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Effective Distribution Theory

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Abstract

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In this thesis we introduce and study a notion of effectivity (or computability) for test functions and for distributions. This is done using the theory of effective (Scott-Ershov) domains and effective domain representations.

To be able to construct effective domain representations of the spaces of test functions considered in distribution theory we need to develop the theory of admissible domain representations over countable pseudobases. This is done in the first paper of the thesis. To construct an effective domain representation of the space of distributions, we introduce and develop a notion of partial continuous function on domains. This is done in the second paper of the thesis. In the third paper we apply the results from the first two papers to develop an effective theory of distributions using effective domains. We prove that the vector space operations on each space, as well as the standard embeddings into the space of distributions effectivise. We also prove that the Fourier transform (as well as its inverse) on the space of tempered distributions is effective. Finally, we show how to use convolution to compute primitives on the space of distributions. In the last paper we investigate the effective properties of a structure theorem for the space of distributions with compact support. We show that each of the four characterisations of the class of compactly supported distributions in the structure theorem gives rise to an effective domain representation of the space. We then use effective reductions (and Turing-reductions) to study the reducibility properties of these four representations. We prove that three of the four representations are effectively equivalent, and furthermore, that all four representations are Turing-equivalent. Finally, we consider a similar structure theorem for the space of distributions supported at 0.

Keywords: Computable mathematics, computable analysis, domain theory, domain representations, distribution theory.

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List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I Dahlgren, F. 2007. Admissible Domain Representations of Inductive Limit Spaces, *U.U.D.M. report 2007:35, Department of Mathematics, Uppsala university*.
- II Dahlgren, F. 2007. Partial Continuous Functions and Admissible Domain Representations, *To appear in the Journal of Logic and Computation*.
- III Dahlgren, F. 2007. A Domain Theoretic Approach to Effective Distribution Theory, *U.U.D.M. report 2007:37, Department of Mathematics, Uppsala university*.
- IV Dahlgren, F. Effective Structure Theorems for Spaces of Compactly Supported Distributions, *U.U.D.M. report 2007:38, Department of Mathematics, Uppsala university*.

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Contents

1. Introduction

Since the days of Gödel, Church, and Turing it is well-known what it means for a function on the natural numbers $\mathbb{N}$ to be computable. The famous Church-Turing thesis states that:

(CT) A function on the natural numbers is computable, or effectively calculable, if and only if it is computable by a Turing machine.

This characterisation extends easily enough to computations on the set of words over a finite (or countably infinite) alphabet. In the more general case when A is a countable set, we may introduce a computability theory on A by the use of numberings. A numbering of A is simply a surjective function $\alpha : \mathbb{N} \to A$. We think of $n \in \mathbb{N}$ as a *name* or *representation* of $\alpha(n) \in A$. We may now "compute on A" by computing directly on the set of names in $\mathbb{N}$. (This idea is well-known in computer science: All modern computers operate on binary representations of natural numbers which are used to represent different types of data.)

1.1 Capturing the notion of approximation

As mathematicians our interests often go beyond the countable, and it is natural to ask if and how the theory of computation on countable sets extends to the uncountable. After all, we have been computing on the real numbers for thousands of years, and so it is only natural to ask how this idea of computation on uncountable sets may be captured formally. By a simple cardinality argument it is clear that we cannot use the natural numbers to represent all the elements in an uncountable set. There are simply to few of them!

One important realisation is that when we compute *on* the set real numbers we are not really computing *with* actual real numbers, but with rational (or floating point) approximations of real numbers. The actual result of a computation on the real numbers is obtained in the limit as we supply better and better approximations of the input to our algorithm. It thus becomes interesting to capture the notion of *approximation* formally. In an attempt to do this we will propose three simple axioms which we require any well-behaved set of approximations to satisfy, and we will show how these three axioms give rise to a structure known as an *algebraic domain* as well as a *domain representation* of the approximated space. This is one way (originally due to Stoltenberg-Hansen) to motivate the theories of algebraic domains and domain representations which are developed and applied in this thesis.

Thus, let X be a set, and let A be a set of approximations of the elements in X. (We think of the elements in A as finite tokens of information describing the elements in X. They may be natural numbers, words over some finite alphabet, or some other finite, computationally well-behaved objects.) It will be convenient to introduce the notation $a \prec x$ for the relation "a approximates x". The structure $(X, A, \prec)$ is called an *approximation structure* if it satisfies the the axioms 1 – 3 given below. Taken together, the axioms 1 – 3 describe how we expect the "concrete" or "finite" approximations in A to relate to the "ideal" elements in X. Before we state each of the axioms, we give a brief explanation of the intuition behind the formulation.

As we have seen, we cannot expect to be able to compute over the ideal elements in X directly, and so we have to resort to computing over the approximations in A. For this reason, it is natural to require that the elements in X can be completely distinguished by their approximations. Thus, we require that $(X, A, \prec)$ satisfies the following separation property:

Axiom 1 (Separation)**.** *If* $\{a \in A;\ a \prec x\} = \{a \in A;\ a \prec y\}$*, then* $x = y$ *in* X.

That is, if we cannot distinguish (or separate) x and y by their approximations, then they must in fact be the same element in X.

If a and b are two approximations in A and all the information in a is also in b, it is reasonable to expect that $\{x \in X;\ a \prec X\} \supseteq \{x \in X;\ b \prec x\}$. To capture this idea of relative information content formally, we say that $a \in A$ *contains no more information than* $b \in A$ (written $a \sqsubseteq b$), if $\{x \in X;\ a \prec x\} \supseteq \{x \in X;\ b \prec x\}$. The order relation $\sqsubseteq$ is usually known as the *information order* on A. It is easy to see that $a \sqsubseteq a$ for all $a \in A$, and that $a \sqsubseteq b \sqsubseteq c$ implies that $a \sqsubseteq c$ for all $a, b, c \in A$. It follows that $\sqsubseteq$ is a preorder on the set A.

To motivate the second axiom, suppose that a and b are two different approximations of the same element $x \in X$. Then a and b contain consistent information (since this information applies to or holds for the element x), and it is reasonable to assume that we can combine the information in a and in b to a single approximation c in A, which contains all the information contained in both a and b. Thus, we assume that

Axiom 2 (Upper bounds)**.** *If a and b both approximate the same element x, there is an approximation c of x such that $a \sqsubseteq c$ and $b \sqsubseteq c$.*

The approximation c contains the information contained in a (since $a \sqsubseteq c$) as well as the information contained in b (since $b \sqsubseteq c$). We note that c may contain more information than a and b do together.

Finally, it will be convenient to assume that there is an approximation $\bot \in A$ which contains no information at all. Clearly, if there isn't one we may simply add one. Thus, we require that

Axiom 3 (Bottom)**.** *There is an approximation $\bot \in A$ which approximates every element in X.*

This means that $\{x \in X;\ \bot \prec x\} = X$, and so in particular, it follows that $\bot \sqsubseteq a$ for all approximations $a \in A$.

We will assume throughout that $(X, A, \prec)$ is an approximation structure. That is, we assume that $(X, A, \prec)$ satisfies the axioms 1 – 3 above.

To approximate an element $x \in X$ we may simply list the approximations of x in A. The set $\{a \in A;\ a \prec x\}$ of approximations of x may be seen as a "generalised sequence" which "converges" to x in A. Note that it follows immediately by the first axiom that the "limit" of $\{a \in A;\ a \prec x\}$ is unique. Now, just as we may construct the real numbers from the rational numbers by adding limits to all rational Cauchy sequences, we may construct a corresponding completion of A by adding limits to all generalised sequences in A. The completion of A will contain (copies of) the concrete approximations in A as well as (copies of) the ideal elements in X. To describe how this completion of is obtained, we will first need to define what we mean by the term generalised sequence.

Let C be a subset of A. We say that C is *directed* if C is nonempty, and whenever $a, b \in C$ there is some $c \in C$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$. Intuitively, since the information contained in the two approximations a and b is also contained in the single approximation c it follows that a and b do not contain contradictory or inconsistent information. Hence, a directed set in A is simply a *consistent* set of approximations. A good example of a directed subset of A is the set of approximations of an element $x \in X$. (That $\{a \in A;\ a \prec x\}$ is directed follows by the second axiom.) Another example is the *down set* of $b \in A$ given by $\downarrow b = \{a \in A;\ a \sqsubseteq b\}$.

We would like to add limits to all directed set in A, but as it turns out, it is enough to add limits to a subclass of "canonical" directed sets to get all limits. Such canonical sets are called *ideals*. An ideal in A is simply a subset of A which is directed and downwards closed. That is, $I \subseteq A$ is an ideal if I is directed, and $a \in I$ whenever $a \sqsubseteq b$ for some $b \in I$. We note that the set $\{a \in A;\ a \prec x\}$ is an ideal, and so is the set $\{a \in A;\ a \sqsubseteq b\}$.

Thus, to add limits to all generalised sequences in A we let $\mathrm{Idl}(A)$ be the set of ideals in A. Since ideals are simply sets of consistent information we may order them after relative information content as before: If I and J are ideals in A we say that *I contains no more information than J* if $I \subseteq J$, and denote this by $I \sqsubseteq J$. Since the order relation $\sqsubseteq$ on $\mathrm{Idl}(A)$ captures the notion of relative information content it is known as the *information order* on $\mathrm{Idl}(A)$. The partially ordered set $(\mathrm{Idl}(A), \sqsubseteq)$ is called the *ideal completion* of A.

We can embed the set A into $\mathrm{Idl}(A)$ as follows: If $b \in A$ we already know that $\downarrow b = \{a \in A;\ a \sqsubseteq b\}$ is an ideal in A. It is not so difficult to show that the map $\downarrow : A \to \mathrm{Idl}(A)$ taking b in A to the ideal $\downarrow b$ preserves the order of information on A in the sense that $a \sqsubseteq b$ in A if and only if $\downarrow a \sqsubseteq \downarrow b$ in $\mathrm{Idl}(A)$ for all $a, b \in A$. It follows that we may identify A with the subset $\{\downarrow a;\ a \in A\}$ of $\mathrm{Idl}(A)$. (Note that $\downarrow : A \to \mathrm{Idl}(A)$ is not one-to-one in general: If $a \sqsubseteq b \sqsubseteq a$ in A then $\downarrow a = \downarrow b$ in $\mathrm{Idl}(A)$. This is not really a problem since a and b have the same information content whenever $a \sqsubseteq b \sqsubseteq a$, and so at least from an information theoretic point of view, they are identical.)

What is more interesting is that we can embed the space X into $\mathrm{Idl}(A)$ as well. If $x \in X$ we let $[x]$ denote the ideal $[x] = \{a \in A;\ a \prec x\}$. That $[x]$ is an ideal for each $x \in X$ follows by Axioms 2 and 3 above. The map $[\cdot] : X \to \mathrm{Idl}(A)$ is one-to-one by Axiom 1 and so defines an embedding from X into $\mathrm{Idl}(A)$. We may thus identify the set X with the subset $\{[x];\ x \in X\}$ of $\mathrm{Idl}(A)$. It follows that the structure $\mathrm{Idl}(A)$ contains both the concrete approximations in A, as well as the ideal elements in X. We think of the elements in $\mathrm{Idl}(A)$ as approximations of the elements in X. Some are concrete and finite (such as the approximations coming from the set A), and some are ideal elements (such as the elements from X).

The ideal completion of A is complete in the following sense: If C is a directed set in $\mathrm{Idl}(A)$ then C has a *least upper bound* J, in the sense that $I \sqsubseteq J$ for each ideal I in C, and if $I \sqsubseteq K$ for all ideals I in C then $J \sqsubseteq K$ as well. The ideal J contains the information contained in the ideals in C *and nothing more*. We write $\bigsqcup C$ for the least upper bound of C. It is easy to show that each ideal J is the least upper bound of the concrete approximations a in A such that $a \sqsubseteq J$. It follows that each ideal J is completely determined by its concrete or finite approximations. In particular, if $x \in X$ then $[x]$ is the least upper bound of the set $\{a \in A;\ \downarrow a \sqsubseteq [x]\} = \{a \in A;\ a \prec x\}$.

1.2 Computation on approximation structures

Suppose that X and Y are two sets, and that $(X, A, \prec_A)$ and $(Y, B, \prec_B)$ are approximation structures. To describe how the concrete approximations in A and B are used to compute on X and Y, we need to be a little more precise about the nature of the elements in A and B. Since the approximations in A and B are thought of as concrete or finite tokens of information it is natural to assume that each approximation in A and B can be completely described by a string over some fixed finite alphabet[1].

Now, we would like to say that a function $f : X \to Y$ from X to Y is *effectively calculable* (or simply *effective*) if there is an algorithm M which takes information about the input x to information about the output $f(x)$. To be a little more precise, we note that if we want to approximate the value of f at x we take the information that we have about x and input this to the algorithm M. In general, this information will be a *consistent set* of approximations of x in A. We expect that the output will be either a single approximation of $f(x)$, or perhaps a *consistent set* of approximations of $f(x)$. Since the input and output of M are consistent sets of approximations, it seems natural to assume that M can be realised as a function from the ideal completion of A to the ideal completion of B.

Thus, to be able to describe how we compute using the concrete approximations in A and B we need to fix a class of functions from $\mathrm{Idl}(A)$ to $\mathrm{Idl}(B)$ which can reasonably be thought of as realisations of a computation. If $\overline{f} : \mathrm{Idl}(A) \to \mathrm{Idl}(B)$ is a realisation of M it is natural to require that $\overline{f}$ *preserves information content* in the following sense:

Axiom 4 (Monotonicity). *If $I \sqsubseteq J$ in $\mathrm{Idl}(A)$, then $\overline{f}(I) \sqsubseteq \overline{f}(J)$ in $\mathrm{Idl}(B)$.*

It is also natural to require that each finite piece of information about the output can be obtained from some finite piece of information about the input. In particular, if x is an element of X, and b is some concrete or finite approximation of $f(x)$ in B we assume that there exists some concrete approximation a of x in A such that $\overline{f}(\downarrow a)$ contains all the information in b. That is, we require that $\overline{f}$ is *continuous* in the sense that:

Axiom 5 (Continuity). *If J is in $\mathrm{Idl}(A)$, b is in B, and $b \sqsubseteq \overline{f}(J)$ there is some $a \in A$ such that $\downarrow a \sqsubseteq J$ and $\downarrow b \sqsubseteq \overline{f}(\downarrow a)$.*

Since $\overline{f}$ is a realisation of M and is supposed to compute the function f it is natural to assume that $\overline{f}$ extends f. Thus, we require that:

[1]This can be made precise using numberings, but we believe that this intuitive idea is enough at the moment.

Axiom 6 (The extension property)**.** *The function* $\overline{f}$ *satisfies* $\overline{f}([x]) = [f(x)]$ *for each* $x \in X$.

Finally, to be able to approximate the value of f at x we need to require that the function $\overline{f}$ can be computed or effectively approximated in some way. Thus, we require that $\overline{f} : \mathrm{Idl}(A) \to \mathrm{Idl}(B)$ is *effective* in the following sense:

Axiom 7 (Effectivity)**.** *There is a Turing machine* $\overline{M}$ *which given* $a \in A$*, lists the set of all approximation* $b \in B$ *such that* $\downarrow b \sqsubseteq \overline{f}(\downarrow a)$.

To obtain information about an element in X we simply list the approximations of x in A. We say that an element $x \in X$ is *computable* if there is a Turing machine which lists all the approximations $a \in A$ such that $a \prec x$. (Note that it follows by a simple cardinality argument that most elements in an uncountable set are *not* computable in this way.) If $x \in X$ is computable we may compute the value of f at x as follows: Let $b \in B$. By 5 and 7 we know that $b \prec f(x)$ if and only if $\downarrow b \sqsubseteq \overline{f}([x])$ if and only if $\downarrow b \sqsubseteq \overline{f}(\downarrow a)$ for some approximation a of x in A. Thus, to approximate $f(x)$ we enumerate-indent the approximations of x, and for each $a \prec x$, we enumerate-indent the set of all $b \in B$ such that $\downarrow b \sqsubseteq \overline{f}(\downarrow a)$. By 4 and 7 we know that any approximation below $\overline{f}(\downarrow a)$ is an approximation of $f(x)$ whenever a approximates x. By 5 we know that any approximation b of x can be obtained in this way. It follows that $f : X \to Y$ takes computable elements in X to computable elements in Y. Furthermore, *if we can compute* x *in* X*, we can also compute the element* $f(x)$ *in* Y.

1.3 Domain theory

The structure $\mathrm{Idl}(A)$ introduced in Section 1.1 is an example of an algebraic domain. (In fact, it turns out that any algebraic domain can be obtained in this way.) Domain theory was introduced by Dana Scott in the late 1960s to construct a denotational semantics for the untyped lambda calculus. In the early 1970s, Yuri Ershov developed the theory of f-spaces (independently from Scott) to study the class of Kleene-Kreisel continuous functionals. It has since been shown that the two theories are closely related to each other.

Intuitively, a domain is a mathematical structure which models the two notions of *approximation* and *convergence*. More precisely, there is an order relation on the domain which is thought of as an information order, and information poor elements further down are thought of as *approximations* of the information rich elements higher up in the domain. The domain also supports a notion of *convergence* for consistent (that is, for directed) sets of information. In technical terms, this translates to the requirement that every directed set of elements in the domain has a least upper bound.

Definition 1.3.1. *Let $(D, \sqsubseteq)$ be a partially ordered set. The set $C \subseteq D$ is* directed *if C is nonempty and given $a, b \in C$ there is some $c \in C$ such that $a, b \sqsubseteq c$.*

A directed complete partial order *(dcpo) is a partially ordered set $(D, \sqsubseteq)$ with a least element $\bot$, where each directed set $C \subseteq D$ has a supremum $\bigsqcup C$ in D.*

Now, suppose that $(D, \sqsubseteq)$ is a dcpo. An element $a \in D$ is compact *(or* finite*) if for each directed set $C \subseteq D$, if $a \sqsubseteq \bigsqcup C$ then $a \sqsubseteq c$ for some $c \in C$. We write D_c for the set of compact elements in D, and if $x \in D$ we let* $\mathrm{approx}(x) = \{a \in D_c;\ a \sqsubseteq x\}$.

The dcpo $(D, \sqsubseteq)$ is an algebraic domain *if* $\mathrm{approx}(x)$ *is directed and* $\bigsqcup \mathrm{approx}(x) = x$ *for each $x \in D$.*

Thus, in an algebraic domain every element is the least upper bound of its compact (or finite) approximations. This important property gives the theory of algebraic domains a strong connection to computability theory.

In the introduction to this section we claimed that the ideal completion of A studied in the previous sections is an algebraic domain. We may now give an argument for why this is so. Suppose that $(X, A, \prec)$ is an approximation structure. We already know that $\mathrm{Idl}(A)$ is a dcpo. To see that $\mathrm{Idl}(A)$ is algebraic, it is enough to note that the compact elements in $\mathrm{Idl}(A)$ are exactly the concrete approximations of the form $\downarrow a$ where a ranges over A.

Something more is true however. Let D be an algebraic domain. Then D may be obtained as the ideal completion of a suitably chosen set of approximations A for some approximation structure $(X, A, \prec)$. To see this, we simply let let $X = D$, $A = D_c$, and define the approximation relation $a \prec x$ by $a \prec x$ if and only if $a \sqsubseteq x$ in D. It is easy to verify that $(X, A, \prec)$ satisfies Axioms 1 – 3, and $D \cong \mathrm{Idl}(D_c) = \mathrm{Idl}(A)$ by Theorem 2.3 in [19].

The structure-preserving maps on domains are those functions which preserve information content. These are the continuous functions.

Definition 1.3.2. *Let D and E be domains and let $f : D \to E$. Then $f : D \to E$ is* monotone *if $f(x) \sqsubseteq_E f(y)$ whenever $x \sqsubseteq_D y$ in D. f is* continuous *if f is monotone and $f(\bigsqcup C) = \bigsqcup f[C]$ for each directed set $C \subseteq D$.*

That is, $f : D \to E$ is continuous if f preserves both the notion of approximation, and the notion of convergence.

If D is a domain we may define a topology on D by taking as subbasis all sets of the form $\uparrow a = \{x \in D;\ a \sqsubseteq x\}$, where a ranges over D_c. The topology obtained in this way is known as the *Scott-topology* on D. One can now show that $f : D \to E$ is continuous in the sense of Definition 1.3.2 if and only if f is continuous with respect to the Scott-topologies on D and E. We also note that $f : \mathrm{Idl}(A) \to \mathrm{Idl}(B)$ is continuous in the sense of Definition 1.3.2 if and only if f satisfies Axioms 4 and 5 from the previous section. Thus, if the reader is convinced by the argument in Sections 1.1 and 1.2 , it follows that the category **Alg** of algebraic domains and continuous functions is the correct setting to study computations on uncountable spaces.

To be able to model computations on higher types, we would like to work over a base category of domains which supports constructions like products and function spaces. That is, we would like the base category of domains to be Cartesian closed. Unfortunately, one can show that there are algebraic domains D and E such that the function space $[D \to E]$ does not form an algebraic domain, and so in particular, it follows that the category **Alg** is not Cartesian closed.

One way to circumvent this problem is to restrict the base category of domains to a subcategory of **Alg** which is Cartesian closed.

Definition 1.3.3. *Let $(D, \sqsubseteq)$ be a dcpo. A subset C of D is* consistent *if it has an upper bound. That is, C is consistent if there is some $x \in D$ such that $c \sqsubseteq x$ for each $c \in C$.*

The dcpo D is consistently complete *if every consistent subset of D has a least upper bound*[2]*, and D is a* Scott-Ershov domain *(or simply* domain*) if D is a consistently complete algebraic domain.*

Let **Dom** be the category of (Scott-Ershov) domains and continuous functions. That the category of domains is Cartesian closed was shown by Scott in the late 1960s. Since every domain is algebraic by definition we may use domains to model computations on uncountable spaces as in Section 1.2, and since the category of domains is Cartesian closed it seems well suited to model computations on higher types. To indicate how this may be achieved, we need to be introduce a notion of computability or effectivity for domains.

In Section 1.2 we required that the concrete approximations could be completely described using finite strings over some fixed alphabet. We will now make this idea a bit more precise using numberings.

Definition 1.3.4. *Let D be a domain and let $\alpha : \mathbb{N} \to \mathscr{D}_c$ be a numbering*[3] *of D_c. Then (D, α) is an* effective *if the relations "$\alpha(m) \sqsubseteq \beta(n)$", "$\alpha(k)$ is the*

[2] This condition is equivalent to the requirement that all consistent pairs $a, b \in D_c$ have a least upper bound in D_c.

[3] A *numbering* of D_c is simply a surjective map from the natural numbers to D_c.

least upper bound of $\alpha(m)$ and $\alpha(n)$", and "$\{\alpha(m),\ \alpha(n)\}$ is consistent" are decidable by a Turing machine.

If (D, α) is an effective domain then $x \in D$ is *computable* if there is a Turing machine which lists the set $\{n \in \mathbb{N};\ \alpha(n) \sqsubseteq x\}$ on the output tape. (That is, $x \in D$ is computable if the set $\{n \in \mathbb{N};\ \alpha(n) \sqsubseteq x\}$ is r.e.) Intuitively, $x \in D$ is computable if we have a method to "approximate x arbitrarily well".

We also required in Section 1.2 that a continuous function which modeled a computation could be computed or effectively approximated in some way. We may now rephrase this requirement using numberings as well

Definition 1.3.5. *Let (D, α) and (E, β) be effective domains, and let $f : D \to E$ be continuous. We say that $f : D \to E$ is* effective *if there is a Turing machine which given $m \in \mathbb{N}$, lists the set of all $n \in \mathbb{N}$ such that $\beta(n) \sqsubseteq f(\alpha(m))$. (That is, $f : D \to E$ is effective if the relation "$\beta(n) \sqsubseteq f(\alpha(m))$" is r.e.)*

Thus, a continuous function $f : D \to E$ from D to E is effective if we have a method which allows us to list approximations of the output of f, whenever we are given a list of approximations of the input. One can show, just as in Section 1.2, that effective continuous functions take computable elements to computable elements.

We let **Eff-Dom** be the category with objects effective domains and morphisms effective continuous functions. The category of effective domains is Cartesian closed as well, and the inclusion functor from **Eff-Dom** to **Dom** preserves this structure. The category of effective domains also supports many other constructions such as inductive and recursive definitions, and least fixed points, which are of interest from the point of view of semantics.

1.4 Domain representations

In the first two sections of this introduction we showed how one may use approximation structures to introduce a notion of computation on an uncountable space X. The approach outlined in Sections 1.1 and 1.2 is well suited for motivational purposes, but it will be convenient to have a more general notion of domain representation which includes the approximation structures as a special case. The correct notion of a domain representation of a topological space is a generalisation both of the notion of a numbering of a countable space, as well as the notion of an approximation structure defined in Section 1.1. More precisely

Definition 1.4.1. *Let X be a topological space. A* domain representation *of X is a triple $(D,\ D^R,\ \delta)$ where D is an algebraic domain, $D^R \subseteq D$, and $\delta : D^R \to X$ is a continuous function from D^R onto X. If the algebraic domain D is effective we say that $(D,\ D^R,\ \delta)$ is an* effective *domain representation of X.*

Informally, we think of the elements in D as approximations of the elements in X. The elements in D^R are approximations which contain enough information to single out a unique element in X. We think of these information rich approximations as *names* or *representations* of the elements in X. The function δ takes each name or representation in D^R to the corresponding element in X. It is natural to require that δ is continuous to connect the notions of approximation and convergence in D to the topology on the space X, as well as surjective, to ensure that each element in X has a representation in D^R. However, we do not require that δ is one-to-one or a homeomorphism, and so in particular, it follows that an element in X may have many different names in D^R.

It is easy to show that the notion of a domain representation is a generalisation of the notion of a numbering introduced above: Let $D_{\mathbb{N}}$ be the domain $\mathbb{N} \cup \{\perp\}$ where $\perp \sqsubseteq n$ for each $n \in \mathbb{N}$, and $m \sqsubseteq n$ if and only if $m = n$. Let $D^R_{\mathbb{N}} = \mathbb{N}$. Now, if $\alpha : \mathbb{N} \to A$ is a numbering of the discrete set A then $(D_{\mathbb{N}},\ D^R_{\mathbb{N}},\ \alpha)$ is a domain representation of A. We can also show that domain representations generalise the notion of an approximation structure: If $(X, A, \prec)$ is an approximation structure, then $\mathrm{Idl}(A)$ is an algebraic domain. We now define $\mathrm{Idl}(A)^R = \{[x];\ x \in X\}$ and let $\alpha : \mathrm{Idl}(A)^R \to X$ be the function given by $[x] \mapsto x$. It follows that $(\mathrm{Idl}(A),\ \mathrm{Idl}(A)^R,\ \alpha)$ is a domain representation of X provided that the topology on X is weaker than the final topology induced by α.

