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Effect of time-periodic boundary temperatures/ body force on Rayleigh–Benard convection in a ferromagnetic fluid

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Summary. We discuss the thermal instability in a layer of a ferromagnetic fluid when the boundaries of the layer are subjected to synchronous/asynchronous imposed time-periodic boundary temperatures (ITBT)/ time-periodic body force (TBF). Only infinitesimal disturbances are considered. The Venezian approach is adopted in arriving at the critical Rayleigh and wave numbers for small amplitudes of ITBT. A perturbation solution in powers of the amplitude of the applied temperature field is obtained. When the ITBT at the two walls are synchronized then, for moderate frequency values, the role of magnetization in inducing sub-critical instabilities is delineated. A similar role is shown to be played by the Prandtl number. The magnetization parameters and Prandtl number have the opposite effect at large frequencies. The system is most stable when the ITBT is asynchronous. The effect of TBF on the onset of convection is found to be qualitatively similar to the effect of an asynchronous ITBT. Low Prandtl number fluids are shown to be more easily vulnerable to destabilization by TBF compared to very large Prandtl number fluids. The problem has relevance in many ferromagnetic fluid applications wherein regulation of thermal convection is called for.

1 Introduction

Ferro fluid technology is the basis of a wide variety of products used for high technology applications in the semiconductor and computer industries. Ferro fluids are also used in a wide variety of thermoelectric cooling modules which prove instrumental for the refrigeration of semiconductor process equipment, laser diodes, medical treatment and optical communication equipment. They are the basis of ingenious new techniques for the separation of materials according to density. Ferro fluids have been found to be an essential element in a nuclear magnetic resonance probe differentiating free and shale oil in oil prospecting. They are used extensively for the study of magnetic domain structures in magnetic tapes, rigid discs, crystalline and amorphous alloys, garnets steels and geological rocks. Other commercial uses are ink jet printing, magneto gravimetric preparations of nonferrous metals, pumping without moving parts and biotechnology. Control of convection is important in many of these non-isothermal applications.

One of the effective mechanisms of hindering convection is through the maintenance of a non-uniform temperature gradient which is only space-dependent. However, in many practical situations non-uniform temperature gradients find their origin in transient heating or cooling at the boundaries, hence warranting the use of a basic temperature profile which is a function of both position and time. Venezian [1] investigated the stability of a horizontal layer of a viscous fluid heated from below when, in addition to a steady temperature difference between the surfaces of the layer, a time-dependent sinusoidal perturbation is applied to the wall temperatures. Subsequently, it was shown by Yih and Li [2] that time-periodic modulation of the wall temperatures has a destabilizing effect on the onset of convection over a wide range of frequencies of modulation although such a modulation is stabilizing for low frequencies. The critical Rayleigh number (corresponding to onset of convection) in these problems depends on the frequency of the imposed temperature modulation, and the study suggests that it is possible to hasten or delay the onset of instability by adjusting this modulation. The works of Lage [5], [6] deal with oscillatory heating and time-dependent vertical density gradient effects on convection. The stability of a non-ferromagnetic fluid layer subjected to an ITBT/TBF has also been studied ([1]–[4], [7], [8] and references therein).

The unmodulated Benard convection in ferromagnetic fluids has been considered by many authors ([9]–[16] and references therein). The problem of control of convection is of relevance and interest in innumerable ferromagnetic fluid applications [17], [18] and is also mathematically quite challenging. It is with this motivation that we study the problem of the ITBT/TBF–means of regulating convection. We determine the onset of convection in a ferromagnetic fluid layer heated from below when, in addition to a fixed temperature difference between the walls, an additional time-periodic perturbation is applied to the wall temperatures or we consider a time-periodic body force. We present below the two cases separately.

2 Time-periodic boundary temperatures

2.1 Mathematical formulation

We consider a ferromagnetic fluid layer confined between two infinite horizontal surfaces, a distance "h" apart. A vertical downward gravity force acts on the fluid together with a uniform, vertical magnetic field $\vec{H}_0$. A Cartesian co-ordinate system is taken with the origin in the lower boundary and the z-axis vertically upwards. The surface temperatures are

$$T_R + \frac{1}{2}\Delta T[1 + \varepsilon \cos \omega t] \quad \text{at } z = 0 \tag{1}$$

and

$$T_R - \frac{1}{2}\Delta T[1 - \varepsilon \cos (\omega t + \phi)] \quad \text{at } z = h, \tag{2}$$

where T_R is a reference temperature, ΔT is the temperature difference between the two surfaces in the unmodulated case, ε is the amplitude of the thermal modulation, ω is the frequency and ϕ is the phase (see Fig. 1). For the velocity we choose the stress-free boundary conditions and an idealized one for the magnetic field (discussed later). We adopt the Boussinesq approximation, and for small departures from T_R the density ρ, as a function of temperature T, is given by

$$\rho = \rho_R[1 - \alpha(T - T_R)], \tag{3}$$

where α is the constant coefficient of thermal expansion and $\rho_R = \rho(T_R)$. The thermal diffusivity κ and the kinematic viscosity ν of the fluid are regarded as constants. The governing equations for a Boussinesq, Newtonian ferromagnetic fluid are

$$\rho_R \frac{D\vec{q}}{Dt} = -\nabla p + \rho\vec{g} + \nabla \cdot \left(\vec{H}\vec{B}\right) + \mu\nabla^2\vec{q}, \tag{4}$$

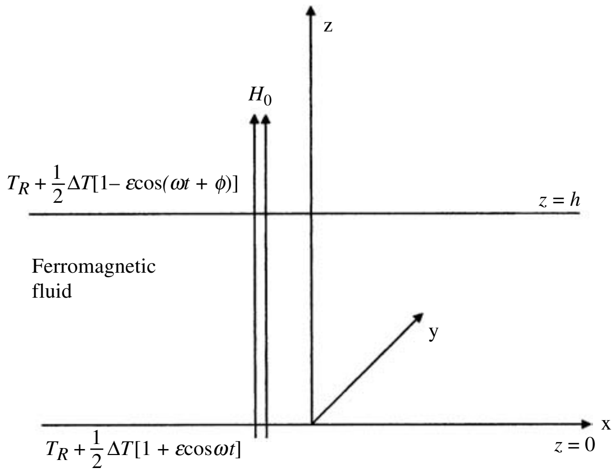

Fig. 1. Physical configuration of the Rayleigh–Benard convection in a ferromagnetic fluid with imposed time periodic boundary temperatures

$$\left[\rho_R C_{VH} - \mu_0 \vec{H} \cdot \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right] \frac{DT}{Dt} + \mu_0 T \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H} \cdot \frac{D\vec{H}}{Dt} = K_1 \nabla^2 T, \tag{5}$$

and

$$\nabla \cdot \vec{q} = 0, \tag{6}$$

where μ is the dynamic coefficient of viscosity, K_1 is the thermal conductivity, μ_0 is the magnetic permeability, $\vec{q}$ is the velocity, C_{VH} is the specific heat at constant volume and constant magnetic field, $\vec{g}$ is the acceleration due to gravity, p is the pressure, $\vec{M}$ is the magnetization, $\vec{B}$ is the magnetic induction and $\vec{H}$ is the magnetic field.

Maxwell's equations, simplified for a non-conducting fluid with no displacement current, become

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = 0 \tag{7}$$

and

$$\vec{B} = \mu_0 \left(\vec{M} + \vec{H}\right). \tag{8}$$

We assume that the magnetization $\vec{M}$ is aligned with the magnetic field, but allows a dependence on the magnitude of the magnetic field as well as the temperature,

$$\vec{M} = \frac{\vec{H}}{H} M(H, T). \tag{9}$$

The magnetic equation of state is linearized about the magnetic field H_0 and an average temperature T_R to give

$$M = M_0 + \chi(H - H_0) - K_m(T - T_R), \tag{10}$$

where χ is the magnetic susceptibility and K_m is the pyromagnetic coefficient.

