Effective Constraints for Quantum Systems

Artur Tsobanjan

work with M. Bojowald, B. Sandhöfer and A. Skirzewski

The Pennsylvania State University

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[1] arXiv:0804.3365, published in Rev. Math. Phys.
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Introduction

- Classically constraints are functions on phase-space Γ_{class}
  - arise directly from the action principle
  - must vanish on the accessible region of Γ_{class}
  - some distinct points of Γ_{class} are physically equivalent
- Follow Dirac-Bergmann algorithm
  - solve constraints—restrict to region where they vanish
  - factor out gauge orbits
  - result: reduced phase-space Γ_{red}
- Γ_{red} generally not a cotangent bundle/no natural polarization —ordinary quantization typically has to be modified
- Avoid this problem using Dirac's prescription
  - quantize the free system
  - promote constraints to operators $\mathbf{C}_{\mathrm{i}}$
  - impose $\mathbf{C}_{\mathrm{i}}|\psi_{\mathrm{phys}}\rangle = 0$

Complications

Condition on physical states: $\boxed{\mathbf{C}|\psi_{\rm phys}\rangle = 0}$

- Construction is clear if $|\psi_{\rm phys}\rangle \in \mathcal{H}_{\rm kin}$
- In general $|\psi_{\rm phys}\rangle$ are distributional
  $\rightarrow$ a separate inner product must be defined to construct $\mathcal{H}_{\rm phys}$
- "Effective" scheme for semiclassical states
  - enlarge $\Gamma_{\rm class}$ to $\Gamma_{\rm Q}$ adding leading order quantum parameters
  - formulate constraints for extra variables on $\Gamma_{\rm Q}$
  - analyze the enlarged system as if it were classical

Sketch of the method

- Take a system that is understood in the absence of constraints
- Expectation values of canonically quantized observables represent the classical phase space Γ_{class}
- This space will be appended by a finite number of quantum fluctuations (or "moments"), representing leading order semiclassical contributions
- The expanded phase-space is equipped with a Poisson structure that follows from the quantum commutator
- A version of Dirac's condition can be formulated directly on our variables as constraint functions on Γ_{Q}
- We are left with a "classical constrained system"

Basic assumptions

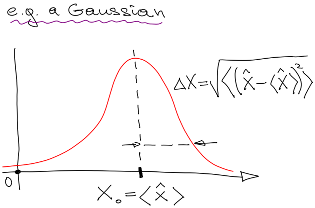
- Assume that the free system has been quantized, in particular
  - we have a sufficiently complete Poisson subalgebra of classical phase space functions $a_i : \Gamma_{\rm class} \to \mathbb{R}$, $i = 1, 2 \dots N$, with $\{a_i, a_j\} = \alpha_{ij}{}^k a_k$
  - it is identified with elements of an associative algebra $\mathbf{a}_i \in \mathscr{A}$, $i = 1, 2 \dots N$, with $[\mathbf{a}_i, \mathbf{a}_j] = \mathbf{a}_i \mathbf{a}_j - \mathbf{a}_j \mathbf{a}_i = i\hbar\alpha_{ij}{}^k \mathbf{a}_k$
  - $\mathscr{A}$ is generated by polynomials in the basic elements $\mathbf{a}_i$
  - there is a single constraint $\mathbf{C} \in \mathscr{A}$
- A state is a linear, complex-valued function on the algebra $\mathscr{A}$ $\rightarrow$ specified by the values assigned to ordered polynomials $\mathbf{a}_1^{n_1} \dots \mathbf{a}_N^{n_N}$ impose $\langle \mathbb{1} \rangle = 1$

What "quantum parameters"?

- Instead of values of ordered polynomials describe a state by
  - N expectation values $\langle \mathbf{a}_\mathrm{i} \rangle$
  - ∞ number of "moments" $\langle (\mathbf{a}_1 - \langle \mathbf{a}_1 \rangle)^{n_1} \ldots (\mathbf{a}_\mathrm{N} - \langle \mathbf{a}_\mathrm{N} \rangle)^{n_N} \rangle_{\mathrm{Weyl}}$

- The value of any polynomial function $\langle f(\mathbf{a}_1, \ldots, \mathbf{a}_\mathrm{N}) \rangle$ may be expressed using these variables

- e.g. a particle on a line, $[\mathbf{x}, \mathbf{p}] = i\hbar$, $\langle (\mathbf{x} - \langle \mathbf{x} \rangle)^2 \rangle$ is the squared spread of the wave-function



- For semiclassical wave-functions "moments" $\propto \hbar^{\frac{1}{2}\sum n_i}$
  $\longrightarrow$ take lower order moments as "quantum parameters"

How do these parameters fit into Γ_{Q}?

