# Effective Choice and Boundedness Principles in Computable Analysis*

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**Abstract.** In this paper we study a new approach to classify mathematical theorems according to their computational content. Basically, we are asking the question which theorems can be continuously or computably transferred into each other? For this purpose theorems are considered via their realizers which are operations with certain input and output data. The technical tool to express continuous or computable relations between such operations is Weihrauch reducibility and the partially ordered degree structure induced by it. We have identified certain choice principles on closed sets which are cornerstones among Weihrauch degrees and it turns out that certain core theorems in analysis can be classified naturally in this structure. In particular, we study theorems such as the Intermediate Value Theorem, the Baire Category Theorem, the Banach Inverse Mapping Theorem, the Closed Graph Theorem and the Uniform Boundedness Theorem. Well-known omniscience principles from constructive mathematics such as $\mathsf{LPO}$ and $\mathsf{LLPO}$ can also naturally be considered as Weihrauch degrees and they play an important role in our classification. Our classification scheme does not require any particular logical framework or axiomatic setting, but it can be carried out in the framework of classical mathematics using tools of topology, computability theory and computable analysis. Finally, we present a number of metatheorems that allow to derive upper bounds for the classification of the Weihrauch degree of many theorems and we discuss the Brouwer Fixed Point Theorem as an example.

**Keywords:** Computable analysis, constructive analysis, reverse mathematics, effective descriptive set theory

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# 1 Introduction

The purpose of this paper is to propose a new approach to classify mathematical theorems according to their computational content and according to their logical complexity[3].

## 1.1 Realizability of theorems and Weihrauch reducibility

The basic idea is to interpret theorems, which are typically $\Pi_2$–theorems of the form

$$(\forall x \in X)(\exists y \in Y)(x, y) \in A,$$

as operations $F :\subseteq X \rightrightarrows Y, x \mapsto \{y \in Y : (x, y) \in A\}$ that map certain input data $X$ into certain output data $Y$. In other words, we are representing theorems by their realizers or multi-valued Skolem functions, which is a very natural approach for many typical theorems. For instance, the Intermediate Value Theorem states that

$$(\forall f \in \mathcal{C}[0,1], f(0) \cdot f(1) < 0)(\exists x \in [0,1])f(x) = 0$$

and hence it is natural to consider the partial multi-valued operation

$$\mathsf{IVT} :\subseteq \mathcal{C}[0,1] \rightrightarrows [0,1], f \mapsto \{x \in [0,1] : f(x) = 0\}$$

with $\mathrm{dom}(\mathsf{IVT}) := \{f \in \mathcal{C}[0,1] : f(0) \cdot f(1) < 0\}$ as a representative of this theorem. It follows from the Intermediate Value Theorem itself that this operation is well-defined. The goal of our study is to understand the computational content of theorems like the Intermediate Value Theorem and to analyze how they compare to other theorems. In order to understand the relation of two theorems $T$ and $T'$ to each other we will ask the question whether a realizer $G$ of $T'$ can be computably or continuously transformed into a realizer $F$ of $T$. In other words, we consider theorems as points in a space (represented by their realizers) and we study whether these points can be computably or continuously transferred into each other. This study is carried out entirely in the domain of classical logic and using tools from topology, computability theory and computable analysis [17].

In fact the technical tool to express the relation of realizers to each other is a reducibility that Weihrauch introduced in the 1990s in two unpublished papers [15, 16] and which since then has been studied by several others (see for instance [11, 2, 3, 12, 10, 6, 13]). Basically, the idea is to say that a single-valued function $F$ is Weihrauch reducible to $G$, in symbols $F \leq_{\mathrm{W}} G$, if there are computable function $H$ and $K$ such that

$$F = H\langle \mathrm{id}, GK \rangle.$$

Here $K$ can be considered as an input adaption and $H$ as an output adaption. The output adaption has direct access to the input, since in many cases the input

[3] This paper is only an extended abstract, but a full version with all definitions and proofs is available for the interested reader [7].

cannot be looped through $G$. Here and in the following $\langle\ \rangle$ denotes suitable finite or infinite tupling functions. This reducibility can be extended to sets of functions and to multi-valued functions on represented spaces. The resulting structure has been studied in [6] and among other things it has been proved that parallelization is a closure operator for Weihrauch reducibility. To parallelize a multi-valued function $F$ just means to consider

$$\widehat{F}\langle p_0, p_1, p_2, ...\rangle := \langle F(p_0) \times F(p_1) \times F(p_2) \times ...\rangle,$$

i.e. to take countably many instances of $F$ in parallel.