We now study the condition for onset of convection in the aforementioned ferromagnetic fluid layer. In the undisturbed state, the temperature T_H, pressure p_H, applied magnetic field $\vec{H}_H$, magnetic induction $\vec{B}_H$ and magnetization $\vec{M}_H$ satisfy the following equations:

$$-\frac{\partial p_H}{\partial z} = \rho_H g - B_H \frac{\partial H_H}{\partial z}, \tag{11}$$

$$\left[\rho_R C_{VH} - \mu_0 \vec{H}_H \cdot \left(\frac{\partial \vec{M}_H}{\partial T}\right)_{V,H}\right] \frac{\partial T_H}{\partial t} = K_1 \nabla^2 T_H \tag{12}$$

and

$$\rho_H = \rho_R[1 - \alpha(T_H - T_R)]. \tag{13}$$

Following Venezian [1], the solution of (12) satisfying the thermal boundary conditions (1) and (2) is

$$T_H = T_R + \frac{\Delta T}{2h}(h - 2z) + \varepsilon \,\mathrm{Re}\left\{\left[a(\lambda)e^{\frac{\lambda z}{h}} + a(-\lambda)e^{\frac{-\lambda z}{h}}\right]e^{-i\omega t}\right\}, \tag{14}$$

where

$$\begin{aligned}&\lambda = (1 - i)\left(\frac{\omega h^2}{2K'}\right)^{\frac{1}{2}}, \quad a(\lambda) = \frac{\Delta T}{2}\left[\frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right],\\ &K' = \frac{K_1}{C_1}, \quad C_1 = \rho_R C_{VH} - \mu_o \vec{H}_H \cdot \left(\frac{\partial \vec{M}_H}{\partial T}\right)_{V,H}\end{aligned} \tag{15}$$

and Re stands for the real part.

2.2 Linear stability analysis

Let the basic state be disturbed by an infinitesimal thermal perturbation. We now have

$$\begin{aligned}&\vec{q} = \vec{q}_H + \vec{q}\,', \quad p = p_H + p', \quad \rho = \rho_H + \rho', \quad T = T_H + \theta,\\ &\vec{H} = H_H \hat{k} + \vec{H}' \quad \text{and} \quad \vec{M} = M_H \hat{k} + \vec{M}'.\end{aligned} \tag{16}$$

The prime indicates that the quantities are infinitesimal perturbations.

Substituting Eqs. (16) into Eqs. (3)–(10) and using the basic state solution, we get the linearized equations governing the infinitesimal perturbations in the form:

$$\rho' = -\alpha \rho_R \theta, \tag{17}$$

$$\begin{aligned}\rho_R \frac{\partial \vec{q}\,'}{\partial t} = &-\nabla p' + \rho' \vec{g} - \frac{\mu_0 K_m}{(1+\chi)}\left(\frac{\Delta T}{h} - \varepsilon \frac{\partial T_1}{\partial z}\right)\left[(1+\chi)H_3' - K_m \theta\right]\\ &+ \mu_0 (M_0 + H_0)\frac{\partial \vec{H}'}{\partial z} + \mu \nabla^2 \vec{q},\end{aligned} \tag{18}$$

$$\begin{aligned}&\left[\rho_R C_0 - \frac{\mu_0 K_m^2}{(1+\chi)}\left\{\frac{\Delta T}{2h}(2z - h) - \varepsilon T_1\right\}\right]\frac{\partial \theta}{\partial t} + \rho_R C_0 w \frac{\partial T_H}{\partial z}\\ &- \mu_0 K_m \left[T_0 - \frac{\Delta T}{2h}(2z - h) + \varepsilon T_1\right]\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial z}\right) + \frac{\mu_0 K_m^2 T_0}{(1+\chi)}\left\{\frac{\Delta T}{h} - \varepsilon \frac{\partial T_1}{\partial z}\right\} w = K_1 \nabla^2 \theta,\end{aligned} \tag{19}$$

$$\nabla \cdot \vec{q}\,' = 0, \tag{20}$$

$$(1+\chi)\frac{\partial^2 \Phi}{\partial z^2} + \left(1 + \frac{M_0}{H_0}\right)\nabla_1^2 \Phi - K_m \frac{\partial \theta}{\partial z} = 0, \tag{21}$$

where Φ is the magnetic potential and $\vec{H}' = \nabla \Phi$.

Substituting Eq. (17) in (18) and operating curl twice on the resulting equation, writing Eqs. (18)–(21) in dimensionless form by setting

$$(x^*, y^*, z^*) = \left(\frac{x}{h}, \frac{y}{h}, \frac{z}{h}\right), \quad W^* = \frac{w}{\left(\frac{K_1}{\rho_R C_0 h}\right)}, \quad D^* = hD,$$
$$t^* = \frac{t}{\left(\frac{\rho_R C_0 h^2}{K_1}\right)}, \quad \theta^* = \frac{\theta}{\left(\frac{\nu K_1}{\rho_R C_0 \alpha g h^3}\right)}, \quad \Phi^* = \frac{\Phi}{\left(\frac{K_m \nu K_1}{\rho_R C_0 (1+\chi) \alpha g h^2}\right)}, \tag{22}$$

we get:

$$\nabla^2\left(\nabla^2 - \frac{1}{\Pr}\frac{\partial}{\partial t}\right)W + \left(1 - M_1\frac{\partial T_H}{\partial t}\right)\nabla_1^2\theta + M_1\frac{\partial T_H}{\partial z}\frac{\partial}{\partial z}\left(\nabla_1^2\Phi\right) = 0, \tag{23}$$

$$(1 + M_2)\frac{\partial \theta}{\partial t} + (1 + M_2)RW\frac{\partial T_H}{\partial z} - M_2\left(1 + \frac{T_H}{T_0}\right)\frac{\partial (D\Phi)}{\partial t} = \nabla^2\theta, \tag{24}$$

$$\left(\frac{\partial^2}{\partial z^2} + M_3\nabla_1^2\right)\Phi - \frac{\partial \theta}{\partial z} = 0, \tag{25}$$

where $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2}$.

In the above equations the asterisks have been dropped for simplicity. The dimensionless parameters are

$$\Pr = \frac{\nu}{\kappa} \quad \text{(Prandtl number)},$$

$$R = \frac{\rho_0 C_0 \alpha g \Delta T h^3}{\nu K_1} \quad \text{(Rayleigh number)},$$

$$M_1 = \frac{\mu_0 K_m^2 \Delta T}{(1+\chi)\alpha g \rho_0 h} \quad \text{(Buoyancy magnetization parameter)},$$

$$M_2 = \frac{\mu_0 K_m^2 T_0}{\rho_0 C_0 (1+\chi)} \quad \text{(Magnetization parameter)},$$

and

$$M_3 = \frac{\left(1 + \frac{M_0}{H_0}\right)}{(1+\chi)} \quad \text{(Non-buoyancy magnetization parameter)}.$$

In Eq. (24), $\frac{\partial T_H}{\partial z}$ is given by

$$\frac{\partial T_H}{\partial z} = -1 + \varepsilon f, \tag{26}$$

where

$$f = \mathrm{Re}\left\{\left(A(\lambda)e^{\lambda z} + A(-\lambda)e^{-\lambda z}\right)e^{-i\omega t}\right\}$$

and

$$A(\lambda) = \frac{\lambda}{2}\left[\frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right].$$

Equations (23)–(25) are solved subject to the following conditions appropriate for stress–free, isothermal and magnetic boundaries:

$$W = \frac{\partial^2 W}{\partial z^2} = \theta = \frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = 0, 1. \tag{27}$$

The last of the conditions in Eq. (27) is based on the assumption of infinite susceptibility with respect to the perturbed magnetic potential. This simple boundary condition (27) though admittedly an artificial one to consider, is of importance since its exact solution is readily obtained, and the essential features of the problem can be disclosed by a discussion of this case.