We now know how to represent both countable as well as uncountable topological spaces using domains, but to be able to talk about computations on topological spaces, we need to understand how to generalise the Axioms 4 – 7 from Section 1.2 to arbitrary domain representations. Since we have not (yet) required the domains D and E to be effective, we will start with Axioms 4 – 6. Thus, suppose that $(D,\ D^R,\ \delta)$ and $(E,\ E^R, \varepsilon)$ are domain representations of the spaces X and Y, and $f : X \to Y$ is a function from X to Y, we say that f is *representable* if there is a continuous function $\overline{f} : D \to E$ which takes δ-names of $x \in X$ to ε-names of $f(x) \in Y$. That is, if there is a continuous function $\overline{f} : D \to E$ which makes the diagram in Figure 1.1 commute. In particular, since $\overline{f}$ is continuous, it follows that $\overline{f}$ takes approximations of x in D to approximations of $f(x)$ in E.

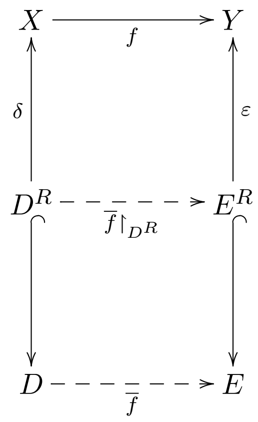

Figure 1.1.

It is a simple exercise in the definitions to show that in the special case when the domain representations $(D,\ D^R,\ \delta)$ and $(E,\ E^R, \varepsilon)$ are constructed from approximation structures as above, then $\overline{f} : D \to E$ represents $f : X \to Y$ if and only if $\overline{f}$ and f together satisfy Axioms 4, 5 and 6 from Section 1.2 .

If $(D,\ D^R,\ \delta)$ is an effective domain representation of the space X we may lift the effective structure on D to get an effective structure on the space X. More precisely, we say that $x \in X$ is *computable* if x has a computable δ-name. That is, if there is a computable element in $r \in D^R$ such that $\delta(r) = x$. It is clear that this corresponds to the notion of computability introduced in Section 1.2. Similarly, if $(E,\ E^R,\ \varepsilon)$ is an effective domain representation of the space Y, and $f : X \to Y$, we say that f is *effective* if f is representable by some effective continuous function $\overline{f} : D \to E$. This requirement corresponds exactly to the Axioms 4 – 7 introduced earlier, and as before, if $x \in X$ is computable and the function $f : X \to Y$ is effective, then $f(x)$ is computable as well. Moreover, given a Turing machine which generates a list of approximations of (a name of) x, we may construct a new Turing machine which lists the approximations of (a name of) $f(x)$.

Thus, given effective domain representations of the topological spaces X and Y, we may intelligently talk about the computable elements in X and Y, as well as computable or effective functions from X to Y, even though both spaces may be uncountable. However, the notion of computability introduced on the spaces X and Y depends heavily on how we choose to represent X and Y. By changing the domain representation of either X or Y we will get a different class of representable maps, as well as a different notion of effectivity. This confusing state of affairs is very different from the classical setting where each of the different models of computation on the natural numbers yields the same class of computable functions.

To make sure that we get the "right" notion of effectivity it would be useful to have some means of determining if a given domain representation is (in some sense) the "best possible" representation of the represented space. To be able to do this we need to be able to quantify or measure how well a given domain representation suits the represented space. The notion of *admissibility*, developed by Schröder in [15], may be seen as a *topological* measure of well-suitedness.

1.5 Admissible domain representations

Suppose that $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ are domain representations of the spaces X and Y. It is natural to ask if every continuous function from X to Y can be represented as a continuous function from D to E. (Clearly, if the domain representations $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ are effective, then any effective map from X to Y is automatically representable. Thus, in order to have as many effective functions from X to Y as is possible, we would like as many representable maps from X to Y as possible.) This problem has been studied in the context of algebraic domains by (among others) Stoltenberg-Hansen, Blanck, and Hamrin. We say that the domain representation $(E,\, E^R,\, \varepsilon)$ is *admissible* if every continuous function from X to Y is representable whenever the domain representation $(D,\, D^R,\, \delta)$ is countably based and dense[4].

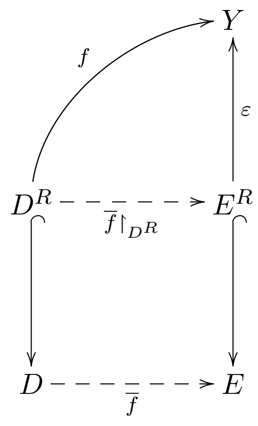

Figure 1.2.

[4]A domain representation $(D,\, D^R,\, \delta)$ is called *countably based* if the compact elements in D form a countable set, and $(D,\, D^R,\, \delta)$ is *dense* if D^R is a dense subset of D.

Definition 1.5.1. *Let* $(E,\ E^R,\ \varepsilon)$ *be a domain representation of the topological space* Y. *Then* $(E,\ E^R,\ \varepsilon)$ *is* admissible *if for each countably based domain* D*, each dense subset* $D^R \subseteq D$*, and each continuous function* $f : D^R \to Y$*, there is a continuous function* $\overline{f} : D \to E$ *such that the diagram in Figure 1.2 commutes.*

Informally, if $(E,\ E^R,\ \varepsilon)$ is an admissible domain representation of the space Y then E encodes just enough information about the elements of Y to ensure that the surjection $\varepsilon : D^R \to E$ is continuous. It follows that we may view admissibility as a notion of (topological) well-behavedness for domain representations. To be sure, there are also more pragmatic reasons why admissible representations are interesting, at least from a representation theoretic point of view. If $(D,\ D^R, \delta)$ is a countably based and dense domain representation of X and $(E,\ E^R,\ \varepsilon)$ is an admissible domain representation of Y, then every continuous function from X to Y is representable as a continuous function from D to E. In fact, one may show that $(E,\ E^R,\ \varepsilon)$ is admissible if and only if this is true for any space X and any countably based and dense domain representation $(D,\ D^R,\ \delta)$ of X.

2. Included papers and descriptions of the main results

The goal of the thesis is to study computable processes in distribution theory. To be sure, this problem has been studied before in different contexts: By Ishihara and Yoshida (in the context of Bishop-style constructive mathematics), Washihara (using computability structures), and Weihrauch and Zhong (in the context of Type Two Effectivity). Our aim is to show that it is possible to introduce a notion of computation in distribution theory using effective domains and effective domain representations.

2.1 Paper I

It is well-known (c.f. [12]) that a topological space X has a countably based and admissible domain representation if and only if X is T_0 and has a countable pseudobasis. More precisely, if $(D,\ D^R, \delta)$ is an admissible domain representation of X then the collection of all sets $\delta[\uparrow a \cap D^R]$, where a ranges over D_c, forms a pseudobasis on X. Conversely, if $\mathscr{B}$ is a pseudobasis on X then we may construct an admissible domain representation over the domain $\mathrm{Idl}(\mathscr{B}, \supseteq)$. The problem with this construction, at least from a computability theoretic point of view, is that there is usually no numbering of the cusl $(\mathscr{B}, \supseteq)$ which makes the relation $\supseteq$ on $\mathscr{B}$ computable. It follows that the domain $\mathrm{Idl}(\mathscr{B}, \supseteq)$ is noneffective in general.

One may ask if it is possible to "perturb" the order relation on $(\mathscr{B}, \supseteq)$ to get a computable cusl which still supports an admissible domain representation of X. In the first part of this paper we show how a natural notion of perturbation gives rise to a Galois correspondence $a \dashv s$ between the ideal completion of $(\mathscr{B}, \supseteq)$ and the ideal completion of the perturbed cusl $(B, \sqsubseteq)$. We also show that it is possible to lift the admissible representation over $\mathrm{Idl}(\mathscr{B})$ along the left adjoint of this Galois correspondence to get an admissible representation over the ideal completion of $(B, \sqsubseteq)$. If $(B, \sqsubseteq)$ is computable, it follows that the resulting domain representation is effective. This construction will be used in the third paper of the thesis to construct effective and admissible domain representations of the space of smooth functions as well as the space of rapidly decreasing functions from $\mathbb{R}$ to $\mathbb{C}$.

To construct a domain representation of $\mathscr{D}$, the space of smooth functions from $\mathbb{R}$ to $\mathbb{C}$ with compact support, or the space of *test functions*, we note that $\mathscr{D}$ is given as the inductive limit of a countable family $(\mathscr{D}_k)_{k\geq 1}$ of metrisable spaces. Each of the spaces $\mathscr{D}_k$ can be given an effective and admissible domain representation D_k using the theory developed in the first part of the paper. Since D_k embeds into D_l whenever $k \leq l$ it is natural to attempt to construct a domain representation of $\mathscr{D}$, the inductive limit of the spaces $(D_k)_{k\geq 1}$, over the inductive limit of the domains $(D_k)_{k\geq 1}$.

To show that this construction yields an effective domain representation of $\mathscr{D}$ we will consider the following more general situation: Suppose that we are given a countable directed system of topological spaces $(X_i)_{i\in I}$ together with a directed system of domains $(D_i)_{i\in I}$, where each of the domains D_i supports a domain representation $(D_i,\, D_i^R,\, \delta_i)$ of X_i. Let X be the inductive limit of the spaces $(X_i)_{i\in I}$ and let D be the inductive limit of the domains $(D_i)_{i\in I}$. We now ask if it is possible in general to construct a domain representation of X over D? In the second part of this paper we show that it is enough to require that the embedding $e_{i,j} : D_i \to D_j$ preserves the notion of approximation on D_i for all $i \leq j$ in I to be able to construct a domain representation of the space X over the domain D. More precisely, we show that

Theorem 2.1.1. *If the embedding $e_{i,j} : D_i \to D_j$ is a faithful extension for all $i \leq j$ in I, then $(D,\, \bigcup_{i\in I} e_i[D_i^R],\, \bigcup_{i\in I}(\delta_i \circ p_i))$ is a domain representation of X, where (e_i, p_i) is the resulting embedding-projection pair for (D_i, D).*

One may also ask if this construction preserves admissibility. That is, if the constructed domain representation of X is admissible if the domain representations of the spaces $(X_i)_{i\in I}$ are all admissible. In the general case, the answer to this question seems to be no. However, if each space X_i is T_1 and the domain representations $(D_i,\, D_i^R,\, \delta_i)$ are "sufficiently well-behaved", then admissibility is preserved.

Theorem 2.1.2. *Suppose that X_i is a T_1-space for each $i \in I$. If the embedding $e_{i,j} : D_i \to D_j$ is a faithful extension for all $i \leq j$ in I and each domain representation $(D_i,\, D_i^R,\, \delta_i)$ is almost standard, then $(D,\, \bigcup_{i\in I} e_i[D_i^R],\, \bigcup_{i\in I}(\delta_i \circ p_i))$ is almost standard as well.*

Since every domain representation which is almost standard is also admissible, it follows by Theorem 2.1.2 that admissibility is preserved in the case when the domain representations of the spaces $(X_i)_{i\in I}$ are almost standard.

It is easy to show that this result is not optimal, but it is general enough to cover the special case when the directed system $(X_i)_{i\in I}$ is the countable family $(\mathscr{D}_k)_{k\geq 1}$ above: Since each space $\mathscr{D}_k$ is metrisable we may easily construct an effective and almost standard domain representation $(D_k,\, D_k^R,\, \delta_k)$ of $\mathscr{D}_k$. Since D_k embeds faithfully into D_l whenever $k \leq l$, we may now apply Theorem 2.1.2 to get an effective and almost standard domain representation of the space of test functions. This construction is described in detail in the third paper of the thesis.

2.2 Paper II

One reason why admissible domain representations are interesting, at least from a representation theoretic point of view, is that they allow us to represent many continuous functions. As we have seen, if $(D,\, D^R,\, \delta)$ is a countably based and dense domain representation of X and $(E,\, E^R,\, \varepsilon)$ is an admissible domain representation of Y, then every continuous function from X to Y is representable as a continuous function from D to E. We may use this fact to construct a domain representation of the space of continuous functions from X to Y (with the compact-open topology) over the domain of continuous functions from D to E. (For details, we refer the reader to [12].) This is interesting for our purposes since a distribution is simply a continuous linear map from $\mathscr{D}$ to $\mathbb{C}$. Since the standard effective domain representation of the complex numbers is admissible, it follows every distribution is representable *if the domain representation of the space of test functions is countably based and dense*.

However, the domain representation of $\mathscr{D}$ which we constructed in the first paper is not dense, and so the result by Hamrin does not apply. Indeed, it is still not known if there is a dense and effective domain representation of the space of test functions. Since there is no construction which takes an arbitrary effective domain representation of X to an effective and dense representation of the same space, we need to generalise the result by Hamrin to arbitrary countably based domain representations. To be able to achieve this generality, we need to introduce the notion of a *partial* continuous function.

Definition 2.2.1. *Let D and E be domains. A* partial continuous function *from D to E is a pair (S, f) where S is a nonempty closed subset of D, and $f : S \to E$ is continuous and strict.*

We write $f : D \rightharpoonup E$ if $(\mathrm{dom}(f), f)$ is a partial continuous function from D to E.

The premise of this paper is the observation that we can generalise the result in [12] to arbitrary countably based domain representations if we allow the representation of a map $f : X \to Y$ to be a *partial* continuous function. More precisely, if $(D,\ D^R,\ \delta)$ is a countably based (but not necessarily dense) domain representation of X and $(E,\ E^R,\ \varepsilon)$ is an admissible domain representation of Y, then every continuous function from X to Y is representable by a *partial* continuous function from D to E in the following sense: For each continuous function $f : X \to Y$ there is a partial continuous function $\overline{f} : D \to E$ such that $D^R \subseteq \mathrm{dom}(\overline{f})$ and the diagram in Figure 2.1 commutes.

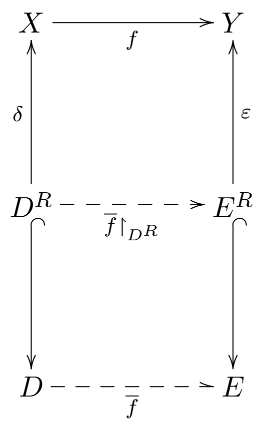

Figure 2.1.

Let $\mathbf{PAdm}$ be the category with objects pairs $((D, D^R, \delta),\ X)$ where D is a countably based Scott-Ershov domain, X is a topological space and $(D,\ D^R,\ \delta)$ is an admissible domain representation of X, and morphisms $f : ((D, D^R, \delta),\ X) \to ((E, E^R, \varepsilon),\ Y)$ functions $f : X \to Y$ which are representable by some partial continuous function from D to E. In this paper we prove that

Theorem 2.2.2. **PAdm** *is Cartesian closed.*

We give a direct proof of Theorem 2.2.2 which also provides an explicit characterisation of the function space construction in **PAdm**. We note that it would also be possible to give an indirect proof by noting that **PAdm** is equivalent to the category of topological spaces with countably based, dense and admissible representations studied by Hamrin, as well as the category of T_0 quotients of countably based spaces introduced by Schröder and Simpson.

In the paper we also introduce a notion of effectivity for partial continuous functions on effective domains, and show that the construction of the partial function space in the underlying category of domains is effective. It follows that the effective counterpart of $\mathbf{PAdm}$ is close to being Cartesian closed. In fact, the only construction which does not preserve effectivity is type conversion which can still be shown to be effective in many interesting cases.

In particular, the theory developed in this paper makes it possible to construct effective and admissible domain representations of the space of distributions, the space of tempered distributions, as well as the space of distributions with compact support.

2.3 Paper III

In this paper, we apply the theory developed in the first two papers in the thesis to introduce and study a notion of effectivity for test functions and distributions. We construct effective and admissible domain representations of the space $\mathscr{E}$ of smooth functions, the space $\mathscr{S}$ of rapidly decreasing functions, and the space $\mathscr{D}$ of test functions (using the methods developed in the first paper). We then apply the ideas developed in the second paper to construct effective and admissible domain representations of the space $\mathscr{D}'$ of distributions, the space $\mathscr{S}'$ of tempered distributions, and the space $\mathscr{E}'$ of distributions with compact support. We show that the vector space structure on these spaces is effective with respect to the introduced representations, and furthermore that the standard embeddings from $\mathscr{E}$, $\mathscr{D}$, and $\mathscr{S}$ into $\mathscr{D}'$ are all effectivise.

One of the main goals of the paper is to prove that the Fourier transform on the space of tempered distributions is effective. If the Fourier transform and its inverse are both effective, we expect to be able to use Fourier techniques to prove effective versions of existence theorems for solutions of partial differential equations.

To be able to to determine if the Fourier transform on the space $\mathscr{S}'$ of tempered distributions is effective, we need to make a preliminary study of the Fourier transform on the space $\mathscr{S}$ of rapidly decreasing smooth functions. The Fourier transform on $\mathscr{S}$ is given by

$$\hat{\varphi}(t) = \frac{1}{\sqrt{2\pi i}} \int e^{-ixt} \varphi(x) dx.$$

The map $\varphi \mapsto \hat{\varphi}$ is continuous, one-to-one, and maps $\mathscr{S}$ onto $\mathscr{S}$. The inverse transform is also continuous and is given by

$$\check{\varphi}(t) = \frac{1}{\sqrt{2\pi i}} \int e^{ixt} \varphi(x) dx.$$

In this paper we prove that

Theorem 2.3.1. *The Fourier transform $\varphi \mapsto \hat{\varphi}$ and the inverse transform $\varphi \mapsto \check{\varphi}$ are effective maps from $\mathscr{S}$ to $\mathscr{S}$.*

The Fourier transform on the space $\mathscr{S}'$ of tempered distributions is defined by extending or lifting the map $\varphi \mapsto \hat{\varphi}$ to $\mathscr{S}'$. More precisely, the Fourier transform $\hat{u}$ of a tempered distribution u is given by $\hat{u}(\varphi) = u(\hat{\varphi})$. The map $u \mapsto \hat{u}$ is a continuous bijection on $\mathscr{S}'$, with a continuous inverse given by $u \mapsto \check{u}$ where $\check{u}$ is the tempered distribution given by $\check{u}(\varphi) = u(\check{\varphi})$.

Thus, to prove that the Fourier transform on $\mathscr{S}'$ is effective, it is enough to show that effectivity is preserved under such extensions. We prove

Theorem 2.3.2. *Let $f : \mathscr{S} \to \mathscr{S}$ be a linear map, and suppose that $F : \mathscr{S}' \to \mathscr{S}'$ given by $F(u)(\varphi) = u(f(\varphi))$ is continuous. If f is effective then so is F.*

Since both the Fourier transform as well as its inverse are defined in this way, we immediately have the following corollary:

Corollary 2.3.3. *The Fourier transform and the inverse Fourier transform on $\mathscr{S}'$ are both effective.*

Finally, we study the convolution operator on spaces of distributions, and use the results obtained to prove that the solution operator for the simple differential equation $u' = v$ has a computable solution operator. Thus we prove

Theorem 2.3.4. *There is an effective map $s : \mathscr{D} \to \mathscr{D}$ such that $s(v)' = v$ for each distribution v.*

2.4 Paper IV

There are a number of interesting spaces of generalised functions apart from the space of distributions. Examples include the space of tempered distributions which is mapped bijectively onto itself by the Fourier transform, as well as the space of compactly supported distributions which is very well-behaved under convolution. It is well-known that the space of compactly supported distributions have many other interesting purely structural properties. In particular, there are (at least) four essentially different characterisations of the class of distributions with compact support. More precisely, we have

Theorem 2.4.1 (The structure theorem for $\mathscr{E}'$)**.** *Let u be a continuous linear functional on $\mathscr{D}$. Then the following are equivalent*

- *(UE)* *u extends (uniquely) to a linear continuous functional on $\mathscr{E}$.*
- *(CS)* *The support of u is a compact subset of $\mathbb{R}$.*
- *(FO)* *u has finite order.*
- *(WC)* *There are compactly supported continuous functions $g_0, g_2, \ldots, g_n$ such that $u = \sum_{k=0}^{n} g_k^{(k)}$.*

Each characterisation $(\star)$ of the space of distributions with compact support yields an effective domain representation $D_\star$ of $\mathscr{E}'$. To study the effective content of the structure theorem for $\mathscr{E}'$ we study the reducibility properties of these four different domain representations. Intuitively, if we can translate or reduce approximations in $D_\star$ to approximations in $D_\dagger$ in an effective way, then we conclude that the corresponding implication $(\star) \implies (\dagger)$ has effective content.

To be more precise, if $(D,\, D^R,\, \delta)$ and $(E,\, E^R, \varepsilon)$ are two domain representations of the same topological space X we say that D is *continuously reducible* to E (and write $D \leq_{\mathrm{c}} E$) if there is a partial continuous function $r : D \rightharpoonup E$ from D to E such that $D^R \subseteq \mathrm{dom}(r)$ and $\delta(x) = \varepsilon(r(x))$ for each representing element $x \in D^R$. If the domains D and E, and the continuous function r are all effective we say that D is *effectively reducible* to E (written $D \leq_{\mathrm{e}} E$) . The corresponding equivalence relations are written $D \equiv_{\mathrm{c}} E$ and $D \equiv_{\mathrm{e}} E$.

We prove in this paper that the domain representations corresponding to the three characterisations (UE), (CS), and (FO) are all effectively equivalent, and furthermore, that the representation corresponding to (WC) is not equivalent to the others. More precisely, we prove that

Theorem 2.4.2. $D_{\mathrm{WC}} <_{\mathrm{e}} D_{\mathrm{UE}} \equiv_{\mathrm{e}} D_{\mathrm{CS}} \equiv_{\mathrm{e}} D_{\mathrm{FO}}$.

We also introduce a weaker notion of effective reducibility called *Turing-reducibility* (written $D \leq_{\mathrm{T}} E$) and show that D_{WC} and D_{UE} are equivalent with respect to this weak reducibility relation. Thus, we have

Theorem 2.4.3. $D_{\mathrm{WC}} \equiv_{\mathrm{T}} D_{\mathrm{UE}} \equiv_{\mathrm{T}} D_{\mathrm{CS}} \equiv_{\mathrm{T}} D_{\mathrm{FO}}$.

Intuitively, this tells us that we can always compute a name of the compactly supported distribution u in $D_\dagger$ from a name of the same distribution in $D_\star$. Thus in particular, it follows that the structure theorem for the space $\mathscr{E}'$ does have effective content, although perhaps not in the way we first expected.

Finally, we consider a similar structure theorem for the space of distribution supported at 0. In this case the classical proofs are structurally simpler and the two resulting effective representations of the space can be shown to be effectively equivalent.

3. Sammanfattning på svenska (Summary in Swedish)

Vad betyder det att beräkna en funktion som har decimaltal (eller *reella tal* som de oftast kallas inom matematik) som in- och utdata? I allmänhet kan vi inte ens ge ett reellt tal som indata till en dator eftersom vi i sådana fall är tvungna att mata in det reella talets oändligt många decimaler. Samma problem uppstår om vi försöker ge en funktion f på de reella talen som indata till en beräkning. För att fullständigt beskriva funktionen är vi tvungna att säga vad funktionen antar för värde i varje punkt. Vi måste alltså för varje reellt tal x säga vad $f(x)$ är. Detta är självklart inte heller möjligt.

En sätt att kringå det här problemet är att arbeta med ändliga approximationer av in och utdata. Det vill säga, i stället för att försöka räkna direkt med reella tal så räknar vi med *approximationer* av reella tal. (Den här idén är förstås inte ny. Det är så man alltid gjort beräkningar med reella tal.) För att detta ska fungera som vi förväntar oss så måste våra approximationer och beräkningsmetoder (eller *algoritmer*) uppfylla vissa naturliga krav. Till exempel förväntar vi oss att vi genom att förse en beräkning med bättre och bättre approximationer av indata, kommer att få bättre och bättre approximationer av utdata tillbaka. Det är också naturligt att kräva att varje reelt tal kan beskrivas godtyckligt nogrannt med hjälp av dess approximationer.

Om vi försöker formalisera approximationsbegreppet som det ser ut för reella tal så får vi en matematisk struktur som kallas för en *domänrepresentation*. Fördelen med domäner och domänrepresentationer är att de kan användas för att studera beräkningsbarhet på andra matematiska strukturer än de reella talen. I den här avhandlingen visar vi hur man kan använda domänrepresentationer för att introducera ett beräkningsbarhetsbegrepp på rum av distributioner. (En *distribution* är en typ av "generaliserad funktion" som uppkommer naturligt som lösningar till vissa differentialekvationer.)

För att visa att detta är möjligt är vi först tvungna att utveckla teorin för admissibla domänrepresentationer. Detta utgör det huvudsakliga innehållet i avhandlingens första och andra del.

I den tredje artikeln visar vi hur man kan tillämpa konstruktionerna från de första två artiklarna för att konstruera domänrepresentationer av de rum av distributioner som vi är intresserade av. Vi visar dessutom att många av de viktiga operationerna på rum av distributioner är beräkningsbara, och ger ett exempel på hur man kan använda domänrepresentationer för att beräkna lösningar till differentialekvationer.

I den sista artikeln använder vi domänrepresentationer för att studera det beräkningsmässiga innehållet i ett strukturellt resultat för rummet av distributioner med kompakt stöd. Dessa distributioner kan karakteriseras på (åtminstone) fyra olika sätt, och vi undersöker här om det är möjligt att beräkna informationen i en sådan karakterisering givet informationen från någon av de tre andra. Det visar sig att svaret beror på vad vi lägger för betydelse i ordet "beräkna".

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Paper I

Admissible Domain Representations of Inductive Limit Spaces

Fredrik Dahlgren

Abstract

In this paper we consider the following separate but related questions: "How do we construct an effective admissible domain representation of a space X from a pseudobasis on X?", and "How do we construct an effective admissible domain representation of the inductive limit of a directed system $(X_i)_{i \in I}$ of topological spaces, given effective admissible domain representations of the individual spaces X_i?". An attempt to answer the first of these questions leads us naturally to the notion of an admissible pair. To answer the second question we employ admissible pairs to characterise a subclass of admissible domain representations which is closed under inductive limits.

