $t = 0$, let us choose the distributions

$$e^\omega = r^2 \qquad \dot{\omega} = 0 \qquad \ddot{\omega} = \omega_0(r), \tag{22}$$

where $\ddot{\omega}$ is initially any desired function of r.

  In accordance with (14), (16) and (18) we shall then have at $t = 0$

$$4\pi\rho = \Lambda - \frac{3}{2}\ddot{\omega}_0 - \frac{1}{2}\ddot{\omega}_0'r \tag{23}$$

$$\frac{\partial \log \rho}{\partial t} = 0 \tag{24}$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda. \tag{25}$$

Hence in this distorted Einstein model, we no longer have the uniform density of dust given by (21), and although the density of dust is initially not changing with time, the density will start to increase at those values of r where it is greater than the simple Einstein value $4\pi\rho = \Lambda$, and to decrease where it is less. This demonstrates a further kind of instability for the Einstein model—in addition to that already discussed by Eddington and others—since the initial behavior is such as to emphasize the existing differences from the uniform Einstein distribution.[4] Furthermore, in regions where the density starts to increase it is evident from the full form of equation (18) that reversal in the process of condensation would not occur short of arrival at a singular state involving infinite density or of the breakdown in our simplified equations. It will also be noted from (25) for the case of naturally flat space, $\Lambda = 0$, that *any* spherically symmetrical stationary distribution of dust would start to condense, in agreement with intuitions developed at the Newtonian level of gravitational theory.

  c. *Non-Static Friedmann Model.*—At $t = 0$ let us choose the distributions

$$e^\omega = e^g r^2 \qquad \dot{\omega} = \dot{g}_0 \qquad \ddot{\omega} = \ddot{g}_0, \tag{26}$$

where g_0, $\dot{g}_0$ and $\ddot{g}_0$ are the instantaneous values corresponding to a certain function $g(t)$. In accordance with (9), (10), (11) and (12) this will lead to the known Friedmann line element for a uniform distribution of expanding or contracting dust

$$ds^2 = -e^{g(t)} \left(\frac{dr^2}{1 - r^2/R_0^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) + dt^2, \qquad (27)$$

where R_0 is a constant the value of which can be obtained with the help of (10), and $g(t)$ has the known form of dependence on t for a homogeneous model containing nothing but dust exerting negligible pressure.

In accordance with (14), (16) and (18) the initial distribution and behavior of the dust in the model would be given by

$$4\pi\rho = \Lambda - \frac{3}{2} \ddot{g}_0 - \frac{3}{4} \dot{g}_0{}^2 \qquad (28)$$

$$\frac{\partial \log \rho}{\partial t} = -\frac{3}{2}\dot{g}_0 \qquad (29)$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda + \frac{1}{3} \left(\frac{\partial \log \rho}{\partial t} \right)^2. \qquad (30)$$

d. Distorted Friedmann Model.—At $t = 0$ let us choose the distributions

$$e^\omega = e^{g_0} r^2 \qquad \dot{\omega} = \dot{g}_0 \qquad \ddot{\omega} = \ddot{\omega}_0(r), \qquad (31)$$

where g_0 and $\dot{g}_0$ are the same quantities as before but $\ddot{\omega}$ is initially any desired function of r.

In accordance with (14), (16) and (18) we shall then have at $t = 0$

$$4\pi\rho = \Lambda - \frac{3}{2}\ddot{\omega}_0 - \frac{1}{2}\ddot{\omega}_0'r - \frac{3}{4}\dot{g}_0{}^2 \qquad (32)$$

$$\frac{\partial \log \rho}{\partial t} = -\frac{3}{2}\dot{g}_0 \qquad (33)$$

$$\frac{\partial^2 \log \rho}{\partial t^2} = 4\pi\rho - \Lambda + \frac{1}{3} \left(\frac{\partial \log \rho}{\partial t} \right)^2. \qquad (34)$$

