SIAM J. Appl. Math.
Vol. 15, No. 5, September, 1967
Printed in U.S.A.

EFFECT OF VIBRATION ON THE ACCURACY OF A VERTICAL REFERENCE PENDULUM*

T. K. CAUGHEY†

1. Introduction. The pendulum has been used from ancient times as a means of establishing a local vertical reference. One of the earliest accounts of the use of the pendulum as a plumb-bob comes from Thebes, Egypt, about 1,100 B.C. [1, p. 481]. As the years passed, the requirements for accuracy have increased until today the accuracy requirements for missile applications are such that second order effects may determine whether or not the pedulum may be used to establish a local vertical reference.

Using the best techniques currently available, it is impossible to eliminate entirely the vibration transmitted through the supporting structure to the point of support of the pendulum. The purpose of this paper is to examine the effect of such residual vibration on the accuracy of a pendulum type vertical reference. It will be shown that if the vertical and horizontal components of acceleration of the point of support of the pendulum are correlated in time, then in general the mean position of the pendulum does not coincide with the local vertical axis. This zero-shift, or "Auswanderungs"-phenomenon, is not new, having received considerable attention in the German literature of the 1930's, notably by Klotter and his associates [2], [3] and Erdélyi [4]. More recently, problems of this type have been studied by Weidenhammer [5] and Conrad and Shellhorn [6]. Except for [6] the analysis was restricted to weak damping and the case of harmonic excitation, where both vertical and horizontal components of the acceleration of the point of support have the same frequency. In the present study the analysis will be for heavily damped systems acted upon by a class of bounded excitations having zero time average.

2. Formulation of the problem. Consider a heavily damped pendulum of length l and mass m whose point of support is subjected to weak oscillatory accelerations $V(t)$ and $H(t)$ in the vertical and horizontal directions, respectively. If θ denotes the angle which the pendulum makes with the vertical axis, and C denotes the damping coefficient of the pendulum, then the equation of motion of the pendulum is

$$(1) \qquad ml^2\ddot{\theta} + C\dot{\theta} + ml[g + V(t)]\sin\theta = mlH(t)\cos\theta, \qquad t \geqq t_0,$$

* Received by the editors August 12, 1966, and in revised form February 15, 1967.
† Division of Engineering and Applied Science, California Institute of Technology, Pasadena, California 91109.

where $\dot{\theta} = d\theta/dt$, etc. Let

$$
\begin{aligned}
g/l &= \omega_0^2, \\
C/ml^2 &= 2\omega_0 z, \\
\omega_0 t &= \tau, \\
V(t)/g &= -\epsilon_1 f_1(\tau), \\
H(t)/g &= \epsilon_2 f_2(\tau).
\end{aligned} \tag{2}
$$

Using (2) we may write (1) as

$$\theta'' + 2z\theta' + [1 - \epsilon_1 f_1(\tau)] \sin\theta = \epsilon_2 f_2(\tau) \cos\theta, \qquad \tau \geqq \tau_0, \tag{3}$$

where $\theta' = d\theta/d\tau$, etc.

2.1. Assumptions. In the following analysis it will be assumed that the following conditions hold:

(4)
- (i) $z > 1$, i.e., the pendulum is supercritically damped;
- (ii) $f_i(\tau)$, $i = 1, 2$, are piecewise continuous and bounded such that $\sup_\tau |f_i(\tau)| = 1$, $i = 1, 2$;
- (iii) $\overline{f_i(\tau)} = \lim_{T\to\infty} (1/T) \int_0^T f_i(\xi)\, d\xi = 0$, $i = 1, 2$;
- (iv) $|\epsilon_i| \ll 1$, $i = 1, 2$.

2.2. Existence and uniqueness of solutions of (3). If (3) is rewritten in vector form,

$$
\begin{aligned}
\bar{y}' &= \bar{F}(\bar{y}, \tau), \\
\bar{y} &= \begin{Bmatrix} \theta \\ \theta' \end{Bmatrix}, \qquad \tau \geqq \tau_0 .
\end{aligned} \tag{5}
$$

Using condition (4) (ii) it is easily shown that the vector function $\bar{F}(\bar{y}, \tau)$ satisfies a Lipschitz condition,

$$\| \bar{F}(\bar{y}_1, \tau) - \bar{F}(\bar{y}_2, \tau) \| \leqq (1 + 2z) \| \bar{y}_1 - \bar{y}_2 \| \quad \text{for all} \quad \tau \geqq \tau_0 . \tag{6}$$

Thus (5) satisfies the conditions of the Cauchy-Lipschitz theorem; hence the existence and uniqueness of the solutions of the initial value problem associated with (3) are assured.