The second result is particularly interesting from the point of view of computable analysis, since it allows us to construct an effective and admissible domain representation of the space of test functions which is of interest in distribution theory.

Key words: Domain theory, domain representations, computability theory, computable analysis.

1 Introduction

To study effective processes in analysis and algebra we need to generalise computability theory on the natural numbers to uncountable spaces. One way of achieving this generality in computable analysis is to extend the computability theory on effective domains to a general topological space X via a domain representation of X. If X is a topological space, a domain representation of X is a triple $(D,\, D^R,\, \delta)$ where D is a domain, D^R is a subset of D, and $\delta : D^R \to D$ is continuous and surjective. If D is effective, the effective structure on D lifts to give a δ-computability theory on the space X. Similarly, if $(E,\, E^R,\, \varepsilon)$ is an effective domain representation of the topological space Y, the notion of an effective continuous function from D to E lifts to a notion of a (D, E)-effective continuous function from X to Y.

To be able to study effective processes of higher type we need to be able to represent the space of continuous functions from X to Y. If every continuous function from X to Y can be represented as a continuous function from D to E we may construct an effective domain representation of the space of continuous functions from X to Y over the domain of continuous functions from D to E. This is not always the case however, and so it becomes interesting to find sufficient conditions on the domain representations $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ to ensure that every continuous function from X to Y has a representation as a continuous function from D to E. In the context of algebraic domains, this problem has been studied by (among others) Stoltenberg-Hansen, Blanck, and Hamrin in [17], [5], and [10].

In [13], Schröder introduced the notion of admissibility in the context of Type Two Effectivity. The corresponding notion in the context of domain representations has been studied by Bauer, Simpson, Menni, and Hamrin. A domain representation $(E,\, E^R,\, \varepsilon)$ of Y is admissible (or ω-projecting) if given any countably based domain D, any dense subset $D^R \subseteq D$, and any continuous function $f : D^R \to Y$, then f factors through ε via some continuous function $\overline{f} : D \to E$ in the sense that $f(x) = (\varepsilon \circ \overline{f})(x)$ for each $x \in D^R$. If we restrict our attention to the case when $(D,\, D^R,\, \delta)$ is a domain representation of the space X and $f : X \to Y$ is continuous, then since $f \circ \delta$ is continuous, $f \circ \delta$ factors through ε via some continuous function $\overline{f} : D \to E$. It follows that every continuous function from X to Y is representable in this way as a continuous function from D to E.

The question of which spaces admit admissible domain representations was answered independently by Bauer [3] and Hamrin [10]. These results are entirely analogous to the corresponding result for admissible type two representations given in [12]. A topological space Y admits a countably based admissible domain representation if and only if Y is T_0 and has a countable pseudobasis, where the notion of a pseudobasis is a generalisation of the notion of a topological basis. More precisely, a collection $\mathscr{B}$ of subsets of a topological space Y is a pseudobasis if and only if $\mathscr{B}$ contains Y, is closed under finite nonempty intersections in the sense that when $B, C \in \mathscr{B}$ and $B \cap C$ is nonempty, then $B \cap C \in \mathscr{B}$, and whenever $(y_n)_n \longrightarrow y$ in Y and U is some open neighbourhood of y, then $y_n \in B$ for almost all $n \in \mathbb{N}$ for some $B \in \mathscr{B}$ such that $B \subseteq U$. It follows that every topological basis on Y is a pseudobasis, but a pseudobasis on Y is not a basis for the topology on Y in general. In fact, if $\mathscr{B}$ is a pseudobasis and $B \in \mathscr{B}$ then B can be open, or closed, or neither.

If $\mathscr{B}$ is a pseudobasis on Y then we may construct an admissible domain representation of Y over $\mathrm{Idl}(\mathscr{B}, \supseteq)$, the ideal completion of $(\mathscr{B}, \supseteq)$. The problem with this construction from a computability theoretic point of view is that inclusion on $\mathscr{B}$ is not computable (with respect to any numbering of $\mathscr{B}$) in general, and so $\mathrm{Idl}(\mathscr{B}, \supseteq)$ is not an effective domain.

This problem is addressed in the first part of this paper. The general idea is the following: To be able to construct an effective domain from the pseudobasis $\mathscr{B}$ we need to be able to "perturb" the cusl $(\mathscr{B}, \supseteq)$ slightly to get a computable cusl $(B, \sqsubseteq)$ which behaves sufficiently like $(\mathscr{B}, \supseteq)$ for the construction of an admissible domain representation of Y to go through with $(B, \sqsubseteq)$ in place of $(\mathscr{B}, \supseteq)$. Intuitively, we may think of the elements of the perturbed cusl B as computationally well-behaved representations of the basis elements in $\mathscr{B}$. We note that since equality on $\mathscr{B}$ is not computable (with respect to any numbering of $\mathscr{B}$) in general we should expect to have more than one representation in B for each basis element in $\mathscr{B}$. If $a : B \to \mathscr{B}$ takes representations in B to the corresponding basis element in $\mathscr{B}$ it is natural to require that a is monotone and surjective. We will show in section 3 that if a is monotone, surjective, and preserves suprema then a extends to the left adjoint of a Galois correspondence $a \dashv s$ between the ideal completions of $(B, \sqsubseteq)$ and $(\mathscr{B}, \supseteq)$. Using general properties of this Galois correspondence we show that it is possible to "lift" the admissible domain representation over $\mathrm{Idl}(\mathscr{B}, \supseteq)$ along a to construct an admissible domain representation over $\mathrm{Idl}(B, \sqsubseteq)$.

In the second part of the paper we apply the theory already developed to the problem of constructing an admissible domain representation of the inductive limit of a directed system of admissibly representable topological spaces. To ensure that the inductive limit is topologically well-behaved, we will need to require that each represented space is T_1. We will show that the inductive limit X of a countable directed system $(X_i)_{i \in I}$ of admissibly representable T_1-spaces has an admissible domain representation. However, the construction implicit in the proof is nonuniform, and so we cannot guarantee that the admissible domain representation of the inductive limit is effective in general.

To give an effective construction we consider the following related question: "Let $(X_i)_{i \in I}$ be a countable directed system of T_1-spaces and suppose that $(D_i,\ D_i^R,\ \delta_i)$ is an admissible domain representation of X_i for each $i \in I$. Suppose further that D_i embeds into D_j whenever $i \leq j$ in I. Is it then possible to construct an admissible domain representation of the inductive limit of the directed system $(X_i)_{i \in I}$ over the inductive limit of the directed system $(D_i)_{i \in I}$ of domains?" We will show that if the domain representations $(D_i,\ D_i^R,\ \delta_i)$ are obtained as liftings as above, and each embedding $D_i \hookrightarrow D_j$ preserves approximation in a precise sense, then it is possible to glue together the individual domain representations $(D_i,\ D_i^R,\ \delta_i)$ to construct an admissible domain representation of the inductive limit of the directed system $(X_i)_{i \in I}$ over the inductive limit of the directed system $(D_i)_{i \in I}$. This is interesting from a computability theoretic point of view since the construction of the inductive limit of a directed system of domains preserves effectivity, and so the theorem gives sufficient conditions for the existence of an effective admissible domain representation of the inductive limit of $(X_i)_{i \in I}$.

To conclude, we will say something about how these results may be applied to construct an effective and admissible domain representation of the space of test functions which is of interest in distribution theory.

2 Preliminaries from domain theory

We begin by recalling some definitions and notation from the theory of Scott-Ershov domains. This section is not meant as an introduction to the subject, but should rather be thought of as a review of some basic definitions and notational conventions. For a thorough treatment of domain theory we recommend the book [15] or the comprehensive survey given in [1].

A *Scott-Ershov domain* (or simply domain) is a consistently complete algebraic cpo. Let D be a domain. Then D_c denotes the set of compact elements in D. Given $x \in D$ we write $\mathrm{approx}(x)$ for the set $\{a \in D_c;\ a \sqsubseteq x\}$. Since D is algebraic, $\mathrm{approx}(x)$ is directed and $\bigsqcup \mathrm{approx}(x) = x$ for each $x \in D$. The *Scott-topology* on D is generated by the collection of all up-sets $\uparrow a = \{x \in D;\ a \sqsubseteq x\}$ where a ranges over D_c. If D and E are domains and $f : D \to E$ then f is continuous with respect to the Scott-topologies on D and E if and only if f is monotone[1] and preserves suprema of directed sets.

A *preordered conditional upper semilattice* (precusl) is a preordered set $(P, \sqsubseteq)$ with a distinguished[2] least element $\bot$, which is consistently complete in the following sense: Whenever p and q are consistent in P there is some $r \in P$ such that $p, q \sqsubseteq r$ and if $p, q \sqsubseteq s$ for some $s \in P$ we have $r \sqsubseteq s$. r is called a *least upper bound* for p and q. It is clear that r need not be unique and so p and q may have many least upper bounds in P. P is a *conditional upper semilattice* (cusl) if $\sqsubseteq$ is a partial order on P. If P is a cusl then least upper bounds in P are unique whenever they exist. If P is a precusl we write $\mathrm{Idl}(P)$ (or sometimes $\mathrm{Idl}(P, \sqsubseteq)$ if the order relation on P is not clear from the context) for the set $\{I \subseteq P;\ I \text{ is an ideal}\}$. If we order $\mathrm{Idl}(P)$ by

$$I \sqsubseteq J \iff I \subseteq J$$

[1] Here we follow the notational conventions from [15]. That is, if D and E are domains and $f : D \to E$, we say that f is *monotone* if $x \sqsubseteq_D y \implies f(x) \sqsubseteq_E f(y)$ for all $x, y \in D$, and that f is *order-preserving* if $x \sqsubseteq_D y \iff f(x) \sqsubseteq_E f(y)$ for all $x, y \in D$.

[2] In general, there may be more than one least element in P. In this case we pick one and denote this least element $\bot$.

then $\mathrm{Idl}(P) = (\mathrm{Idl}(P), \sqsubseteq, \{\bot\})$ is a domain. The domain $\mathrm{Idl}(P)$ is called the *ideal completion* of P. If D is a domain then D_c is a cusl and $D \cong \mathrm{Idl}(D_c)$.

The domain D is *effective* if there is a total numbering $\alpha : \mathbb{N} \to D_c$ of the set D_c such that the relations "$\alpha(m) \sqsubseteq \alpha(n)$", "$\alpha(k) = \alpha(m) \sqcup \alpha(n)$", and "$\alpha(m)$ and $\alpha(n)$ are consistent" are recursive. When (D, α) and (E, β) are effective domains, then $x \in D$ is called *α-computable* (or simply computable) if the relation $\alpha(m) \sqsubseteq x$ is r.e. and if $f : D \to E$ is continuous then f is *(α, β)-effective* (or simply effective) if the relation $\beta(n) \sqsubseteq_E f(\alpha(m))$ is r.e.

2.1 Inductive limits of directed systems of domains

For the second part of the paper we will need to review some well-known results about inductive limits of directed systems of domains. We state these results without proofs. For a more thorough introduction to inductive limit constructions in categories of domains, see [1] or [15].

Let D and E be domains. Then (e, p) is an *embedding-projection pair* for (D, E) if $e : D \to E$ and $p : E \to D$ are continuous functions such that $p \circ e = \mathrm{id}_D$ and $e \circ p \sqsubseteq \mathrm{id}_E$. If (e, p) is an embedding-projection pair for (D, E) then $e : D \to E$ is called an *embedding* and $p : E \to D$ is called a *projection*.

If (e, p) is an embedding-projection pair then (e, p) is a Galois correspondence by definition. It follows that we only need to specify the embedding $e : D \to E$ of the embedding-projection pair (e, p) since p is uniquely determined by e. If D and E are domains then $e : D \to E$ is an embedding if e satisfies (all of the) conditions 1 – 4 below.

1. e is strict ($e(\bot_D) = \bot_E$).
2. e is order-preserving.
3. $e[D_c] \subseteq E_c$.
4. Whenever $e(x)$ and $e(y)$ are consistent in E then x and y are consistent in D and $e(x \sqcup y) = e(x) \sqcup e(y)$.

The following strengthening of 3 above follows easily from the definitions: $e(c)$ is compact in E if and only if c is compact in D.

We let $\mathbf{Dom}$ be the category of domains with morphisms embedding-projection pairs. It is well-known that $\mathbf{Dom}$ has inductive limits of directed systems (see [1] or [15]). We give a brief review of the general construction.

Let $(I, \leq)$ be a directed partial order and let $(D_i)_{i \in I}$ be a family of domains indexed by I. Suppose that $e_{i,j} : D_i \to D_j$ is an embedding for all $i, j \in I$ such that $i \leq j$. The pair $((D_i)_{i \in I}, (e_{i,j})_{i \leq j \in I})$ is called a *directed system* if

1. $e_{i,i} : D_i \to D_i$ is the identity on D_i for each $i \in I$, and

2. $e_{i,k} = e_{j,k} \circ e_{i,j}$ for all $i, j, k \in I$ such that $i \leq j \leq k$.

Let $((D_i)_{i \in I},\ (e_{i,j})_{i \leq j \in I})$ be a directed system and let $p_{i,j} : D_j \to D_i$ be the projection corresponding to $e_{i,j} : D_i \to D_j$ for all $i \leq j$ in I. We let D be the set $\{x \in \prod_{i \in I} D_i;\ p_{i,j}(x_j) = x_i \text{ for all } i \leq j \in I\}$, and order D by $x \sqsubseteq y$ if and only if $x_i \sqsubseteq_i y_i$ for all $i \in I$. Then $(D,\ \sqsubseteq,\ \lambda i.\bot_i)$ is a domain. Let $p_i : D \to D_i$ be the projection given by $x \mapsto x_i$ and let $e_i : D_i \to D$ be the corresponding embedding. Then $(D,\ (e_i)_{i \in I})$ is an inductive limit of $((D_i)_{i \in I},\ (e_{i,j})_{i \leq j \in I})$, and $c \in D$ is compact if and only if $c = e_i(c_i)$ for some $i \in I$ and some compact $c_i \in D_i$.

2.2 Admissible domain representations

Let X be a topological space. A *domain representation* of X is a triple $(D,\ D^R,\ \delta)$ where D is a domain, D^R is a subset of D, and $\delta : D^R \to X$ is a continuous function from D^R onto X. For $r \in D$ and $x \in X$ we write $r \prec x$ if $x \in \delta[\uparrow r \cap D^R]$. Thus, $r \prec x$ if and only if there is some $s \in D^R$ such that $r \sqsubseteq s$ and $\delta(s) = x$.

When D is effective then $(D,\ D^R,\ \delta)$ is an *effective domain representation* of X. If D is effective via $\alpha : \mathbb{N} \to D_c$ we have the following situation:

$$\mathbb{N} \xrightarrow{\ \alpha\ } D_c \hookrightarrow D \hookleftarrow D^R \xrightarrow{\ \delta\ } X.$$

Suppose $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ are effective domain representations of X and Y respectively. We say that $x \in X$ is *D-computable* (or simply computable if the representation $(D,\ D^R,\ \delta)$ is clear from the context) if there is some computable $r \in D$ such that $\delta(r) = x$. A continuous function $f : X \to Y$ is (D, E)-*representable* (or simply representable when the domain representations $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ are clear from the context) if there is a continuous function $\overline{f} : D \to E$ such that $\overline{f}[D^R] \subseteq E^R$ and $f(\delta(r)) = \varepsilon(\overline{f}(r))$ for each $r \in D^R$. f is (D, E)-*effective* (or simply effective) if $\overline{f}$ is effective. Finally, a domain representation $(D,\ D^R,\ \delta)$ of a space X is called *dense* if the set D^R of representing elements is dense in D with respect to the Scott-topology on D.

Definition 2.1. Let $(E,\ E^R,\ \varepsilon)$ be a domain representation of Y. $(E,\ E^R,\ \varepsilon)$ is called *admissible*[3] if for each triple $(D,\ D^R,\ \delta)$ where D is a countably based domain, D^R is a dense subset of D, and $\delta : D^R \to Y$ is a continuous

[3] This notion of an admissible representation corresponds to that of an ω-admissible representation found in [10], and to that of an ω-projecting equilogical space in [3], and [11].

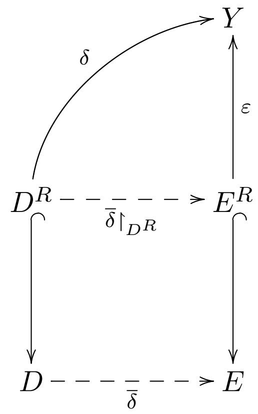

Figure 1.

function from D^R to Y, there is a continuous function $\overline{\delta} : D \to E$ which maps D^R into E^R and satisfies $\delta(r) = \varepsilon(\overline{\delta}(r))$ for each $r \in D^R$. That is, the diagram in Figure 1 commutes.

The following simple observation indicates why admissibility is interesting from a purely representation theoretic point of view.

Theorem 2.2. *Suppose* $(D,\, D^R,\, \delta)$ *is a countably based dense representation of* X *and* $(E,\, E^R,\, \varepsilon)$ *is an admissible representation of* Y*. Then every sequentially continuous function from* X *to* Y *is representable.*

(For a proof of Theorem 2.2 see [10].)

Theorem 2.2 can be used as a tool in constructing a representation of the space of continuous functions from X to Y (with the topology corresponding to the convergence relation given by $(f_n)_n \longrightarrow f$ if and only if $(f_n(x_n))_n \longrightarrow f(x)$ for each convergent sequence $(x_n)_n \longrightarrow x$ in X) over the domain $[D \to E]$ of continuous functions from D to E.

Definition 2.3. Let X be a topological space. a *pseudobasis* for X is a family $\mathscr{B} \subseteq \mathscr{P}(X)$ of subsets of X such that X is in $\mathscr{B}$, if $B, C \in \mathscr{B}$ and $B \cap C$ is nonempty, then $B \cap C \in \mathscr{B}$, and if $(x_n)_n \longrightarrow x$ in X and U is an open neighbourhood of x, then there is a $B \in \mathscr{B}$ such that $x \in B \subseteq U$ and $x_n \in B$ for almost all n.

If $\mathscr{B}$ is a basis for the topology on X which is closed under finite nonempty intersections, then $\mathscr{B}$ is also a pseudobasis for X. Moreover, if $\mathscr{B}$ is a pseudobasis for X and $\mathscr{C}$ is any subset of $\mathscr{P}(X)$ which is closed under finite nonempty intersections and contains $\mathscr{B}$, then $\mathscr{C}$ is also a pseudobasis for X. Pseudobases are related to admissibility in the following way:

Theorem 2.4. *Let X be a topological space. Then X has a countably based admissible domain representation if and only if X is T_0 and has a countable pseudobasis.*

(For a proof of Theorem 2.4, see [10].)

More explicitly, if $(D,\ D^R,\ \delta)$ is a countably based admissible domain representation of X, then the collection of all finite intersections of sets of the form $\{x \in X;\ c \prec x\}$, where c ranges over D_c form a countable pseudobasis on X.

Conversely, if $\mathscr{B}$ is a countable pseudobasis on X, let D be the ideal completion of $(\mathscr{B},\ \supseteq)$. Then D is countably based by construction. If $I \in D$ we say that I *converges* to $x \in X$ (written $I \longrightarrow x$) if $x \in B$ for each $B \in I$, and for each open set U containing x there is a $B \in I$ such that $B \subseteq U$. I in D is *convergent* if $I \longrightarrow x$ for some $x \in X$. We let D^R be the set of convergent ideals in D and define $\delta : D^R \longrightarrow X$ by $\delta(I) = x$ if and only if $I \longrightarrow x$. δ is well-defined since X is T_0 and continuous and onto X since $\mathscr{B}$ is a pseudobasis. It follows that $(D,\ D^R,\ \delta)$ is a countably based domain representation of X. $(D,\ D^R,\ \delta)$ is admissible by Theorem 6.8 in [10]. The representation $(D,\ D^R,\ \delta)$ is called the *standard representation of X over $\mathscr{B}$*.

The problem with this construction from a computability theoretic standpoint is that it is almost always impossible to find a numbering of $\mathscr{B}$ such that the relation $\supseteq$ is computable, and so $\mathrm{Idl}(\mathscr{B}, \supseteq)$ is noneffective in general. To remedy this we need a slightly more flexible construction.

3 Admissible pairs

Let X be a topological space and suppose that $\mathscr{B}$ is a countable pseudobasis on X. If $(\mathscr{B}, \supseteq)$ is a computable cusl, then $\mathrm{Idl}(\mathscr{B}, \supseteq)$ is an effective domain and we may construct an effective and admissible domain representation of X over $\mathrm{Idl}(\mathscr{B}, \supseteq)$ as above. However, since we cannot expect to be able to compute inclusion on $\mathscr{B}$, the domain $\mathrm{Idl}(\mathscr{B}, \supseteq)$ is noneffective in general.

In the general case when $(\mathscr{B}, \supseteq)$ is not computable we need a more flexible construction which allows us to perturb $\mathscr{B}$ slightly to get a computable cusl. Luckily, it turns out that the construction of an admissible domain representation from a pseudobasis as outlined above is stable under slight "perturbations" of the cusl $(\mathscr{B}, \supseteq)$. To give an indication of what we have in mind we consider the following example:

Example 3.1. Let B be a recursive subset of the natural numbers and let $\beta : B \to \mathscr{B}$ be a numbering of the pseudobasis $\mathscr{B}$. In most cases we will not be able to decide when two elements in B represent the same basis element, and so an element in $\mathscr{B}$ may have more than one representation in B. Since inclusion on $\mathscr{B}$ is undecidable in general we cannot expect to find a decidable order relation $\sqsubseteq$ on B such that $m \sqsubseteq n \iff \beta(m) \supseteq \beta(n)$. However, it may be possible to find a decidable partial order $\sqsubseteq$ on B such that $m \sqsubseteq n \implies \beta(m) \supseteq \beta(n)$. Now, if $(B, \sqsubseteq)$ is a cusl and β preserves suprema in the following sense:

1. If $\beta(m)$ and $\beta(n)$ are consistent in $(\mathscr{B}, \supseteq)$ then m and n are consistent in $(B, \sqsubseteq)$ and $\beta(m \sqcup n) = \beta(m) \cap \beta(n)$.

then β extends to the left adjoint of a Galois correspondence $\beta \dashv s$ between the ideal completions of $(B, \sqsubseteq)$ and $(\mathscr{B}, \supseteq)$. Using general properties of the Galois correspondence $\beta \dashv s$ it turns out to be possible to "lift" the admissible domain representation over $\mathrm{Idl}(\mathscr{B}, \supseteq)$ along β to construct an admissible domain representation over the *effective* domain $\mathrm{Idl}(B, \sqsubseteq)$. (This is Theorem 3.14.)

The following definition generalises the situation described in this motivating example.

Definition 3.2. Let D and E be domains. Then (a, s) is an *admissible pair*[4] for (D, E) if $a : D \to E$ and $s : E \to D$ are continuous functions such that $a \circ s = \mathrm{id}_E$ and $id_D \sqsubseteq s \circ a$.

The next lemma summarises some simple but important properties of admissible pairs. (Note in particular that 3.3.3 – 3.3.5 correspond to the conditions imposed on the numbering β in the motivating example above.)

Lemma 3.3. *Let D and E be domains and let (a, s) be an admissible pair for (D, E). Then*

1. $x \sqsubseteq s(y) \iff a(x) \sqsubseteq y$ *for all* $x \in D$ *and* $y \in E$.
2. $x \sqsubseteq y \iff s(x) \sqsubseteq s(y)$ *for all* $x, y \in E$.
3. a *is surjective.*
4. $a[D_c] \subseteq E_c$.
5. *if* $x, y \in D$ *and* $\{a(x), a(y)\}$ *is consistent in* E *then* $\{x, y\}$ *is consistent in* D *and* $a(x \sqcup y) = a(x) \sqcup a(y)$.

[4] Admissible pairs are mentioned briefly under the name of continuous insertion-closure pairs (or i-c pairs) in [1] p. 25.

Proof. 1. Suppose $x \in D$ and $y \in E$. Then $x \sqsubseteq s(y)$ implies that $a(x) \sqsubseteq (a \circ s)(y) = y$ and conversely, if $a(x) \sqsubseteq y$ then $x \sqsubseteq (s \circ a)(x) \sqsubseteq s(y)$.

2. s is monotone since s is continuous and conversely, if $s(x) \sqsubseteq s(y)$ then $x = (a \circ s)(x) \sqsubseteq (a \circ s)(y) = y$, and so s is order preserving.

3. This is clear since $a(s(y)) = y$ for each $y \in E$.

4. Let $c \in D_c$ and suppose $B \subseteq E$ is directed. Now, if $a(c) \sqsubseteq \bigsqcup B$ then $c \sqsubseteq (s \circ a)(c) \sqsubseteq s(\bigsqcup B) = \bigsqcup s[B]$ and so $c \sqsubseteq s(b)$ for some $b \in B$. Thus $a(c) \sqsubseteq b$ using 1. Since B was arbitrary $a(c)$ is compact.

5. Let $x, y \in D$ and suppose $\{a(x),\, a(y)\}$ is consistent in E. Then $a(x) \sqcup a(y)$ exists in E and $x,\, y \sqsubseteq s(a(x) \sqcup a(y))$ using 1. It follows that $\{x, y\}$ is consistent in D. Now, $a(x),\, a(y) \sqsubseteq a(x \sqcup y)$ and if $a(x),\, a(y) \sqsubseteq z$ in E then $x,\, y \sqsubseteq s(z)$ by 1 and so $x \sqcup y \sqsubseteq s(z)$ and $a(x \sqcup y) \sqsubseteq (a \circ s)(z) = z$. Thus $a(x \sqcup y) = a(x) \sqcup a(y)$ and a preserves binary suprema. □

Remark 3.4. We note that we may strengthen 3.3.4 somewhat by observing that a actually maps D_c *onto* E_c. (This follows from Proposition 3.12.)

If (a, s) is an admissible pair for (D, E) then (a, s) is a Galois correspondence by 1, and so s is uniquely determined by a (and similarly, a is uniquely determined by s). To be more precise, we have $s(y) = \bigsqcup\{c \in D_c;\ a(c) \sqsubseteq y\}$ for all $y \in E$.

The raison d'être for introducing admissible pairs is that they will allow us to lift an admissible representation over E along the left adjoint a of the admissible pair and thus construct a new admissible representation of X over D.