- Γ_{class} comes with a Poisson bracket, crucial for dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}O = \{O, H\} + \frac{\partial}{\partial t}O$$

- Poisson structure on Γ_{Q} inspired by Ehrenfest's theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{O} \rangle = \frac{1}{i\hbar}\langle [\mathbf{O}, \mathbf{H}] \rangle + \frac{\partial}{\partial t}\langle \mathbf{O} \rangle$$

- Define $\{\langle \mathbf{A} \rangle, \langle \mathbf{B} \rangle\} := \frac{1}{i\hbar}\langle [\mathbf{A}, \mathbf{B}] \rangle$
  brackets for moments follow from linearity and Leibnitz rule

- $\langle \mathbf{H} \rangle$ generates quantum evolution, Schrödinger equation takes the form [M. Bojowald, A. Skirzewski 2006]

$$\frac{\mathrm{d}}{\mathrm{d}t}X = \{X, \langle \mathbf{H} \rangle\} \rightarrow \ \infty \text{ number of coupled ODE} - \mathrm{s}$$

Implementing Dirac's prescription

- Physical states must satisfy $\mathbf{C}|\psi\rangle = 0$
  this implies $\langle\phi|\mathbf{C}|\psi\rangle = 0, \ \forall\ |\phi\rangle$

- One condition on our variables $\langle\psi|\mathbf{C}|\psi\rangle = \langle\mathbf{C}\rangle = 0$
  —still need infinitely many more

- For normalizable $|\psi\rangle$ and $|\phi\rangle$ there is some $\mathbf{A}$ s.t. $\langle\phi| = \langle\psi|\mathbf{A}$
  $\rightarrow$ the condition is equivalent to $\langle\psi|\mathbf{AC}|\psi\rangle = \langle\mathbf{AC}\rangle = 0, \ \forall\mathbf{A}$

- We use this form of the condition enforcing it systematically as

$$\boxed{\langle\mathbf{a}_1^{n_1}\ldots\mathbf{a}_{\mathrm{N}}^{n_N}\mathbf{C}\rangle = 0}$$

The constraint element is always on the right!

Gauge flows

- The elements $\mathbf{a}_1^{n_1} \dots \mathbf{a}_N^{n_N} \mathbf{C}$ are closed with respect to the commutator
  $\rightarrow$ constraint functions $\langle \mathbf{a}_1^{n_1} \dots \mathbf{a}_N^{n_N} \mathbf{C} \rangle$ form a closed Poisson algebra (1st class)
- Dirac-Bergman analysis may be followed with minor adjustments
- Flows generated by constraints are treated as gauge—generated by the action of $\mathbf{C}$ to the left
- Expectation values and moments of the Dirac observables are recovered by constructing gauge invariant functions on Γ_Q
- After truncation, the constraint system is 1st class only up to the relevant order in moments

Constraints on free relativistic particle

- Free system: two canonical pairs $\{\mathbf{q},\mathbf{p};\mathbf{t},\mathbf{p}_t\}$ subject to $[\mathbf{q},\mathbf{p}]=i\hbar=[\mathbf{t},\mathbf{p}_t]$
- Let $\mathscr{A}$ consist of identity and all ordered polynomials in the canonical variables subject to the canonical commutation relations
- A state is completely determined by the values $\langle\mathbf{q}^k\mathbf{p}^l\mathbf{t}^m\mathbf{p}_t^n\rangle$
- Introduce constraint: $\boxed{\mathbf{C}=\mathbf{p}_t^2-\mathbf{p}^2-m^2\mathbb{1}}$, classically—a relativistic particle in 1+1-dimensional Minkowski space
- Systematically impose constraints order by order: $C_{\mathbf{q}^k\mathbf{p}^l\mathbf{t}^m\mathbf{p}_t^n}=\langle\mathbf{q}^k\mathbf{p}^l\mathbf{t}^m\mathbf{p}_t^n\mathbf{C}\rangle=0$ —infinitely many

