#### Effective Categoricity of Injection Structures

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The Incomputable

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#### Outline



- Spectrum Questions
- 3 Decidability

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#### Outline



- 2 Spectrum Questions
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#### Index Sets

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#### Outline



- 2 Spectrum Questions
- 3 Decidability
- Index Sets
- **5**  $\Sigma_1^0$  and Π<sup>0</sup> Structures

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- 5  $\Sigma_1^0$  and Π<sup>0</sup> Structures
- 6 The Difference Hierarchy

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#### Collaborators

This is joint work with

Valentina Harizanov

and

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# Background

- A computable structure A is said to be *computably categorical* if any computable structure B which is isomorphic to A is computably isomorphic to A
- A computable structure A is said to be Δ<sup>0</sup><sub>2</sub>categorical if any computable structure B which is isomorphic to A is Δ<sup>0</sup><sub>2</sub> isomorphic to A.
- The computable categoricity of many interesting structures has been studied.
- Computable categoricity of abelian groups by Goncharov (Algebra and Logic 1975)
- Computable categoricity of linear orderings (Proc AMS 1981) and of Boolean algebras (JSL 1981) by Remmel

# Previous Work on $\Delta_2^0$ Categoricity

- McCoy studied Boolean algebras and linear orderings (APAL 2003)
- Calvert, Cenzer, Harizanov and Morozov studied equivalence structures (APAL 2006) and Abelian groups (APAL 2009)

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### **Countable Injection Structures**

•  $\mathcal{A} = (A, f)$ where A is a set and  $f : A \rightarrow A$  is an injection

The orbit of a is

$$\mathcal{O}_{f}(a) = \{b \in A : (\exists n \in \mathbb{N})(f^{n}(a) = b \lor f^{n}(b) = a)\}.$$

• The order of a is

$$|a|_f = card(Or_f(a))$$

• The *character* of  $\mathcal{A}$  is

 $\chi(\mathcal{A}) = \{(k, n) : \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}.$ 

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## Infinite Orbits

- Injection structures (A, f) may have two types of infinite orbits
- Z-orbits are isomorphic to (Z, S) in which every element is in the range of f
- $\omega$ -orbits are isomorphic to  $(\omega, S)$  and have the form  $\mathcal{O}_f(a) = \{f^n(a) : n \in \mathbb{N}\}$  for some  $a \notin Rng(f)$ .
- Thus injection structures are characterized (up to isomorphism) by the character as well as the number of orbits of types Z and ω.

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#### Complexity of the Orbits and Character

- Let  $\mathcal{A} = (\omega, f)$  be a computable injection structure
- Each infinite orbit is a  $\Sigma_1^0$  set
- Lemma

(a) 
$$\{(k, a) : a \in Rng(f^k)\}$$
 is a  $\Sigma_1^0$  set,

(b) 
$$\{(a, k) : card(\mathcal{O}_{f}(a)) \ge k\}$$
 is a  $\Sigma_{1}^{0}$  set,

(c) 
$$\{a : \mathcal{O}_f(a) \text{ is infinite}\}\$$
 is a  $\Pi_1^0$  set,

(d) 
$$\{a: \mathcal{O}_f(a) \text{ has type } Z\}$$
 is a  $\Pi_2^0$  set,

(e) 
$$\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$$
 is a  $\Sigma_2^{\overline{0}}$  set, and

(f) 
$$\chi(\mathcal{A})$$
 is a  $\Sigma_1^0$  set.

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• **Proposition** For any  $\Sigma_1^0$  character *K*, there is a computable injection structure  $\mathcal{A}$  such that

$$\bigcirc \chi(A) = K$$

- 2 Fin(A) is computable.
- A may have any specified countable number of orbits of types ω and Z.