Theorem 3.5. *Let D and E be domains and suppose $(E,\, E^R,\, \varepsilon)$ is an admissible domain representation of X. Let (a, s) be an admissible pair for (D, E). Then $(D,\, s[E^R],\, (\varepsilon \circ a)\restriction_{s[E^R]})$ is an admissible domain representation of X.*

Proof. Let $D^R = s[E^R]$ and $\delta = (\varepsilon \circ a)\restriction_{D^R}$. Then δ is continuous by definition and surjective since $a \circ s = \mathrm{id}_E$. Thus $(D,\, D^R,\, \delta)$ is a domain representation of X.

To show that $(D,\, D^R,\, \delta)$ is admissible, let F be a countably based domain and suppose F^R is dense in F. Let $\varphi : F^R \to X$ be continuous. Choose $\overline{\varphi} : F \to E$ such that $\overline{\varphi}[F^R] \subseteq E^R$ and $\varphi(x) = (\varepsilon \circ \overline{\varphi})(x)$ for each $x \in F^R$. By definition we have $s[E^R] \subseteq D^R$ and $\varepsilon(y) = (\delta \circ s)(y)$ for each $y \in E^R$. It follows that the diagram in Figure 2 commutes. Thus $(s \circ \overline{\varphi})[F^R] \subseteq D^R$ and $\varphi(x) = (\varepsilon \circ \overline{\varphi})(x) = (\delta \circ s \circ \overline{\varphi})(x)$ for each $x \in F^R$. Since $(F,\, F^R,\, \varphi)$ was arbitrary, $(D,\, D^R,\, \delta)$ is admissible. □

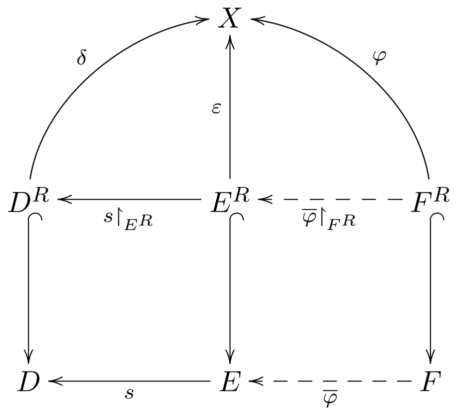

Figure 2.

Remark 3.6. In fact, the result can be strengthened somewhat by noting that it is enough to choose D^R such that $s[E^R] \subseteq D^R \subseteq a^{-1}[E^R]$. This flexibility in the choice of the totality on D is often useful.

This idea of "lifting" the domain representation over E along a is important enough to merit a general definition of a lifting of a domain representation.

Definition 3.7. If (a, s) is an admissible pair for (D, E) we say that the representation $(D,\, D^R,\, \delta)$ is an *admissible lifting* of the representation $(E,\, E^R,\, \varepsilon)$ along a if $(D,\, D^R,\, \delta)$ satisfies $s[E^R] \subseteq D^R \subseteq a^{-1}[E^R]$ and $\delta = (\varepsilon \circ a)\restriction_{D^R}$.

It follows from Theorem 3.5 (and the remark immediately following the theorem) that admissibility is preserved under admissible lifting.

By Lemma 3.3, whenever (a, s) is an admissible pair for (D, E) we know that s is uniquely determined by a. Hence if we know that a is the left adjoint of an admissible pair we can recover s as above. It thus becomes interesting to find sufficient conditions on $a : D \to E$ such that a is the left adjoint of an admissible pair (a, s) for (D, E). We have

Proposition 3.8. *Let D and E be domains and let $a : D \to E$ be a continuous function from D to E. Then a is the left adjoint of an admissible pair if and only if a satisfies the properties 3 – 5 of Lemma 3.3.*

Proof. The implication from left to right is Lemma 3.3. To prove the converse, let $a : D \to E$ be a continuous function from D to E which satisfies 3 – 5 of Lemma 3.3. Since the right adjoint of a (if it exists) is uniquely determined by a we have no choice but to define $s : E \to D$ by $s(y) = \bigsqcup\{c \in D_c;\ a(c) \sqsubseteq y\}$.

s is well-defined: Let $y \in E$. To show that s is well-defined it is enough to show that the set $C_y = \{c \in D_c;\ a(c) \sqsubseteq y\}$ is directed. Since a is surjective there is some $x \in D$ such that $a(x) = \bot$. Thus $a(\bot) \sqsubseteq a(x) = \bot \sqsubseteq y$, and $\bot \in C_y$ so C_y is nonempty. If $b, c \in C_y$ then $a(b)$, $a(c) \sqsubseteq y$ and so $\{a(b),\ a(c)\}$ is consistent in E. Thus $\{b, c\}$ is consistent in D using 3.3.5 and $a(b \sqcup c) = a(b) \sqcup a(c) \sqsubseteq y$. We conclude that $b \sqcup c \in C_y$ and C_y is directed.

s is continuous: It is clear that s is monotone. To show that s is continuous we only need to show that s satisfies $s(\bigsqcup B) \sqsubseteq \bigsqcup s[B]$ for each directed set $B \subseteq E$. Let $B \subseteq E$ be directed and suppose d is compact in D and $d \sqsubseteq s(\bigsqcup B)$. Then $d \sqsubseteq \bigsqcup\{c \in D_c;\ a(c) \sqsubseteq \bigsqcup B\}$. Since d is compact there is a compact element c in D such that $d \sqsubseteq c$ and $a(c) \sqsubseteq \bigsqcup B$. It follows that $a(d) \sqsubseteq \bigsqcup B$. Since $a[D_c] \subseteq E_c$ there is a $b \in B$ such that $a(d) \sqsubseteq b$.

Now, $d \sqsubseteq \bigsqcup\{c \in D_c;\ a(c) \sqsubseteq b\} = s(b)$ and so $d \sqsubseteq \bigsqcup s[B]$. Since d was arbitrary we must have $\mathrm{approx}(s(\bigsqcup B)) \subseteq \mathrm{approx}(\bigsqcup s[B])$ and hence $s(\bigsqcup B) \sqsubseteq \bigsqcup s[B]$ as required.

(a, s) is an admissible pair for (D, E): Let $y \in E$. We now have $(a \circ s)(y) = a(\bigsqcup\{c \in D_c;\ a(c) \sqsubseteq y\}) = \bigsqcup\{a(c);\ c \in D_c \text{ and } a(c) \sqsubseteq y\} \sqsubseteq y$. Since a is surjective there is an $x \in D$ such that $a(x) = y$. Thus $\mathrm{approx}(x) \subseteq \{c \in D_c;\ a(c) \sqsubseteq y\}$ and so $x \sqsubseteq s(y)$. Now $y = a(x) \sqsubseteq (a \circ s)(y)$ and so $(a \circ s)(y) = y$ for each $y \in E$. Conversely, if $x \in D$ then $a(c) \sqsubseteq a(x)$ for each $c \in \mathrm{approx}(x)$ since a is monotone and so $x = \bigsqcup \mathrm{approx}(x) \sqsubseteq \bigsqcup\{c \in D_c;\ a(c) \sqsubseteq a(x)\} = (s \circ a)(x)$ as required. □

Definition 3.9. Let D and E be domains. Then $a : D \to E$ is *admissible* if a is continuous and satisfies conditions 3 – 5 of Lemma 3.3.

It follows immediately that a is admissible if and only if a is the left adjoint of an admissible pair.

We now return to the situation in Theorem 2.4. Let X be a T_0-space and suppose $\mathscr{B} \subseteq \mathscr{P}(X)$ is a pseudobasis on X. Now, using Theorem 2.4 we can construct an admissible representation of X over the ideal completion of $\mathscr{B}$, but as we have already noted, it is not in general well suited if our goal is to introduce a viable computability theory on X. This situation can sometimes be remedied without the loss of admissibility using the theory of admissible pairs. If D is an effective domain and $a : D \to \mathrm{Idl}(\mathscr{B}, \supseteq)$ is admissible we can lift the admissible representation over $\mathrm{Idl}(\mathscr{B}, \supseteq)$ along a to construct an *effective* admissible representation over D.

In applications it is often easier and more conceptually clear to work directly with the pseudobasis $\mathscr{B}$ on X (as in Example 3.1) rather than with the ideal completion of $\mathscr{B}$. To facilitate this it will be useful to have a version of Theorem 3.5 for precusls[5]. This is Theorem 3.14.

[5] We note that if $(B, \sqsubseteq)$ is a computable precusl then $(B, \sqsubseteq)/\!\sim$, where $b \sim c \iff b \sqsubseteq c \sqsubseteq b$, is a computable cusl. Thus, from a computability theoretic standpoint it is enough to

Definition 3.10. If P and Q are precusls and $a : P \to Q$ is a monotone function from P to Q then a is *pre-admissible* if

1. For each $c \in Q$ there is $b \in P$ such that $c \sqsubseteq a(b) \sqsubseteq c$.
2. If $b, c \in P$ and $\{a(b), a(c)\}$ is consistent in Q then $\{b, c\}$ is consistent in P and if d is a least upper bound for b and c in P then $a(d)$ is a least upper bound for $a(b)$ and $a(c)$ in Q.

Remark 3.11. To show that a preserves least upper bounds as in 2 it is enough to show that $a(d)$ is a least upper bound for $a(b)$ and $a(c)$ for some least upper bound d of b and c. The general result follows by monotonicity.

The next proposition relates the two notions of a pre-admissible function and an admissible function.

Proposition 3.12. *Let P and Q be precusls and suppose $a : P \to Q$ is pre-admissible. Then $\overline{a} : \mathrm{Idl}(P) \to \mathrm{Idl}(Q)$ is admissible. Furthermore, any admissible function from $\mathrm{Idl}(P)$ to $\mathrm{Idl}(Q)$ can be obtained in this way.*

Proof. Let $a : P \to Q$ be pre-admissible. Since a is monotone a extends uniquely to a continuous function $\overline{a}$ from $\mathrm{Idl}(P)$ to $\mathrm{Idl}(Q)$. We would like to show that $\overline{a}$ satisfies 3 – 5 of Lemma 3.3.

$\overline{a}$ is surjective: Let $J \in \mathrm{Idl}(Q)$ and let $I = \{c \in P;\ a(c) \in J\}$. Now I is nonempty by 3.10.1 and an ideal by 3.10.2. $\overline{a}(I) \sqsubseteq J$ by construction and $\overline{a}(I) = J$ since for each $c \in J$ there is some $b \in P$ such that $c \sqsubseteq a(b) \sqsubseteq c$. Thus $\overline{a}$ is surjective.

It is clear that $\overline{a}$ satisfies 4 of Lemma 3.3. To show that $\overline{a}$ preserves suprema as in 5 of Lemma 3.3, let I and J be ideals in P and suppose $\overline{a}(I)$ and $\overline{a}(J)$ are consistent in $\mathrm{Idl}(Q)$. If $\overline{a}(I), \overline{a}(J) \subseteq K$ in $\mathrm{Idl}(Q)$ then $a^{-1}[K]$ is an ideal in P by 3.10.1 and 3.10.2 above and $I, J \subseteq a^{-1}[K]$ by construction. Thus $\{I, J\}$ is consistent in $\mathrm{Idl}(P)$.

Now, $\overline{a}(I), \overline{a}(J) \subseteq \overline{a}(I \sqcup J)$ since $\overline{a}$ is monotone, and since $I \sqcup J = \downarrow\{d \in P;\ d$ is a least upper bound for some $b \in I$ and $c \in J\}$, if $e \in \overline{a}(I \sqcup J)$ then $e \sqsubseteq a(d)$ where $d \in P$ is a least upper bound for some $b \in I$ and $c \in J$. Thus e lies below some least upper bound for $a(b)$ and $a(c)$ in Q by 3.10.2 and so $e \in \overline{a}(I) \sqcup \overline{a}(J)$. Hence $\overline{a}(I \sqcup J) \subseteq \overline{a}(I) \sqcup \overline{a}(J)$ and so $\overline{a}(I \sqcup J) = \overline{a}(I) \sqcup \overline{a}(J)$ and we are done.

Finally, to show that every admissible function from $\mathrm{Idl}(P)$ to $\mathrm{Idl}(Q)$ can be obtained in this way, note that every compact element in $\mathrm{Idl}(Q)$ is of the

consider cusls. In practice however it may often be more convenient to work over the precusl $(B, \sqsubseteq)$ rather than the quotient structure, and so we will consider the more general case.

form $\downarrow b$ for some $b \in Q$. However, if the order relation on Q is not a partial order there may be many $c \in Q$ such that $\downarrow c = \downarrow b$. Let $m : \mathrm{Idl}(Q)_c \to Q$ be a function from $\mathrm{Idl}(Q)_c$ to Q such that $m(\downarrow b) =$ some maximal element in $\downarrow b$, for each $b \in Q$. It is easy to see that m must be monotone.

Now, let $a : \mathrm{Idl}(P) \to \mathrm{Idl}(Q)$ be admissible and define $f : P \to Q$ as $f(b) = m(a(\downarrow b))$. Then f is well-defined since $\downarrow b$ is compact in $\mathrm{Idl}(P)$ for each $b \in P$ and so $a(\downarrow b) \in \mathrm{Idl}(Q)_c$ for each $b \in P$. f is monotone by construction and since $\downarrow f(b) = a(\downarrow b)$ for each $b \in P$ we have $\overline{f} = a$. We show that f is pre-admissible.

Thus, let $c \in Q$ and choose $I \in \mathrm{Idl}(P)$ such that $a(I) = \downarrow c$. Now $a(\downarrow b) \subseteq \downarrow c$ for each $b \in I$ and conversely, since $\downarrow c$ is compact and $\bigsqcup_{b \in I} a(\downarrow b) = a(I) = \downarrow c$ there is some $b \in I$ such that $\downarrow c \subseteq a(\downarrow b)$. Thus $a(\downarrow b) = \downarrow c$ and so $c \sqsubseteq f(b) \sqsubseteq c$.

It remains to show that f preserves suprema. Suppose $b, c \in P$ and that $\{f(b),\ f(c)\}$ is consistent in Q. Then $a(\downarrow b)$ and $a(\downarrow c)$ are consistent in $\mathrm{Idl}(Q)$ and so $\downarrow b$ and $\downarrow c$ are consistent in $\mathrm{Idl}(P)$ since a is admissible. We conclude that b and c must be consistent in P. If d is a least upper bound for b and c then $(\downarrow b) \sqcup (\downarrow c) = \downarrow d$ in $\mathrm{Idl}(P)$. We thus conclude that $f(d) = m(a(\downarrow d))$ is a least upper bound for $f(b)$ and $f(c)$ in Q since a is admissible. □

Remark 3.13. It follows by Proposition 3.12 that $a : D \to E$ is admissible if and only if $a[D_c] \subseteq E_c$ and $a\restriction_{D_c} : D_c \to E_c$ is pre-admissible.

The following theorem summarises the general construction of an admissible domain representation of a space X over a precusl B, given a pseudobasis $\mathscr{B}$ on X and a pre-admissible function $a : B \to \mathscr{B}$.

Theorem 3.14. *Let X be a T_0 space and let $\mathscr{B}$ be a countable pseudobasis on X. Let $(B, \sqsubseteq)$ be a precusl and suppose $a : (B, \sqsubseteq) \to (\mathscr{B}, \supseteq)$ is pre-admissible. Let $D = \mathrm{Idl}(B, \sqsubseteq)$ and assume that $D^R \subseteq D$ satisfies*

1. *$a^{-1}[J] \in D^R$ for each convergent ideal $J \in \mathrm{Idl}(\mathscr{B}, \supseteq)$,*

2. *$a[I]$ is convergent for each $I \in D^R$,*

and we define $\delta : D^R \longrightarrow X$ by

3. *$\delta(I) = x$ if and only if $I \in D^R$ and $a[I] \longrightarrow x$ in $\mathrm{Idl}(\mathscr{B}, \supseteq)$.*

Then $(D,\ D^R,\ \delta)$ is an admissible domain representation of X.

Proof. Let $(E,\ E^R,\ \varepsilon)$ be the standard representation of X over $\mathscr{B}$. Then E is admissible by Theorem 6.8 in [Ham05]. Let $D = \mathrm{Idl}(B, \sqsubseteq)$. Since

$a : B \to \mathscr{B}$ is pre-admissible a extends to an admissible function $\overline{a} : D \to E$. Let $s : E \to D$ be the right adjoint of $\overline{a}$. To show that $(D,\, D^R,\, \delta)$ is admissible we note that D^R satisfies $s(J) = \bigsqcup\{c \in D_c;\ \overline{a}(c) \sqsubseteq J\} = \{c \in B;\ a(c) \in J\} = a^{-1}[J] \in D^R$ for each $J \in E^R$ by 1. By 2, we must have $\overline{a}(I) = a[I] \in E^R$ for each $I \in D^R$, and so $s[E^R] \subseteq D^R \subseteq \overline{a}^{-1}[E^R]$. Furthermore, $\delta = (\varepsilon \circ \overline{a})\lceil_{D^R}$ by 3. It follows that $(D,\, D^R,\, \delta)$ is admissible by Theorem 3.5. □

4 Admissible domain representations of inductive limit spaces

We now turn to the problem of constructing an admissible domain representation of the inductive limit $\varinjlim(X_i)_{i\in I}$ of a directed system $(X_i)_{i\in I}$ of admissibly represented spaces. First some definitions.

Let I be a nonempty index set and let $(X_i)_{i\in I}$ be a family of topological spaces indexed by I. $(X_i)_{i\in I}$ is a *directed system* if $I = (I, \leq)$ is a directed partial order and X_i is a subspace of X_j whenever $i \leq j$ in I. $(X_i)_{i\in I}$ is *countable* if the index set I is countable. If $(X_i)_{i\in I}$ is a directed system we define the *inductive limit* of the directed system $(X_i)_{i\in I}$ as the space $\varinjlim(X_i)_{i\in I} = (X, \tau)$ where $X = \bigcup_{i\in I} X_i$ and τ is the finest topology on X such that all of the inclusions $X_i \hookrightarrow X$ are continuous. More explicitly, a set $U \subseteq X$ is open in X if and only if $U \cap X_i$ is open in X_i for each $i \in I$.

Lemma 4.1. *Let $(X_i)_{i\in I}$ be a countable directed system and let X be the inductive limit of $(X_i)_{i\in I}$. Then*

1. *X_i is a subspace of X for each $i \in I$.*
2. *If X_i is T_1 for each $i \in I$ and $(x_n)_n \longrightarrow x$ in X, there is a $j \in I$ such that $\{x_n\}_n \cup \{x\} \subseteq X_j$ and $(x_n)_n \longrightarrow x$ in X_j.*

Proof. The proof for $I = \omega$ is given in [13]. The result follows in the more general case since every countable directed set has a cofinal ω-chain. □

It is well-known, and easy to show, that separation properties up to and including T_4 are preserved under inductive limits. Thus in particular, if each X_i is T_0 then X is T_0 and if X_i is T_1 for each $i \in I$ then X is also T_1. We can now show that the inductive limit of a countable directed system of admissibly represented T_1-spaces admits an admissible domain representation.

Theorem 4.2. *Let $(X_i)_{i\in I}$ be a countable directed system of T_1-spaces and let X be the inductive limit of $(X_i)_{i\in I}$. If $\mathscr{B}_i$ is a pseudobasis in X_i for each $i \in I$, and $\mathscr{B} \subseteq \mathscr{P}(X)$ is given by*

1. *$X \in \mathscr{B}$.*
2. *If $B \in \mathscr{B}_i$ and $B \neq X_i$ then $B \in \mathscr{B}$.*
3. *If $\mathrm{card}(X_i) = 1$ then $X_i \in \mathscr{B}$.*
4. *If $B, C \in \mathscr{B}$ and $B \cap C$ is nonempty, then $B \cap C \in \mathscr{B}$.*

then $\mathscr{B}$ is a pseudobasis in X.

Proof. It is clear that $X \in \mathscr{B}$ and that $\mathscr{B}$ is closed under finite nonempty intersections. Let U be open in X and suppose that $(x_n)_n \longrightarrow x \in U$ in X. By Lemma 4.1 there is a $j \in I$ such that $\{x_n\}_n \cup \{x\} \subseteq X_j$ and $(x_n)_n \longrightarrow x$ in X_j. $U \cap X_j$ is open in X_j by the definition of the topology on X. Since $x \in U \cap X_j$ there is a $B \in \mathscr{B}_j$ and $N \in \mathbb{N}$ such that $x \in B \subseteq U \cap X_j \subseteq U$ and $x_n \in B$ for each $n \geq N$. Since X_j is T_1 we may assume that $B \neq X_j$ whenever X_j has more than one point. It follows that $\mathscr{B}$ is a pseudobasis on X. □

In particular, it follows by Theorem 4.2 that $\bigcup_{i \in I} \mathscr{B}_i$ is a psedobasis on X. (This was first proved by Schröder in [13].) As an immediate corollary of Theorem 4.2 we have

Corollary 4.3. *Let $(X_i)_{i \in I}$ be a countable directed system of T_1-spaces. If X_i has a countably based admissible domain representation for each $i \in I$ then $\varinjlim(X_i)_{i \in I}$ also admits a countably based admissible domain representation.*

Proof. Let $\mathscr{B}_i$ be a countable pseudobasis in X_i. $\mathscr{B}_i$ exists for each $i \in I$ by Theorem 2.4. Let $\mathscr{B}$ be the countable pseudobasis in $\varinjlim(X_i)_{i \in I}$ obtained in Theorem 4.2. Since X is T_1 (and so in particular T_0) we conclude that $\varinjlim(X_k)_{i \in I}$ has a countably based domain representation by Theorem 2.4. □

The construction of the admissible representation of $\varinjlim(X_i)_{i \in I}$ implicit in Corollary 4.3 is clearly noneffective in general. To give an effective construction we will need to assume that the spaces $(X_i)_{i \in I}$ can be represented in some uniform way.

4.1 Almost standard domain representations and faithful extensions

It is well-known that both **Dom** and **Top** are closed under inductive limits. Let $(X_i)_{i \in I}$ be a directed system in **Top**. Let (D_i, D_i^R, δ) is an admissible domain representation of X_i for each $i \in I$, and suppose that D_i embeds into

D_j via $e_{i,j} : D_i \to D_j$ whenever $i \leq j$ in I. It is natural to ask if (or perhaps when) it is possible to construct an admissible domain representation of the inductive limit of $(X_i)_{i \in I}$ over the inductive limit of $((D_i)_{i \in I}, (e_{i,j})_{i \leq j \in I})$. We will show that if the domain representations (D_i, D_i^R, δ_i) are *almost standard* and the embeddings $e_{i,j} : D_i \to D_j$ are *faithful extensions* (both terms are defined below), then it is possible to construct an almost standard domain representation of the inductive limit of $(X_i)_{i \in I}$ over the inductive limit of $((D_i)_{i \in I}, (e_{i,j})_{i \leq j \in I})$. Since every domain representation which is almost standard is admissible this gives sufficient conditions on the directed system $((D_i)_{i \in I}, (e_{i,j})_{i \leq j \in I})$.

Definition 4.4. Let X be a T_0-space and let (D, D^R, δ) be a domain representation of X. (D, D^R, δ) is called *almost standard* if there is a pseudobasis $\mathscr{B}$ on X and an admissible pair (a, s) for $(D, \mathrm{Idl}(\mathscr{B}))$ such that (D, D^R, δ) is an admissible lifting of the standard representation of X over $\mathscr{B}$ along a.

We note that the domain representation (D, D^R, δ) from Theorem 3.14 is almost standard by construction.

If X is a T_0-space and $\mathscr{B}$ is a pseudobasis on X, then the standard representation of X over $\mathrm{Idl}(\mathscr{B})$ is almost standard and so by Theorem 2.4 we have:

Theorem 4.5. *X has a countably based almost standard representation if and only if X is T_0 and has a countable pseudobasis.*

The following elementary lemma which relates approximation in an almost standard representation of X to the pseudobasis on X will be instrumental in what follows.

Lemma 4.6. *Let X be a T_0-space and let (D, D^R, δ) be a domain representation of X. Let $\mathscr{B}$ be a pseudobasis on X and suppose that $a : D \to \mathrm{Idl}(\mathscr{B})$ is an admissible continuous function. If (D, D^R, δ) is an admissible lifting of the standard representation of X over $\mathscr{B}$ along $a : D \to \mathrm{Idl}(\mathscr{B})$ then*

1. *$\bigcap a(r) = \{x \in X;\ r \prec x\}$ for each $r \in D$.*
2. *$B \in \mathscr{B}$ if and only if $B = \{x \in X;\ c \prec x\}$ for some compact $c \in D$.*
3. *$a(c)$ = the principal ideal generated by $\{x \in X;\ c \prec x\}$ for each compact $c \in D$.*

Proof. 1. Let $r \in D$ and suppose $r \prec x$. Choose $s \in D^R$ such that $r \sqsubseteq s$ and $\delta(s) = x$. Then $a(s)$ converges to x and so in particular, we must have

$x \in B$ for each $B \in a(s)$ and hence $x \in \bigcap a(s)$. Now since a is monotone $a(r) \subseteq a(s)$ and so $x \in \bigcap a(r)$.

Conversely, if $x \in \bigcap a(r)$ then $a(r) \subseteq I_x = \{B \in \mathscr{B};\ x \in B\}$. Since $\mathscr{B}$ is closed under finite nonempty intersections it follows that I_x is an ideal, and by the definition of the standard representation over $\mathscr{B}$, we have $I_x \longrightarrow x$. Let s be the right adjoint of a. Then $r \sqsubseteq s(a(r)) \sqsubseteq s(I_x)$ since s is monotone, and $s(I_x) \in D^R$ and $\delta(s(I_x)) = x$ since $(D,\ D^R,\ \delta)$ is an admissible lifting of the standard representation of X over $\mathscr{B}$. Thus $r \prec x$ as required.

2. Suppose that $B \in \mathscr{B}$ and let $J \in \mathrm{Idl}(\mathscr{B})$ be the principal ideal generated by B. Since J is compact in $\mathrm{Idl}(\mathscr{B})$ and $a[D_c] = \mathrm{Idl}(\mathscr{B})_c$ by Remark 3.4 it follows that $a(c) = J$ for some compact $c \in D$. Thus $B = \bigcap J = \bigcap a(c) = \{x \in X;\ c \prec x\}$.