Combining Eqs. (23)–(25) we obtain an equation for the vertical component of velocity W in the form:

$$\nabla^2\left(\nabla^2 - \frac{1}{\Pr}\frac{\partial}{\partial t}\right)\left[\left(\nabla^2 - \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3\right) + M_2\left(1 + \frac{T_H}{T_0}\right)\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial t}\right]W$$
$$= -R\frac{\partial T_H}{\partial z}\left[\left(1 - M_1\frac{\partial T_H}{\partial z}\right)\left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3\right) + M_1\frac{\partial T_H}{\partial z}\frac{\partial^2}{\partial z^2}\right]\nabla_1^2 W. \tag{28}$$

In dimensionless form, the velocity boundary conditions are (see [19])

$$W = \frac{\partial^2 W}{\partial z^2} = \frac{\partial^4 W}{\partial z^4} = \frac{\partial^6 W}{\partial z^6} = 0 \quad \text{at } z = 0, 1, \tag{29}$$

where the sixth-order condition has been derived from the governing equations.

2.3 Stability analysis

Let us now seek the eigenfunctions W and the eigenvalues R of Eq. (28) for the basic temperature distribution (26) which departs from the linear profile $\frac{\partial T_H}{\partial z} = -1$ by quantities of order ε. Thus, the eigenvalues of the present problem differ from those of ordinary Benard convection by quantities of order ε. Since the adopted technique is based on small-amplitudes, ε has to be less than unity. We seek a solution of (28) in the form:

$$W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \cdots\cdots\cdots,$$
$$R = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots\cdots\cdots, \tag{30}$$

where R_0 is the critical Rayleigh number for the unmodulated Rayleigh–Benard convection in ferromagnetic fluids. Substituting Eq. (30) into Eq. (28) and equating powers of ε , we obtain the following system of equations:

$$LW_0 = 0, \tag{31}$$

$$LW_1 = R_1\left(\frac{\partial^2}{\partial z^2} + M_3(1+M_1)\nabla_1^2\right)\nabla_1^2 W_0 - R_0 f\left[M_1 M_3 \nabla_1^2 + \left(\frac{\partial^{2\cdot}}{\partial z^2} + M_3(1+M_1)\nabla_1^2\right)\right]\nabla_1^2 W_0, \tag{32}$$

$$LW_2 = R_0 f\left[\nabla_1^4 M_1 M_3 (f\, W_0 - W_1) - \left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3(1+M_1)\right)\nabla_1^2 W_1\right]$$
$$- R_1\left[M_1 M_3 f \nabla_1^4 W_0 + \left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3(1+M_1)\right)\nabla_1^2 (f\, W_0 - W_1)\right]$$
$$+ R_2\left[\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3(1+M_1)\right]\nabla_1^2 W_0, \tag{33}$$

where

$$L = \nabla^2\left(\nabla^2 - \frac{1}{\Pr}\frac{\partial}{\partial t}\right)\left[\left(\nabla^2 - \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3\right) + M_2\left(1 + \frac{T_H}{T_0}\right)\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial t}\right]$$
$$- \nabla_1^2\left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3(1 + M_1)\right)R_0. \tag{34}$$

We now perform some simplifications in Eq. (34). To this end we note that typical values of M_2 are 10^{-6} (see [9]). Hence, we consider $M_2 = 0$ in Eq. (34) and proceed further. Justification for the neglect of this term can also be provided computationally (see Results and Discussion).

The function W_0 is the solution of the unmodulated Rayleigh–Benard problem in ferromagnetic fluids [9]. The marginally stable solution for that problem is

$$W_0 = \exp\{i(k_x x + k_y y)\}\sin \pi z, \tag{35}$$

corresponding to the lowest mode of convection with the Rayleigh number given by [9]

$$R_0 = \frac{\left(\pi^2 + a^2\right)^3}{a^2\left[1 + M_1 - \frac{M_1\pi^2}{(\pi^2 + a^2 M_3)}\right]}. \tag{36}$$

Equation (32) on using Eq. (35) becomes

$$LW_1 = R_1 a^2\left[\pi^2 + a^2 M_3 + a^2 M_1 M_3\right]\sin \pi z - R_o f a^2\left[\pi^2 + a^2 M_3 + 2a^2 M_1 M_3\right]\sin \pi z. \tag{37}$$

If the above equation is to have a solution, then the right hand side must be orthogonal to the null space of the operator L. This implies that the time-independent part of the right hand side of Eq. (37) must be orthogonal to $\sin \pi z$. Since f varies sinusoidally in time, the only steady term is $R_1 a^2\left[\pi^2 + a^2 M_3 + a^2 M_1 M_3\right]\sin \pi z$, so that R_1 is zero. This result could have been anticipated because changing the sign of ε merely amounts to a shift in the time origin by half a period. Since such a shift does not affect the stability problem, it follows that all the odd co-efficients $R_1, R_3, \ldots\ldots$ in Eq. (30) must vanish.

To solve Eq. (37) we expand the right hand side in a Fourier series and obtain W_1 by inverting the operator L term by term. Following Venezian [1], we arrive at the following expression for R_2:

$$R_2 = -\frac{R_0^2 a^2}{2}\frac{\left[\pi^2 + a^2 M_3 + 2a^2 M_1 M_3\right]}{\left[\pi^2 + a^2 M_3 + a^2 M_1 M_3\right]}\sum_{n=1}^{\infty}\left\{\left(n^2\pi^2 + a^2 M_3 + 2a^2 M_1 M_3\right)|B_n(\lambda)|^2\frac{C_n}{d_n}\right\}. \tag{38}$$

2.4 *Minimum Rayleigh number for convection*

The value of R obtained by this procedure is the eigenvalue corresponding to the eigenfunction W which, though oscillating, remains bounded in time. Since R is a function of the horizontal wave number a and the amplitude of perturbation ε, we may take

$$R(a, \varepsilon) = R_0(a) + \varepsilon^2 R_2(a) + \cdots\cdots\cdots \tag{39}$$

It was shown by Venezian [1] that the critical value of R, i.e., R_c to evaluate the critical value of R_2 is determined to $O(\varepsilon^2)$ by evaluating R_0 and R_2 at $a = a_0$. It is only when one wishes to evaluate R_4 that a_2 must be taken into account where $a = a_2$ minimizes R_2. To evaluate the critical value of R_2 (denoted by R_{2c}) one has to substitute $a = a_0$ in R_2, where a_0 is the value at which R_0 given by Eq. (36) is minimum. We evaluate R_{2c} in the following three cases:

(a) When the oscillating temperature field is symmetric so that the wall temperatures are modulated in phase (with $\phi = 0$). In this case $B_n(\lambda) = b_n$ or 0 according to whether n is even or odd.