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# Categoricity

- Let  $\mathcal{A}$  be a computable structure.
  - A is computably categorical if, any computable structure which is isomorphic to A is computably isomorphic to A;
  - 2  $\mathcal{A}$  is  $\Delta^0_{\alpha}$ -categorical if any computable  $\mathcal{B}$  isomorphic to  $\mathcal{A}$  is  $\Delta^0_{\alpha}$  isomorphic to  $\mathcal{A}$ ;
  - A is relatively *computably categorical* if, for any computable B which is isomorphic to A, there exists an isomorphism between A and B which is computable from B.
  - $\mathcal{A}$  is relatively  $\Delta^0_{\alpha}$ -categorical if, any structure  $\mathcal{B} \cong \mathcal{A}$ , there is an isomorphism which is  $\Delta^0_{\alpha}$ -computable from  $\mathcal{B}$ .
- Relative categoricity implies categoricity but the converse does not hold in general

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## **Computably Categorical Structures**

- **Theorem 1** A is computably categorical if and only A has only finitely many infinite orbits.
- Sketch: The categoricity follows from the following fact: If  $\mathcal{A}$  has finitely many infinite orbits then both  $Fin(\mathcal{A})$  and  $Inf(\mathcal{A})$  are  $\Sigma_1^0$  and hence both are computable.

The other direction is sketched below

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## **Scott Families**

- A Scott family for a structure A is a countable family  $\Phi$  of  $L_{\omega_1\omega}$  formulas, possibly with finitely many fixed parameters from A, such that:
  - (i) Each finite tuple in  $\mathcal{A}$  satisfies some  $\psi \in \Phi$ ;
  - (ii) If  $\vec{a}$ ,  $\vec{b}$  are tuples in  $\mathcal{A}$ , of the same length, satisfying the same formula in  $\Phi$ , then there is an automorphism of  $\mathcal{A}$  that maps  $\vec{a}$  to  $\vec{b}$ .
- **Theorem** Let A be a computable structure. Then the following are equivalent:
  - (a)  $\mathcal{A}$  is relatively  $\Delta^0_{\alpha}$  categorical;
  - (b) A has a c.e. Scott family consisting of computable  $\Sigma_{\alpha}$  formulas.

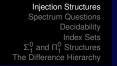
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## **Relative Computable Categoricity**

- **Theorem 2** A computable injection structure A is relatively computably categorical if and only A has finitely many infinite orbits.
- Sketch: For parameters take one element from each of the infinite orbits. The Scott formula for a sequence (a<sub>1</sub>,..., a<sub>m</sub>) of elements states whether a<sub>j</sub> in one of those infinite orbits or has finite order and also for any i ∈ ω, whether f<sup>i</sup>(a) = b for a and b taken from a<sub>1</sub>,..., a<sub>m</sub> plus the parameters.

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## The Other Direction

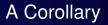
- If A infinitely many infinite orbits, then in fact A is not computably categorical. There are two cases.
- First suppose that A = (ω, f) has infinitely many orbits of type ω.

We may assume that Rng(f) is a computable set. Then we build  $\mathcal{B} = (\omega, g)$  isomorphic to  $\mathcal{A}$  such that Rng(g) is not computable.

Next suppose that A has infinitely many orbits of type Z.
 We may assume that each orbit of A is computable.
 Then we build B isomorphic to A with a particular non-computable orbit.

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• **Corollary** A computable injection structure A is relatively computably categorical iff A is computably categorical.

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# $\Delta_2^0$ categorical structures

- Theorem 3 Suppose A either does not have infinitely many orbits of type ω or does not have infinitely many orbits of type Z. Then A is relatively Δ<sup>0</sup><sub>2</sub> categorical.
- Sketch: Since *Fin*(*A*) is a c. e. set and each infinite orbit is also a c. e. set, there is a Δ<sub>2</sub><sup>0</sup> partition of *A* into three sets: *Fin*(*A*), the orbits of type ω (*A*<sub>ω</sub>), and the orbits of type Z (*A*<sub>Z</sub>); similarly partition *B*.
- We can construct isomorphisms between the three parts of A and the corresponding parts of B.
- For the orbits of type ω, note that the set of beginning elements of orbits is simply ω \ Rng(f) and is therefore a Π<sup>0</sup><sub>1</sub> set.