Conversely, if c is compact in D then $a(c)$ is compact in $\mathrm{Idl}(\mathscr{B})$ and so $a(c) =$ the principal ideal generated by B for some $B \in \mathscr{B}$. It follows that $\{x \in X;\ c \prec x\} = \bigcap a(c) = B \in \mathscr{B}$.

3. Let c be compact in D. Then $a(c)$ is compact in $\mathrm{Idl}(\mathscr{B})$. Choose $B \in \mathscr{B}$ such that $a(c) =$ the principal ideal generated by B. Then $B = \{x \in X;\ c \prec x\}$ as above and so $a(c) =$ the principal ideal generated by $\{x \in X;\ c \prec x\}$ as required. □

In particular, it follows by Lemma 4.6 that the pseudobasis $\mathscr{B}$ as well as the admissible function $a : D \to \mathrm{Idl}(\mathscr{B})$ are both completely determined by the approximation relation $r \prec x$. on $D \times X$.

Definition 4.7. Let $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ be domain representations of the spaces X and Y and suppose that $X \subseteq Y$. Let $e : D \to E$ be an embedding. Then e is a *faithful extension* if

1. $r \in D^R \iff e(r) \in E^R$ for each $r \in D$.
2. $e : D \to E$ represents the inclusion $X \hookrightarrow Y$.
3. If $c \neq \bot_E$ is compact in E and $c \in e[D]$, then $\{y \in Y;\ c \prec y\} \subseteq X$.
4. If c is compact in E, $\{y \in Y;\ c \prec y\}$ is nonempty, and $\{y \in Y;\ c \prec y\} \subseteq X$, then $c \in e[D]$.

Intuitively, if $e : D \to E$ is a faithful extension then e preserves both the domain structure on D (since e is an embedding) as well as the domain representation over D and the approximation relation $\prec$ on D.

Before we go on we stop to consider one important example of a faithful extension.

Example 4.8. Let (D, D^R, δ) be a domain representation of the space X and suppose that $X \subseteq Y$. To construct a domain representation of Y it is natural to attempt to extend the domain representation of X over D by adding new approximations of the elements in $Y \setminus X$ to D. Let (E, E^R, ε) be a domain representation of Y which is obtained in this way. We assume that

1. $D \subseteq E$ and $x \sqsubseteq_D y \iff x \sqsubseteq_E y$ for all $x, y \in D$.

2. $\bot_D = \bot_E$ and $D_c = E_c \cap D$.

3. $D^R = E^R \cap D$ and $\delta = \varepsilon\restriction_{D^R}$.

Since we only need to add new approximations to elements which are not in X, it is also natural to assume that

4. If $c \neq \bot_D$ is compact in D, d is compact in E, and $c \sqsubseteq_E d$. Then $d \in D$.

(Intuitively, if c is in D and $c \neq \bot_D$ then any approximation above c in E will only approximate elements in X, and so it must be in D already.) For the same reason, we require that

5. If d is compact in E and $d \notin D$ then $d \prec y$ for some $y \in Y \setminus X$.

Now, we claim that the inclusion of D into E is a faithful extension.

Proof. Let $e : D \to E$ be the inclusion of D into E. It is an easy exercise in domain theory to show that $e : D \to E$ is an embedding. We have $r \in D^R$ if and only if $e(r) \in E^R$ using 3 and it is clear that e represents the inclusion $X \hookrightarrow Y$. Furthermore, if $c \neq \bot$ is compact in E and $c \in D$ then $\uparrow c \subseteq D$ by 4. It follows immediately that $\{y \in Y;\ c \prec_E y\} \subseteq X$. Conversely, if c is compact in E and $\{y \in Y;\ c \prec_E y\} \subseteq X$ then $c \in D$ using 5. We conclude that $D \hookrightarrow E$ is a faithful extension. □

The following lemma shows that faithful extensions preserve the relation $c \prec x$ for compact elements $c \neq \bot$. (The result actually holds for arbitrary elements $\neq \bot$ in the domain, but this is all we will need. Since embeddings are strict by definition, this is the best we can expect.)

Lemma 4.9. *Let (D, D^R, δ) and (E, E^R, ε) be domain representations of the spaces X and Y and suppose that $X \subseteq Y$. If $e : D \to E$ is a faithful extension then $\{x \in X;\ c \prec x\} = \{y \in Y;\ e(c) \prec y\}$ for each compact $c \neq \bot$ in D.*

Proof. Let c be compact in D and suppose that $c \neq \bot$. Then $e(c) \neq \bot$. If $c \prec x$ then $e(c) \prec x$ since e represents the inclusion $X \hookrightarrow Y$. It follows that $\{x \in X;\ c \prec x\} \subseteq \{y \in Y;\ e(c) \prec y\}$.

Conversely, suppose that $e(c) \prec y$ and choose $s \in E^R$ such that $e(c) \sqsubseteq s$ and $\varepsilon(s) = y$. Let $A = \{a \in D_c;\ e(a) \sqsubseteq s\}$. A is nonempty since $c \in A$ and A is directed since e is an embedding. Let $r = \bigsqcup A$. Then $e(r) = \bigsqcup e[A] \sqsubseteq s$. On the other hand, if b is compact in E and $e(c) \sqsubseteq b \sqsubseteq s$ then $b \in \{y \in Y;\ b \prec y\}$, and $\{y \in Y;\ b \prec y\} \subseteq \{y \in Y;\ e(c) \prec y\} \subseteq X$ and so $e(a) = b$ for some compact $a \in D$. Since $e(c)$ is compact in E it follows that $e[A]$ is cofinal in $\mathrm{approx}(s)$ and so $e(r) = \bigsqcup e[A] = s$.

Since $c \in A$ we have $c \sqsubseteq r$ and since $e(r) = s \in E^R$ and e is faithful we have $r \in D^R$. Now, $\delta(r) = \varepsilon(e(r)) = y$ since e represents the inclusion $X \hookrightarrow Y$. It follows that $c \prec y$.

Since y was arbitrary we have $\{y \in Y;\ e(c) \prec y\} \subseteq \{x \in X;\ c \prec x\}$ and so $\{x \in X;\ c \prec x\} = \{y \in Y;\ e(c) \prec y\}$ as required. □

4.2 An almost standard domain representation of the inductive limit of $(X_i)_{i\in I}$

We now return to the problem of constructing a domain representation of the inductive limit of a directed system $(X_i)_{i\in I}$ of topological spaces over the inductive limit of the directed system $((D_i)_{i\in I},\ (e_{i,j})_{i\leq j\in I})$ of domains.

More precisely, suppose that X is the inductive limit of the directed system $(X_i)_{i\in I}$ and let D be the inductive limit of the directed system $((D_i)_{i\in I},\ (e_{i,j})_{i\leq j\in I})$. Suppose further that $(D_i,\ D_i^R,\ \delta_i)$ is a domain representation of X_i for each $i \in I$. To construct a domain representation of X over D we need to identify the representing elements in D, and define a continuous surjection from the set of representing elements onto X.

Since the inductive limit of the spaces $(X_i)_{i\in I}$ is simply the union of the sets X_i, a natural idea would be to let the representing elements in D be the "union" of the representing elements in each D_i. To be more precise, we let $D^R = \bigcup_{i\in I} e_i[D_i^R]$ and define $\delta : D^R \to X$ by $\delta(e_i(r_i)) = \delta_i(r_i)$.

It is easy to show that δ is well-defined if we assume that each of the embeddings $e_{i,j} : D_i \to D_j$ represents the inclusion of X_i into X_j, and δ is surjective since each δ_i is surjective. However, since the forgetful functor U : **Dom** $\to$ **Top** does not preserve inductive limits we will need stronger requirements on the embeddings $(e_{i,j})_{i\leq j\in I}$ to ensure that $\delta : D^R \to X$ is continuous.

Theorem 4.10. *Let $(X_i)_{i\in I}$ be a countable directed system of topological spaces and let $((D_i)_{i\in I},\ (e_{i,j})_{i\leq j\in I})$ be a directed system where*

1. $(D_i,\, D_i^R,\, \delta_i)$ *is a domain representation of* X_i *for each* $i \in I$.

2. $e_{i,j}$ *is a faithful extension for all* $i \leq j \in I$.

Then

3. $(D,\, D^R,\, \delta)$ *is a domain representation of* X.

4. $e_i : D_i \to D$ *is a faithful extension for each* $i \in I$.

Proof. 3. We need to show that $\delta : D^R \to X$ is well-defined, surjective, and continuous. To show that δ is well-defined, suppose that $r_i \in D_i^R$, $r_j \in D_j^R$, and $e_i(r_i) = e_j(r_j)$. We would like to show that $\delta_i(r_i) = \delta_j(r_j)$. Choose $k \in I$ such that $i, j \leq k$. Then $e_k(e_{i,k}(r_i)) = e_i(r_i) = e_j(r_j) = e_k(e_{j,k}(r_j))$ and so $e_{i,k}(r_i) = e_{j,k}(r_j)$ since e_k is an embedding. Since $e_{i,k}$ and $e_{j,k}$ represent the inclusions $X_i \hookrightarrow X_k$ and $X_j \hookrightarrow X_k$ we have $\delta_i(r_i) = \delta_k(e_{i,k}(r_i)) = \delta_k(e_{j,k}(r_j)) = \delta_j(r_j)$ as required.

It is clear that δ is surjective: If $x \in X$ then $x \in X_i$ for some $i \in I$. Choose $r_i \in D_i^R$ such that $\delta_i(r_i) = x$. Then $e_i(r_i) \in D^R$ and $\delta(e_i(r_i)) = \delta_i(r_i) = x$.

To show that δ is continuous let U be an open subset of X and suppose that $\delta(r) = x \in U$. Choose $i \in I$ and $r_i \in D_i^R$ such that $e_i(r_i) = r$. Then $\delta_i(r_i) = x$. Since $U_i = U \cap X_i$ is open in X_i and $\delta_i : D_i^R \to X_i$ is continuous there is some compact c_i in D_i such that $c_i \sqsubseteq_i r_i$ and $\uparrow c_i \cap D_i^R \subseteq \delta_i^{-1}[U_i]$. We will show that $r \in \uparrow e_i(c_i) \cap D^R \subseteq \delta^{-1}[U]$.

Suppose that $s \in D^R$ and $e_i(c_i) \sqsubseteq s$. Choose $j \in I$ and $s_j \in D_j^R$ such that $e_j(s_j) = s$. Since I is directed we may assume that $i \leq j$. Since $e_j : D_j \to D$ is order preserving and $e_j(e_{i,j}(c_i)) = e_i(c_i) \sqsubseteq e_j(s_j)$ we have $e_{i,j}(c_i) \sqsubseteq_j s_j$. Since $e_{i,j}$ is a faithful extension we must have $\delta(s) = \delta_j(s_j) \in \{x \in X_j;\ e_{i,j}(c_i) \prec_j x\} = \{x \in X_i;\ c_i \prec_i x\} \subseteq U_i$. It follows that $\uparrow e_i(c_i) \cap D^R$ is an open neighbourhood of r contained in $\delta^{-1}[U]$. Since $r \in \delta^{-1}[U]$ was arbitrary we conclude that $\delta^{-1}[U]$ is open in D^R and $\delta : D^R \to X$ is continuous.

4. Let $i \in I$. It follows immediately by the definition of D^R that $e_i(r_i) \in D^R$ whenever $r_i \in D_i^R$. Conversely, suppose that $e_i(r_i) \in D^R$ for some $r_i \in D_i$. Then $e_i(r_i) = e_j(r_j)$ for some $j \in I$ and some $r_j \in D_j^R$. Since I is directed and each embedding $e_{i,j}$ represents the inclusion of X_j into X_k we may assume that $i \leq j$. Thus $e_j(e_{i,j}(r_i)) = e_i(r_i) = e_j(r_j)$ and so $e_{i,j}(r_i) = r_j \in D_j^R$. Since $e_{i,j}$ is a faithful extension we conclude that $r_i \in D_i^R$ as required.

By definition we have $e_i(r_i) \in D^R$ and $\delta(e_i(r_i)) = \delta_i(r_i)$ for each $r_i \in D_i^R$. It follows that $e_i : D_i \to D$ represents the inclusion $X_i \hookrightarrow X$.

Now, let c be compact in D and suppose that $c \neq \bot$. If $c \in e_i[D_i]$ then $c = e_i(c_i)$ for some compact $c_i \in D_i$. Suppose that $r \in D^R$ and $c \sqsubseteq r$ in D. Choose $j \in I$ such that $i \leq j$ and $r_j \in D_j^R$ such that $e_j(r_j) = r$. Now $e_j(e_{i,j}(c_i)) = e_i(c_i) \sqsubseteq e_j(r_j)$ as before and so $e_{i,j}(c_i) \sqsubseteq r_j$. It follows that $\delta(r) = \delta_j(r_j) \in \{x \in X_j;\ e_{i,j}(c_i) \prec_j x\} = \{x \in X_i;\ c_i \prec_i x\} \subseteq X_i$. We thus conclude that $\{x \in X;\ c \prec x\} \subseteq X_i$.

Conversely, let $c \neq \bot$ be compact in D and suppose that $\emptyset \neq \{x \in X;\ c \prec x\} \subseteq X_i$. We would like to show that $c \in e_i[D_i]$. Choose $j \in I$ such that $i \leq j$, and c_j compact in D_j such that $e_j(c_j) = c$. Since $\{x \in X;\ c \prec x\} \neq \emptyset$ there is some $r \in D^R$ such that $c \sqsubseteq r$. Choose $k \in I$ and $r_k \in D_k^R$ such that $e_k(r_k) = r$. We may assume that $j \leq k$. Let $c_k = e_{j,k}(c_j)$. Then $e_k(c_k) = e_j(c_j) = c \sqsubseteq r = e_k(r_k)$, and so $c_k \sqsubseteq_k r_k$. Since $r_k \in D_k^R$ it follows that $\{x \in X_k;\ c_k \prec_k x\}$ is nonempty. Since $e_k : D_k \to D$ represents the inclusion of $X_k \hookrightarrow X$ we have $\{x \in X_k;\ c_k \prec_k x\} \subseteq \{x \in X;\ c \prec x\} \subseteq X_i$. Since $e_{i,k}$ is a faithful extension it follows that $c_k \in e_{i,k}[D_i]$. Choose c_i compact in D_i such that $e_{i,k}(c_i) = c_k$. Then $e_i(c_i) = e_k(e_{i,k}(c_k)) = e_k(c_k) = c$, and so $c \in e_i[D_i]$ as required. This concludes the proof. □

If in addition to the assumptions in the previous theorem we assume that each space X_i is T_1 and each of the domain representations $(D_i,\ D_i^R,\ \delta_i)$ is almost standard, then this is enough to ensure that the domain representation $(D,\ D^R,\ \delta)$ defined above is almost standard as well.

Theorem 4.11. *Suppose that $(X_i)_{i \in I}$ is a countable directed system of T_1-spaces and let $((D_i)_{i \in I},\ (e_{i,j})_{i \leq j \in I})$ be a directed system of domains where*

1. *$(D_i,\ D_i^R,\ \delta_i)$ is an almost standard domain representation of X_i for each $i \in I$.*
2. *$e_{i,j}$ is a faithful extension for $i \leq j \in I$.*

Then $(D,\ D^R,\ \delta)$ is an almost standard domain representation of X.

Proof. $(D,\ D^R,\ \delta)$ is a domain representation by the previous theorem. To show that $(D,\ D^R,\ \delta)$ is almost standard, let $\mathscr{B}_i$ be a pseudobasis on X_i and suppose that (a_i, s_i) is an admissible pair for $(D_i,\ \mathrm{Idl}(\mathscr{B}_i))$ such that the domain representation $(D_i,\ D_i^R,\ \delta)$ is an admissible lifting of the standard representation over $\mathrm{Idl}(\mathscr{B}_i)$ along the admissible function $a_i : D_i \to \mathrm{Idl}(\mathscr{B}_i)$. It will be convenient to consider the case when $\bot_i \in D_i^R$ for some $i \in I$ separately, so for now, we assume that $\bot_i \notin D_i^R$ for all i.

For each $i \in I$ we let $\mathscr{C}_i$ denote the collection of all sets $\{x \in X_i;\ c_i \prec_i x\}$ where c_i ranges over the set of compact elements in D_i such that $c_i \neq \bot_i$. By

Lemma 4.6 we have $\mathscr{B}_i = \mathscr{C}_i \cup \{X_i\}$ for each $i \in I$. Thus in particular, it follows that $\mathscr{C}_i$ is closed under finite nonempty intersections for each $i \in I$. We let $\mathscr{C} = \bigcup_{i \in I} \mathscr{C}_i \cup \{X\}$.

Claim 1. $\mathscr{C}$ is a pseudobasis on X.

Proof of Claim 1. To show that $\mathscr{C}$ is a pseudobasis on X we let $\mathscr{B}$ be the pseudobasis on X from Theorem 4.2. We claim that $\mathscr{B} \subseteq \mathscr{C}$. We note that X is in $\mathscr{C}$ by construction. Let $B \in \mathscr{B}_i$ for some $i \in I$ and suppose that $B \neq X_i$. Then $B = \{x \in X_i;\ c_i \prec_i x\}$ for some compact element $c_i \neq \bot_i$ in D_i by Lemma 4.6. It follows that $B \in \mathscr{C}_i \subseteq \mathscr{C}$. Suppose that $\mathrm{card}(X_i) = 1$. Choose $r_i \in D_i^R$. Since $r_i \neq \bot_i$ there is a compact element $c_i \in D_i$ such that $c_i \neq \bot_i$ and $c_i \sqsubseteq_i r_i$. It follows that $\{x \in X_i;\ c_i \prec x\} = X_i$, and so $X_i \in \mathscr{C}_i \subseteq \mathscr{C}$. To complete the proof, it is enough to show that $\mathscr{C}$ is closed under finite nonempty intersections. Thus, let $B, C \in \mathscr{C}$ and suppose that $B \cap C$ is nonempty. If either B or C is equal to X it is clear that $B \cap C \in \mathscr{C}$, so suppose that $B, C \neq X$. Choose $i, j \in I$, and compact elements $b_i \in D_i$, and $c_i \in D_j$ such that $b_i \neq \bot_i$ and $B = \{x \in X_i;\ b_i \prec_i x\}$, and $c_j \neq \bot_j$ and $C = \{x \in X_j;\ c_j \prec_j x\}$. Since I is directed there is a $k \in I$ such that $i, j \leq k$. Since $e_{i,k}$ is a faithful extension, we have $e_{i,k}(b_i) \neq \bot_k$, and $B = \{x \in X_i;\ b_i \prec_i x\} = \{x \in X_k;\ e_{i,k}(b_i) \prec_k x\} \in \mathscr{C}_k$. It follows in exactly the same way that $C \in \mathscr{C}_k$. Since $\mathscr{C}_k$ is closed under finite nonempty intersections we have $B \cap C \in \mathscr{C}_k \subseteq \mathscr{C}$ as required.

We will now define an admissible function $a : D \to \mathrm{Idl}(\mathscr{C})$ from D to $\mathrm{Idl}(\mathscr{C})$ and show that (D, D^R, δ) is an admissible lifting of the standard representation of X over $\mathscr{C}$ along a. To define a we first define a pre-admissible function $a : D_c \to \mathscr{C}$ from D_c to $\mathscr{C}$ and then extend a to a continuous function from D to $\mathrm{Idl}(\mathscr{C})$. By Lemma 4.6 we have no choice but to define $a : D_c \to \mathscr{C}$ by $a(c) = \{x \in X;\ c \prec x\}$.

Claim 2. The function $a : D_c \to \mathscr{C}$ is pre-admissible.

Proof of Claim 2. We need to show that a is well-defined, monotone, surjective, and that a preserves suprema.

$a : D_c \to \mathscr{B}$ is well-defined: We have to show that $a(c) \in \mathscr{C}$ for each compact c in D. Thus, suppose that c is compact in D. If $c = \bot$ then $a(c) = X \in \mathscr{C}$. If $c \neq \bot$ we choose $i \in I$ and some compact $c_i \in D_i$ such that $e_i(c_i) = c$. Since $c \neq \bot$ we have $c_i \neq \bot_i$. It follows that $\{x \in X_i;\ c_i \prec_i x\} \in \mathscr{C}_i \subseteq \mathscr{C}$. By an application of Theorem 4.10 we have $\{x \in X;\ c \prec x\} =$

$\{x \in X;\ e_i(c_i) \prec x\} = \{x \in X_i; c_i \prec_i x\} \in \mathscr{C}$. Since c was arbitrary, we conclude that $a : D_c \to \mathscr{C}$ is well-defined.

$a : D_c \to \mathscr{C}$ is pre-admissible: It is clear that a is monotone. To show that a is surjective, let $B \in \mathscr{C}$. If $B = X$ then $a(\bot) = B$. Suppose on the other hand that $B = \{x \in X_i;\ c_i \prec_i x\}$ for some $i \in I$ and some compact $c_i \in D_i$ such that $c_i \neq \bot_i$. Then $a(e_i(c_i)) = \{x \in X;\ e_i(c_i) \prec x\} = \{x \in X_i;\ c_i \prec_i x\}$ $= B$, and a is surjective as required.

Finally, to show that a preserves suprema, let b and c be compact in D and suppose that the set $\{a(b),\ a(c)\}$ is consistent in $\mathscr{C}$. If either b or c is equal to $\bot$ it is clear that $\{b, c\}$ is consistent and that $a(b \sqcup c) = a(b) \sqcup a(c)$. We may thus assume that $b, c \neq \bot$. Choose i and j in I, and b_i and c_j compact in D_i and D_j such that $e_i(b_i) = b$ and $e_j(c_j) = c$. Since $b, c \neq \bot$ we have $b_i \neq \bot_i$ and $c_i \neq \bot_j$. Let k be an upper bound for i and j in I. Now, $e_k(e_{i,k}(b_i)) = b$ and $e_k(e_{j,k}(c_j)) = c$. Since $\{a(b),\ a(c)\}$ is consistent in $\mathscr{C}$ and the embedding $e_k : D_k \to D$ is a faithful extension we must have $\{x \in X_k;\ e_{i,k}(b_i) \prec_k x\}$ $\cap \{x \in X_k;\ e_{j,k}(c_j) \prec_k x\} \neq \emptyset$. Since $\mathscr{B}_k$ is closed under nonempty intersections we have $\{x \in X_k;\ e_{i,k}(b_i) \prec_k x\} \cap \{x \in X_k;\ e_{j,k}(c_j) \prec_k x\} \in \mathscr{B}_k$. It follows that $a_k(e_{i,k}(b_i))$ and $a_k(e_{j,k}(c_j))$ are consistent in $\mathrm{Idl}(\mathscr{B}_k)$. Since $a_k : D_k \to \mathrm{Idl}(\mathscr{B}_k)$ is admissible we conclude that $e_{i,k}(b_i)$ and $e_{j,k}(c_j)$ are consistent in D_k. Let $d_k = e_{i,k}(b_i) \sqcup e_{j,k}(c_j)$ and let $d = e_k(d_k)$. Then $d = b \sqcup c$ since e_k preserves suprema. Two applications of Lemma 4.9 give us

$$a(d) = \{x \in X;\ d \prec x\} = \{x \in X_k;\ d_k \prec_k x\} =$$

$$\{x \in X_k;\ e_{i,k}(b_i) \prec_k x\} \cap \{x \in X_k;\ e_{j,k}(c_j) \prec_k x\} =$$

$$\{x \in X; b \prec x\} \cap \{x \in X;\ c \prec x\} = a(b) \sqcup a(c)$$

as required. It follows that $a : D_c \to \mathscr{C}$ is preadmissible.

The function $a : D_c \to \mathscr{C}$ extends to an admissible function $a : D \to \mathrm{Idl}(\mathscr{C})$ from D to $\mathrm{Idl}(\mathscr{C})$. To show that the domain representation $(D,\ D^R,\ \delta)$ is almost standard via $a : D \to \mathrm{Idl}(\mathscr{C})$ we need the following two results:

Claim 3. Let $f_i : \mathscr{B}_i \to \mathscr{C}$ be the monotone function given by $f_i(B) = B$ for each $B \in \mathscr{B}_i$ such that $B \neq X_i$, $f_i(X_i) = X_i$ if $X_i \in \mathscr{C}$, and $f_i(X_i) = X$ otherwise. Since f_i is monotone f_i extends to a continuous function $f_i : \mathrm{Idl}(\mathscr{B}_i) \to \mathrm{Idl}(\mathscr{C})$. We claim that $a(e_i(r_i)) = f_i(a_i(r_i))$ for each $r_i \in D_i^R$.

Proof of Claim 3. If c_i is compact in D_i and $c_i \neq \bot_i$ then $a(e_i(c_i)) =$ the principal ideal in $\mathscr{C}$ generated by $\{x \in X;\ e_i(c_i) \prec x\}$ = the principal ideal in $\mathscr{C}$ generated by $\{x \in X_i;\ c_i \prec_i x\} = f_i($the principal ideal in $\mathscr{B}_i$ generated by $\{x \in X_i;\ c_i \prec_i x\}) = f_i(a_i(c_i))$. The claim follows for arbitrary $r_i \in D_i^R$ since the maps a, e_i, f_i, and a_i are continuous, and $r_i = \bigsqcup\big(\text{approx}(r_i) \setminus \{\bot_i\}\big)$ for each $r_i \in D_i^R$.

(Note that as we have $r_i = \bigsqcup\big(\text{approx}(r_i) \setminus \{\bot_i\}\big)$ for each $r_i \neq \bot_i$ in D_i it actually follows by the proof of Claim 3 that $a(e_i(r_i)) = f_i(a_i(r_i))$ for each $r_i \neq \bot_i$ in D_i. This will be useful in the proof of Claim 4.)

Claim 4. Let $s : \text{Idl}(\mathscr{B}) \to D$ be the right adjoint of $a : D \to \text{Idl}(\mathscr{C})$. Then $e_i(s_i(J)) = s(f_i(J))$ for each convergent ideal $J \in \text{Idl}(\mathscr{B}_i)$.