Corrections up to 2nd order

- Degrees of freedom: 4 expectation values $a = \langle \mathbf{a} \rangle$; 4 spreads $(\Delta a)^2 = \langle (\mathbf{a} - a)^2 \rangle$ and 6 variances $\Delta(ab) = \langle (\mathbf{a} - a)(\mathbf{b} - b) \rangle_{\rm Weyl}$
- 5 non-trivial constraints left:

$$\langle \mathbf{C} \rangle = p_t^2 - p^2 - m^2 + (\Delta p_t)^2 - (\Delta p)^2 = 0, \qquad \langle (\mathbf{t} - \langle \mathbf{t} \rangle)\mathbf{C} \rangle = 2p_t \Delta(tp_t) + i\hbar p_t - 2p\Delta(tp) = 0$$
$$\langle (\mathbf{p}_t - \langle \mathbf{p}_t \rangle)\mathbf{C} \rangle = 2p_t(\Delta p_t)^2 - 2p\Delta(p_t p) = 0, \qquad \langle (\mathbf{q} - \langle \mathbf{q} \rangle)\mathbf{C} \rangle = 2p_t \Delta(p_t q) - 2p\Delta(qp) - i\hbar p = 0$$
$$\langle (\mathbf{p} - \langle \mathbf{p} \rangle)\mathbf{C} \rangle = 2p_t \Delta(p_t p) - 2p(\Delta p)^2 = 0$$

- There are two corresponding surfaces compatible with the semiclassical approximation

$$p_t = \pm E, \quad \Delta(tp_t) = \pm \frac{p}{E}\Delta(tp) - \frac{i\hbar}{2}, \quad (\Delta p_t)^2 = p^2 + m^2 + (\Delta p)^2 - E^2$$
$$\Delta(p_t q) = \pm \frac{p}{E}\left(\Delta(qp) + \frac{i\hbar}{2}\right), \quad \Delta(p_t p) = \pm \frac{p}{E}(\Delta p)^2$$

where

$$E = \frac{1}{\sqrt{2}}\sqrt{\left(p^2 + m^2 + (\Delta p)^2 + \sqrt{(p^2 + m^2 + (\Delta p)^2)^2 - 4p^2(\Delta p)^2}\right)}$$

Deparametrization gauge

- Idea: $\mathbf{p}_t$ is a derivative operator with respect to the evolution parameter $\mathbf{t}$

$$i\hbar\frac{\partial}{\partial t}\psi = \pm\sqrt{\mathbf{p}^2 + m^2\mathbb{1}}$$

- Mimic this process:
  - use constraints to eliminate variables involving p_t
  - gauge-fix all moments of t to vanish
  - treat the remaining gauge as the evolution of variables generated by $\mathbf{q}$ and $\mathbf{p}$ along parameter $t = \langle \mathbf{t} \rangle$

- Resulting evolution along t is generated by

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{O} \rangle = \{\langle \mathbf{O} \rangle, p_t + E\} = \{\langle \mathbf{O} \rangle, E\} + \frac{\partial \langle \mathbf{O} \rangle}{\partial t}$$

- Enforce positivity of the state with respect to $\mathbf{q}$ and $\mathbf{p}$ by imposing
  - reality: $q, p, (\Delta q)^2, (\Delta p)^2, \Delta(qp) \in \mathbb{R}$
  - positivity: $(\Delta p)^2, (\Delta q)^2 \geq 0$
  - inequality: $(\Delta q)^2(\Delta p)^2 - (\Delta(qp))^2 \geq \frac{1}{4}\hbar^2$

Some conclusions

- These steps can be repeated for a particle in an external potential
- For constraints of the form $\mathbf{C} = \mathbf{p}_t^2 - H(\mathbf{q}, \mathbf{p})^2$, evolution in the reparametrization gauge is generated by $\pm\langle|H(\mathbf{q}, \mathbf{p})|\rangle$
- This still holds if the variation of H in time is "slow"—more generally, a different gauge choice is needed
- Our results strengthen the case for using "effective square-root hamiltonians" in quantum cosmology

Outlook

- Constructed a method for deriving semiclassical corrections for constrained quantum-mechanical systems
- Applied to Newtonian and relativistic particle in a potential, recovering the usual deparameterized dynamics
- In the cases where deparametrizadtion is not possible, the gauge choices still need to be better understood
- Construction should generalize to non-canonical Poisson algebras —various cosmological models are to be analyzed next