### 1.2 Effective choice and boundedness principles

A characterization of the Weihrauch degree of theorems is typically achieved by showing that the degree is identical to the degree of some other known principle. We have identified certain choice principles that turned out to be crucial cornerstones in our classification. These principles are *co-finite choice*, *discrete choice*, *interval choice*, *compact choice*, *closed choice*, and are exposed in Sect. 3.

Often it is more convenient to consider these choice principles as boundedness principles and in particular the principles of interval choice have equivalent boundedness versions. In Sect. 3 we will present some boundedness principles that correspond to the above mentioned choice principles.

In Sect. 3 we will show the equivalence of certain choice and boundedness principles and we will compare them to omniscience principles. Omniscience principles have been introduced by Brouwer and Bishop [1, 9] as non-acceptable principles in the intuitionistic framework of constructive analysis.

- ($\mathsf{LPO}$) For any sequence $p \in \mathbb{N}^{\mathbb{N}}$ there exists an $n \in \mathbb{N}$ such that $p(n) = 0$ or $p(n) \neq 0$ for all $n \in \mathbb{N}$.
- ($\mathsf{LLPO}$) For any sequence $p \in \mathbb{N}^{\mathbb{N}}$ such that $p(k) \neq 0$ for at most one $k \in \mathbb{N}$, it follows $p(2n) = 0$ for all $n \in \mathbb{N}$ or $p(2n+1) = 0$ for all $n \in \mathbb{N}$.

The abbreviations stand for *limited principle of omniscience* and *lesser limited principle of omniscience*. The realizers of these statements correspond to discontinuous operations of different degree of discontinuity [16].

The parallelizations $\widehat{\mathsf{LPO}}$ and $\widehat{\mathsf{LLPO}}$ turned out to be particularly important cornerstones in our classification scheme, since $\widehat{\mathsf{LPO}}$ is a $\mathbf{\Sigma}^0_2$–complete operation in the effective Borel hierarchy [3], i.e. it is complete among all limit computable operations with respect to Weihrauch reducibility and similarly $\widehat{\mathsf{LLPO}}$ is complete among all weakly computable operations [10, 6]. Limit computable operations are exactly the effectively $\mathbf{\Sigma}^0_2$–measurable operations and these are exactly those operations that can be computed on a Turing machine that is allowed to revise its output. We have defined weakly computable operations exactly by the above mentioned completeness property in [6]. In Sect. 3 we will show how the choice and boundedness principles are related to the omniscience principles and their parallelizations.

Figure 1 illustrates the relation between the choice principles and other results discussed in this paper.

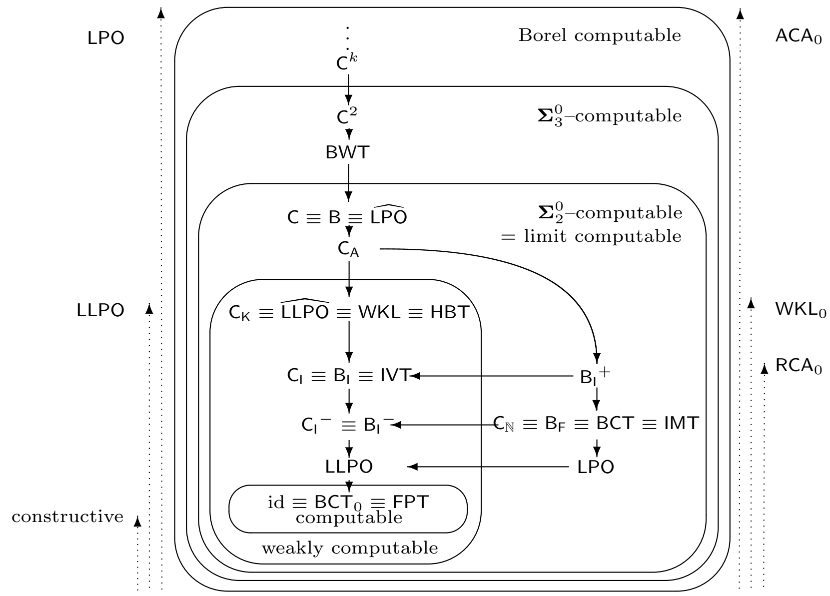


**Fig. 1.** Constructive, computable and reverse mathematics

### 1.3 Theorems in functional analysis

As a case study we analyze a number of theorems from analysis and functional analysis and we classify their Weihrauch degree. In particular, we will consider in Sections 4, 5, 6 and 7 the following theorems:

- ($\mathsf{BCT}_0$) Given a sequence $(A_i)_{i\in\mathbb{N}}$ of closed nowhere dense subsets of a complete separable metric space $X$, there exists a point $x \in X \setminus \bigcup_{i\in\mathbb{N}} A_i$ (Baire Category Theorem).
- ($\mathsf{BCT}$) Given a sequence $(A_i)_{i\in\mathbb{N}}$ of closed subsets of a complete separable metric space $X$ with $X = \bigcup_{i=0}^{\infty} A_i$, there is some $n \in \mathbb{N}$ such that $A_n$ is somewhere dense (Baire Category Theorem).
- ($\mathsf{IMT}$) Any bijective linear bounded operator $T : X \to Y$ on separable Banach spaces $X$ and $Y$ has a bounded inverse $T^{-1} : Y \to X$ (Banach Inverse Mapping Theorem).
- ($\mathsf{OMT}$) Any surjective linear bounded operator $T : X \to Y$ on separable Banach spaces $X$ and $Y$ is open, i.e. $T(U)$ is open for any open $U \subseteq X$ (Open Mapping Theorem).
- ($\mathsf{CGT}$) Any linear operator $T : X \to Y$ with a closed graph$(T) \subseteq X \times Y$ is bounded (Closed Graph Theorem).
- ($\mathsf{UBT}$) Any sequence $(T_i)_{i\in\mathbb{N}}$ of linear bounded operators that is pointwise bounded, i.e. such that $\sup\{||T_i x|| : i \in \mathbb{N}\}$ exists for all $x \in X$, is uniformly bounded, i.e. $\sup\{||T_i|| : i \in \mathbb{N}\}$ exists (Uniform Boundedness Theorem).

- ($\mathsf{HBT}$) Any bounded linear functional $f : Y \to \mathbb{R}$, defined on some closed subspace $Y$ of a Banach space $X$ has a bounded linear extension $g : X \to \mathbb{R}$ with the same norm $||g|| = ||f||$ (Hahn-Banach Theorem).
- ($\mathsf{IVT}$) For any continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) \cdot f(1) < 0$ there exists a $x \in [0,1]$ with $f(x) = 0$ (Intermediate Value Theorem).
- ($\mathsf{BFT}$) Any continuous function $f : [0,1]^n \to [0,1]^n$ has a fixed point $x \in [0,1]^n$, i.e. $f(x) = x$ (Brouwer Fixed Point Theorem).
- ($\mathsf{BWT}$) Any sequence $(x_i)_{i\in\mathbb{N}}$ of numbers in $[0,1]^n$ has a convergent subsequence (Bolzano-Weierstraß Theorem).
- ($\mathsf{WAT}$) For any continuous function $f : [0,1] \to \mathbb{R}$ and any $n \in \mathbb{N}$ there exists a rational polynomial $p \in \mathbb{Q}[x]$ such that $||f-p|| = \sup_{x\in[0,1]} |f(x)-p(x)| < 2^{-n}$ (Weierstraß Approximation Theorem).
- ($\mathsf{WKL}$) Any infinite binary tree has an infinite path (Weak Kőnig's Lemma).

The Baire Category Theorem is an example of a theorem for which it matters which version is realized. In the formulation $\mathsf{BCT}_0$ it leads to a continuous and even computable realizer, whereas the version $\mathsf{BCT}$ is discontinuous. The realizers of the given theorems are operations of different degree of discontinuity and our aim is classify the computational Weihrauch degree of these results. The benefit of such a classification is that practically all purely computability theoretic questions of interest about a theorem in computable analysis can be answered by such a classification. Typical questions are:

1. Is the theorem uniformly computable, i.e. can we compute the output information $y \in Y$ uniformly from the input information $x \in X$?
2. Is the theorem non-uniformly computable, i.e. does there exist a computable output information $y \in Y$ for any computable input information $x \in X$?
3. If there is no uniform solution, is there a uniform computation of a certain effective Borel complexity?
4. If there is no non-uniform computable solution, is there always a non-uniform result of a certain arithmetical complexity or Turing degree?

Answers to questions of this type can be derived from the classification of the Weihrauch degree of a theorem. In the diagram of Fig. 1 we summarize some of our results. The arrows in the diagram are pointing into the direction of computations and implicit logical implications and hence in the inverse direction of the corresponding reductions. No arrow in the diagram can be inverted and no arrows can be added (except those that follow by transitivity).