Proof of Claim 4. Let K = the principal ideal in $\mathscr{B}_i$ generated by $\{x \in X;\ b_i \prec_i x\}$ where $b_i \neq \bot_i$ is compact in D_i, and $\{x \in X;\ b_i \prec x\} \neq \emptyset$. Since a_i is admissible, it follows that $\{d_i \in D_i;\ d_i \text{ is compact and } a_i(d_i) \sqsubseteq K\}$ is directed, and

$$e_i(s_i(K)) = e_i(\bigsqcup\{d_i \in D_i;\ d_i \text{ is compact and } a_i(d_i) \sqsubseteq K\}) =$$

$$\bigsqcup\{e_i(d_i);\ d_i \in D_i \text{ is compact and } a_i(d_i) \sqsubseteq K\} = \bigsqcup A_K.$$

On the other hand, $s(f_i(K)) = \bigsqcup\{d \in D;\ d \text{ is compact and } a(d) \sqsubseteq f_i(K)\} = \bigsqcup B_K$. (Here, A_K and B_K are introduced as names for the sets over which the least upper bounds are taken.)

Let d_i be compact in D_i and suppose that $a_i(d_i) \sqsubseteq K$. If $d_i = \bot_i$ then $a(e_i(d_i)) = a(\bot) = X$, and so $a(e_i(d_i)) \sqsubseteq f_i(K)$. If $d_i \neq \bot_i$ then $a(e_i(d_i)) = f_i(a_i(d_i)) \sqsubseteq f_i(K)$ as in the proof of Claim 3. It follows that $A_K \subseteq B_K$, and so $e_i(s_i(K)) \sqsubseteq s(f_i(K))$.

To show that $s(f_i(K)) \sqsubseteq e_i(s_i(K))$ it is enough to show that A_K is cofinal in B_K. Let c be compact in D and suppose that $a(c) \sqsubseteq f_i(K)$. We need to show that there is a compact element $d_i \in D_i$ such that $a_i(d_i) \sqsubseteq K$, and $c \sqsubseteq e_i(d_i)$. Choose $j \in I$ and c_j compact in D_j such that $c = e_j(c_j)$. If $c_j = \bot_j$ then $c = \bot$ and we may choose $d_i = \bot_i$. Thus, we may assume that $c_j \neq \bot_j$, and since I is directed, we may also assume that $i \leq j$. Let $b_j = e_{i,j}(b_i)$. Then $b_j \neq \bot_j$ and $f_j(a_j(c_j)) = a(e_j(c_j)) = a(c) \sqsubseteq f_i(K) = f_i(a_i(b_i)) = a(e_i(b_i)) = a(e_j(b_j)) = f_j(a_j(b_j))$ as in the proof of Claim 3. Since f_j is order preserving it follows that $a_j(c_j) \sqsubseteq a_j(b_j)$ in $\text{Idl}(\mathscr{B}_j)$. Thus in particular, $a_j(b_j)$ and $a_j(c_j)$ are consistent in $\text{Idl}(\mathscr{B}_j)$. Since a_j is admissible we conclude that b_j and c_j are consistent in D_j. Let $d_j = b_j \sqcup c_j$. We will show that $d_j \in e_{i,j}[D_i]$.

We have $\{x \in X_j;\ b_j \prec_j x\} = \{x \in X_i;\ b_i \prec_i x\}$ since $e_{i,j} : D_i \to D_j$ is a faithful extension. Furthermore, since $a_j(c_j) \sqsubseteq a_j(b_j)$ we have $\{x \in X_j;\ c_j \prec_j x\} = \bigcap a_j(c_j) \supseteq \bigcap a_j(b_j) = \{x \in X_j;\ b_j \prec_j x\} = \{x \in X_i;\ b_i \prec_i x\}$. It follows that d_j satisfies $\{x \in X_j;\ d_j \prec_j x\} = \{x \in X_j;\ b_j \prec_j x\} \cap \{x \in X_j;\ c_j \prec_j x\} = \{x \in X_i;\ b_i \prec_i x\} \subseteq X_i$. Since $e_{i,j} : D_i \to D_j$ is a faithful extension and the set $\{x \in X_i;\ b_i \prec_i x\}$ is nonempty we conclude that $d_j = e_{i,j}(d_i)$ for some compact $d_i \in D_i$. We note that $b_i \sqsubseteq d_i$ since $b_j \sqsubseteq d_j$ and $e_{i,j}$ is order preserving. In particular, we have $d_i \neq \bot_i$.

Since $\{x \in X_i;\ d_i \prec_i x\} = \{x \in X_j;\ d_j \prec_j x\} = \{x \in X_i;\ b_i \prec_i x\}$ we conclude that $f_i(a_i(d_i)) = a(e_i(d_i)) =$ the principal ideal in $\mathscr{C}$ generated by $\{x \in X_i;\ d_i \prec_i x\} =$ the principal ideal in $\mathscr{C}$ generated by $\{x \in X_i;\ b_i \prec_i x\} = a(e_i(b_i)) = f_i(a_i(b_i))$. Since f_i is order preserving it follows that $a_i(d_i) = a_i(b_i) = K$ and so $a_i(d_i) \sqsubseteq K$. Finally, since $c_j \sqsubseteq_j d_j$ we have $c = e_j(c_j) \sqsubseteq e_j(d_j) = e_j(e_{i,j}(d_i)) = e_i(d_i)$. It follows that A_K is cofinal in B_K as required, and $s(f_i(K)) = e_i(s_i(K))$.

Let $J \longrightarrow x$ in $\mathrm{Idl}(\mathscr{B}_i)$, and let C_J denote the collection of all $a_i(b_i) \in \mathrm{Idl}(\mathscr{B}_i)$ where b_i ranges over all compact elements $b_i \neq \bot_i$ in D_i such that $\{x \in X_i;\ b_i \prec x\}$ is nonempty and $a_i(b_i) \sqsubseteq J$. We claim that $J = \bigsqcup C_J$. If $J \neq \{X_i\}$ this is clear since each compact ideal $K \neq \{X_i\}$ below J is equal to $a_i(b_i)$ for some compact $b_i \neq \bot_i$ such that $b_i \prec x$. Suppose that $J = \{X_i\}$. Since X_i is T_1 it follows that $\mathrm{card}(X_i) = 1$ and $X_i = \{x\}$. Let $r_i \in D_i^R$. Then $r_i \neq \bot_i$ since $\bot_i \notin D_i^R$. Choose b_i compact in D_i such that $b_i \neq \bot_i$ and $b_i \sqsubseteq r_i$. Then $a(b_i) =$ the principal ideal generated by $\{x \in X_i;\ b_i \prec_i x\} =$ the principal ideal generated by $\{x\} = J$. It follows that $C_J = \mathrm{approx}(J)$, and so $J = \bigsqcup C_J$ as required.

Now, since the maps s, f_i, e_i, and s_i are continuous and $s(f_i(K)) = e_i(s_i(K))$ for each K in C_J, we conclude that $s(f_i(J)) = e_i(s_i(J))$.

Claim 5. $(D,\ D^R,\ \delta)$ is an almost standard domain representation of X.

Proof of Claim 5. To show that $(D,\ D^R,\ \delta)$ is almost standard, suppose that $r \in D^R$ and $\delta(r) = x$. We would like to show that $a(r) \longrightarrow x$ in $\mathrm{Idl}(\mathscr{C})$. Choose $i \in I$ and $r_i \in D_i^R$ such that $r = e_i(r_i)$ and $\delta_i(r_i) = x$. Since $(D_i,\ D_i^R,\ \delta_i)$ is almost standard we know that $a_i(r_i) \longrightarrow x$ in $\mathrm{Idl}(\mathscr{B}_i)$.

Since $r_i \in D_i^R$ we have $a(r) = a(e_i(r_i)) = f_i(a_i(r_i))$. Thus, for each $C \in a(r)$ there is some $B \in a_i(r_i)$ such that $B \subseteq C$. It follows that $x \in C$ for each $C \in a(r)$. Let V be an open neighbourhood of x in X. Then $U = V \cap X_i$

is open in X_i and $x \in U$. We now choose $B \in a_i(r_i)$ such that $B \subseteq U$. We may actually assume that $B = \{x \in X_i;\ b_i \prec_i x\}$ for some compact $b_i \neq \bot_i$ in D_i. (If $\mathrm{card}(X_i) > 1$ we can assume that $B \neq X_i$. If $\mathrm{card}(X_i) = 1$ and $B = X_i$ there is a compact element $b_i \in D_i$ such that $b_i \neq \bot_i$ and $\{x \in X_i;\ b_i \prec_i x\} = X_i$ as above.) It follows that $B \in \mathscr{C}_i \subseteq \mathscr{C}$ and $B \in f_i(a_i(r_i)) = a(r)$ as required. Since $B \subseteq U \subseteq V$ we conclude that $a(r) \longrightarrow x$.

Conversely, let $J \longrightarrow x$ in $\mathrm{Idl}(\mathscr{C})$. We first assume that $J \neq \{X\}$. If $C \in J$ and $C \neq X$, then $C \in \mathscr{C}_i$ for some $i \in I$. It follows that $C = \{x \in X_i;\ c_i \prec_i x\}$ for some compact $c_i \in D_i$ such that $c_i \neq \bot_i$. We claim that $\{B \in J;\ B \subseteq C\} \subseteq \mathscr{C}_i$. Suppose that $B \in J$ and $B \subseteq C$. Choose $j \in I$ such that $B \in \mathscr{C}_j$. Since $B \in \mathscr{C}_j$ there is a compact element $b_j \neq \bot_j$ in D_j such that $B = \{x \in X_j;\ b_j \prec_j x\}$. Since I is directed we may assume that $i \leq j$ in I. Since $e_{i,j} : D_i \to D_j$ is a faithful extension and $\{x \in X_j;\ b_j \prec_j x\} = B \subseteq C \subseteq X_i$ we conclude that $b_j \in e_{i,j}[D_i]$. Since $e_{i,j}$ is an embedding $b_j = e_{i,j}(b_i)$ for some compact element $b_i \neq \bot_i$ in D_i. By Lemma 4.9 we have $B = \{x \in X_i;\ b_i \prec_i x\}$ and so $B \in \mathscr{C}_i$ as required.

Let $K = (J \cap \mathscr{C}_i) \cup \{X_i\}$. Then $K \in \mathrm{Idl}(\mathscr{B}_i)$ and $f_i(K) = J$ since $B \in K$ for each $B \in J$ such that $B \subseteq C$. We will show that $K \longrightarrow x$ in $\mathrm{Idl}(\mathscr{B}_i)$. Since $f_i(K) = J$ it is clear that $x \in B$ for each $B \in K$. Let U be an open neighbourhood of x in X_i. Then $U = V \cap X_i$ for some open set V in X. Since $J \longrightarrow x$ in $\mathrm{Idl}(\mathscr{C})$ there is some $B \in J$ such that $B \subseteq V$. Let A be some upper bound of B and C in J. Then $A \in K$ and $A \subseteq B \cap C \subseteq V \cap X_i = U$. Since U was arbitrary we conclude that $K \longrightarrow x$ in $\mathrm{Idl}(\mathscr{B}_i)$. The domain representation (D_i, D_i^R, δ_i) of X_i is almost standard via (a_i, s_i) and so $s_i(K) \in D_i^R$. Now, since $f_i(K) = J$ we have $s(J) = s(f_i(K)) = e_i(s_i(K)) \in D^R$ as required.

If $J = \{X\}$ then $\mathrm{card}(X) = 1$ since X is T_1. Choose $i \in I$ such that $X_i = X$. Then $J \in \mathrm{Idl}(\mathscr{B}_i)$ and J is convergent in $\mathrm{Idl}(\mathscr{B}_i)$. It follows that $s_i(J) \in D_i^R$, and furthermore, that $s(J) = s(f_i(J)) = e_i(s_i(J)) \in D^R$.

Finally, we consider the case when $\bot_i \in D_i^R$ for some $i \in I$. By the next claim, this forces the cardinality of X to be equal to 1.

Claim 6. If $\bot_i \in D_i^R$ for some $i \in I$, then $\mathrm{card}(X) = 1$.

Proof of Claim 6. Suppose that $\bot_i \in D_i^R$. Then $\bot = e_i(\bot_i) \in D^R$. Let $r \in D^R$ and suppose that $\delta(r) \neq \delta(\bot)$. Since X is T_1 there is an open neighbourhood of $\delta(\bot)$ which does not contain $\delta(r)$. It follows that $\bot \in \delta^{-1}[U]$ but $r \notin \delta^{-1}[U]$. This is a contradiction since $\delta^{-1}[U]$ is upwards closed in D^R.

Suppose that $X = \{x\}$ and let $\mathscr{C} = \{X, \emptyset\}$ if there is some compact element $c \in D$ such that $c \not\prec x$, and $\mathscr{C} = \{X\}$ otherwise. We define $a : D_c \to \mathscr{B}$ by $a(c) = X$ if $c \prec x$, and $a(c) = \emptyset$ if $c \not\prec x$. It is clear that a is monotone and surjective. That a preserves suprema follows exactly as above. It follows that a is pre-admissible.

Since $a : D_c \to \mathscr{C}$ is pre-admissible a extends to an admissible function $a : D \to \mathrm{Idl}(\mathscr{C})$ as before. As usual, we denote the right adjoint of a by $s : \mathrm{Idl}(\mathscr{C}) \to D$. We now claim that $(D,\, D^R,\, \delta)$ is almost standard via (a, s).

Claim 7. $(D,\, D^R,\, \delta)$ is an almost standard domain representation of X.

Proof of Claim 7. Suppose that $r \in D^R$. Then $a(r) = \{X\}$. It follows that $a(r) \longrightarrow x$. Conversely, let $J \longrightarrow x$ in $\mathrm{Idl}(\mathscr{C})$. Then $J = \{X\}$. Choose $i \in I$ such that $X_i = X$. Then $J \in \mathrm{Idl}(\mathscr{B}_i)$, and $J \longrightarrow x$ in $\mathrm{Idl}(\mathscr{B}_i)$. Let $r_i = s_i(J)$ and let $r = e_i(r_i)$. Then $r \in D^R$ since e_i is a faithful extension. Since $a(r) = J$ we have $r \sqsubseteq s(a(r)) = s(J)$. Let $c \in D$ be compact, and suppose that $c \sqsubseteq s(J)$. Choose $j \in I$ and $c_j \in D_j$ such that $i \leq j$ and $e_j(c_j) = c$. Since $\{x \in X_j;\ c_j \prec_j x\} = \{x \in X;\ c \prec x\} = X$ we have $\{x \in X;\ c_j \prec_j x\} \neq \emptyset$ and $\{x \in X;\ c_j \prec_j x\} \subseteq X_i$. It follows that $c_j = e_{i,j}(c_i)$ for some compact $c_i \in D_i$. Since $\{x \in X_i;\ c_i \prec_i x\} = X_i = X$ we have $a_i(c_i) = J$, and $c_i \sqsubseteq_i s_i(a(c_i)) = s_i(J) = r_i$. It follows that $c = e_j(c_j) = e_j(e_{i,j}(c_i)) = e_i(c_i) \sqsubseteq e_i(r_i) = r$. Since c was arbitrary, we conclude that $\mathrm{approx}(s(J)) \subseteq \mathrm{approx}(r)$, and $s(J) = r \in D^R$ as required. This concludes the proof. □

Since the construction of the inductive limit in $\mathbf{Dom}$ preserves effectivity, this theorem will allow us to construct an effective admissible domain representation of the inductive limit of the directed system $(X_i)_{i \in I}$, provided that the domain representations $(D_i,\, D_i^R,\, \delta_i)$ of the individual spaces X_i are uniformly effective For details, see [15].

5 Concluding remarks

Most spaces which we encounter in analysis are at least T_0. We know from Theorem 2.4 that if X is T_0 then X has a countably based admissible domain representation if and only if X admits a countable pseudobasis. Given a pseudobasis on X it is in many cases possible to employ the theory of admissible pairs developed in section 3 to construct an effective admissible domain representation $(D,\, D^R,\, \delta)$ of X. This allows us to transfer the computability theory on the effective domain D to a δ-computability theory on the space X. Moreover, since the domain representation $(D,\, D^R, \delta)$ is admissible we may construct an effective type structure over X to study computability in higher types.

As shown in section 4 we may also use admissible pairs to give a uniform construction of an admissible domain representation of the inductive limit of a directed system $(X_i)_{i\in I}$ of admissibly representable T_1-spaces. More precisely, if $(D_i,\, D_i^R,\, \delta_i)$ is an admissible domain representation of X_i for each $i \in I$ and $e_{i,j} : D_i \to D_j$ is an embedding whenever $i \leq j \in I$ it is sometimes possible to "glue together" the individual domain representations $(D_i,\, D_i^R,\, \delta_i)$ to construct an admissible domain representation of the inductive limit of $(X_i)_{i\in I}$ over "the inductive limit" of the representations $(D_i,\, D_i^R,\, \delta_i)_{i\in I}$. For the construction to go through we need to require that

1. Each embedding $e_{i,j} : D_i \to D_j$ preserves the notion of approximation on the domain D_i.

In addition, if we want the domain representation of the inductive limit to be admissible we also need to require that

2. Each domain representation $(D_i,\, D_i^R,\, \delta_i)$ is "almost" the standard representation over a pseudobasis on X_i.

The second condition can be made precise using admissible pairs. Now, since the construction of the inductive limit in the category $\mathbf{Dom}$ preserves effectivity, this construction can be used to construct an effective admissible domain representation of the inductive limit of $(X_i)_{i\in I}$ given effective admissible domain representations of the individual spaces X_i.

As an application of the theory developed in this paper, we note that the construction outlined in Section 4 can be employed to construct an effective and admissible domain representation of the space $\mathscr{D}$ of test functions studied in distribution theory: Formally, the space of test functions is the inductive limit of the spaces $(\mathscr{D}_k)_{k\geq 1}$, where $\mathscr{D}_k$ is the space of smooth functions from $\mathbb{R}$ to $\mathbb{C}$ with compact support contained in the interval $[-k, k]$. Each of the spaces $\mathscr{D}_k$ is a separable metric space, and it is easy to construct an effective and almost standard representation $(D_k,\, D_k^R,\, \delta_k)$ of $\mathscr{D}_k$.

With a modicum of care, it turns out that D_k embeds faithfully into D_l whenever $k \leq l$. It follows that we may apply Theorem 4.11 to construct an effective and admissible domain representation of the inductive limit $\mathscr{D}$ of the spaces $(\mathscr{D}_k)_{k\geq 1}$.

Now, given an effective and admissible domain representation we may introduce a notion of computation on $\mathscr{D}$. This gives a framework in which to study computable processes on the space of test functions, but it is also an important stepping stone towards a general theory of computation for distributions. For more details we refer the reader to [7].

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Paper II

Partial Continuous Functions and Admissible Domain Representations

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Abstract

It is well-known that to be able to represent continuous functions between domain representable spaces it is critical that the domain representations of the spaces we consider are dense. In this article we show how to develop a representation theory over a category of domains with morphisms partial continuous functions. The raison d'être for introducing partial continuous functions is that by passing to partial maps, we are free to consider totalities which are not dense. We show that the category of admissibly representable spaces with morphisms functions which are representable by a partial continuous function is Cartesian closed. Finally, we consider the question of effectivity.

Key words: Domain theory, domain representations, computability theory, computable analysis.

1 Introduction

One way of studying computability on uncountable spaces is through effective domain representations. A domain representation of a space X is a domain D together with a continuous function $\delta : D^R \to X$ onto X where D^R is some nonempty subset of D. When D is an effective domain the computability theory on D lifts to a δ-computability theory on the space X. If $(E,\ E^R, \varepsilon)$ is a domain representation of the space Y we say that $f : X \to Y$ is representable if there is a continuous function $\overline{f} : D \to E$ which takes representations of $x \in X$ to representations of $f(x)$ for each $x \in X$. If every continuous function from X to Y is representable we may construct a domain representation of the space of continuous functions from X to Y over $[D \to E]$.

It thus becomes interesting to find necessary and sufficient conditions on the domain representations $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ to ensure that every continuous function from X to Y is representable. In the context of algebraic domains, this problem has been studied by (among others) Stoltenberg-Hansen, Blanck and Hamrin (c.f. [18], [4] and [10]). It turns out that it is often important that the representation $(D,\ D^R,\ \delta)$ is dense. That is, that the set D^R of representing elements is dense in D with respect to the Scott-topology on D. However, if $(D,\ D^R,\ \delta)$ is not dense there is no general effective construction which given $(D,\ D^R,\ \delta)$ yields a dense and effective representation of X. This is perhaps not so problematic as long as we are interested in building type structures over $\mathbb{R}^n$ or $\mathbb{C}^n$, but if we would like to study computability on more complex topological spaces such as the space $C^\infty(\mathbb{R})$ of smooth functions from $\mathbb{R}$ to $\mathbb{C}$, or the space $\mathscr{D}$ of smooth functions with compact support which are considered in distribution theory, the requirement of denseness becomes a rather daunting exercise in computability theory. Indeed, it is still not known if there is an effective dense domain representation of the space $\mathscr{D}$ of smooth functions with compact support. The natural candidate for an effective domain representation of $\mathscr{D}$ is not dense, and so the standard arguments showing that every distribution is representable fail.

One way to effectively circumvent the problem of finding dense representations of the spaces under consideration is to represent continuous functions from X to Y by partial continuous functions from D to E. Here, a partial continuous function from D to E is a pair (S, f) where S is a nonempty subset of D and f is a (total) continuous function from S to E.

To make sure that enough continuous functions are representable, and to be able to lift the order-theoretic characterisations of continuity to partial continuous functions, we need to place some restrictions on the domain $S \subseteq D$ of a partial continuous function f from D to E. As we shall see, by a careful analysis of which properties we require of S we get a category of domains with morphisms partial continuous functions which is well suited for representation theory in general, and an effective theory of distributions in particular. The category obtained is closely related to the category $\mathrm{Mod}(\mathbb{V})$ of modest sets over the partial combinatory algebra $\mathbb{V}$ introduced in [2].

2 Preliminaries from domain theory

A *Scott-Ershov domain* (or simply domain) is a consistently complete algebraic cpo. Let D be a domain. Then D_c denotes the set of compact elements in D. Given $x \in D$ we write $\mathrm{approx}(x)$ for the set $\{a \in D_c;\ a \sqsubseteq x\}$. Since D is algebraic, $\mathrm{approx}(x)$ is directed and $\bigsqcup \mathrm{approx}(x) = x$ for each $x \in D$. The *Scott-topology* on D is generated by the sets $\uparrow c$ where c is compact in D.

D is *effective* if there is a numbering[1] (that is, a surjective map) $\alpha : \mathbb{N} \to D_c$ of the set D_c such that $(D_c, \sqsubseteq, \sqcup, \text{cons}, \bot)$ is a computable structure[2] with respect to α. When (D, α) and (E, β) are effective domains then $x \in D$ is *α-computable* if the relation $\alpha(m) \sqsubseteq x$ is r.e. and if $f : D \to E$ is continuous then f is *(α, β)-effective* if the relation $\beta(n) \sqsubseteq_E f(\alpha(m))$ is r.e. We usually leave out the prefixes α, β and (α, β), if the numberings α and β are either clear from the context or not important.

If (D, α) is an effective domain we can encode all the computational information about the structure $(D_c, \sqsubseteq, \sqcup, \text{cons}, \bot)$ as a natural number $\ulcorner (D, \alpha) \urcorner$. More precisely, if $a, b, c, d \in \mathbb{N}$ are characteristic indices for the set $\text{dom}(\alpha)$, and relations "$\alpha(m) \sqsubseteq \alpha(n)$", "$\alpha(k) = \alpha(m) \sqcup \alpha(n)$", "$\alpha(m)$ and $\alpha(n)$) are consistent", and "$\alpha(n) = \bot$", we may define $\ulcorner (D, \alpha) \urcorner$ as $\langle a, b, c, d \rangle$, where $(a, b, c, d) \mapsto \langle a, b, c, d \rangle$ is some standard total recursive bijection from $\mathbb{N}^4$ to $\mathbb{N}$. The number $\ulcorner (D, \alpha) \urcorner$ is called an *index* for the effective domain (D, α).

Let X be a topological space. A *domain representation* of X is a triple $(D,\ D^R,\ \delta)$ where D is a domain, D^R a nonempty subset of D, and $\delta : D^R \to X$ a continuous function from D^R onto X. We assume throughout that $\bot \notin D^R$. For $r \in D$ and $x \in X$ we write $r \prec x$ if $x \in \delta[\uparrow r \cap D^R]$. Thus, $r \prec x$ if and only if there is some $s \in D^R$ such that $r \sqsubseteq s$ and $\delta(s) = x$.

When D is effective then $(D,\ D^R,\ \delta)$ is an effective domain representation of X. Suppose $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ are effective domain representations of X and Y respectively. We say that $x \in X$ is *D-computable* (or simply computable if the representation $(D,\ D^R,\ \delta)$ is clear from the context) if there is some computable $r \in D^R$ such that $\delta(r) = x$. A continuous function $f : X \to Y$ is *(D, E)-representable* (or just representable) if there is a continuous function $\overline{f} : D \to E$ such that $\overline{f}[D^R] \subseteq E^R$ and $f(\delta(r)) = \varepsilon(\overline{f}(r))$ for each $r \in D^R$. f is *(D, E)-effective* (or simply effective) if $\overline{f}$ is effective. A domain representation $(D,\ D^R,\ \delta)$ of a space X is *dense* if the set D^R of representing elements is dense in D with respect to the Scott-topology on D.

We are now ready to introduce the important notion of admissibility. Admissible representations of topological spaces where originally studied in the context of Type Two Effectivity by Schröder (c.f. [13] and [14]). The corresponding notion for equilogical spaces has been studied by Menni and Simpson [11], and Bauer [2]. The definition used here is taken from [10].

Definition 2.1. Let $(E,\ E^R,\ \varepsilon)$ be a domain representation of Y. $(E,\ E^R,\ \varepsilon)$ is called *admissible* if for each triple $(D,\ D^R,\ \delta)$ where D is a countably based domain, $D^R \subseteq D$ is dense in D, and $\delta : D^R \to Y$ is a continuous function

[1]We will write $\alpha : \mathbb{N} \to D_c$ even though the domain of α may be a proper (decidable) subset of the natural numbers. For an introduction to the theory of numberings c.f. [7, Ers75].

[2]For definitions we refer the reader to either [7, Ers75], or [16].

from D^R to Y, there is a continuous function $\overline{\delta} : D \to E$ such that $\overline{\delta}[D^R] \subseteq E^R$ and $\delta(r) = \varepsilon(\overline{\delta}(r))$ for each $r \in D^R$.

The following simple observation indicates why admissibility is interesting from a purely representation-theoretic point of view.

Theorem 2.1. *Suppose $(D,\, D^R,\, \delta)$ is a countably based dense representation of X and $(E,\, E^R,\, \varepsilon)$ is an admissible representation of Y. Then every sequentially continuous function from X to Y is representable.*

For a proof of Theorem 2.1 see [10].