2.3. Ultimate boundedness of solutions. It will be shown that under weak restrictions on the initial conditions, all solutions of (3) are ultimately bounded, and that in particular

$$\lim_{\tau_0 \to -\infty} \{\sup_{\tau > 0} |\theta|\} \leqq 2 |\epsilon_2| / \sqrt{1 - 3|\epsilon_1|}. \tag{7}$$

Consider the function V defined by

(8) $$V = [2z + 1/z][1 - \cos\theta] + \theta' \sin\theta + \theta'^2/2z.$$

Define the compact set Ω_2 by

(9) $$\Omega_2 : V \leqq 3z/2 + 1/z.$$

The function V has the following properties:

(10)
- (i) $V(\theta, \theta')$ together with its first partial derivatives is continuous in Ω_2.
- (ii) $V(0, 0) = 0$.
- (iii) $V(\theta, \theta')$ is nonnegative in Ω_2 and vanishes only at the origin. The origin is an isolated minimum of V.
- (iv) The time derivative V' of V along any trajectory of (3) satisfies the inequality[1]

$$V' \leqq -\tfrac{1}{2}[\sin^2\theta + \theta'^2][1 - 3\,|\,\epsilon_1\,|\,] + \epsilon_2^2.$$

Denote the compact set Ω_0 by

(11) $$\Omega_0 : \sin^2\theta + \theta'^2 \leqq 2\epsilon_2^2/[1 - 3\,|\,\epsilon_1\,|], \qquad |\,\theta\,| < \pi/2.$$

Denote the compact set Ω_1 by

(12) $$\Omega_1 : V \leqq V_1 = 2\epsilon_2^2[z + 1/z]/[1 - 3\,|\,\epsilon_1\,|],$$

where V_1 is so chosen that

(13)
$$\text{(a)} \quad \Omega_0 \subset \Omega_1 \subset \Omega_2,$$
$$\text{(b)} \quad V' \leqq -\delta < 0 \quad \text{for} \quad [\theta, \theta'] \in (\Omega_2 - \Omega_1).$$

This choice of V_1 gives

(14) $$\delta \sim O[\epsilon_2^2(\sqrt{1 + z^2} - z)^2].$$

LEMMA 1. *Each solution of* (3) *which at some time* $\tau_1 \geqq \tau_0$ *is in* Ω_1 *can never thereafter leave* Ω_1.

Proof. Let $[\theta(\tau), \theta'(\tau)]$ be a solution of (3) which at time $\tau_1 \geqq \tau_0$ is in Ω_1. Suppose that at some later time τ_3, $[\theta(\tau_3), \theta'(\tau_3)]$ is in $(\Omega_2 - \Omega_1)$. Then there exists a τ_2, $\tau_1 < \tau_2 < \tau_3$, such that $[\theta(\tau), \theta'(\tau)] \in (\Omega_2 - \Omega_1)$ for $\tau_2 < \tau < \tau_3$, and τ_2 is the smallest number with this property. This implies that $[\theta(\tau_2), \theta'(\tau_2)] \in \Omega_1$, and therefore $V[\theta(\tau_2), \theta'(\tau_2)] < V[\theta(\tau_3), \theta'(\tau_3)]$. But this is impossible since $V(\theta, \theta')$ is nonincreasing for τ in the interval $\tau_2 < \tau < \tau_3$. Hence if $[\theta(\tau_1), \theta'(\tau_1)] \in \Omega_1$, it can never thereafter leave Ω_1.

[1] This inequality is easily established using (3) and (4) and the inequality $|\,xy\,| \leqq [x^2 + y^2]/2$.

LEMMA 2. *Each solution of* (3) *starting in* $(\Omega_2 - \Omega_1)$ *is ultimately in* Ω_1.

Proof. Since $V(\theta, \theta')$ is positive definite in Ω_2 and has an isolated minimum at the origin and $V' \leqq -\delta < 0$ for every point in $(\Omega_2 - \Omega_1)$, it follows that every solution starting in $(\Omega_2 - \Omega_1)$ is ultimately in Ω_1. By Lemma 1 it must thereafter remain in Ω_1.