In Sect. 7 we provide a number of metatheorems that allow to determine upper bounds of the Weihrauch degree of many theorems straightforwardly, just because of the mere topological form of the statement. For instance, any classical result of the form

$$(\forall x \in X)(\exists y \in Y)(x,y) \in A$$

with a co-c.e. closed $A \subseteq X \times Y$ and a co-c.e. compact $Y$ has a realizer that is reducible to compact choice $\mathsf{C_K}$. The table in Fig. 2 summarizes the topological types of metatheorems and the corresponding version of computability. We il-

| metatheorem | computability | unique case |
|---|---|---|
| open | computable | computable |
| compact | weakly computable | computable |
| locally compact | limit computable | non-uniformly computable |

**Fig. 2.** Types of metatheorems, choice and computability

lustrate that these metatheorems are useful and we show that one directly gets upper bounds for theorems such as the Brouwer Fixed Point Theorem and the Peano Existence Theorem for the initial value problem of ordinary differential equations.

## 2 Weihrauch reducibility, omniscience principles and weak computability

In this section we briefly recall some definitions from [6] on Weihrauch reducibility. We assume that the reader has some basic familiarity with concepts from computable analysis and otherwise we refer the reader for all undefined concepts to [17]. In a first step we define Weihrauch reducibility for sets of functions on Baire space, as it was already considered by Weihrauch [15, 16].

**Definition 1 (Weihrauch reducibility).** Let $\mathcal{F}$ and $\mathcal{G}$ be sets of functions of type $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. We say that $\mathcal{F}$ is *Weihrauch reducible* to $\mathcal{G}$, in symbols $\mathcal{F} \leq_{\mathrm{W}} \mathcal{G}$, if there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$(\forall G \in \mathcal{G})(\exists F \in \mathcal{F})\ F = H\langle \mathrm{id}, GK\rangle.$$

Analogously, we define $\mathcal{F} \leq_{\mathrm{sW}} \mathcal{G}$ using the equation $F = HGK$ and in this case we say that $\mathcal{F}$ is *strongly Weihrauch reducible* to $\mathcal{G}$.

We denote the induced equivalence relations by $\equiv_{\mathrm{W}}$ and $\equiv_{\mathrm{sW}}$, respectively.

In the next step we define the concept of a realizer of a multi-valued function as it is used in computable analysis [17]. We recall that a *representation* $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ of a set $X$ is a surjective (and potentially partial) map. In general, the inclusion symbol "$\subseteq$" indicates partiality in this paper. In this situation we say that $(X, \delta_X)$ is a *represented space*.

**Definition 2 (Realizer).** Let $(X, \delta_X)$ and $(Y, \delta_Y)$ be represented spaces and let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function. Then $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called *realizer* of $f$ with respect to $(\delta_X, \delta_Y)$, in symbols $F \vdash f$, if

$$\delta_Y F(p) \in f\delta_X(p)$$

for all $p \in \mathrm{dom}(f\delta_X)$.

Usually, we do not mention the representations explicitly since they will be clear from the context. A multi-valued function $f :\subseteq X \rightrightarrows Y$ on represented spaces is called *continuous* or *computable*, if it has a continuous or computable realizer, respectively. Using reducibility for sets and the concept of a realizer we can now define Weihrauch reducibility for multi-valued functions.

**Definition 3 (Realizer reducibility).** Let $f$ and $g$ be multi-valued functions on represented spaces. Then $f$ is said to be *Weihrauch reducible* to $g$, in symbols $f \leq_{\mathrm{W}} g$, if and only if $\{F : F \vdash f\} \leq_{\mathrm{W}} \{G : G \vdash g\}$. Analogously, we define $f \leq_{\mathrm{sW}} g$ with the help of $\leq_{\mathrm{sW}}$ on sets.

That is, $f \leq_{\mathrm{W}} g$ holds if any realizer of $g$ computes some realizer of $f$ with some fixed uniform translations $H$ and $K$. It is clear that Weihrauch reducibility and its strong version form preorders, i.e. both relations are reflexive and transitive.

One can show that the product of multi-valued functions $f \times g$ and the direct sum $f \oplus g$ are both monotone operations with respect to strong and ordinary Weihrauch reducibility and hence both operations can be extended to Weihrauch degrees. This turns the structure of partially ordered Weihrauch degrees into a lower-semi lattice with the direct sum operation as greatest lower bound operation. It turns out that a very important operation on this lower semi-lattice is parallelization, which can be understood as countably infinite product operation.