(b) When the wall temperature field is antisymmetric corresponding to out-of-phase modulation (with $\phi = \pi$). In this case $B_n(\lambda) = 0$ or b_n according to whether n is even or odd.
(c) When only the temperature of the bottom wall is modulated, the upper plate being held at a constant temperature. This case corresponds to $\phi = -i\infty$. Here $B_n(\lambda) = (1/2)b_n$ for integer values of *n,* where

$$b_n = \frac{-4n\pi^2\lambda^2}{\left[\lambda^2 + (n+1)^2\pi^2\right]\left[\lambda^2 + (n-1)^2\pi^2\right]},$$

$$\lambda = (1-i)\left(\frac{\omega}{2}\right)^{\frac{1}{2}},$$

$$|b_n|^2 = \frac{16n^2\pi^4\omega^2}{\left[\omega^2 + (n+1)^4\pi^4\right]\left[\omega^2 + (n-1)^4\pi^4\right]},$$

$$C_n = \left\{ \begin{array}{l} (n^2\pi^2 + a^2)(n^2\pi^2 + M_3a^2)\left((n^2\pi^2 + a^2)^2 - \frac{\omega^2}{\mathrm{Pr}}\right) \\ -a^2R_0\left[(1+M_1)(n^2\pi^2 + M_3a^2) - M_1\pi^2\right] \end{array} \right\}$$

and

$$d_n = \left\{ \begin{array}{l} (n^2\pi^2 + a^2)(n^2\pi^2 + M_3a^2)\left((n^2\pi^2 + a^2)^2 - \frac{\omega^2}{\mathrm{Pr}}\right) \\ -a^2R_0\left[(1+M_1)(n^2\pi^2 + M_3a^2) - M_1\pi^2\right] \end{array} \right\}^2 + \left\{\omega\left(1 + \frac{1}{\mathrm{Pr}}\right)(n^2\pi^2 + a^2)^2(n^2\pi^2 + M_3a^2)\right\}^2.$$

Following Venezian [1], we get the expression for R_{2c} in the form

$$R_{2c} = -\frac{R_0^2a^2}{2}\frac{\left[\pi^2 + a^2M_3 + 2a^2M_1M_3\right]}{\left[\pi^2 + a^2M_3 + a^2M_1M_3\right]}\sum\left[n^2\pi^2 + a^2M_3 + 2a^2M_1M_3\right]|b_n|^2\frac{C_n}{d_n}. \tag{40}$$

In Eq. (40), the summation extends over even values of *n* for case (a), odd values of *n* for case (b) and all integer values for case (c). The infinite series (40) converges in all cases for 5 terms.

3 Time-periodic body force

3.1 Mathematical formulation

We consider a Boussinesq ferromagnetic fluid layer confined between two infinite horizontal walls, a distance "h" apart. A periodically varying vertical gravity field acts on the fluid and is taken as

$$\vec{g} = g_0[1 + \varepsilon\cos\,\omega t]\hat{k}, \tag{41}$$

where g_0 is the mean gravity, ε is the amplitude of the TBF, ω is the frequency, t is the time and $\hat{k}$ is the unit vector in the vertical direction. The TBF is also referred to as g-jitter and can be generated by vertically oscillating the fluid layer, rhythmically, thus causing a cosinusoidal modulation of the gravitational field [8].

The governing equations are essentially those used in case A except Eq. (4) in place of which we take the following equation:

$$\rho_0 \frac{D\vec{q}}{Dt} = -\nabla p + \rho g_0 (1 + \varepsilon \cos \omega t)\hat{k} + \nabla.\left(\vec{H}\vec{B}\right) + \mu \nabla^2 \vec{q}, \tag{42}$$

where all the quantities have their usual meaning as defined in Sect. 1.

3.2 Basic state

The basic state and the perturbations are as indicated in part A. Following the analysis of Sect. 1, we get the equation for the vertical component of the velocity W in the nondimensional form as

$$\begin{aligned}&\nabla^2\left(\nabla^2 - \frac{\partial}{\partial t}\right)\left(\nabla^2 - \Pr\frac{\partial}{\partial t}\right)\left[\left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3\right)\left(\Pr\frac{\partial}{\partial t} - \nabla^2\right)\right]W \\ &\quad = R\left[(1 + \varepsilon \cos \omega t + M_1)\left\{\left(\frac{\partial^2}{\partial z^2} + \nabla_1^2 M_3\right)\left(\Pr\frac{\partial}{\partial t} - \nabla^2\right)\right\} - M_1 \frac{\partial^2}{\partial z^2}\left(\Pr\frac{\partial}{\partial t} - \nabla^2\right)\right]\nabla_1^2 W,\end{aligned} \tag{43}$$

and the velocity boundary conditions are Eq. (29).

3.3 Stability analysis

As in Sect. 1, we seek the eigenfunctions W and the eigenvalues R of Eq. (43) for small amplitude of the modulation ($\varepsilon < 1$). The eigenvalues of the present problem differ from those of ordinary Rayleigh–Benard convection by quantities of order ε.

Following the approach given in Sect. 1, we get

$$R_{2c} = \frac{R_0^2 a^2}{2} \frac{\left[\pi^2 + a^2 M_3\right]}{\left[\pi^2 + a^2 M_3 + a^2 M_1 M_3\right]} \sum \left[n^2\pi^2 + a^2 M_3\right] C_n, \tag{44}$$

where

$$C_n = \left\{ \begin{array}{l} \left(n^2\pi^2 + a^2\right)\left(n^2\pi^2 + M_3 a^2\right)\left(\left(n^2\pi^2 + a^2\right)^2 - \Pr\omega^2\right) \\ -a^2 R_0\left[(1 + M_1)\left(n^2\pi^2 + M_3 a^2\right) - M_1 n^2\pi^2\right] \end{array} \right\} / d_n,$$

$$\begin{aligned} d_n = &\left\{ \begin{array}{l} \left(n^2\pi^2 + a^2\right)\left(n^2\pi^2 + M_3 a^2\right)\left(\left(n^2\pi^2 + a^2\right)^2 - \Pr\omega^2\right) \\ -a^2 R_0\left[(1 + M_1)\left(n^2\pi^2 + M_3 a^2\right) - M_1 n^2\pi^2\right] \end{array} \right\}^2 \\ &+ \left\{\omega(1 + \Pr)\left(n^2\pi^2 + a^2\right)^2\left(n^2\pi^2 + M_3 a^2\right)\right\}^2. \end{aligned}$$

The expression for R_0, the eigenvalue of the unmodulated problem, is the same as Eq. (36). We now discuss about the roles played by ITBT/TBF and magnetization parameters on Rayleigh–Benard convection in Newtonian ferromagnetic Boussinesq fluids.

4 Results and discussion

We note here that the applied uniform magnetic field affects convection in the electrically non-conducting fluid essentially due to the micron-sized ferrite particles which are suspended in the carrier fluid. The micron-sized magnetic particles make the fluid magnetically-responding in addition to being thermally responding. Before we embark on a discussion of results depicted by the graphs, we must note that the presence of suspended ferrite particles in the carrier fluid is to increase its viscosity. This follows from the well-known Einstein relation for suspended particles,

$$\mu = \mu_0(1 + 2.5\alpha\phi),$$

where μ and μ_0 are the viscosities of ferromagnetic suspension (i.e. carrier fluid + suspended ferrite particles) and carrier fluid, respectively, α is the shape factor and ϕ is the volume fraction of the suspended particles. In view of this we consider values of the Prandtl number of ferromagnetic fluids higher than those of carrier fluids without suspended particles. We first discuss the results of ITBT followed by those of TBF.

ITBT

A note on the role played by ITBT is also to be mentioned here. In this case R_{2c} is a crucial quantity which determines whether ITBT leads to sub-critical instability or not. The study of the behaviour of R_{2c} is of some interest in the limiting cases of very small and very large frequencies. We find that when $\omega \ll 1, R_{2c}$ depends weakly on the magnetization parameters M_1 and M_3 but when $\omega \gg 1, R_{2c}$ tends to zero, so that the effects of ITBT and the magnetization parameters become small. For moderate values of ω, magnetization parameters will affect R_{2c}. In the paper we consider two types of ITBT:

(i) Synchronous ITBT which means that the two ITBTs are in-phase. ($\phi = 0$)
(ii) Asynchronous ITBT which means that the two ITBTs are out-of-phase. In this case we consider two sub-cases:

Type i: There is phase difference between the two ITBTs ($\phi = \pi$) and
Type ii: Only one wall, say the lower one, is ITBT-affected. ($\phi \to -i\infty$)

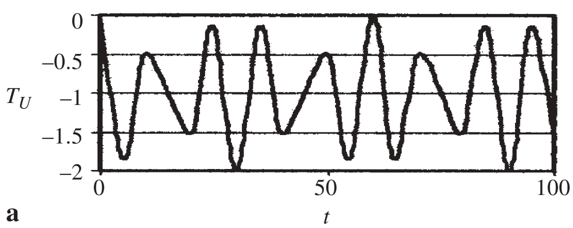

Fig. 2a. Plot of T_U versus t for $\phi = 0$

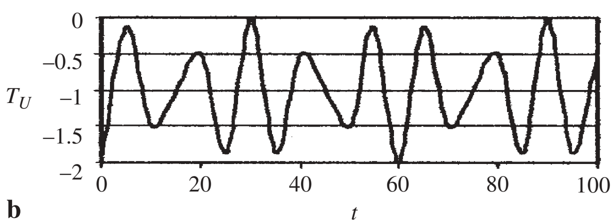

Fig. 2b. Plot of T_U versus t for $\phi = \pi$

Exhaustive computation reveals that the term $M_2\left(1+\frac{T_H}{T_0}\right)\frac{\partial(D\Phi)}{\partial t}$ in Eq. (34) makes a contribution in the fifth decimal digit to the eigenvalue and hence warrants neglect.