THEOREM 1. *Given* (3), *conditions* (4) *and the compact sets* Ω_1 *and* Ω_2 *defined in* (9) *and* (13), *then*

(i) *any solution of* (3) *starting in* Ω_2 *is ultimately bounded in* Ω_1,

(ii) *as* τ_0 *tends to minus infinity, then for all* τ *greater than zero the maximum angular displacement is given by*

$$\lim_{\tau_0 \to -\infty} \{\sup_{\tau > 0} |\theta|\} \leqq 2|\epsilon_2|/\sqrt{1 - 3|\epsilon_1|}.$$

Proof. The proof of (i) follows directly upon application of Lemmas 1 and 2. The proof of (ii) follows from (i) and some simple but tedious algebra.

2.4. Solution of (3). In this study we are interested in the solutions of (3) for large time, and since by Theorem 1 any solution of (3) starting in Ω_2 is ultimately in Ω_1, there is no loss of generality in assuming that solutions start in Ω_1 at $\tau = \tau_0$. Furthermore, by (ii), solutions in Ω_1 are bounded by $2|\epsilon_2|/\sqrt{1 - 3|\epsilon_1|}$.

It is convenient for the analysis which follows to rewrite (3) in the form of a nonlinear integral equation:[2]

$$\begin{aligned} \theta(\tau) &= U(\tau - \tau_0)\theta_0 + V(\tau - \tau_0)\theta'_0 \\ &\quad + \int_{\tau_0}^{\tau} V(\tau - \xi)[\epsilon_2 f_2(\xi) + \epsilon_1 f_1(\xi)\theta(\xi)]\, d\xi \\ &\quad + \int_{\tau_0}^{\tau} V(\tau - \xi) G(\theta(\xi), \xi)\, d\xi, \qquad [\theta_0, \theta'_0] \in \Omega_1, \end{aligned} \tag{15}$$

$$G(\theta(\xi), \xi) = [1 - \epsilon_1 f_1(\xi)][\theta(\xi) - \sin\theta(\xi)] - \epsilon_2 f_2(\xi)[1 - \cos\theta(\xi)],$$

where

$$\begin{aligned} U(\tau) &= \frac{\lambda_2 e^{-\lambda_1 \tau} - \lambda_1 e^{-\lambda_2 \tau}}{\lambda_2 - \lambda_1}, \\ V(\tau) &= \frac{e^{-\lambda_1 \tau} - e^{-\lambda_2 \tau}}{\lambda_2 - \lambda_1}, \end{aligned} \tag{16}$$

λ_1, λ_2 are the roots of $\lambda^2 - 2z\lambda + 1 = 0$.

It may be shown by standard techniques that solutions of (15) exist for all τ greater than τ_0 and that such solutions are unique.

[2] See Appendix for derivation.

Since our interest is in the behavior of solutions for large time, let τ_0 tend to minus infinity.

Equation (15) becomes

$$
\begin{aligned}
\theta(\tau) &= \int_0^\infty V(\xi)[\epsilon_2 f_2(\tau - \xi)]\, d\xi + \int_0^\infty V(\xi)[\epsilon_1 f_1(\tau - \xi)\theta(\tau - \xi)]\, d\xi \\
&\qquad + \int_0^\infty V(\xi) G[\theta(\tau - \xi), \tau - \xi]\, d\xi.
\end{aligned} \tag{17}
$$

Denote the right-hand side of (17) by $\phi_0(\tau)$.

Substituting into the second and third integrals in (17), in place of $\theta(\tau)$, the value as given by the equation itself, we obtain

$$
\begin{aligned}
\theta(\tau) &= \sum_{k=0}^{2} \epsilon_2 \epsilon_1^{\,k} \psi_k(\tau) + R_3(\tau) + \epsilon_1 \chi_{1,0}(\tau) \\
&\qquad + \int_0^\infty V(\xi) G[\phi_0(\tau - \xi), \tau - \xi]\, d\xi,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
\psi_k(\tau) &= \int_0^\infty \underset{(k+1)\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{k+1} V(\xi_i)\, d\xi_i \prod_{i=1}^{k} f_1(\tau - \eta_i) f_2(\tau - \eta_{k+1}), \\
\chi_{k,l}(\tau) &= \int_0^\infty \underset{(k+1)\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{k+1} V(\xi_i)\, d\xi_i \\
&\qquad \cdot \prod_{i=1}^{k} f_1(\tau - \eta_i) G(\phi_l(\tau - \eta_{k+1}), \tau - \eta_{k+1}), \\
R_{k+1} &= \epsilon_1^{\,k} \int_0^\infty \underset{(k+1)\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{k+1} V(\xi_i)\, d\xi_i \prod_{i=1}^{k+1} f_1(\tau - \eta_i) \theta(\tau - \eta_{k+1}), \\
\eta_i &= \sum_{j=1}^{i} \xi_j .
\end{aligned} \tag{19}
$$