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# Non- $\Delta_2^0$ Categorical Structures

- **Theorem 4** If A has infinitely many orbits of type  $\omega$  and infinitely many orbits of type  $\mathbb{Z}$ , then A is not  $\Delta_2^0$  categorical.
- Sketch: Consider structures with infinitely many orbits of type  $\omega$ , infinitely many orbits of type  $\mathbb{Z}$ , and no finite orbits. There is a computable structure  $\mathcal{A}$  such that  $\mathcal{A}_{\omega}$  and  $\mathcal{A}_{\mathbb{Z}}$  are computable sets.

Build a computable structure  $\mathcal{B}$  such that  $\mathcal{B}_{\omega}$  is *not*  $\Delta_2^0$ . Then  $\mathcal{A}$  and  $\mathcal{B}$  are not  $\Delta_2^0$  isomorphic.

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### The Construction

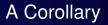
- Let *C* be an arbitrary  $\Sigma_2^0$  set.
- We will define g such that  $\mathcal{O}_g(2i+1)$  has type  $\omega$  if and only if  $i \in C$ .
- The orbits of  $\mathcal{B} = (B, g)$  will be exactly  $\{\mathcal{O}_g(2i+1) : i \in \mathbb{N}\}$ .
- There is a computable function  $\phi$  such that  $i \in C$  iff  $W_{f(i)}$  is finite.

Every time a new element comes into  $W_{f(i)}$  extend the orbit of 2i + 1 to the left.

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• **Corollary** A computable injection structure  $\mathcal{A}$  is relatively  $\Delta_2^0$  categorical iff  $\mathcal{A}$  is  $\Delta_2^0$  categorical.

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• **Theorem** Every computable injection structure is relatively  $\Delta_3^0$  categorical.

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#### **Computably Enumerable Degrees**

- The proof of Theorem 1 has the following corollaries.
- Corollary Let d be a c. e. degree.
  - If  $\mathcal{A}$  is a computable injection structure which has infinitely many orbits of type  $\omega$ , then there is a computable injection structure  $\mathcal{B} = (B, g)$  isomorphic to  $\mathcal{A}$  in which Rng(g) is a c. e. set of degree **d**.
  - 2 If  $\mathcal{A}$  is a computable injection structure which has infinitely many infinite orbits of type  $\mathbb{Z}$ , then there is a computable injection structure  $\mathcal{B} = (B, g)$  isomorphic to  $\mathcal{A}$  in which  $\mathcal{O}_g(1)$  is of type  $\mathbb{Z}$  and is a c. e. set of degree **d**.

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The Complexity of Fin(A)

- Fin(A) is always a c.e. set but emphcannot be an arbitrary c.e. set
- If A has an infinite orbit, then this orbit will be an infinite c.e. set in the complement of *Fin*(A).
- Thus Fin(A) cannot be a simple c.e. set.
- There is a c.e. set which is not simple but is an orbit.

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## The Degree of $Fin(\mathcal{A})$

• Theorem Let c be a c. e. degree.

Let  $\mathcal{A} = (A, f)$  be a computable injection structure such that  $Fin(\mathcal{A})$  is infinite,  $\mathcal{A}$  has infinitely many orbits of size k for every  $k \in \omega$ , and  $\mathcal{A}$  has infinitely many infinite orbits. Then there is a computable injection structure  $\mathcal{B} = (B, g)$  such that  $\mathcal{B}$  is isomorphic to  $\mathcal{A}$  and  $Fin(\mathcal{B})$  is of degree **c**.

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#### Some Spectrum Results

- **Theorem** For any infiinite co-infinite c. e. set *C* and any c. e. character,
  - (i) There is a computable injection structure  $(\mathbb{N}, g)$  with Rng(g) = C consisting of infinitely many orbits of type  $\omega$ .
  - (ii) If *C* is not simple, then there is a computable injection structure  $(\mathbb{N}, h)$  with character *K*, with  $Rng(h) \equiv_T C$ , and with an arbitrary number of orbits of type  $\mathbb{Z}$ .
- **Theorem** For any c. e. set *C* and any computable injection structure A with infinitely many infinite orbits, there is a computable injection structure B isomorphic to A such that *C* is 1 1 reducible to an orbit of *A*.