Figures 2 and 3 are plots of $T_U = -1 + \varepsilon\cos(\omega t + \phi)$ and $T_L = 1 + \varepsilon\cos(\omega t)$ versus t for synchronous ITBT and type (i) of asynchronous ITBT. Type (ii) essentially means there is no ITBT at the upper surface. The lower surface ITBT is as in Fig. 3. We now discuss the results arrived at in the paper.

Figure 4 is the plot of R_{2c} versus ω for different values of buoyancy magnetization parameter M_1 (the Prandtl number Pr and non-buoyancy magnetization parameter M_3 being fixed) with respect to synchronous ITBT. The buoyancy magnetization parameter M_1 is the ratio of the magnetic to gravitational forces. It can be seen that for synchronous ITBT, R_{2c} increases with an increase in M_1 at a given frequency ω. Hence, M_1 has a stabilizing effect on the flow. It is also interesting to see from the figures that for a given M_1, R_{2c} first decreases with an increase in ω, reaches a minimum at $\omega = 20$ and then increases with an increase in ω. This shows that for a ferromagnetic fluid the flow is destabilized for small values of ω and stabilized for large ω. This

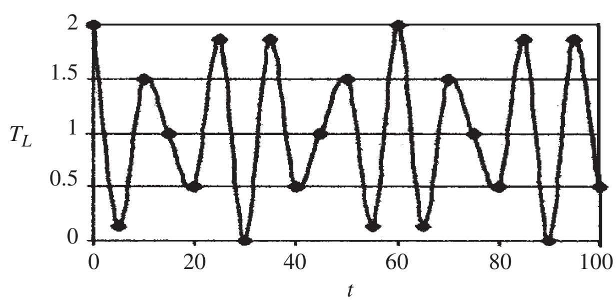

Fig. 3. Plot of T_L versus t

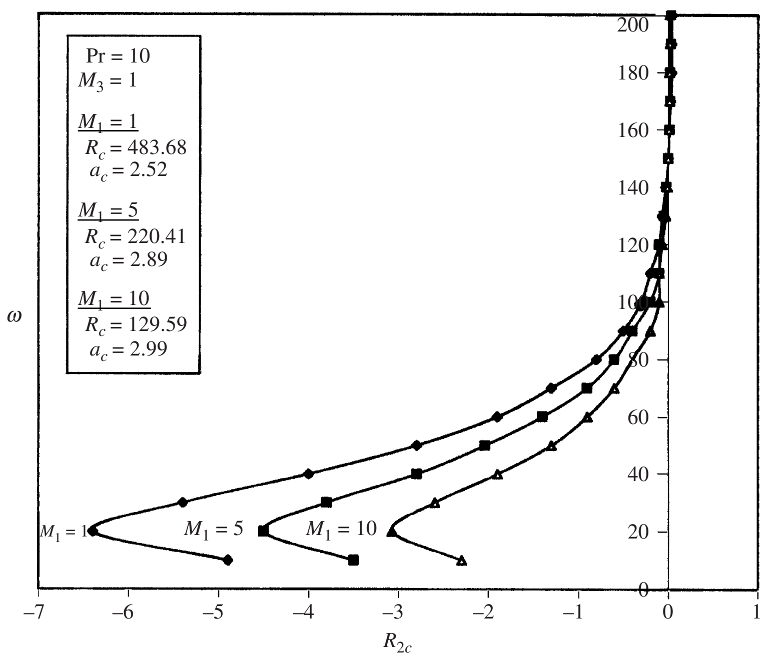

Fig. 4. R_{2c} as a function of ω when the ITBT's are synchronous for different values of M_1

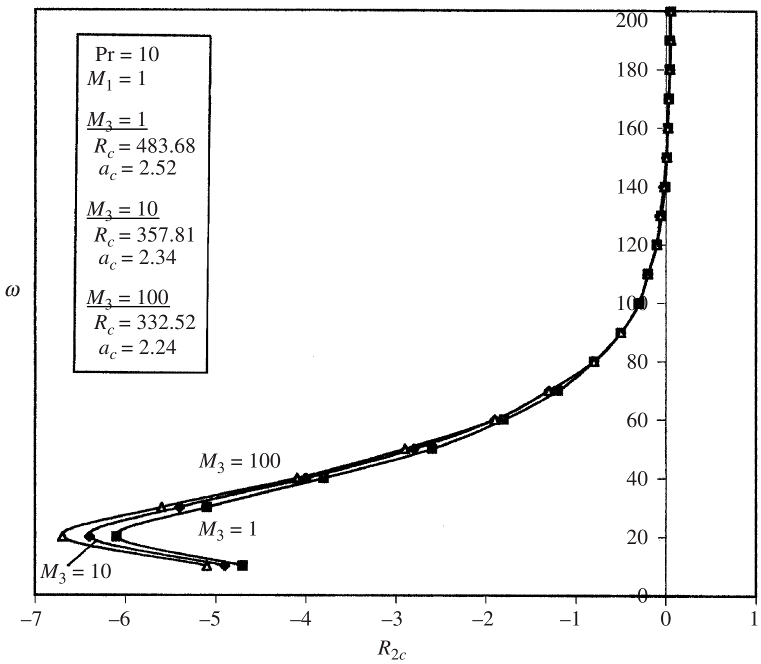

Fig. 5. R_{2c} as a function of ω when the ITBT's are synchronous for different values of M_3

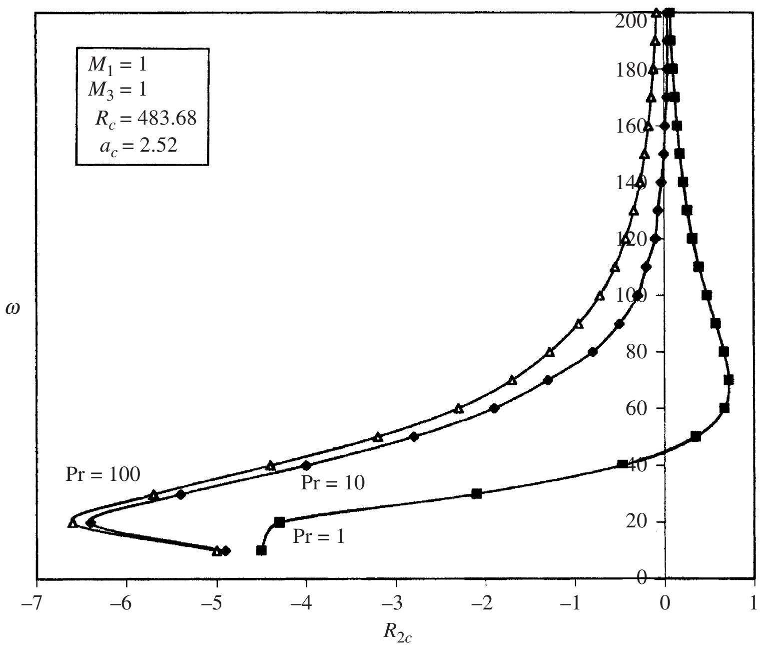

Fig. 6. R_{2c} as a function of ω in respect of synchronous ITBT for different values of Pr

is due to the fact that when the frequency of modulation is low, the effect of ITBT is felt throughout the fluid. For synchronous ITBT of the fluid, the temperature profiles consist of the steady line section plus a parabolic profile which oscillates in time. As the amplitude of the