**Definition 4 (Parallelization).** Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function. Then we define the *parallelization* $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$ of $f$ by

$$\widehat{f}(x_i)_{i\in\mathbb{N}} := \bigtimes_{i=0}^{\infty} f(x_i)$$

for all $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$.

We mention that parallelization acts as a closure operator with respect to Weihrauch reducibility.

## 3 Choice and boundedness principles

In this section we study choice principles and boundedness principles. Both types of principles are closely related to each other and they are also related to the omniscience principles mentioned earlier. In some sense most of the boundedness principles are just variants of the choice principles that are more convenient for some applications.

By $\mathcal{A}(X)$ or $\mathcal{A}_-(X)$ we denote the set of closed subsets of a metric space $X$. The index "$-$" indicates that we assume that the hyperspace $\mathcal{A}_-(X)$ is equipped with the lower Fell topology and a corresponding negative information representation $\psi_-$ (see [8] for details). All choice principles are restrictions of the multi-valued choice map

$$\mathrm{Choice} :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A,$$

which is defined for non-empty closed sets $A \subseteq X$ and maps any such set in a multi-valued way to the set of its members. That is, the input is a non-empty closed set $A \in \mathcal{A}_-(X)$ and the output is one of the (possibly many) points $x \in A$. We can define restrictions of the choice map by specifying the respective domains and ranges.

**Definition 5 (Choice principles).** We define multi-valued operations as restrictions of the respective choice maps as follows:

1. $\mathsf{C_F} :\subseteq \mathcal{A}_-(\mathbb{N}) \rightrightarrows \mathbb{N}$, $\mathrm{dom}(\mathsf{C_F}) := \{A \subseteq \mathbb{N} : A \text{ co-finite}\}$.
2. $\mathsf{C}_\mathbb{N} :\subseteq \mathcal{A}_-(\mathbb{N}) \rightrightarrows \mathbb{N}$, $\mathrm{dom}(\mathsf{C}_\mathbb{N}) := \{A \subseteq \mathbb{N} : A \neq \emptyset\}$.
3. $\mathsf{C_I} :\subseteq \mathcal{A}_-[0,1] \rightrightarrows [0,1]$, $\mathrm{dom}(\mathsf{C_I}) := \{[a,b] : 0 \leq a \leq b \leq 1\}$.
4. $\mathsf{C_I}^- :\subseteq \mathcal{A}_-[0,1] \rightrightarrows [0,1]$, $\mathrm{dom}(\mathsf{C_I}^-) := \{[a,b] : 0 \leq a < b \leq 1\}$.
5. $\mathsf{C_K} :\subseteq \mathcal{A}_-([0,1]) \rightrightarrows [0,1]$, $\mathrm{dom}(\mathsf{C_K}) := \{K \subseteq [0,1] : K \neq \emptyset \text{ compact}\}$.
6. $\mathsf{C_A} :\subseteq \mathcal{A}_-(\mathbb{R}) \rightrightarrows \mathbb{R}$, $\mathrm{dom}(\mathsf{C_A}) := \{A \subseteq \mathbb{R} : A \neq \emptyset \text{ closed}\}$.

We refer to these operations as *co-finite choice*, *discrete choice*, *interval choice*, *proper interval choice*, *compact choice* and *closed choice*, respectively.

For practical purposes it is often more convenient to handle these choice principles in form of the closely related boundedness principles that we define now.

**Definition 6 (Boundedness principles).** We define the following multi-valued operations:

1. $\mathsf{B_F} : \mathbb{R}_< \rightrightarrows \mathbb{R}, x \mapsto [x, \infty)$.
2. $\mathsf{B_I} :\subseteq \mathbb{R}_< \times \mathbb{R}_> \rightrightarrows \mathbb{R}, (x,y) \mapsto [x,y]$, $\mathrm{dom}(\mathsf{B_I}) := \{(x,y) : x \leq y\}$.
3. $\mathsf{B_I}^- :\subseteq \mathbb{R}_< \times \mathbb{R}_> \rightrightarrows \mathbb{R}, (x,y) \mapsto [x,y]$, $\mathrm{dom}(\mathsf{B_I}^-) := \{(x,y) : x < y\}$.
4. $\mathsf{B_I}^+ :\subseteq \mathbb{R}_< \times \overline{\mathbb{R}_>} \to \mathbb{R}, (x,y) \mapsto [x,y]$, $\mathrm{dom}(\mathsf{B_I}^+) := \{(x,y) : x \leq y\}$.
5. $\mathsf{B} : \mathbb{R}_< \to \mathbb{R}, x \mapsto x$.