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#### Character versus Theory

- *Th*(*A*) is the first-order theory of *A*
- *FTh*(A) is the elementary diagram of A
   A is *decidable* if *FTh*(A) is computable.
- Proposition χ(A) is many-one reducible to Th(A).
   If Th(A) is decidable, then χ(A) is computable.

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# Computing $FTh(\mathcal{A})$

- For an injection structure  $\mathcal{A} = (\mathcal{A}, f)$ , let  $\mathcal{R}^{\mathcal{A}}(n, a) \iff (\exists x) f^{n}(x) = a$
- **Theorem** *Fth*(*A* is computable from *R*<sup>*A*</sup> together with *A*. Sketch: Use quantifier elimination as in the theory of successor.

First add the relations  $\gamma_n$  where  $\gamma_n(a) \iff R^A(a, n)$ .

Theorem For any B, there exists A isomorphic to B, such that A and R<sup>A</sup> are computable from χ(A).

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# Decidability of $Th(\mathcal{A})$

- Theorem For any injection structure A, Th(A) and χ(A) have the same Turing degree.
   Thus Th(A) is decidable if and only if χ(A) is computable.
- **Theorem** If  $\chi(\mathcal{B})$  is computable, then there is a decidable  $\mathcal{A}$  isomorphic to  $\mathcal{B}$ . (Hence  $Th(\mathcal{B})$  is decidable.)
- **Corollary** If *A* has bounded character, then *Th*(*A*) is decidable.

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## Decidability of Computably Categorical Structures

• **Proposition** There is a computably categorical injection structure A such that Th(A) is not decidable.

Sketch: Let *W* be a non-computable c.e. set and let A have character  $\{(n, 1) : n \in W\}$  and no infinite orbits.

This contrasts with the result for equivalence structures.

Proposition For any computable character K, there is a decidable injection structure A with character K and with any number of orbits of types ω and Z. Furthermore, {a : O<sub>f</sub>(a) is finite} is computable.

Sketch: There is a computable structure  $\mathcal{B}$  with character K and thus there is a decidable structure  $\mathcal{A}$  isomorphic to  $\mathcal{B}$ .

• Corollary If *B* has computable character and no infinite orbits, then *B* is decidable.

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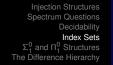
## **Being Injective**

Theorem Inj = {e : A<sub>e</sub> is an injection structure} and the set of indices of finitary injection structures are Π<sub>2</sub><sup>0</sup> complete sets.

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## The Infinite Orbits

 Inj<sub>m</sub> = {e : A<sub>e</sub> structure with exactly *m* orbits of type ω}. Inj<sub>≤m</sub> = {e : A<sub>e</sub> structure with ≤ *m* orbits of type ω} and similarly define Inj<sub><m</sub>, Inj<sub>>m</sub> and Inj<sub>≥m</sub>.

 $Inj^n = \{e : A_e \text{ structure with exactly } n \text{ orbits of type } \mathbb{Z}\}$ and similarly define  $Inj^{\leq n}$ ,  $Inj^{< n}$ ,  $Inj^{>n}$  and  $Inj^{\geq n}$ . Combine these to define for example  $Inj_m^n$  to be  $\{e : A_e \text{ has } m \text{ orbits of type } \omega \text{ and } n \text{ orbits of type } \mathbb{Z}\}.$ 

• **Theorem**  $Inj_{\leq m}$  is  $\Pi_2^0$  complete,  $Inj_{>m}$  is  $\Sigma_2^0$  complete, and  $Inj_{m+1}$  is  $D_2^0$  complete.