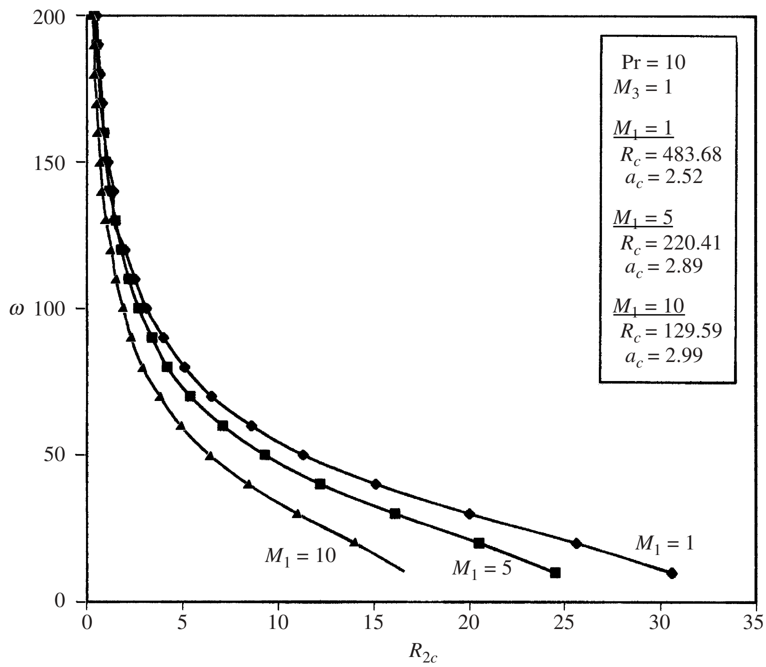

Fig. 7. R_{2c} as a function of ω with respect to asynchronous ITBT (with $\phi = \pi$) for different values of M_1

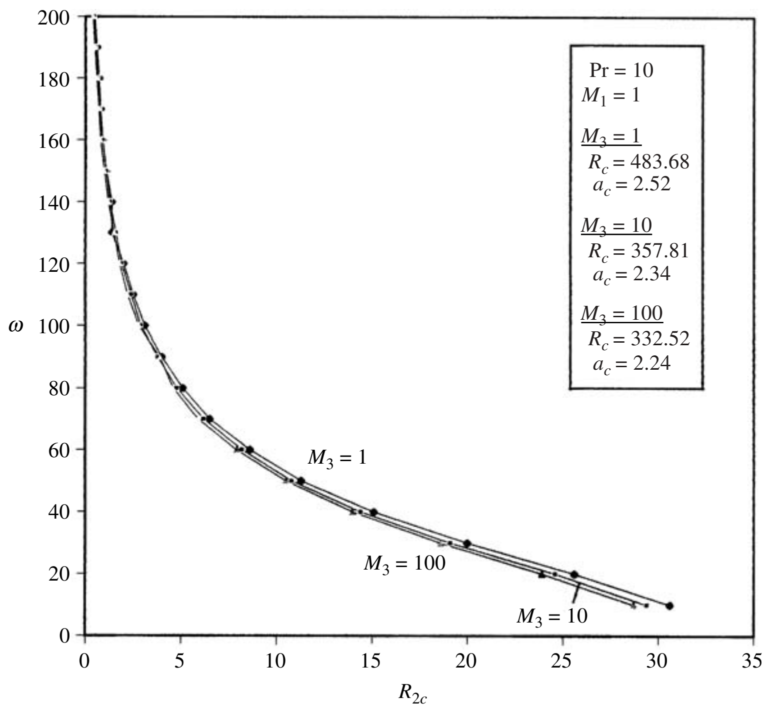

Fig. 8. R_{2c} as a function of ω with respect to asynchronous ITBT (with $\phi = \pi$) for different values of M_3

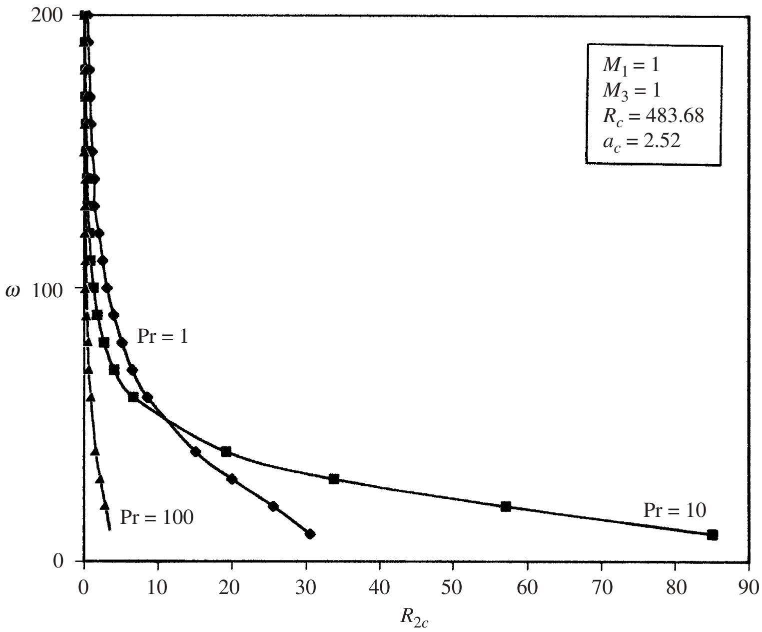

Fig. 9. R_{2c} as a function of ω with respect to asynchronous ITBT (with $\phi = \pi$) for different values of Pr

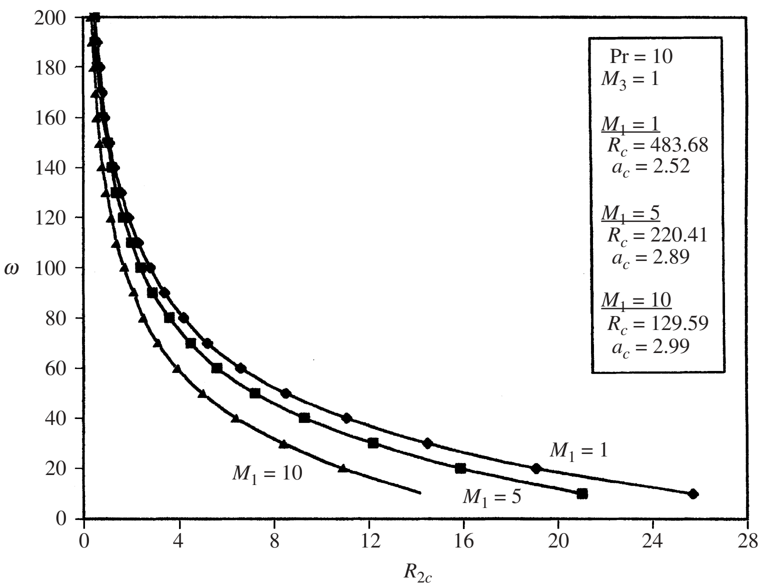

Fig. 10. R_{2c} as a function of ω when only the lower wall is ITBT affected for different values of M_1

modulation increases the parabolic part of the profile becomes more and more significant. It is known that a parabolic profile is subject to finite amplitude instabilities so that convection occurs at lower Rayleigh numbers than those predicted by the linear theory. There is also a value of ω for which the stabilizing influence is minimum, and this minimum decreases with an increase in M_1.