**Proposition 1 (Discrete choice).** $\mathsf{B_F} \equiv_{\mathrm{sW}} \mathsf{C_F} \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}$.

**Proposition 2 (Interval choice).** $\mathsf{B_I} \equiv_{\mathrm{sW}} \mathsf{C_I}$, $\mathsf{B_I}^- \equiv_{\mathrm{sW}} \mathsf{C_I}^-$, $\mathsf{B_I}^+ \leq_{\mathrm{sW}} \mathsf{C_A}$.

We recall that it is known that $\mathsf{B}$ is equivalent to $\mathsf{C} : A \mapsto \mathrm{cf}_A$, $\mathrm{dom}(\mathsf{C}) = \mathcal{A}_-(\mathbb{N})$ (which can be considered as countable closed choice).

**Proposition 3 (Countable closed choice).** $\mathsf{B} \equiv_{\mathrm{W}} \mathsf{C} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}}$.

We have identified two chains of choice principles that are related in the given way.

**Corollary 1 (Choice hierarchies).** *We obtain*

1. $\mathsf{LLPO} <_{\mathrm{W}} \mathsf{C_I}^- <_{\mathrm{W}} \mathsf{C_I} <_{\mathrm{W}} \mathsf{C_K} \equiv_{\mathrm{W}} \widehat{\mathsf{LLPO}} \leq_{\mathrm{W}} \mathsf{C_A}$.
2. $\mathsf{LPO} <_{\mathrm{W}} \mathsf{C}_\mathbb{N} <_{\mathrm{W}} \mathsf{B_I}^+ <_{\mathrm{W}} \mathsf{C_A} <_{\mathrm{W}} \mathsf{C} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}}$.
3. $\mathsf{LLPO} <_{\mathrm{W}} \mathsf{LPO}$, $\mathsf{C_I}^- <_{\mathrm{W}} \mathsf{C}_\mathbb{N}$, $\mathsf{C_I} <_{\mathrm{W}} \mathsf{B_I}^+$.

**Corollary 2 (Countable choice principles).** *We obtain the following two equivalence classes:* $\widehat{\mathsf{LLPO}} \equiv_{\mathrm{W}} \widehat{\mathsf{C_I}^-} \equiv_{\mathrm{W}} \widehat{\mathsf{C_I}} \equiv_{\mathrm{W}} \widehat{\mathsf{C_K}} <_{\mathrm{W}} \widehat{\mathsf{LPO}} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_\mathbb{N}} \equiv_{\mathrm{W}} \widehat{\mathsf{C_A}}$.

## 4 Discrete Choice and the Baire Category Theorem

In this section we want to classify the Weihrauch degree of the Baire Category Theorem and some core theorems from functional analysis such as the Banach Inverse Mapping Theorem, the Open Mapping Theorem, the Closed Graph Theorem and the Uniform Boundedness Theorem.

**Theorem 1 (Baire Category Theorem).** *Let $X$ be a non-empty complete computable metric space. Then* $\mathsf{BCT}_X \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

**Theorem 2 (Banach Inverse Mapping Theorem).** *Let $X, Y$ be computable Banach spaces. Then* $\mathsf{IMT}_{X,Y} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{IMT}_{\ell_2,\ell_2}$.

**Theorem 3 (Open Mapping Theorem).** *Let $X, Y$ be computable Banach spaces. Then* $\mathsf{OMT}_{X,Y} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{OMT}_{\ell_2,\ell_2}$.

**Theorem 4 (Closed Graph Theorem).** *Let $X, Y$ be computable Banach spaces. Then* $\mathsf{CGT}_{X,Y} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{CGT}_{\ell_2,\ell_2}$.

**Theorem 5 (Uniform Boundedness Theorem).** *Let $X, Y$ be computable Banach spaces different from $\{0\}$. Then* $\mathsf{UBT}_{X,Y} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

A common feature of all the theorems discussed in this section that are equivalent to $\mathsf{C}_{\mathbb{N}}$ are:

1. They are discontinuous and hence non-computable (since $\mathsf{C}_{\mathbb{N}}$ is so).
2. They admit non-uniform computable solutions (since $\mathsf{C}_{\mathbb{N}}$ has a realizer that maps computable inputs to computable outputs).
3. They have $\Delta^0_2$–complete sequential counterexamples (since $\widehat{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}$, any realizer maps some computable sequence to some $\Delta^0_2$–complete sequence in the arithmetical hierarchy).