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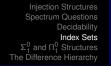


## Proof Sketch

Sketch: First define *f* so A<sub>f(e)</sub> has all finite orbits if e ∈ Inf and otherwise has one infinite orbit of type ω together with a finite number of finite orbits.
φ<sup>s</sup> with domain {0,1,..., s - 1} having some finite orbits of sizes k<sub>0</sub>, k<sub>1</sub>,..., k<sub>s</sub>, where s is the cardinality of W<sub>e,s</sub>.
φ<sup>s</sup>(x) = x + 1 except for φ<sup>s</sup>(k<sub>0</sub> - 1) = 0 and, for i < s, φ<sup>s</sup>(k<sub>0</sub> + k<sub>1</sub> + ··· + k<sub>i</sub> - 1) = k<sub>0</sub> + k<sub>1</sub> + ··· + k<sub>i-1</sub>. If a new element comes into W<sub>e,s+1</sub>, let φ<sup>s+1</sup>(s) = k<sub>0</sub> + k<sub>1</sub> + ··· + k<sub>s-1</sub>, thus closing off the last orbit. Otherwise φ<sup>s+1</sup>(s) = s + 1.

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## Sketch Continued

- If W<sub>e</sub> is finite and no new elements come in after stage s, then we have φ<sub>f(e)</sub>(x) = x + 1 for all x > s, and thus exactly one infinite orbit, of type ω.
   If W<sub>e</sub> is infinite, then all orbits are finite.
- Reduce the  $D_2^0$  complete set

 $D = \{ \langle a, b \rangle : a \in Fin \& b \in Inf \}.$ 

Let *f* and *g* be as above so that  $e \in Fin$  if and only if  $\mathcal{A}_{f(e)}$  has exactly one infinite orbit of type  $\omega$  and all other orbits finite, and  $e \in Inf$  if and only if all orbits of  $\mathcal{A}_e$  are finite.

Let A<sub>g(e)</sub> be two copies of A<sub>f(e)</sub>. Now let A<sub>h(a,b)</sub> consist of a copy of A<sub>f(a)</sub> together with a copy of A<sub>g(b)</sub>. It can be checked that A<sub>h(a,b)</sub> has exactly one infinite orbit of type ω if and only if ⟨a, b⟩ ∈ D.



## Orbits of Type $\ensuremath{\mathbb{Z}}$

- **Theorem**  $Inj^{\leq n}$  is  $\Pi_3^0$  complete,  $Inj^{>n}$  is  $\Sigma_3^0$  complete, and  $Inj^{n+1}$  is  $D_3^0$  complete.
- Sketch: Reduce the  $\Sigma_3^0$  complete set
  - $Cof = \{e : W_e \text{ is cofinite}\} \text{ to } Inj^1.$

Start to build  $\omega$  chains going forward from each number 2m + 1 by mapping *x* to 2x.

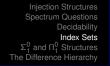
When some *m* comes into  $W_{e,s+1}$ , find the longest

sequence  $k, k + 1, \ldots, m, m + 1, \ldots, n$  including m.

Put the chains from 2(m + 1) + 1 to 2n + 1 at the end of the 2m + 1 chain and fix them there.

If k < m, put the newly expanded 2m + 1 chain at the end of the 2k + 1 chain.

Add an element to the beginning of the 2k + 1 chain.



#### Sketch Continued

• If  $W_e$  is cofinite, there will be a least m such that every  $n \ge m$  belongs to  $W_e$ . In that case, the orbit of 2m + 1 will be a  $\mathbb{Z}$  chain, all of the 2n + 1 chains for n > m will be included in this orbit, and there will be finitely many orbits of type  $\omega$  for the numbers k < m.

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#### **Computable Categoricity**

Theorem The property of computable categoricity is Σ<sup>0</sup><sub>3</sub> complete (that is,

 $CCI = \{e : A_e \text{ has finitely many infinite orbits}\}$ 

is a  $\Sigma_3^0$  complete set).

Sketch: Reduce the  $\Sigma_3^0$  complete set

 $Cof = \{e : W_e \text{ is cofinite}\}.$ 

Define *f* such that for any *e*,  $A_{f(e)}$  will have finitely many infinite orbits if and only if  $W_e$  is cofinite.

The orbits of  $A_{f(e)}$  will be exactly the orbits O(2i + 1) for  $i \in \omega$  and the even numbers will be used in order to fill out the orbits.

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 Theorem The property of Δ<sup>0</sup><sub>2</sub> categoricity is Σ<sup>0</sup><sub>4</sub> complete. Sketch: Fix a Π<sup>0</sup><sub>4</sub> set *C* and define a reduction *f* such that for any *e*, *A*<sub>f(e)</sub> has only infinite orbits and has infinitely many orbits of type Z if and only if *e* ∈ *C*.