Denote the right-hand side of (18) by $\phi_1(\tau)$.

Substituting into the second, third and fourth terms in (18), in place of $\theta(\tau)$, the value as given by the equation itself, we obtain

$$
\begin{aligned}
\theta(\tau) &= \sum_{k=0}^{4} \epsilon_2 \epsilon_1^{\,k} \psi_k(\tau) + R_5(\tau) + \sum_{k=2}^{3} \epsilon_1^{\,k} \chi_k(\tau) \\
&\qquad + \int_0^\infty V(\xi) G[\phi_1(\tau - \xi), \tau - \xi]\, d\xi.
\end{aligned} \tag{20}
$$

The method of successive substitutions can be carried on indefinitely; however, (20) is sufficient to determine $\theta(\tau)$ up to terms of order ϵ^4, where ϵ is defined by

$$\epsilon = \max[|\epsilon_1|, |\epsilon_2|]. \tag{21}$$

Using Theorem 1, condition (4) (ii), (15) and (16) yields

$$\begin{aligned} &\text{(i)}\ \sup_\tau |\theta| \sim O(\epsilon), \\ &\text{(ii)}\ |G[\theta(\tau), \tau]| \sim O(\epsilon^3), \\ &\text{(iii)}\ |\chi_k| \leqq 1, \\ &\text{(iv)}\ |R_{k+1}| \sim O(\epsilon^{k+1}). \end{aligned} \tag{22}$$

Using (17) and (18) and (22), we see readily that

$$\phi_1(\tau) = \sum_{k=0}^{1} \epsilon_2 \epsilon_1^k \psi_k(\tau) + O(\epsilon^3). \tag{23}$$

By substituting (23) into (20), it may be shown that

$$\begin{aligned} \theta(\tau) = \sum_{k=0}^{2} \epsilon_2 \epsilon_1^k \psi_k(\tau) &+ \frac{\epsilon_2^3}{6} \int_0^\infty \underset{4\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{4} V(\xi_i)\, d\xi_i \\ &\cdot \prod_{i=2}^{4} f_2(\tau - \xi_1 - \xi_i) \\ &- \frac{\epsilon_2^3}{2} \int_0^\infty \underset{3\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{3} V(\xi_i)\, d\xi_i \\ &\cdot \prod_{i=2}^{3} f_2(\tau - \xi_1 - \xi_i) f_2(\tau - \xi_i) + O(\epsilon^4). \end{aligned} \tag{24}$$

All integrals appearing in (24) are absolutely convergent; furthermore,

$$\sup_\tau |\theta(\tau)| \sim O(\epsilon). \tag{25}$$

3. Zero-shift of pendulum. In order to obtain the mean deflection or zero-shift of the pendulum, the displacement must be averaged with respect to time.

Thus

$$\bar{\theta} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \theta(\tau)\, d\tau. \tag{26}$$

Using (24) we obtain

$$\begin{aligned} \bar{\theta} = \sum_{k=0}^{2} \epsilon_2 \epsilon_1^k \bar{\psi}_k &+ \frac{\epsilon_2^3}{6} \int_0^\infty \underset{4\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{4} V(\xi_i)\, d\xi_i \overline{\prod_{i=2}^{4} f_2(\tau - \xi_1 - \xi_i)} \\ &- \frac{\epsilon_2^3}{2} \int_0^\infty \underset{3\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{3} V(\xi_i)\, d\xi_i \overline{\prod_{i=2}^{3} f_2(\tau - \xi_1 - \xi_i) f_2(\tau - \xi_1)} \\ &+ O(\epsilon^4), \end{aligned} \tag{27}$$

where

$$(28)\quad \bar{\psi}_k = \int_0^\infty \underset{(k+1)\text{-fold}}{\cdots} \int_0^\infty \prod_{i=1}^{k+1} V(\xi_i)\, d\xi_i\, \overline{\prod_{i=1}^{k} f_1(\tau - \eta_i) f_2(\tau - \eta_{k+1})}.$$

Equation (27) is applicable to two separate classes of problems:
(i) $f_1(\tau)$ and $f_2(\tau)$ deterministic processes,
(ii) $f_1(\tau)$ and $f_2(\tau)$ stochastic processes.