All the properties mentioned here are degree theoretic properties and any theorem equivalent to $\mathsf{C}_{\mathbb{N}}$ will be of the same category.

## 5 Interval Choice and the Intermediate Value Theorem

**Theorem 6 (Intermediate Value Theorem).** $\mathsf{IVT} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathsf{I}}$.

We list some common features of all theorems that are equivalent to $\mathsf{C}_{\mathsf{I}}$.

1. They are discontinuous and hence non-computable (since $\mathsf{C}_{\mathsf{I}}$ is so).
2. They admit non-uniform computable solutions (since $\mathsf{C}_{\mathsf{I}}$ has a realizer that maps computable inputs to computable outputs).
3. They are uniformly computable under all classical conditions where the solution is uniquely determined (since $\mathsf{C}_{\mathsf{I}}$ is weakly computable).
4. They have limit computable sequential counterexamples of any basis type (since $\widehat{\mathsf{C}_{\mathsf{I}}} \equiv_{\mathrm{W}} \mathsf{WKL}$).

By a basis type we mean any set $B \subseteq \mathbb{N}^{\mathbb{N}}$ that forms a basis for $\Pi^0_1$ subsets of Cantor space $\{0,1\}^{\mathbb{N}}$, such as the set of low points.

## 6 Compact Choice and the Hahn-Banach Theorem

**Theorem 7 (Hahn-Banach Theorem).** $\mathsf{HBT} \equiv_{\mathrm{W}} \mathsf{C_K}$.

Common features of all theorems equivalent to $\mathsf{C_K}$ are:

1. They are discontinuous and hence non-computable (since $\mathsf{C_K}$ is so).
2. They are uniformly computable under all classical conditions where the solution is uniquely determined (since $\mathsf{C_K}$ is weakly computable).
3. They have limit computable counterexamples of any basis type (since we have that $\mathsf{C_K} \equiv_{\mathrm{W}} \mathsf{WKL}$).

## 7 Metatheorems and Applications

In this section we want to discuss a number of metatheorems that allow some conclusions on the status of theorems merely regarding the logical form of these theorems. Essentially, we are trying to identify the computational status of $\Pi_2$–theorems, i.e. theorems of the form

$$(\forall x \in X)(\exists y \in Y)\ (x, y) \in A,$$

where depending on the properties of $Y$ and $A$ automatically certain computable versions of realizers of these theorems exist. In many cases this allows to get some upper bound on the Weihrauch degree of the corresponding theorem straightforwardly.

**Theorem 8 (Open Metatheorem).** *Let $X, Y$ be computable metric spaces and let $U \subseteq X \times Y$ be c.e. open. If*

$$(\forall x \in X)(\exists y \in Y)(x, y) \in U,$$

*then $R : X \rightrightarrows Y, x \mapsto \{y \in Y : (x, y) \in U\}$ is computable.*

**Corollary 3 (Weierstraß Approximation Theorem).** $\mathsf{WAT} \equiv_{\mathrm{W}} \mathrm{id}$.

The next metatheorem is a similar observation for co-c.e. closed predicates and co-c.e. compact $Y$.

**Theorem 9 (Compact Metatheorem).** *Let $X, Y$ be computable metric spaces and let $Y$ be co-c.e. compact and $A \subseteq X \times Y$ co-c.e. closed. If*

$$(\forall x \in X)(\exists y \in Y)(x, y) \in A,$$

*then $R : X \rightrightarrows Y, x \mapsto \{y \in Y : (x, y) \in A\}$ is weakly computable, i.e. $R \leq_{\mathrm{W}} \mathsf{C_K}$.*

**Corollary 4 (Brouwer Fixed Point Theorem).** $\mathsf{BFT} \leq_{\mathrm{W}} \mathsf{WKL}$.

Many other theorems of analysis that have to do with the solution of equations in compact spaces fall into the same category. This applies for instance to the Schauder Fixed Point Theorem and also to the Intermediate Value Theorem. Sometimes it is not immediately clear that a theorem is of this form. In case of the Peano Existence Theorem for solutions of initial value problems of ordinary differential equations it is easy to see that it can be reduced to the Schauder Fixed Point Theorem (see [14]). Another example of this type is the Hahn-Banach Theorem. As it is usually formulated, is not of the form of an equation with a solution in a compact space. However, using the Banach-Alaoglu Theorem, it can be brought into this form (see [4, 10]). Whenever a theorem that falls under the Compact Metatheorem has a unique solution, then that solution is automatically computable by Corollary 8.8 in [6].