Comparing this initial state with that for the Friedmann case, we see, although the rate of expansion or contraction in all parts of the model has been chosen the same as before, that the density of dust and the second derivative for its change with time are no longer uniform in different parts of the model. Indeed, by comparing (30) and (34) we can write for the two models at $t = 0$

$$\frac{\partial^2 \log \rho_D}{\partial t^2} - \frac{\partial^2 \log \rho_F}{\partial t^2} = 4\pi(\rho_D - \rho_F), \qquad (35)$$

where the subscripts distinguish between the Friedmann and this distorted Friedmann model. Hence at those values of r where the density in the distorted model is different from that in the Friedmann model there is at least an initial tendency for the differences to be emphasized, and from the full form of (18) it is evident in cases where condensation is taking place that the discrepancies will continue to increase until we reach a singular state involving infinite density or reach a breakdown in the simplified equations. This demonstrates a type of instability also for the Friedmann model.[4]

e. *Combination of Uniform Distributions.*—As a final application of the equations, we may consider an initial distribution at $t = 0$ which corresponds in a given zone, say from 0 to r_a, to the instantaneous conditions in a particular Friedmann model in accordance with the equations

$$e^\omega = e^{g_1} r^2 \qquad \dot\omega = \dot g_1 \qquad \ddot\omega = \ddot g_1. \qquad (36)$$

This can then be surrounded by a transition zone from r_a to r_b where the values change to a new set which correspond for a further range from r_b to r_c to a different Friedmann model in accordance with the equations

$$e^\omega = e^{g_2} r^2 \qquad \dot\omega = \dot g_2 \qquad \ddot\omega = \ddot g_2, \qquad (37)$$

and this can be followed by such additional transition zones and Friedmann zones as may be desired.

In accordance with (9), (10), (11) and (12), the dust in each Friedmann zone will then behave as in some particular completely homogeneous model without reference to the behavior of other parts of the model.[5]

5. *Conclusion.*—The foregoing results demonstrate the lack of existence of any general kind of gravitational action which would necessarily lead to the disappearance of inhomogeneities in cosmological models. This is shown both by the discovery of cases where disturbances away from an originally uniform static or non-static distribution of density would tend to increase with time, and by the possibility for models with non-interacting zones in which the behavior would agree with that for quite different homogeneous distributions.

In applying these findings to the phenomena of the actual universe, the highly simplified character of the models must of course be recognized. In the first place, although the models permit lack of homogeneity, for purposes of mathematical simplification they still retain spherical symmetry around a particular origin. Hence the phenomena of the actual universe will be affected by a more drastic kind of inhomogeneities than those here considered. In the second place, the fluid in the models was taken as dust exerting negligible pressure. Hence no allowance is made for effects such as thermal flow from one portion of matter to another

which in the actual universe might provide a *non-gravitational* kind of action which would tend to iron out inhomogeneities.

In view of the lack of complete correspondence between our models and reality we must not be too dogmatic in making assertions as to the actual universe. Nevertheless, it is at least evident from the results obtained, that we must proceed with caution in applying to the actual universe any *wide* extrapolations—either spatial or temporal—of results deduced from strictly homogeneous models. In agreement with the possibility for zones where the behavior would correspond to any desired homogeneous model, it is of course proper to treat the phenomena, in *our own neighborhood* out say to 10^8 light-years and over a *limited range of time* say 10^8 years, as approximately represented by the line element for an appropriate homogeneous model. To assert, however, that this same line element would apply to the universe as a whole, or that a homogeneous model would remain a suitable approximation at times of great condensation would not be necessarily sound. Hence, it would appear wise at the present stage of theoretical development, to envisage the possibility that regions of the universe beyond the range of our present telescopes might be contracting rather than expanding and contain matter with a density and stage of evolutionary development quite different from those with which we are familiar. It would also appear wise not to draw too definite conclusions from the behavior of homogeneous models as to a supposed initial state of the whole universe.

[1] Lemaître, *Ann. de la Soc. Scient. de Bruxelles*, **A53**, 51 (1933).

[2] These Proceedings, **19**, 559 (1933).

[3] Allowing for the difference in nomenclature, equations (9), (10), (11) and (15) are, respectively, equivalent to the equations of Lemaître, loc. cit., (8.1), (8.4), (8.2) and (8.3).

[4] Professor Dingle has been good enough to send me proof of a forthcoming article, received as this note was being completed, in which the question of the stability of homogeneous models is treated in a different manner from that here employed.

[5] This agrees with the procedure of Lemaître, loc. cit., in treating a condensing nebula surrounded by an expanding universe as analogous to two different Friedmann zones. In the present considerations we are contemplating different zones all of which are large enough to contain many nebulae.