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• Injection structures (*A*, *f*) where *A* is c.e. and *f* is the restriction to *A* of a partial computable function.

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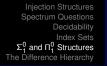
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### Complexity of the Orbits and Character

- Each infinite orbit is a  $\Sigma_1^0$  set
- $\{(k, a) : a \in Rng(f^k)\}$  is a  $\Sigma_1^0$  set
- $\{(a, k) : card(\mathcal{O}_f(a)) \ge k\}$  is a  $D_1^0$  set, the intersection of a  $\Pi_1^0$  set with A
- $\{a: \mathcal{O}_f(a) \text{ is infinite}\}\$  is a  $\Pi_1^0$  set
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$  is a  $\Pi_2^0$  set
- {a : O<sub>f</sub>(a) has type ω} is a Σ<sub>2</sub><sup>0</sup> set
- $\chi(\mathcal{A})$  is a  $\Sigma_1^0$  set
- It follows that any Σ<sup>0</sup><sub>1</sub> injection structure is isomorphic to a computable structure

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#### Isomorphisms

- **Theorem** For any  $\Sigma_1^0$  injection structure  $\mathcal{A}$ , there exists a computable injection structure  $\mathcal{B}$  and a computable isomorphism from  $\mathcal{B}$  onto  $\mathcal{A}$ .
- **Theorem** If  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are isomorphic  $\Sigma_1^0$  injection structures with finitely many infinite orbits, then there is an isomorphism  $\psi : \mathcal{A}_1 \to \mathcal{A}_2$  such that both  $\psi$  and  $\psi^{-1}$  are partial computable.
- Theorem If A<sub>1</sub> and A<sub>2</sub> are isomorphic Σ<sub>1</sub><sup>0</sup> injection structures with either finitely many orbits of type Z or finitely many orbits of type ω, then there is a Δ<sub>2</sub><sup>0</sup> isomorphism ψ : A<sub>1</sub> → A<sub>2</sub>.
- Theorem If A<sub>1</sub> and A<sub>2</sub> are isomorphic Σ<sup>0</sup><sub>1</sub> injection structures, then there is a Δ<sup>0</sup><sub>3</sub> isomorphism ψ<sub>2</sub>: A<sub>1</sub> → A<sub>2</sub>.

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# $Inf(\mathcal{A})$

- Theorem For any d.c.e. set B, there is a Σ<sub>1</sub><sup>0</sup> injection structure A such that B is 1 1 reducible to Inf(A).
- **Proof** Let B = C D, where C and D are c.e. sets and  $D \subset C$ . Let  $A = \{2n + 1 : n \in C\} \cup \{2n : n \in \mathbb{N}\}.$
- For each *n*, we begin to define the orbit of 2n + 1 in A by setting f(2n + 1) = 2(2n + 1), f(2(2n + 1)) = 4(2n + 1) and so on, until we see that  $n \in D$  at some stage s + 1. Then let  $f(2^{s}(2n + 1)) = 2n + 1$  and for t > s, let  $f(2^{t}(2n + 1)) = 2^{t+1}(2n + 1)$ . It follows that for each  $n, n \in B$  IFF  $2n \in Inf(A, f)$ .

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## Complexity of $\Pi_1^0$ structures

- Each infinite orbit is a Σ<sup>0</sup><sub>1</sub> set
- $\{(k, a) : a \in Rng(f^k)\}$  is a  $\Sigma_2^0$  set
- {(a, k) : card(O<sub>f</sub>(a)) ≥ k} is a Π<sup>0</sup><sub>1</sub> set,
- {a: O<sub>f</sub>(a) is finite} is a D<sub>1</sub><sup>0</sup> set, the intersection of A with a c.e. set
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$  is a  $\Pi_3^0$  set
- $\{a: \mathcal{O}_f(a) \text{ has type } \omega\}$  is a  $\Sigma_3^0$  set
- *χ*(*A*) is a Σ<sup>0</sup><sub>2</sub> set

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- Proposition For any Σ<sub>2</sub><sup>0</sup> character K, which is infinite and coinfinite, there is a computable injection structure A with χ(A) = K and with any specified countable number of orbits of types ω and Z.
- **Proposition** For any d.c.e. set *B*, there is a  $\Pi_1^0$  injection structure  $\mathcal{A}$  such that *B* is 1 1 reducible to  $Fin(\mathcal{A})$ .