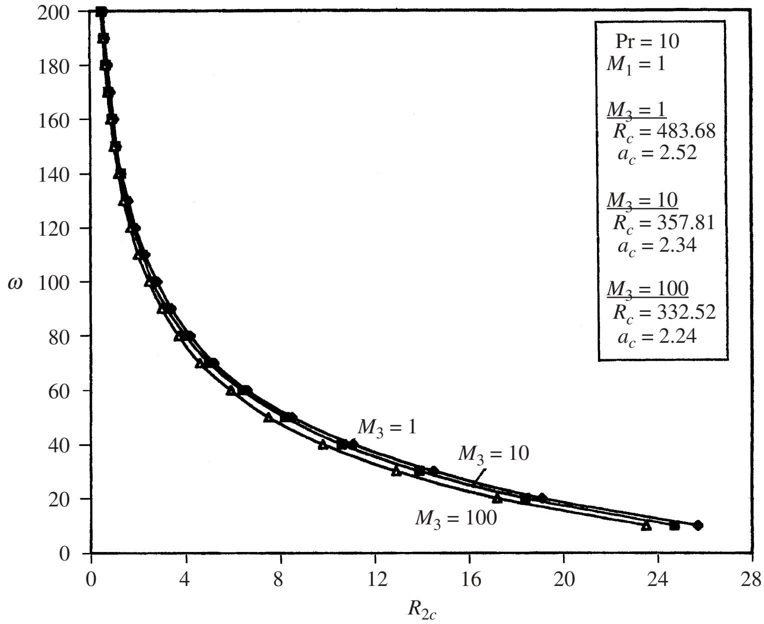

Fig. 11. R_{2c} as a function of ω when only the lower wall is ITBT affected for different values of M_3

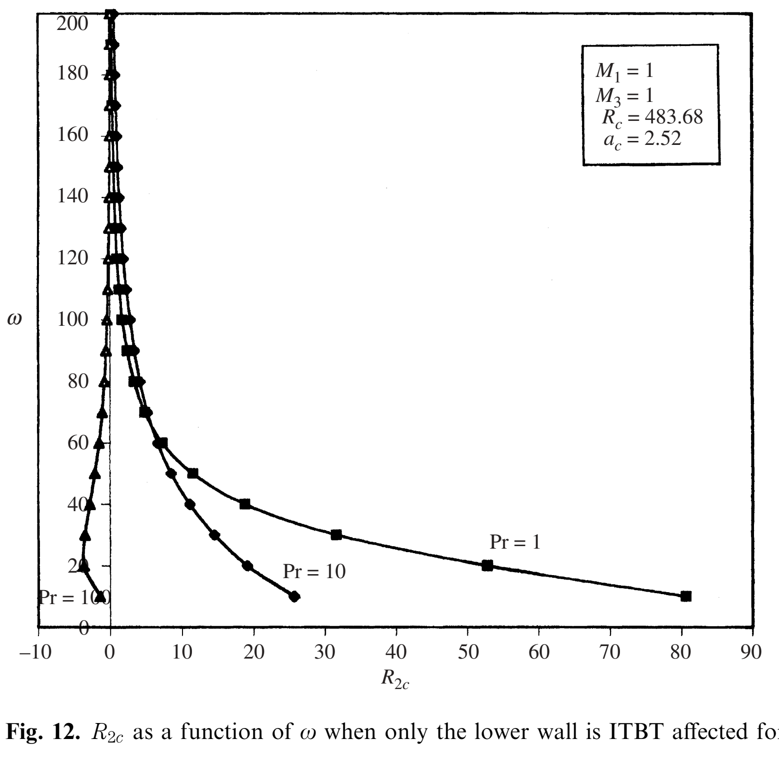

Fig. 12. R_{2c} as a function of ω when only the lower wall is ITBT affected for different values of Pr

Figure 5 is the plot of R_{2c} versus ω for different values of M_3 (Pr and M_1 being fixed) with respect to synchronous ITBT. The non-buoyancy magnetization parameter M_3 measures the departure of linearity in the magnetic equation of state. It can be seen that for synchronous ITBT, R_{2c} decreases with an increase in M_3 at a given frequency ω. Thus M_3 has a destabilizing effect on the flow. It is also interesting to see from the figures that for a given M_3, R_{2c} first decreases with an increase in ω, reaches a minimum at $\omega = 20$ and then increases with an increase in ω. This shows that for a ferromagnetic fluid the flow is destabilized for small values of ω and stabilized for large ω. The observed invariance of ω_c with change in M_1 and M_3 (also seen in the other graphs) is quite intriguing and presently inexplicable.

Figure 6 shows the variation of R_{2c} with ω for different values of Pr (with M_1 and M_3 fixed) in the case of synchronous ITBT. It is appropriate to note here that Pr does not affect the R_0 – part of R. It affects only R_2. We also observe here that the increase in Pr is due to increased concentration of ferrite suspended particles. It may be noticed that for moderate values of frequency R_{2c} decreases with an increase in Pr. We can infer from this that the effect of increasing Pr is to destabilize the system. It is also observed that for low concentration of the suspended ferrite particles (i.e., Pr $\leq$ 3) supercritical motion is possible and for high concentration only (i.e., Pr $>$ 3) subcritical motion is possible. Thus, in the case of fluids with suspended particles subcritical motions are more likely than supercritical motions, for $\omega < 200$. In this graph we

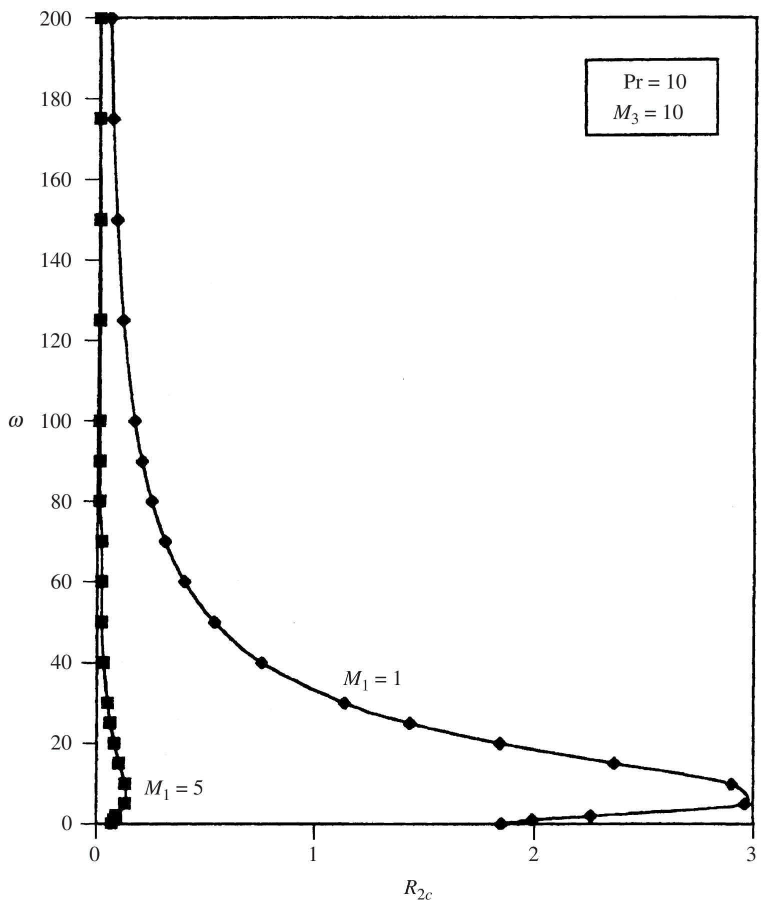

Fig. 13. R_{2c} as a function of ω with respect to TBF for different values of M_1

considered the most destabilizing ($M_1 = 1$) and most stabilizing ($M_3 = 1$) cases. The case of most stabilizing ($M_1 = 1$) and most destabilizing ($M_3 = 100$) was considered in Fig. 5. Earlier in Fig. 4 we also considered the most stabilizing ($M_1 = 10$) and most destabilizing ($M_3 = 1$) cases.