Suppose that $f_1(\tau)$ and $f_2(\tau)$ are member functions of stochastic processes $\{f_1(\tau)\}$ and $\{f_2(\tau)\}$ respectively, where $\{f_1(\tau)\}$ and $\{f_2(\tau)\}$ have the following properties:
(a) they are strictly stationary;
(b) they have mean zero;
(c) they are bounded and so satisfy (4) (ii);
(d) they possess an ergodic property guaranteeing strict equality of time averages and process expectations with probability one.

In this case (27) is applicable if the time averages appearing on the right-hand side of (27) are replaced by the process expectations. Equations (27) and (28) show quite clearly that if any of the time averages appearing on the right-hand side of (27) are nonzero, then in general the mean position of the pendulum will not coincide with the local vertical axis. In particular it will be observed that the greatest zero-shift will occur if $f_1(\tau)$ and $f_2(\tau)$ are correlated in time; however, zero-shift may still occur even if $f_1(\tau)$ and $f_2(\tau)$ are uncorrelated.

4. Examples. To illustrate the results of the present study, consider the following examples.

Example 1. Suppose that $f_1(\tau) = \sin \omega\tau, f_2(\tau) = \sin(\omega\tau + \phi)$. Then

$$(29)\quad \begin{aligned} &\overline{f_2(\tau)} = 0,\\ &\overline{f_1(\tau - \xi_1) f_2(\tau - \xi_1 - \xi_2)} = \tfrac{1}{2}\cos(\omega\xi_2 - \phi),\\ &\overline{\prod_{i=2}^{4} f_2(\tau - \xi_1 - \xi_i)} = \overline{\prod_{i=2}^{3} f_2(\tau - \xi_1 - \xi_i) f_2(\tau - \xi_1)} = 0,\\ &\overline{\prod_{i=1}^{2} f_1(\tau - \eta_i) f_2(\tau - \tau_3)} = 0. \end{aligned}$$

Substituting (29) into (27) and carrying out the elementary integrations yields

$$(30)\quad \begin{aligned} \bar{\theta} = {}& (\epsilon_1\epsilon_2/2)[(1 - \omega^2)\cos\phi + 2\omega z \sin\phi]\\ &\cdot[(1 - \omega^2)^2 + (2z\omega)^2]^{-1} + O(\epsilon^4). \end{aligned}$$

In (30) if $\phi = 0, \omega = \frac{1}{2}, z = 2, \epsilon = \epsilon_1 = \epsilon_2 = 10^{-2}$, then

$$\bar{\theta} = 5 + O(2 \times 10^{-2}) \text{ seconds of arc.} \tag{31}$$

Example 2. Suppose that $f_1(\tau) = \sin \omega\tau$, $f_2(\tau) = \sin 2\omega\tau$. Then

$$\begin{aligned}
&\overline{f_2(\tau)} = 0,\\
&\overline{f_1(\tau - \xi_1)f_2(\tau - \xi_1 - \xi_2)} = 0,\\
&\overline{\prod_{i=2}^{4} f_2(\tau - \xi_1 - \xi_i)} = \overline{\prod_{i=2}^{3} f_2(\tau - \xi_1 - \xi_i) f_2(\tau - \xi_1)} = 0,\\
&\overline{\prod_{i=1}^{2} f_1(\tau - \eta_i) f_2(\tau - \eta_3)} = \tfrac{1}{4}\,[\sin \omega(\xi_2 + 2\xi_3)].
\end{aligned} \tag{32}$$

Substituting (32) into (27) and carrying out the elementary integrations gives

$$\begin{aligned}
\bar{\theta} = \epsilon_1^2 \epsilon_2 [3\omega z(1 - 2\omega^2)][(1 - \omega^2)^2 + (2z\omega)^2]^{-1}[(1 - 4\omega^2)^2 + (4\omega z)^2]^{-1}\\
+ O(\epsilon^4).
\end{aligned} \tag{33}$$