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 Theorem If A and B are isomorphic Π<sup>0</sup><sub>1</sub> injection structures with only finitely many infinite orbits, then A and B are Δ<sup>0</sup><sub>2</sub> isomorphic.

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#### Infinite Orbits: Case One

- Lemma For any Π<sup>0</sup><sub>1</sub> injection structure A, the relation R(a, b), defined by R(a, b) if and only if a and b are in the same orbit, is a D<sup>0</sup><sub>2</sub> set.
- **Theorem** If the  $\Pi_1^0$  injection structure  $\mathcal{A} = (A, f)$  has only finitely many orbits of type  $\omega$ , then  $\mathcal{A}$  is  $\Delta_2^0$  isomorphic to a computable structure.
- **Corollary** If  $\mathcal{A}$  and  $\mathcal{B}$  are isomorphic  $\Pi_1^0$  injection structures with only finitely many orbits of type  $\omega$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\Delta_2^0$  isomorphic.

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#### Infinite Orbits: Case Two

- **Proposition** For any infinite, co-infinite  $\Sigma_2^0$  set *C*, there is a  $\Pi_1^0$  injection structure  $\mathcal{A} = (A, f)$  with all orbits of type  $\omega$  such that  $C \leq_T Ran(f)$ .
- **Theorem** For any  $\Sigma_1^0$  character K, there is a  $\Pi_1^0$  injection structure  $\mathcal{B}$  with character K, with infinitely many orbits of type  $\omega$  and with an arbitrary number of orbits of type Z, such that  $\mathcal{B}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  injection structure.

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#### Injection Structures in the Ershov Hierarchy

- $\mathcal{A} = (\mathcal{A}, f)$  where  $\mathcal{A}$  is an *n*-c.e. set and f is a computable function
- Each infinite orbit is a  $\Sigma_1^0$  set
- $\{(k, a) : a \in Rng(f^k)\}$  is a  $\Sigma_2^0$  set
- $\{(a,k): card(\mathcal{O}_f(a)) \ge k\}$  is an *n*-c.e. set,
- { $a : O_f(a)$  is infinite} is the intersection of A with a  $\Pi_1^0$  set, so is *n*-c.e. if *n* is even and N + 1-c.e. if *n* is odd.
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$  is a  $\Pi_3^0$  set
- {a : O<sub>f</sub>(a) has type ω} is a Σ<sub>3</sub><sup>0</sup> set
- *χ*(*A*) is a Σ<sup>0</sup><sub>2</sub> set

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## Back to $\Pi_1^0$ structures

- Lemma For any n ∈ N and any infinite n-c.e. set B, there is a Π<sub>1</sub><sup>0</sup> set A and a total computable 1 − 1 function φ mapping A onto B.
- **Proposition** For any *n*-c.e. injection structure  $\mathcal{A}$ , there exist a  $\Pi_1^0$  structure  $\mathcal{B}$  and a computable injection  $\phi : \mathbb{N} \to \mathbb{N}$  that maps  $\mathcal{B}$  onto  $\mathcal{A}$ .
- Corollary If A and B are isomorphic *n*-c.e. injection structures with only finitely many orbits of type ω, then A and B are Δ<sup>0</sup><sub>2</sub> isomorphic.