Theorem 2.1 can be used as a tool in constructing a representation of the space of continuous functions from X to Y over the domain $[D \to E]$ of continuous functions from D to E. However, if the representation $(D,\, D^R,\, \delta)$ is not dense Theorem 2.1 does not tell us anything.

3 Partial continuous functions

In the case when $(D,\, D^R,\, \delta)$ is a countably based and dense representation of the space X and $(E,\, E^R, \varepsilon)$ is an admissible representation of Y Theorem 2.1 tells that every sequentially continuous function $f : X \to Y$ from X to Y lifts to a continuous function $\overline{f} : D \to E$ such that $\varepsilon(\overline{f}(x)) = f(\delta(x))$ for each $x \in D^R$. In the case when the representation $(D,\, D^R,\, \delta)$ of X is not dense there is a standard construction in domain theory which constructs a dense representation of X from $(D,\, D^R,\, \delta)$: Let

$$D_c^D = \{a \in D_c;\ a \sqsubseteq x \text{ for some } x \in D^R\}$$

and

$$D^D = \{\bigsqcup A;\ A \subseteq D_c^D \text{ is directed}\}.$$

D^D is sometimes called the *dense part* of D. It is not difficult to show that $D^D = (D^D,\, \sqsubseteq_D,\, \bot_D)$ is a domain and that $D^R \subseteq D^D$. In fact, D^R is a subspace of D^D. Thus, $(D^D,\, D^R,\, \delta)$ is a domain representation of X and D^R is dense in D^D by construction.

We may now apply Theorem 2.1 to show that every sequentially continuous function $f : X \to Y$ from X to Y has a continuous representation $\overline{f} : D^D \to E$ from D^D to E. However, there is no a priori reason why the relation $(\exists x \in D^R)(a \sqsubseteq x)$ on D_c should be decidable, even when $(D,\, D^R,\, \delta)$ is effective, and thus $(D^D,\, D^R, \delta)$ is noneffective in general.

3.1 Partial continuous functions with Scott-closed domains

A reasonable alternative to working over D^D would be to view $(D^D, \overline{f})$ as a partial function from D to E. Categories of domains with morphisms partial continuous functions have been studied before by (among others) Plotkin and Fiore (c.f. [12] and [9]). Here a partial continuous function from D to E is a pair (U, f) where $U \subseteq D$ is a Scott-open subset of D and $f : U \to E$ is a continuous function from U to E. However, this notion of a partial continuous function is not the appropriate one for our purposes for essentially two different reasons:

1. D^D is not an open subset of D in general. (In fact, it is easy to see that D^D is open if and only if $D^D = D$.)

2. Every partial continuous function (U, f) from D to E with open domain U extends to a total continuous function $f : D \to E$. Thus nothing is gained from a representation theoretic point of view when going to partial continuous functions with open domains.

Thus, as a first step we would like to distinguish the appropriate notion of partial continuous function for our setting. As a first step, we note that D^D is a nonempty Scott-closed subset of D.

Lemma 3.1. *Let D be a domain and let S be a nonempty subset of D. Then the closure of S is the set $\{\bigsqcup A;\ A \subseteq \downarrow S \text{ is directed}\}$.*

Thus in particular, it follows that D^D is closed in D since D^D_c is downwards closed.

Proof. The lemma follows since $T \subseteq D$ is closed if and only if T is both downwards closed and closed under suprema of directed sets. □

Together with the observation above that every sequentially continuous function is representable by a continuous function $\overline{f} : D^D \to E$, this suggests that it is enough to consider partial continuous functions (S, f) where S is a nonempty Scott-closed subset of D, and $f : S \to E$ is a continuous function from S to E. However, to be able to compose partial continuous functions, it will be convenient to require that the function f is strict (that is, that f takes $\perp_D$ to $\perp_E$).

Lemma 3.2. *Let D and E be domains, and let $S \subseteq D$ and $T \subseteq E$ be nonempty Scott-closed subsets of D and E. Let $f : S \to E$ be a strict continuous function from S into E. Then $f^{-1}[T]$ is nonempty and Scott-closed in D.*

Proof. The set $f^{-1}[T]$ nonempty since f is strict, and so $f(\bot_D) = \bot_E \in T$. $f^{-1}[T]$ is closed since f is continuous and $S \subseteq D$ is closed. □

Thus, given domains D, E, and F, two nonempty closed sets $S \subseteq D$, and $T \subseteq E$, and strict continuous functions $f : S \to E$, and $g : T \to F$, we may define the composite of (S, f) and (T, g) as the pair $(f^{-1}[T], g \circ f)$. The set $f^{-1}[T]$ is a nonempty closed subset of D by the previous lemma, and $g \circ f$ is a strict continuous function from $f^{-1}[T]$ to F. Since identities preserve the least element, it follows that domains with morphisms pairs $(S, f) : D \to E$ where S is a nonempty closed subset of D and $f : S \to E$ is a strict continuous function from S to E form a category. Inspired by this we make the following definition:

Definition 3.1. Let D and E be domains. A *partial continuous function* from D to E is a pair (S, f) where $S \subseteq D$ is a nonempty closed subset of D and f is a strict continuous function from S to E.

We write $f : D \rightharpoonup E$ if $(\mathrm{dom}(f), f)$ is a partial continuous function from D to E and we write $[D \rightharpoonup E]$ for the set of partial continuous functions from D to E. We denote the set of strict continuous functions from D to E by $[D \to_\bot E]$.

If S is nonempty and Scott-closed in D, and $f : S \to E$ is continuous then f is a partial function from D to E provided that f is strict. If not, we are always free to redefine f on $\bot_D$ to get a strict continuous function from S to E. More precisely, we define $f_\bot : S \to E$ as

$$f_\bot(x) = \begin{cases} f(x) & \text{if } x \neq \bot_D \\ \bot_E & \text{if } x = \bot_D. \end{cases}$$

Then $f_\bot$ is a strict continuous function from S to E which agrees with f except possibly on $\bot_D$. Moreover, the map $f \mapsto f_\bot$ is continuous.

Lemma 3.3. *Let D and E be domains. Then $f \mapsto f_\bot$ is a continuous function from $[D \to E]$ to $[D \to_\bot E]$.*

Proof. It is clear that the map $f \mapsto f_\bot$ is monotone. It is continuous since we have $(\bigsqcup A)_\bot(x) = \bigsqcup\{f_\bot(x);\ f \in A\}$ for each $x \in D$. □

As a motivation for Definition 3.1 we give a new characterisation of admissibility in terms of partial continuous functions.

Theorem 3.4. *Let (E, E^R, ε) be a domain representation of Y. Then (E, E^R, ε) is admissible if and only if for every triple (D, D^R, δ) where D is a countably based domain, $D^R \subseteq D$, and $\delta : D^R \to Y$ is a continuous* *function from* D^R *to* Y, *there is a partial continuous function* $\overline{\delta} : D \rightharpoonup E$ *such that*

1. $D^R \subseteq \mathrm{dom}(\overline{\delta})$.

2. $\overline{\delta}[D^R] \subseteq E^R$.

3. $\delta(x) = \varepsilon(\overline{\delta}(x))$ *for each* $x \in D^R$.

Proof. Suppose first that $(E,\ E^R, \varepsilon)$ is an admissible representation of Y and let D be a domain, $D^R \subseteq D$, and suppose $\delta : D^R \to Y$ is continuous. Now D^R is dense in D^D and so there is a (total) continuous function $\overline{\delta} : D^D \to E$ such that $\overline{\delta}[D^R] \subseteq E^R$ and $\delta(x) = \varepsilon(\overline{\delta}(x))$ for each $x \in D^R$. Since D^D is closed in D, $(D^D, \overline{\delta})$ is the required partial function from D to E.

Conversely, suppose $(E,\ E^R,\ \varepsilon)$ is a domain representation of Y such that for each triple $(D,\ D^R,\ \delta)$ where D is a domain, $D^R \subseteq D$, and $\delta : D^R \to Y$ is continuous, there is a partial continuous function $\overline{\delta} : D \rightharpoonup E$ satisfying 1 – 3 above. To show that $(E,\ E^R,\ \varepsilon)$ is admissible, choose $(D,\ D^R,\ \delta)$ where D^R is dense in D and let $\overline{\delta} : D \rightharpoonup E$ be as above. It is clearly enough to show that $\overline{\delta}$ is in fact total.

Since $\mathrm{dom}(\overline{\delta})$ is is downwards closed and $D^R \subseteq \mathrm{dom}(\overline{\delta})$ we must have $D^D_c \subseteq \mathrm{dom}(\overline{\delta})$. Since $\mathrm{dom}(\overline{\delta})$ is closed under suprema of directed sets we have $D^D \subseteq \mathrm{dom}(\overline{\delta})$. Finally, since D^R is dense in D we have $D^D = D$ and so $\mathrm{dom}(\overline{\delta}) = D$ and $\overline{\delta}$ is total as required. □

Theorem 3.4 suggests representing continuous functions in the category of topological spaces using partial continuous functions on domains. We now make this idea more precise.

Definition 3.2. Let $(D,\ D^R,\ \delta)$ is a domain representation X and let $(E,\ E^R,\ \varepsilon)$ a domain representation of Y. We say that $\overline{f} : D \rightharpoonup E$ *represents* the continuous function $f : X \to Y$ if

1. $D^R \subseteq \mathrm{dom}(\overline{f})$.

2. $\overline{f}[D^R] \subseteq E^R$.

3. $f(\delta(x)) = \varepsilon(\overline{f}(x))$ for each $x \in D^R$.

$f : X \to Y$ is called (D, E)-*representable* (or simply representable) if there is a partial continuous function $\overline{f} : D \rightharpoonup E$ satisfying the conditions 1 – 3 above. (If we would like to distinguish between partial representations and total representations we say that f is *partially representable* if the continuous function

$\overline{f}$ representing f is a partial continuous function and f is *totally representable* if $\overline{f}$ is a total function.)

Now Theorem 3.4 immediately yields a version of Theorem 2.1 for partial continuous functions.

Theorem 3.5. *Let* $(D,\, D^R,\, \delta)$ *be a countably based domain representation of* X *and let* $(E,\, E^R,\, \varepsilon)$ *be an admissible representation of* Y. *Then every sequentially continuous function* $f : X \to Y$ *is representable by a partial continuous function* $\overline{f} : D \rightharpoonup E$.

Proof. If $f : X \to Y$ is sequentially continuous then $f \circ \delta$ is continuous since D is countably based. The result now follows from the previous theorem. □

Just as in the case when the representation $(D,\, D^R,\, \delta)$ is dense we have the following characterisation of the sequentially continuous functions from X to Y. (For a thorough study of the dense and total case, see [10].)

Theorem 3.6. *Let* $(D,\, D^R,\, \delta)$ *be a countably based admissible domain representation of* X *and let* $(E,\, E^R,\, \varepsilon)$ *be a domain representation of* Y. *If* $f : X \to Y$ *is representable then* f *is sequentially continuous.*

Proof. Let $(x_n)_n \longrightarrow x$ in X. Let S be the domain $(\omega + 1,\, \leq,\, 0)$ and let $S^R = S - \{0\}$. Let $s : S^R \to X$ be the continuous function given by $s(n) = x_n$ for $0 < n < \omega$ and $s(\omega) = x$. Let $\overline{s} : S \to D$ represent s. Now $f(x_n) = \varepsilon(\overline{f}(\overline{s}(n)))$ for each $0 < n < \omega$ and $f(x) = \varepsilon(\overline{f}(\overline{s}(\omega)))$ since $D^R \subseteq \mathrm{dom}(\overline{f})$.

We would like to show that $(f(x_n))_n \longrightarrow f(x)$ in Y. Let $U \subseteq Y$ be an open set in Y containing $f(x)$. Since $\varepsilon : E^R \to Y$ is continuous there is a compact element $e \in E$ such that $e \sqsubseteq \overline{f}(\overline{s}(\omega))$ and $y \in U$ whenever $e \prec y$. Since $e \sqsubseteq \overline{f}(\overline{s}(\omega))$ there is some $N \in S$ such that $e \sqsubseteq \overline{f}(\overline{s}(n))$ whenever $n \geq N$. We now have $f(x_n) = \varepsilon(\overline{f}(\overline{s}(n))) \in U$ for each $n \geq N$ and so $(f(x_n))_n \longrightarrow f(x)$ as required. □

Corollary 3.7. *Let* $(D,\, D^R,\, \delta)$ *be a countably based admissible domain representation of* X *and let* $(E,\, E^R,\, \varepsilon)$ *be an admissible domain representation of* Y. *Then* $f : X \to Y$ *is representable if and only if* f *is sequentially continuous.*

3.2 The domain of partial continuous functions

It is well-known that the category of domains with morphisms total continuous functions forms a Cartesian closed category (c.f. [16]). This convenient circumstance is employed in representation theory to build domain representable type structures over domain representable spaces. It is only natural to expect that much of this categorical structure will be lost when going to partial continuous functions. However, as we shall see not all is lost.

As a first step we show that the Scott-closed subsets of a domain D have a natural domain structure.

Definition 3.3. Let $\mathrm{cl}(D)$ be the collection of all nonempty Scott-closed subsets of D. We order $\mathrm{cl}(D)$ by $S \sqsubseteq T \iff S \subseteq T$.

Now, $\sqsubseteq$ is a partial order on D with least element $\{\bot\}$. More is true however.

Theorem 3.8. *Let D be a domain. Then* $\mathrm{cl}(D) = (\mathrm{cl}(D), \sqsubseteq, \{\bot\})$ *is a domain and* $S \in \mathrm{cl}(D)$ *is compact if and only if* $S = (\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$ *for some* $n \in \mathbb{N}$ *and compact* $a_0, a_1, \ldots a_n$ *in D.*

In fact, this result is well-known (c.f. [1]). We include a proof for the sake of completeness.

Proof. To show that $\mathrm{cl}(D)$ is a cpo, let $B \subseteq \mathrm{cl}(D)$ be directed. Then $\bigsqcup B = \overline{\bigcup B}$. (Note that we do not need the hypothesis that B is directed, it is enough to require that B is nonempty for $\bigsqcup B$ to be well-defined.) By Lemma 3.1, if B is a directed subset of $\mathrm{cl}(D)$ then $\bigsqcup B = \{\bigsqcup A;\ A \subseteq \bigcup B$ and A is directed$\}$.

To characterise the compact elements in $\mathrm{cl}(D)$ let $S = (\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$ for some $n \in \mathbb{N}$ and compact $a_0, a_1, \ldots a_n$ in D. We would like to show that S is compact in $\mathrm{cl}(D)$. Thus, let B be a directed subset of $\mathrm{cl}(D)$ and suppose that $S \sqsubseteq \bigsqcup B$ in $\mathrm{cl}(D)$. Then for each $0 \le i \le n$ there is a directed set $A_i \subseteq \bigcup B$ such that $a_i = \bigsqcup A_i$ in D. It follows that $a_i \sqsubseteq b_i$ for some $b_i \in A_i$ and so $a_i \in \bigcup B$ for each $0 \le i \le n$ since $\bigcup B$ is downwards closed. Since B is directed there is some $T \in B$ such that $a_i \in T$ for all i and then $S \sqsubseteq T$ since T is downwards closed. Since B was arbitrary we conclude that S is compact in $\mathrm{cl}(D)$.

Conversely, suppose that S is compact in $\mathrm{cl}(D)$. Let $B \subseteq \mathrm{cl}(D)$ be the set of all $(\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$ where $n \in \mathbb{N}$ and $a_0, a_1, \ldots a_n \in D_c \cap S$. B is directed since B is closed under finite unions, and $\bigsqcup B = S$ since S is a domain and $S_c = D_c \cap S$. Since S is compact there are $a_0, a_1, \ldots a_n \in S_c$ such that $S \sqsubseteq (\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$, and so $S = (\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$ as required.

To show that $\mathrm{cl}(D)$ is algebraic, if $S \in \mathrm{cl}(D)$ then $\mathrm{approx}(S)$ is directed since it is closed under finite unions and $\bigsqcup \mathrm{approx}(S) = S$ since S is a domain and $S_c = D_c \cap S$.

Finally, to show that $\mathrm{cl}(D)$ is consistently complete it is enough to note that $\mathrm{cl}(D)_c$ is closed under finite unions. Thus $\mathrm{cl}(D)$ is a domain with compact elements $(\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$ for $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in D_c$ which was what we wanted to show. $\square$

Remark 3.1. If $S = (\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_m)$ and $T = (\downarrow b_0) \cup (\downarrow b_1) \cup \ldots \cup (\downarrow b_n)$ are compact in $\mathrm{cl}(D)$, then $S \sqsubseteq T \iff$ for each a_i there is some b_j such that $a_i \sqsubseteq_D b_j$. It follows that $\mathrm{cl}(D)$ is isomorphic to the Hoare power domain over D (c.f. [1]) .

Remark 3.2. Note that if D is countably based then so is $\mathrm{cl}(D)$.

To show that $[D \rightharpoonup E]$ admits a natural domain structure we define a partial order on $[D \rightharpoonup E]$ as follows:

Definition 3.4. If D and E are domains and $f, g : D \rightharpoonup E$, we let $f \sqsubseteq g \iff \mathrm{dom}(f) \sqsubseteq \mathrm{dom}(g)$ in $\mathrm{cl}(D)$, and $f(x) \sqsubseteq_E g(x)$ for each $x \in \mathrm{dom}(f)$.

It follows that $\sqsubseteq$ is a partial order on $[D \rightharpoonup E]$ with a least element given by $\bot_{[D \rightharpoonup E]} = (\{\bot_D\}, \lambda x.\bot_E)$. To show that $[D \rightharpoonup E]$ is a domain we need to review a result for total continuous functions on domains.

Let D and E be domains. For $c \in D_c$ and $d \in E_c$ we let $(c \searrow d) : D \to E$ be the continuous function from D to E given by

$$(c \searrow d)(x) = \begin{cases} d & \text{if } c \sqsubseteq x \\ \bot_E & \text{otherwise.} \end{cases}$$

$(c \searrow d)$ is called a *step function* from D to E. It is a well-known fact in domain theory that a strict continuous function $g : D \to E$ is compact in $[D \to_\bot E]$ if and only if there are step functions $(c_1 \searrow d_1), (c_2 \searrow d_2), \ldots, (c_n \searrow d_n)$ from D to E such that $d_i = \bot_E$ whenever $c_i = \bot_D$ for each $1 \le i \le n$, and $g = (c_1 \searrow d_1) \sqcup (c_2 \searrow d_2) \sqcup \ldots \sqcup (c_n \searrow d_n)$. (For a proof of this result c.f. [16].) We can now show that $[D \rightharpoonup E]$ is a domain.

Theorem 3.9. *Let D and E be domains. Then $[D \rightharpoonup E] = ([D \rightharpoonup E], \sqsubseteq, \bot_{[D \rightharpoonup E]})$ is a domain and $g : D \rightharpoonup E$ is compact in $[D \rightharpoonup E]$ if and only if $\mathrm{dom}(g)$ is compact in $\mathrm{cl}(D)$ and g is compact in $[\mathrm{dom}(g) \to_\bot E]$.*

Proof. To show that $[D \rightharpoonup E]$ is a cpo, let $A \subseteq [D \rightharpoonup E]$ be directed. Then $B = \{\mathrm{dom}(f);\ f \in A\}$ is directed. Let $S = \bigsqcup B \subseteq D$. If a is compact in S then $\downarrow a$ is compact in $\mathrm{cl}(D)$ and so $a \in \mathrm{dom}(f)$ for some $f \in A$. Since $a \in \mathrm{dom}(f)$ and $f \sqsubseteq g$ implies that $a \in \mathrm{dom}(g)$, it follows that the set $\{f(a);\ f \in A \text{ and } a \in \mathrm{dom}(f)\}$ is directed for each $a \in S_c$. Define $F : S_c \to E$ by $F(a) = \bigsqcup\{f(a);\ f \in A \text{ and } a \in \mathrm{dom}(f)\}$. Then $F : S_c \to E$ is monotone and so F extends to a continuous function $F : S \to E$. F is strict since each $f \in A$ is strict and thus F is a partial continuous function from D to E.

If $f \in A$ then $\mathrm{dom}(f) \subseteq \mathrm{dom}(F)$ and if $a \in \mathrm{dom}(f)$ is compact then $f(a) \sqsubseteq_E F(a)$. Thus $f(x) \sqsubseteq_E F(x)$ for each $x \in \mathrm{dom}(f)$ since f and F are continuous and $\mathrm{dom}(f)$ is a domain. Conversely, if $f \sqsubseteq g$ for each $f \in A$ then $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ for each $f \in A$ and so $\mathrm{dom}(F) \subseteq \mathrm{dom}(g)$. If $a \in \mathrm{dom}(F)$ is compact then $a \in \mathrm{dom}(f)$ for some $f \in A$, and since g is an upper bound for A we have $f(a) \sqsubseteq_E g(a)$ for each $f \in A$ such that $a \in \mathrm{dom}(f)$. Thus $F(a) = \bigsqcup\{f(a);\ f \in A \text{ and } a \in \mathrm{dom}(f)\} \sqsubseteq_E g(a)$ for each compact $a \in \mathrm{dom}(F)$. By continuity we have $F \sqsubseteq g$ as before. We conclude that $F = \bigsqcup A$ and so $[D \rightharpoonup E]$ is a cpo.

Now, suppose $g : D \rightharpoonup E$ is a partial continuous function from D to E with compact domain such that g is compact in $[\mathrm{dom}(g) \to_\perp E]$. Choose $a_0, a_1, \ldots, a_m \in D_c$ such that $\mathrm{dom}(g) = (\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_m)$ and suppose $g = (c_0 \searrow d_0) \sqcup (c_1 \searrow d_1) \sqcup \ldots \sqcup (c_n \searrow d_n) \in [\mathrm{dom}(g) \to_\perp E]_c$. If $A \subseteq [D \rightharpoonup E]$ is directed and $g \sqsubseteq F = \bigsqcup A$ then $\mathrm{dom}(g) \subseteq \mathrm{dom}(F) = \bigsqcup\{\mathrm{dom}(f);\ f \in A\}$. Thus $\mathrm{dom}(g) \sqsubseteq \mathrm{dom}(f)$ for some $f \in A$. Let $B = \{f\lceil_{\mathrm{dom}(g)} : \mathrm{dom}(g) \to E;\ f \in A \text{ and } \mathrm{dom}(g) \sqsubseteq \mathrm{dom}(f)\}$. Then $B \subseteq [\mathrm{dom}(g) \to_\perp E]$ is directed and $g \sqsubseteq \bigsqcup B$. Since g is compact in $[\mathrm{dom}(g) \to_\perp E]$, $g \sqsubseteq f\lceil_{\mathrm{dom}(g)}$ for some $f \in A$ such that $\mathrm{dom}(g) \sqsubseteq \mathrm{dom}(f)$. Thus $g \sqsubseteq f$ in $[D \rightharpoonup E]$ and so g is compact in $[D \rightharpoonup E]$.

Conversely, suppose $g : D \rightharpoonup E$ is compact in $[D \rightharpoonup E]$. For each finite sequence $\vec{a} = a_0, a_1, \ldots, a_n$ of compact elements in $\mathrm{dom}(g)$ let $g_{\vec{a}}$ be the restriction of g to $(\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$. Then $g_{\vec{a}} \in [D \rightharpoonup E]$ for each finite sequence $\vec{a}$ and $\mathrm{dom}(g_{\vec{a}})$ is compact in $\mathrm{cl}(D)$. Let $A = \{g_{\vec{a}} : D \rightharpoonup E;\ \vec{a} = a_0, a_1, \ldots, a_n \in (\mathrm{dom}(g))_c\}$.

If $\vec{a}$ and $\vec{c}$ are finite sequences of compact elements in $\mathrm{dom}(g)$ we write $\vec{a} * \vec{c}$ for the concatenation of $\vec{a}$ and $\vec{c}$. We now note that A is directed since $g_{\vec{a}*\vec{c}} \in A$ for all $g_{\vec{a}}, g_{\vec{c}} \in A$, and $\bigsqcup A = g$. Since g is compact, $g \sqsubseteq g_{\vec{a}}$ for some finite sequence $\vec{a} = a_0, a_1, \ldots, a_n \in (\mathrm{dom}(g))_c$. It follows that $\mathrm{dom}(g) = (\downarrow a_0) \cup (\downarrow a_1) \cup \ldots \cup (\downarrow a_n)$.

Each strict continuous function $f : \mathrm{dom}(g) \to E$ is a partial continuous function from D to E. Thus in particular, $B = \{f : \mathrm{dom}(g) \to E;\ f$ is compact in $[\mathrm{dom}(g) \to_\perp E]$ and $f \sqsubseteq g\}$ is a directed subset of $[D \rightharpoonup E]$ and $\bigsqcup B = g$ in $[D \rightharpoonup E]$. Thus $g = f$ for some $f \in B$ and so g is compact in $[\mathrm{dom}(g) \to_\perp E]$. We conclude that $g : D \rightharpoonup E$ is compact in $[D \rightharpoonup E]$ if and only if $\mathrm{dom}(g)$ is compact in $\mathrm{cl}(D)$ and g is compact in $[\mathrm{dom}(g) \to_\perp E]$.