Example 3. Suppose that $f_1(\tau)$ is equal to $f_2(\tau)$ and that $f_1(\tau)$ is a random telegraph signal with average rate of zero crossing ν. Using the statistical properties of the random telegraph signal [7], it is easily shown that

$$\begin{aligned}
&\overline{f_2(\tau)} = E[f_2(\tau)] = 0,\\
&\overline{f_1(\tau - \xi_1)f_2(\tau - \xi_1 - \xi_2)} = E[f_1(\tau - \xi_1)f_2(\tau - \xi_1 - \xi_2)]\\
&\qquad = \exp\,[-\,2\nu\xi_2],\\
&\overline{\prod_{i=2}^{4} f_2(\tau - \xi_1 - \xi_i)} = E\left[\prod_{i=2}^{4} f_2(\tau - \xi_1 - \xi_i)\right] = 0,\\
&\overline{\prod_{i=2}^{3} f_2(\tau - \xi_1 - \xi_i) f_2(\tau - \xi_1)} = E\left[\prod_{i=2}^{3} f_2(\tau - \xi_1 - \xi_i) f_2(\tau - \xi_1)\right] = 0,\\
&\overline{\prod_{i=1}^{2} f_1(\tau - \eta_i) f_2(\tau - \eta_3)} = E\left[\prod_{i=1}^{2} f_1(\tau - \eta_i) f_2(\tau - \eta_3)\right] = 0.
\end{aligned} \tag{34}$$

Substituting (34) into (27) and carrying out the elementary integrations yields

$$\bar{\theta} = \epsilon_1 \epsilon_2 [1 + (2\nu)(2z) + (2\nu)^2]^{-1} + O(\epsilon^4). \tag{35}$$

The above examples serve to illustrate the effect of vibration on the accuracy of a vertical reference pendulum. Examples 1 and 3 show quite clearly that if the horizontal and vertical components of acceleration of the point of support of the pendulum are correlated in time, then in general the mean position of the pendulum does not coincide with the vertical axis.

That this is a sufficient, but not a necessary condition for zero-shift of the pendulum is clearly illustrated in Example 2, where $f_1(\tau)$ and $f_2(\tau)$ are uncorrelated to first order, i.e., $\overline{f_1(\tau)f_2(\tau + \alpha)} \equiv 0$, yet the pendulum exhibits a shift in its mean position. In this example, the zero-shift is caused by the fact that the quantity $\overline{f_1(\tau)f_1(\tau + \alpha)f_2(\tau + \beta)}$ is not identically zero.

It should be noted that in all three cases the zero-shift $\bar{\theta}$ decreases as z and ω (or ν) increase. This suggests that the errors introduced by vibration may, to some extent, be reduced by making the natural frequency of the pendulum as low as possible, and by making the damping as large as possible.

Appendix. Derivation of (15) and (17). Equation (3) may be rewritten as

$$\begin{aligned} &\theta'' + 2z\theta' + \theta = \epsilon_2 f_2(\tau) + \epsilon_1 f_1(\tau)\theta + G(\theta(\tau), \tau), \\ &\theta(\tau_0) = \theta_0, \qquad \theta'(\tau_0) = \theta_0', \qquad [\theta_0, \theta_0'] \in \Omega_1, \end{aligned} \tag{A1}$$

where $G(\theta(\tau), \tau)$ is defined in (15).

Equation (A1) may be written in the form of an integral equation:

$$\begin{aligned} \theta(\tau) = {} & U(\tau - \tau_0)\theta_0 + V(\tau - \tau_0)\theta_0' \\ & + \int_{\tau_0}^{\tau} V(\tau - \xi)[\epsilon_2 f_2(\xi) + \epsilon_1 f_1(\xi)\theta(\xi) + G(\theta(\xi), \xi)]\, d\xi, \end{aligned} \tag{A2}$$

where U and V are defined by (16).

To show that $\lim_{(\tau-\tau_0)\to\infty} \theta(\tau)$ is independent of θ_0 and θ_0', consider the solution of (A2) with initial conditions

$$\theta(\tau_0) = \phi(\tau_0) = 0, \qquad \theta'(\tau_0) = \phi'(\tau_0) = 0.$$

Thus,

$$\phi(\tau) = \int_{\tau_0}^{\tau} V(\tau - \xi)[\epsilon_2 f_2(\xi) + \epsilon_1 f_1(\xi)\phi(\xi) + G(\phi(\xi), \xi)]\, d\xi. \tag{A3}$$

For $[\phi, \phi'] \in \Omega_1$, $[\theta, \theta'] \in \Omega_1$, (15) shows that

$$| G(\theta(\xi), \xi) - G(\phi(\xi), \xi)| \leqq \epsilon^2 K \, | \theta(\xi) - \phi(\xi) |, \tag{A4}$$

where $K \sim O(1)$.