Figures 7, 8 and 9 show the plot of R_{2c} versus ω for different values of M_1, M_3 and Pr (with other corresponding parameters fixed) in the case of asynchronous ITBT with a phase difference between the two ITBTs. We observe that R_{2c} decreases with increasing M_1, M_3 and Pr. Thus in these cases the effect is one of stabilization decreasing with increasing frequency.

We observe that for asynchronous ITBT with a phase difference between the two ITBTs, even though R_{2c} decreases with an increase in M_1, M_3 and Pr it does not become negative. Thus subcritical motions are ruled out in this case. The above results are due to the fact that in the case of asynchronous ITBT the temperature field has essentially a linear gradient varying in time, so that the instantaneous Rayleigh number is supercritical for half a cycle and subcritical during the other half cycle [1]. We also observe that M_1 and Pr have opposing influences in synchronous and asynchronous ITBT whereas M_3 has an identical influence on R_{2c} in both synchronous and asynchronous ITBT.

For asynchronous ITBT where only the lower wall is ITBT-affected we observe from Figs. 10, 11 and 12 that the effect of the various parameters on R_{2c} is qualitatively similar to the previous case of asynchronous ITBT with a phase difference between the two ITBTs. A point to be noted in this case is that for very high values of the Prandtl number Pr, sub-critical motions are possible for low and moderate values of the frequency.

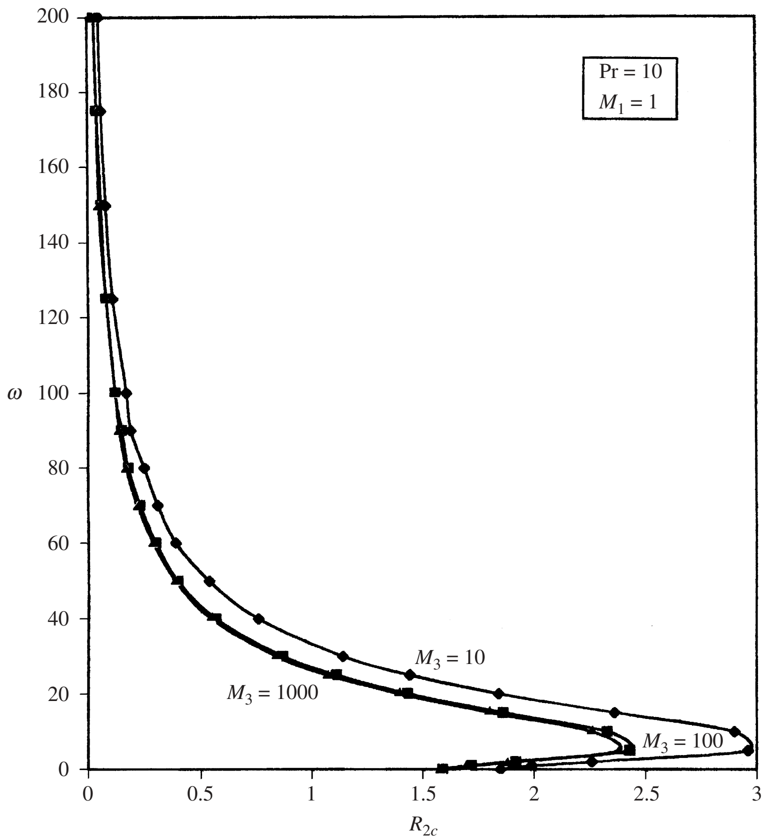

Fig. 14. R_{2c} as a function of ω with respect to TBF for different values of M_3

TBF

The solution obtained in Sect. 3 is based on the assumption that the amplitude of TBF is small. When $\omega < 1$, i.e., the period of TBF is large, the TBF affects the entire volume of fluid and hence the disturbances grow large. On the other hand, the effect of TBF disappears for large frequencies as is the case of ITBT. This is due to the fact that the buoyancy force takes a mean value leading to the equilibrium state of the non-TBF case. In the TBF problem R_{2c} is a crucial quantity which determines whether TBF leads to sub – critical instability or not.

In Fig. 13, R_{2c} is plotted against ω for different values of M_1 (the Prandtl number Pr and M_3 being fixed). It can be seen that R_{2c} decreases with an increase in M_1 at a given frequency ω. M_1 has a destabilizing effect on the flow.

In Fig. 14, R_{2c} is plotted against ω for different values of M_3 (the Prandtl number Pr and M_1 being fixed). It can be seen that R_{2c} decreases with an increase in M_3 at a given frequency ω. M_3 has a destabilizing influence on the flow. It is also interesting to see from Figs. 13, 14 that for a given M_1 and M_3, R_{2c} first increases with an increase in ω, reaches a maximum at $\omega = 7$ and then decreases with an increase in ω. This shows that for a ferromagnetic fluid the flow is stabilized for small values of ω and destabilized for large ω.

Figure 15 shows the variation of R_{2c} with ω for different values of the Prandtl number Pr (with M_1 and M_3 fixed). It may be noticed that for moderate values of frequency R_{2c} decreases with an increase in Pr. Since R_0 is independent of Pr, we may infer that an increase in Pr has a

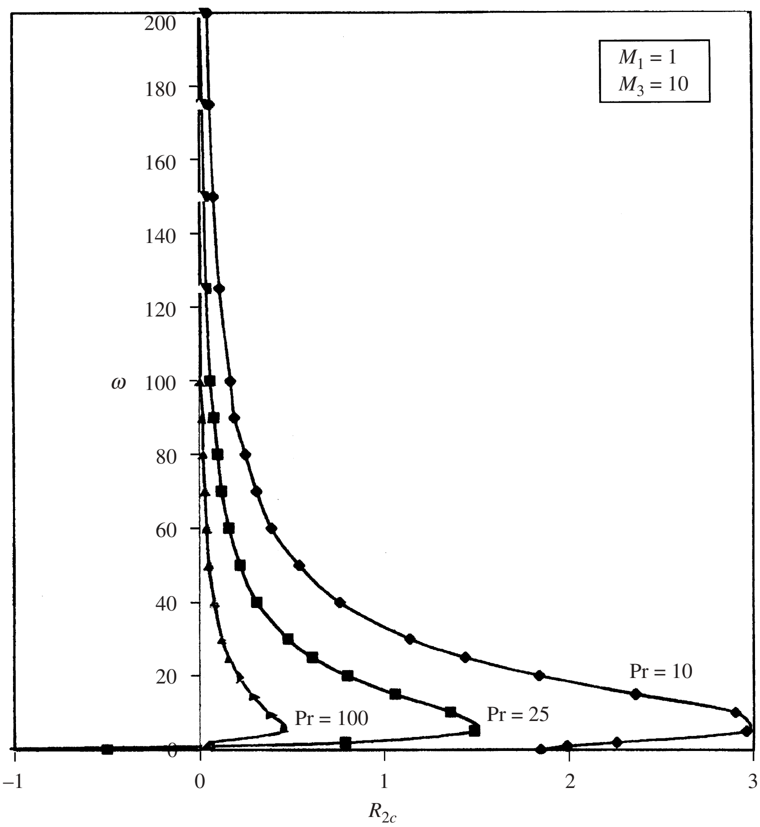

Fig. 15. R_{2c} as a function of ω with respect to TBF for different values of Pr

destabilizing effect on the flow for moderate values of ω. It is interesting to note that at large Prandtl number R_{2c} can become negative and for lower values of the Prandtl number we get only supercritical motions.

5 Conclusion

The results of the study indicate that ITBT can give rise to sub-critical motion and TBF leads to delayed convection. It is also observed that for large frequencies the effects of ITBT/TBF disappear. The problem throws light on an external means of controlling convection in ferromagnetic fluids which is quite important from the application point of view [17], [18]. Presently work is under progress to consider the effect of a time-periodic boundary magnetic potential on convection.

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