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#### $\alpha\text{-c.e.}$ functions

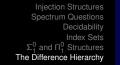
- Let  $g(x) = \lim_{s \to a} f(x, s)$ , where *f* is a computable function.
  - (i) g is an *n-c.e.* function if for all  $x \in \omega$ , card({ $s: f(x, s) \neq f(x, s+1)$ }) < n.
  - (ii) g is an ω-c.e. function if there is a computable function g such that for all x ∈ ω,
    - $1 \leq card(\{s: f(x,s) \neq f(x,s+1)\}) \leq g(x).$
- Proposition For any nonempty Σ<sub>2</sub><sup>0</sup> set A there is a 2-c.e. function whose range is A.
- A function *f* is *graph-α-c.e.* if the graph of *α* is an *α*-c.e. set.
- Proposition
  - (a) For every  $n \in \omega$  there exists an (n+1)-c.e. function that is not graph-*n*-c.e.
  - (b) There is a graph-2-c.e. function that is not an ω-c.e. function.



## 2-c.e. Categoricity: Case One

- **Theorem** There exist computable injection structures, each consisting of infinitely many orbits of type  $\omega$ , which is not 2-c.e. isomorphic.
- Sketch: For a strucure A = (ω, f), define the set E<sup>A</sup> to be those elements of the form f<sup>2n</sup>(a) where a ∉ Ran(f). In the standard structure, E<sup>A</sup> will be a computable set. We can build a computable copy in which E<sup>A</sup> is *n*-c.e. or even ω-c.e. complete. Each orbit of A contains exactly one even number 2e and this orbit O(2e), will be used to defeat the eth ω-c.e. set C<sub>e</sub>. That is, begin with 2e ∉ Ran(f) and whenever e goes into or out of C<sub>e</sub>, add an element to the beginning of O(2e), so that 2e ∈ E<sup>A</sup> IFF e ∉ C<sub>e</sub>.

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### 2-c.e. Categoricity: Case Two

- Theorem There exist computable injection structures, each consisting of infinitely many orbits of type Z, which is not ω-c.e. isomorphic.
- Sketch: Here we will diagonalize against the possible 2-c.e. isomorphisms h<sub>e</sub> from the standard structure A to our structure B, by having a pair of elements a<sub>e</sub> and b<sub>e</sub> in different orbits in A but having h<sub>e</sub>(a<sub>e</sub>) in the same orbit with h<sub>e</sub>(b<sub>e</sub>) in B. We build B with infinitely many orbits of type Z by extending our finite orbits in both directions at each stage and by adding new orbits at each stage. When h<sub>e</sub>(a<sub>e</sub>) and h<sub>e</sub>(b<sub>e</sub>) are defined (or redefined) and we have them in different orbits, we simply combine those into one orbit.

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## 2-c.e. Injections

- $\mathcal{A} = (\omega, f)$  where *f* is an *n*-c.e. function
- Each infinite orbit is a Δ<sup>0</sup><sub>2</sub> set
- $\{(k, a) : a \in Rng(f^k)\}$  is a  $\Sigma_2^0$  set
- {(a, k) : card(O<sub>f</sub>(a)) ≥ k} is a Δ<sub>2</sub><sup>0</sup> set,
- {*a* : O<sub>f</sub>(*a*) is finite} is Σ<sup>0</sup><sub>2</sub>
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$  is a  $\Pi_3^0$  set
- {a : O<sub>f</sub>(a) has type ω} is a Σ<sub>3</sub><sup>0</sup> set
- $\chi(\mathcal{A})$  is a  $\Sigma_3^0$  set
- So every 2-c.e. injection is isomorphic to a Π<sup>0</sup><sub>1</sub> injection

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#### Existence

- **Theorem** Let *K* be a  $\Sigma_2^0$  character.
  - There is a 2-c.e. injection *f* such that  $(\omega, f)$  has character *K* and has infinite orbits.
  - 2 If K possesses an  $s_1$  function, then there is a 2-c.e. injection f such that  $(\omega, f)$  has character K and has no infinite orbits.
- Sketch: Let B = (ω, E) be a computable equivalence structure with character K (in the second case, B has infinitely many infinite orbits)
   Build f so that each equivalence class becomes an orbit
- **Question** Is the *s*<sub>1</sub> function necessary?

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#### Future Work

- Structures (*A*, *f*) where *f* is finite-to-one.
- In particular 2 to 1 or  $\leq$  2 to 1.
- These are much more complicated.

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