Subtracting (A3) from (A2) and using (16) with $z \geqq 2$ yields

$$\begin{aligned} | \theta(\tau) - \phi(\tau)| \leqq {} & \frac{\lambda_2}{\lambda_2 - \lambda_1} \exp\left(-\lambda_1(\tau - \tau_0)\right)[| \theta_0 | + | \theta'_0 |] \\ & + \frac{\epsilon + \epsilon^2 K}{\lambda_2 - \lambda_1} \int_{\tau_0}^{\tau} \exp\left[-\lambda_1(\tau - \xi)\right] | \theta(\xi) - \phi(\xi)|\, d\xi. \end{aligned} \tag{A5}$$

After a simple rearrangement, application of Gronwall's lemma yields

$$
\text{(A6)}\quad \begin{aligned} |\theta(\tau) - \phi(\tau)| \leqq \frac{\lambda_2}{\lambda_2 - \lambda_1} [|\theta_0| + [\theta'_0|] \\ \cdot \exp\left[-\left(\lambda_1 - \frac{\epsilon + \epsilon^2 K}{\lambda_2 - \lambda_1}\right)(\tau - \tau_0)\right] \end{aligned}
$$

for $\epsilon \ll 1$, $z \geqq 2$, $\epsilon + \epsilon^2 K < \lambda_1(\lambda_2 - \lambda_1)$. Therefore, $|\theta(\tau) - \phi(\tau)|$ tends uniformly to zero as $(\tau - \tau_0)$ tends to infinity. Hence,

$$
\lim_{\substack{\tau_0 \to \infty \\ \tau > 0}} \theta(\tau) = \lim_{\substack{\tau_0 \to -\infty \\ \tau > 0}} \phi(\tau),
$$

independent of θ_0 and θ_0'.

Thus if $\tau_0 \to -\infty$, $\tau > 0$, (A2) becomes

$$
\text{(A7)}\quad \theta(\tau) = \int_{-\infty}^{\tau} V(\tau - \xi)[\epsilon_2 f_2(\xi) + \epsilon_1 f_1(\xi)\theta(\xi) + G(\theta(\xi), \xi)]\, d\xi.
$$

The transformation $\xi = \tau - \eta$ reduces (A7) to

$$
\text{(A8)}\quad \begin{aligned} \theta(\tau) = \int_0^{\infty} V(\eta)[\epsilon_2 f_2(\tau - \eta) + \epsilon_1 f_1(\tau - \eta)\theta(\tau - \eta) \\ + G(\theta(\tau - \eta), \tau - \eta)]\, d\eta. \end{aligned}
$$

Equation (17) is obtained by replacing η by ξ in (A8).

REFERENCES

[1] C. Singer, E. J. Holmyard and A. R. Hall, *A History of Technology*, vol. I, Oxford University Press, New York, 1954.

[2] K. Klotter, *Über die Fehlweisungen von Messgeraten unter der Wirkung von Erschütterungen*, Jber. Deutsch. Luft-Forschung, 1939, pp. III, 3–11.

[3] K. Klotter and G. Kotowski, *Über die Stabilität der Bewegungen des Pendels mit oszillierendem Aufhängepunkt*, Z. Angew. Math. Mech., 19 (1939), pp. 289–296.

[4] A. Erdélyi, *Über die kleinen Schwingungen eines Pendels mit oszillierendem Aufhängepunkt*, Ibid., 14 (1934), pp. 235–247.

[5] F. Weidenhammer, *Auswanderungserscheinungen in Schwingungsmessgeräten*, Ing.-Arch., 26 (1958), pp. 43–60.

[6] D. A. Conrad and D. G. Shellhorn, *Zero shift in galvanometers subjected to mechanical vibration*, Ibid., 31 (1962), pp. 444–452.

[7] W. M. Wonham and A. T. Fuller, *Probability densities of the smooth random telegraph signal*, J. Electronics Control, 5 (1958), pp. 567–576.