Thus, under all (perhaps purely classical) conditions under which the Brouwer Fixed Point Theorem, the Intermediate Valued Theorem, the Hahn-Banach Theorem or the Peano Existence Theorem have unique solutions, they are already automatically fully computable.

**Theorem 10 (Locally Compact Metatheorem).** *Let $X, Y$ be computable metric spaces, let $Y$ be effectively locally compact and let $A \subseteq X \times Y$ be co-c.e. closed. If*

$$(\forall x \in X)(\exists y \in Y)(x, y) \in A,$$

*then $R : X \rightrightarrows Y, x \mapsto \{y \in Y : (x, y) \in A\}$ satisfies $R \leq_{\mathrm{W}} \mathsf{C}_{\mathcal{A}(Y)}$, where $\mathsf{C}_{\mathcal{A}(Y)}$ is defined like $\mathsf{C}_A$ with $Y$ instead of $\mathbb{R}$. In particular, $R$ is limit computable.*

**Corollary 5.** *Let $X$ be a computable metric space and let $Y$ be an effectively locally compact metric space. If $f : X \to Y$ is a function with a co-c.e. closed graph* $\mathrm{graph}(f) = \{(x, y) \in X \times Y : f(x) = y\}$, *then $f$ is limit computable. In particular, the inverse $g^{-1} : X \to Y$ of any computable bijective function $g : Y \to X$ is limit computable.*

## References


1. Bishop, E., Bridges, D. S.: Constructive Analysis. Volume 279 of Grundlehren der Mathematischen Wissenschaften. Springer, Berlin (1985)
2. Brattka, V.: Computable invariance. Theoretical Computer Science **210** (1999) 3–20
3. Brattka, V.: Effective Borel measurability and reducibility of functions. Mathematical Logic Quarterly **51**(1) (2005) 19–44
4. Brattka, V.: Borel complexity and computability of the Hahn-Banach Theorem. Archive for Mathematical Logic **46**(7–8) (2008) 547–564
5. Brattka, V., Gherardi, G.: Borel complexity of topological operations on computable metric spaces. Journal of Logic and Computation **19**(1) (2009) 45–76
6. Brattka, V., Gherardi, G.: Weihrauch degrees, omniscience principles and weak computability. http://arxiv.org/abs/0905.4679 (preliminary version)
7. Brattka, V., Gherardi, G.: Effective choice and boundedness principles in computable analysis. http://arxiv.org/abs/0905.4685 (preliminary version)
8. Brattka, V., Presser, G.: Computability on subsets of metric spaces. Theoretical Computer Science **305** (2003) 43–76

9. Bridges, D. S., Richman, F.: Varieties of Constructive Mathematics. Volume 97 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (1987)
10. Gherardi, G., Marcone, A.: How incomputable is the separable Hahn-Banach theorem? Notre Dame Journal of Formal Logic (to appear). Extended abstract version in CCA 2008 Proceedings ENTCS **221** (2008) 85–102
11. Hertling, P.: Unstetigkeitsgrade von Funktionen in der effektiven Analysis. Dissertation. Volume 208 of Informatik Berichte 208. FernUniversität Hagen, Hagen (1996)
12. Mylatz, U.: Vergleich unstetiger Funktionen: "Principle of Omniscience" und Vollständigkeit in der $C$–hierarchie. PhD Thesis. Faculty for Mathematics and Computer Science. University Hagen. Germany (2006)
13. Pauly, A.: On the (semi)lattices induced by continuous reducibilities. http://arxiv.org/abs/0903.2177 (preliminary version)
14. Simpson, S. G.: Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic. Springer, Berlin (1999)
15. Weihrauch, K.: The degrees of discontinuity of some translators between representations of the real numbers. Technical Report TR-92-050. International Computer Science Institute. Berkeley. July (1992)
16. Weihrauch, K.: The TTE-interpretation of three hierarchies of omniscience principles. Volume 130 of Informatik Berichte FernUniversität Hagen, Hagen (1992)
17. Weihrauch, K.: Computable Analysis. Springer, Berlin (2000)