To show that $[D \rightharpoonup E]$ is consistently complete, let $f : D \rightharpoonup E$. If $g, g' \sqsubseteq f$ are compact, let $(c_1 \searrow d_1), (c_2 \searrow d_2), \ldots, (c_k \searrow d_k), (c'_1 \searrow d'_1), (c'_2 \searrow d'_2),$

..., $(c'_l \searrow d'_l)$, be step functions from D to E such that $g = (c_1 \searrow d_1) \sqcup (c_2 \searrow d_2) \sqcup \ldots \sqcup (c_k \searrow d_k)$ in $[\mathrm{dom}(g) \to_\perp E]$, and $g' = (c'_1 \searrow d'_1) \sqcup (c'_2 \searrow d'_2) \sqcup \ldots \sqcup (c'_l \searrow d'_l)$ in $[\mathrm{dom}(g') \to_\perp E]$. Define $h : D \rightharpoonup E$ by $\mathrm{dom}(h) = \mathrm{dom}(g) \cup \mathrm{dom}(g')$ and

$$h = \bigsqcup \Big(\{(c_i \searrow e_i);\ 1 \le i \le k\} \cup \{(c'_i \searrow e'_i);\ 1 \le i \le l\} \Big)$$

where e_i and e'_i are given by

$$e_i = \bigsqcup \Big(\{d_i\} \cup \{d'_j;\ c'_j \sqsubseteq_D c_i\} \Big) \text{ and } e'_i = \bigsqcup \Big(\{d'_i\} \cup \{d_j;\ c_j \sqsubseteq_D c'_i\} \Big)$$

for all i. e_i is well-defined for each $1 \le i \le k$ since E is consistently complete and the set $\{d_i\} \cup \{d'_j \in E_c;\ c'_j \sqsubseteq_D c_i\}$ is bounded above by $f(c_i)$. The proof that e'_i is well-defined for each $0 \le i \le l$ is identical. Since each e_i and e'_i is the supremum of finitely many compact elements in E, we have $e_i,\ e'_i \in E_c$ for each i. Finally, to show that h is well-defined it is enough to show that $\{e_i\}_{i \in I} \cup \{e'_i\}_{i \in I'}$ is consistent in E whenever $\{c_i\}_{i \in I} \cup \{c'_i\}_{i \in I'}$ is consistent in $\mathrm{dom}(h)$, for all finite index sets $I \subseteq \{1,\ 2,\ \ldots k\}$ and $I' \subseteq \{1,\ 2,\ \ldots,\ l\}$.

Let $I \subseteq \{1,\ 2,\ \ldots k\}$ and $I' \subseteq \{1,\ 2,\ \ldots,\ l\}$ and suppose $\{c_i\}_{i \in I} \cup \{c'_i\}_{i \in I'}$ is consistent in $\mathrm{dom}(h)$. Let x be an upper bound for the set $\{c_i\}_{i \in I} \cup \{c'_i\}_{i \in I'}$ in $\mathrm{dom}(h)$. Then $x \in \mathrm{dom}(f)$ and $d_j,\ d'_j \sqsubseteq_E f(x)$ for each j such that $c_j,\ c'_j \sqsubseteq_D x$. Thus in particular, if $c_i \sqsubseteq_D x$ then $f(x)$ is an upper bound for the set $\{d_i\} \cup \{d'_j \in E_c;\ c'_j \sqsubseteq c_i\}$ in E and so $e_i \sqsubseteq_E f(x)$ by the definition of e_i. Similarly, if $c'_i \sqsubseteq_D x$ then $e'_i \sqsubseteq_E f(x)$. We conclude that $f(x)$ is an upper bound for $\{e_i\}_{i \in I} \cup \{e'_i\}_{i \in I'}$ in E and so h is well-defined. Thus $h \in [D \rightharpoonup E]_c$.

If $k : D \rightharpoonup E$ is a partial continuous function from D to E and $g,\ g' \sqsubseteq k$ then $\{d_i\} \cup \{d'_j \in E_c;\ c'_j \sqsubseteq_D c_i\}$ is bounded above by $k(c_i)$ for each i. It follows that $h(c_i) = e_i \sqsubseteq_E k(c_i)$ for each i. Similarly, $h(c'_i) \sqsubseteq_E k(c'_i)$ for each i and so $h \sqsubseteq k$ in $[D \rightharpoonup E]$. We conclude that h is the least upper bound for g and g'.

Finally, To show that $[D \rightharpoonup E]$ is algebraic, we note that $\mathrm{approx}(f)$ is directed for each $f : D \rightharpoonup E$ since $[D \rightharpoonup E]$ is consistently complete.

It is clear that $F = \bigsqcup \mathrm{approx}(f) \sqsubseteq f$ for each $f : D \rightharpoonup E$. To prove the converse, let c be compact in $\mathrm{dom}(f)$ and let $d \in \mathrm{approx}(f(c))$. Let $f_{c,d} : D \rightharpoonup E$ be the partial continuous function given by $\mathrm{dom}(f_{c,d}) = \downarrow c$ and $f_{c,d} = (c \searrow d)$ on $\downarrow c$. Let $A = \{f_{c,d} : D \rightharpoonup E;\ c \text{ is compact in } \mathrm{dom}(f) \text{ and } d \in \mathrm{approx}(f(c))\}$. Then $A \subseteq \mathrm{approx}(f)$ and so $c \in \mathrm{dom}(F)$ for each compact $c \in \mathrm{dom}(f)$. Thus $\mathrm{dom}(F) = \mathrm{dom}(f)$. Furthermore, since F is an upper bound for A in

$[D \rightharpoonup E]$, $f_{c,d} \sqsubseteq F$ for each $f_{c,d} \in A$. We conclude that $d \sqsubseteq_E F(c)$ for each compact c in $\mathrm{dom}(f)$ and each $d \in \mathrm{approx}(f(c))$ and so $F(c)$ is an upper bound for $\mathrm{approx}(f(c))$ in E. It follows that $f(c) \sqsubseteq_E F(c)$ for each compact $c \in \mathrm{dom}(f)$ and so $f \sqsubseteq F$. This concludes the proof. $\square$

Remark 3.3. It follows from the proof of Theorem 3.9 that the map $f \mapsto \mathrm{dom}(f)$ taking a partial continuous function from D to E to its domain in $\mathrm{cl}(D)$ is continuous.

Remark 3.4. We also note that it follows immediately by the characterisation of $[D \rightharpoonup E]_c$ that $[D \rightharpoonup E]$ is countably based whenever D and E are countably based.

We now apply Corollary 3.7 and Theorem 3.9 to construct a countably based and admissible domain representation of the space of sequentially continuous functions from X to Y, given countably based and admissible representations of the spaces X and Y.

Let X and Y be topological spaces. We write $[X \to_\omega Y]$ for the space of sequentially continuous functions from X to Y. If $A \subseteq X$ and $B \subseteq Y$ we let $M(A, B) = \{f \in [X \to_\omega Y];\ f[A] \subseteq B\}$. The collection of all sets $M(\{x_n\}_n \cup \{x\}, U)$ where $(x_n)_n \longrightarrow x$ in X and U is an open subset of Y form a subbasis for a topology τ_ω on $[X \to_\omega Y]$. The topology τ_ω is a natural generalisation of the topology of pointwise convergence on $[X \to_\omega Y]$, the latter being generated by the collection of all finite intersection of sets $M(\{x\}, U)$ where $x \in X$ and U is open in Y. The following lemma characterises the convergence relation on $([X \to_\omega Y], \tau_\omega)$.

Lemma 3.10. *Let X and Y be topological spaces. $(f_m)_m \longrightarrow f$ in $([X \to_\omega Y], \tau_\omega)$ if and only if $(f_n(x_n))_n \longrightarrow f(x)$ in Y for each convergent sequence $(x_n)_n \longrightarrow x$ in X.*

Proof. Suppose that $(f_m)_m \longrightarrow f$ in $([X \to_\omega Y], \tau_\omega)$, and let $(x_n)_n \longrightarrow x$ in X. Let U be an open neighbourhood of $f(x)$ in Y, and choose $N \in \mathbb{N}$ such that $f(x_n) \in U$ for each $n \geq N$. It follows that $f \in M(\{x_n;\ n \geq N\} \cup \{x\}, U)$, and since $(f_n)_n \longrightarrow f$ in $([X \to_\omega Y], \tau_\omega)$, that $f_m \in M(\{x_n;\ n \geq N\} \cup \{x\}, U)$ for all natural numbers m greater than some $M \in \mathbb{N}$. Thus in particular, if $n \geq \max(M, N)$ we have $f_n(x_n) \in U$. Since U was arbitrary, we must have $(f_n(x_n))_n \longrightarrow f(x)$ in Y as required.

Conversely, suppose that $(f_n(x_n))_n \longrightarrow f(x)$ whenever $(x_n)_n \longrightarrow x$ in X. We need to show that $(f_m)_m \longrightarrow f$ in $([X \to_\omega Y], \tau_\omega)$. Thus, let $(y_n)_n \longrightarrow y$ in X, let $U \subset Y$ be open, and suppose that $f \in M(\{y_n\}_n \cup \{y\}, U)$. We would like to show that $f_m \in M(\{y_n\}_n \cup \{y\}, U)$ for almost all m.

Suppose for a contradiction that for each $N \in \mathbb{N}$ there are $m, n \geq N$ such that $f_m(y_n) \notin U$. It follows that there are strictly monotone functions $h, k : \mathbb{N} \to \mathbb{N}$ such that $f_{h(n)}(y_{k(n)}) \notin U$ for all n. Let $l : \mathbb{N} \to \mathbb{N}$ be the function $l(n) =$ the least N such that $h(N) \geq n$, and let $(x_n)_n$ be the sequence given by $x_n = y_{k(l(n))}$ for all $n \in \mathbb{N}$. Since $l(m) \leq l(n)$ whenever $m \leq n$ in $\mathbb{N}$, and $(l(n))_n \longrightarrow \infty$, it follows that $(x_n)_n \longrightarrow y$ in X, and so $(f_n(x_n))_n \longrightarrow f(y)$ in Y. On the other hand, since $l(h(n)) = n$ for all n we have $f_{h(n)}(x_{h(n)}) = f_{h(n)}(y_{k(l(h(n)))}) = f_{h(n)}(y_{k(n)}) \notin U$ for all n. This is our desired contradiction.

Now, choose N such that $f_m(y_n) \in U$ whenever $m, n \geq N$. Since $f_m(y_n) \longrightarrow f(y_n) \in U$ for each n, there is some M such that $f_m(y_n) \in U$ for each $n \leq N$, whenever $m \geq M$. It follows that $f_m \in M(\{y_n\}_n \cup \{y\}, U)$ whenever $m \geq \max(M, N)$. This concludes the proof. □

Now, suppose $(D,\, D^R,\, \delta)$ is a countably based admissible domain representation of X and $(E,\, E^R,\, \varepsilon)$ is a countably based admissible representation of Y. By Corollary 3.7, every sequentially continuous function $f : X \to Y$ from X to Y is representable by a partial continuous function $\overline{f} : D \rightharpoonup E$ from D to E. Let $[D \rightharpoonup E]^R$ be the set $\{\overline{f} : D \rightharpoonup E;\ \overline{f}$ represents some function $f : X \to Y\}$ and define $[\delta \rightharpoonup \varepsilon] : [D \rightharpoonup E]^R \to [X \to_\omega Y]$ by

$$[\delta \rightharpoonup \varepsilon](\overline{f}) = f \iff \overline{f} \text{ represents } f.$$

Then $[\delta \rightharpoonup \varepsilon]$ is well-defined and $[\delta \rightharpoonup \varepsilon]$ is surjective by Corollary 3.7.

Theorem 3.11. *Let $(D,\, D^R, \delta)$ be a countably based admissible domain representation of X and let $(E,\, E^R,\, \varepsilon)$ be a countably based admissible domain representation of Y. Then $([D \rightharpoonup E],\, [D \rightharpoonup E]^R,\, [\delta \rightharpoonup \varepsilon])$ is a countably based admissible domain representation of* $([X \to_\omega Y], \tau_\omega)$.

Proof. Since $[D \rightharpoonup E]$ is countably based, to show that $[\delta \rightharpoonup \varepsilon]$ is continuous it is enough to show that $(f_n)_n \longrightarrow f$ for each convergent sequence $(\overline{f}_n)_n \longrightarrow \overline{f}$ in $[D \rightharpoonup E]^R$. By Lemma 3.10 it is enough to show that $(f_n(x_n))_n \longrightarrow f(x)$ in Y for every convergent sequence $(x_n) \longrightarrow x$ in X.

Thus, suppose $(x_n) \longrightarrow x$ in X. Let S be the domain $(\omega + 1, \leq, 0)$ and let $S^R = S - \{0\}$. Let $s : S^R \to X$ be the continuous function given by $s(n) = x_n$ for each $0 < n < \omega$ and $s(\omega) = x$. Since $(D,\, D^R,\, \delta)$ is admissible there is a partial continuous function $\overline{s} : S \to D$ such that $S^R \subseteq \mathrm{dom}(\overline{s})$ and $s(n) = (\delta \circ \overline{s})(n)$ for each $n \in S^R$. Let $r_n = \overline{s}(n)$ and let $r = \overline{s}(\omega)$. Then $(r_n)_n \longrightarrow r$ in D^R, $\delta(r_n) = x_n$ for each $0 < n < \omega$, and $\delta(r) = x$.

Choose $d \in \mathrm{approx}(\overline{f}(r))$. Since $\overline{f} : D \rightharpoonup E$ is continuous there is a compact $c \in \mathrm{approx}(r)$ such that $d \sqsubseteq_E \overline{f}(c)$. Since $\uparrow c$ is open in D, $c \sqsubseteq_D r_n$

for almost all n. Similarly, $(\downarrow c, (c \searrow d)) \sqsubseteq \overline{f}_n$ for almost all n. Choose $N \in \mathbb{N}$ such that $c \sqsubseteq_D r_n$ and $(\downarrow c, (c \searrow d)) \sqsubseteq \overline{f}_n$ for all $n \geq N$. Then $d \sqsubseteq_E \overline{f}_n(r_n)$ for all $n \geq N$ and thus $(\overline{f}_n(r_n))_n \longrightarrow \overline{f}(r)$. Since ε is continuous it follows immediately that $(f_n(x_n))_n \longrightarrow f(x)$ in Y and so $(f_n)_n \longrightarrow f$ by Lemma 3.10.

To show that $([D \rightharpoonup E], [D \rightharpoonup E]^R, [\delta \rightharpoonup \varepsilon])$ is admissible, let F be a countably based domain, F^R a nonempty subset of F and let $\varphi : F^R \rightarrow [X \rightarrow_\omega Y]$ be a continuous function from F^R to $[X \rightarrow_\omega Y]$. Let $\psi : F^R \times D^R \rightarrow Y$ be the function given by $(r, s) \mapsto \varphi(r)(\delta(s))$. The function ψ is continuous by Lemma 3.10 since $F^R \times D^R$ is countably based. Since the domain representation (E, E^R, ε) is admissible, there is a partial continuous function $\overline{\psi} : F \times D \rightharpoonup E$ such that $F^R \times D^R \subseteq \mathrm{dom}(\overline{\psi})$, $\overline{\psi}[F^R \times D^R] \subseteq E^R$, and $\varepsilon(\overline{\psi}(r, s)) = \psi(r, s)$ for each pair $(r, s) \in F^R \times D^R$. We may assume that $\mathrm{dom}(\overline{\psi}) = F^D \times D^D$ (where F^D is the closure of F^R in F), and that $\overline{\psi}(f, \bot_D) = \overline{\psi}(\bot_F, d) = \bot_E$ for each $f \in F^D$ and $d \in D^D$. Since $\overline{\psi}$ is a total continuous function from $F^D \times D^D$ to E, there is a continuous function $(\overline{\psi})^* : F^D \rightarrow [D^D \rightarrow E]$ such that $(\overline{\psi})^*(f)(d) = \overline{\psi}(f, d)$ for each pair $(f, d) \in F^D \times D^D$. We let $\overline{\varphi} = (\overline{\psi})^*_\bot$. Since $\overline{\psi}(f, \bot_D) = \bot_E$ for each $f \in F$ it follows that $\overline{\varphi}$ is a partial continuous function from F to $[D \rightharpoonup E]$.

Since $\mathrm{dom}(\overline{\varphi}) = F^D$ we have $F^R \subseteq \mathrm{dom}(\overline{\varphi})$, and if $r \in F^R$ then $\overline{\varphi}(r)(s) = \overline{\psi}(r, s)$ for each $s \in D^R$. Thus, if $s \in D^R$ then $\varepsilon(\overline{\varphi}(r)(s)) = \varepsilon(\overline{\psi}(r, s)) = \psi(r, s) = \varphi(r)(\delta(s))$. It follows that $\overline{\varphi}(r)$ represents $\varphi(r) \in [X \rightarrow_\omega Y]$ for each $r \in F^R$. Thus in particular, we have $\overline{\varphi}(r) \in [D \rightharpoonup E]^R$, and $[\delta \rightharpoonup \varepsilon](\overline{\varphi}(r)) = \varphi(r)$ for each $r \in F^R$. Since (F, F^R, φ) was arbitrary, $([D \rightharpoonup E], [D \rightharpoonup E]^R, [\delta \rightharpoonup \varepsilon])$ is admissible. □

Before we go on to study evaluation and type conversion we take note of the following fact. (For a proof, see [10].)

Fact 3.1. Let (D, D^R, δ) and (E, E^R, ε) be countably based and admissible domain representations of the spaces X and Y. Then $(D \times E, D^R \times E^R, \delta \times \varepsilon)$ is a countably based and admissible domain representation of $X \times Y$. Moreover, the projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are sequentially continuous and thus representable by Corollary 3.7.

Since evaluation $(f, x) \mapsto f(x)$ is a sequentially continuous by Lemma 3.10, and $([D \rightharpoonup E], [D \rightharpoonup E]^R, [\delta \rightharpoonup \varepsilon])$ is admissible, it follows immediately by Fact 3.1 and Corollary 3.7 that

Proposition 3.12. $(f, x) \mapsto f(x)$ *is representable.*

Let X, Y, and Z be topological spaces. If $f : X \times Y \to Z$ is sequentially continuous then $y \mapsto f(x, y)$ is sequentially continuous for each $x \in X$. We write $f^* : X \to [Y \to_\omega Z]$ for the map $x \mapsto f(x, \cdot)$.

Proposition 3.13. *Let* $(D,\ D^R,\ \delta)$, $(E,\ E^R,\ \varepsilon)$, *and* $(F,\ F^R,\ \varphi)$ *be countably based and admissible domain representations of the spaces* X, Y *and* Z *and suppose* $f : X \times Y \to Z$ *is representable. Then* $f^* : X \to [Y \to_\omega Z]$ *is representable.*

Proof. By Corollary 3.7 it is enough to show that $f^* : X \to [Y \to_\omega Z]$ is sequentially continuous. Suppose $(x_n)_n \longrightarrow x$ in X. We would like to show that $(f^*(x_n))_n \longrightarrow f^*(x)$ in $[Y \to_\omega Z]$. By Lemma 3.10 it is enough to show that $(f^*(x_n)(y_n))_n \longrightarrow f^*(x)(y)$ for each convergent sequence $(y_n)_n \longrightarrow y$ in Y.

Suppose $(y_n) \longrightarrow y$ in Y. Then $((x_n, y_n))_n \longrightarrow (x, y)$ in $X \times Y$ and so $f^*(x_n)(y_n) = f(x_n, y_n) \longrightarrow f(x, y) = f^*(x)(y)$ in Y since f is sequentially continuous. It follows that f^* is sequentially continuous by Lemma 3.10. □

The results in this section may be summarised in the following way: Let $\mathbf{PAdm}$ be the category with objects ordered pairs (D, X) where D is a countably based admissible domain representation of the space X, and morphisms $f : (D, X) \to (E, Y)$ functions from X to Y which are representable by some partial continuous function $\overline{f} : D \rightharpoonup E$. We now have the following theorem:

Theorem 3.14. $\mathbf{PAdm}$ *is Cartesian closed.*

4 Effectivity

In this section we will show that the constructions from the previous section are effective. We first consider the domain of Scott-closed subsets of an effective domain D.

Definition 4.1. Let (D, α) be an effective domain. We define $\mathrm{cl}(\alpha) : \mathbb{N} \to \mathrm{cl}(D)_c$ by $\mathrm{cl}(\alpha)(k) = (\downarrow a_1) \cup (\downarrow a_2) \cup \ldots \cup (\downarrow a_n)$ if $k = \langle k_1, k_2, \ldots k_n \rangle$, where $k_i \in \mathrm{dom}(\alpha)$ and $\alpha(k_i) = a_i$ for each $1 \leq i \leq n$.

It is clear that $\mathrm{cl}(\alpha)$ is surjective, and so $\mathrm{cl}(\alpha)$ is a numbering of $\mathrm{cl}(D)_c$.

Theorem 4.1. *Let* (D, α) *be an effective domain. Then* $(\mathrm{cl}(D), \mathrm{cl}(\alpha))$ *is effective.*

Proof. $\mathrm{dom}(\mathrm{cl}(\alpha))$ is clearly decidable since (D, α) is effective. That the order relation "$\mathrm{cl}(\alpha)(k) \sqsubseteq \mathrm{cl}(\alpha)(l)$" is decidable follows since

$\mathrm{cl}(\alpha)(\langle k_1, k_2, \ldots, k_m\rangle) \sqsubseteq \mathrm{cl}(\alpha)(\langle l_1, l_2, \ldots, l_n\rangle)$ if and only if for each $1 \le i \le m$ there is a $1 \le j \le n$ such that $\alpha(k_i) \sqsubseteq_D \alpha(l_j)$. This is decidable since (D, α) is effective.

The consistency relation on $\mathrm{cl}(D)$ is trivial since $\mathrm{cl}(D)$ has a greatest element. Since $S \sqcup T = S \cup T$ if S and T are compact in $\mathrm{cl}(D)$, we have $\mathrm{cl}(\alpha)(k) \sqcup \mathrm{cl}(\alpha)(l) = \mathrm{cl}(\alpha)(k * l)$ where $(k, l) \mapsto k * l$ is concatenation on $\mathbb{N}$. It follows that $(S, T) \mapsto S \sqcup T$ is computable. Finally, $\mathrm{cl}(\alpha)(\langle k\rangle) = \{\bot\}$ if and only if $\alpha(k) = \bot$. This concludes the proof. □

Remark 4.1. We could give an even shorter proof by noting that the theorem follows immediately by Remark 3.1 and the observation that the Hoare power domain construction preserves effectivity.

Remark 4.2. It follows from the proof of Theorem 4.1 that an index for $(\mathrm{cl}(D), \mathrm{cl}(\alpha))$ is obtained uniformly from an index for (D, α).

Before we go on to show that the construction of the domain of partial continuous is effective we give a characterisation of the computable Scott-closed subsets of D.

Proposition 4.2. *Let (D, α) be an effective domain and suppose $S \subseteq D$ is Scott-closed in D. Then S is $\mathrm{cl}(\alpha)$-computable if and only if the relation "$\alpha(m) \in S$" is r.e. in m.*

Proof. Suppose first that S is $\mathrm{cl}(\alpha)$-computable. Since $a \in S$ if and only if $\downarrow a \subseteq S$ it follows immediately that the relation "$\alpha(m) \in S$" is r.e.

Conversely, suppose that the relation "$\alpha(m) \in S$" is r.e. If $a_1, a_2, \ldots, a_n \in D_c$ then $(\downarrow a_1) \cup (\downarrow a_2) \cup \ldots \cup (\downarrow a_n) \subseteq S$ if and only if $a_1, a_2, \ldots, a_n \in S$. This is r.e. and so S is $\mathrm{cl}(\alpha)$-computable. □

If S is compact in $\mathrm{cl}(D)$ we define $\alpha\lceil_S : \mathbb{N} \to S$ by $\mathrm{dom}(\alpha\lceil_S) = \{n \in \mathrm{dom}(\alpha); \alpha(n) \in S\}$ and $\alpha\lceil_S(n) = \alpha(n)$ for each $n \in \mathrm{dom}(\alpha\lceil_S)$. We will write $D\lceil_k$ for the Scott-closed set $\mathrm{cl}(\alpha)(k) \subseteq D$ and $\alpha\lceil_k$ for the numbering $\alpha\lceil_{\mathrm{cl}(\alpha)(k)} : \mathbb{N} \to D\lceil_k$.

Lemma 4.3. *Let (D, α) be an effective domain. Then an index for $(D\lceil_k, \alpha\lceil_k)$ can be obtained uniformly from k and an index for (D, α).*

Proof. $\mathrm{dom}(\alpha\lceil_k)$ is decidable uniformly in k and an index for (D, α): Suppose $k = \langle k_1, k_2, \ldots, k_n\rangle$. Now $m \in \mathrm{dom}(\alpha\lceil_k)$ if and only if $m \in \mathrm{dom}(\alpha)$ and $\alpha(m) \sqsubseteq_D \alpha(k_i)$ for some $1 \le i \le n$. This is decidable since (D, α) is effective.

To complete the proof it is clearly enough to show that we can decide the relations $(a \sqsubseteq b)$, $\mathrm{cons}(a, b)$, $(c = a \sqcup b)$ and $(a = \bot)$ on $D\!\restriction_k$ uniformly in k and an index for (D, α).

We show that $\mathrm{cons}(a, b) \iff$ "a and b are consistent in $D\!\restriction_k$" is decidable uniformly in k and an index for (D, α): If $l, m \in \mathrm{dom}(\alpha\!\restriction_k)$ then $\{\alpha(l), \alpha(m)\}$ is consistent in $D\!\restriction_k$ if and only if $\alpha(l), \alpha(m) \sqsubseteq_D \alpha(k_i)$ for some $1 \leq i \leq n$. This is decidable since (D, α) is effective. □

Let (D, α) and (E, β) be effective domains and let $[\alpha \to_\bot \beta] : \mathbb{N} \to [D \to_\bot E]_c$ be the standard numbering of $[D \to_\bot E]_c$. More precisely, k is in the domain of $[\alpha \to_\bot \beta]$ if and only if $k = \langle\langle l_1, m_1\rangle, \langle l_2, m_2\rangle, \ldots, \langle l_n, m_n\rangle\rangle$ where

1. $l_1, l_2, \ldots, l_n \in \mathrm{dom}(\alpha)$ and $m_1, m_2, \ldots, m_n \in \mathrm{dom}(\beta)$.
2. For each nonempty subset $I \subseteq \{1, 2, \ldots, n\}$, if $\{\alpha(l_i);\ i \in I\}$ is consistent in D then $\{\beta(m_i);\ i \in I\}$ is consistent in E.
3. If $\alpha(l_i) = \bot_D$ then $\beta(m_i) = \bot_E$.

Now $[\alpha \to_\bot \beta](k) = \bigsqcup\{(\alpha(l_i) \searrow \beta(m_i)); 1 \leq i \leq n\}$. We define a numbering $[\alpha \rightharpoonup \beta] : \mathbb{N} \to [D \rightharpoonup E]_c$ of $[D \rightharpoonup E]_c$ as follows:

Definition 4.2. Let (D, α) and (E, β) be effective domains, and let $[\alpha \rightharpoonup \beta] : \mathbb{N} \to [D \rightharpoonup E]_c$ be the numbering given by $k \in \mathrm{dom}([\alpha \rightharpoonup \beta])$ if and only if $k = \langle l, m\rangle$ where $l \in \mathrm{dom}(\mathrm{cl}(\alpha))$ and $m \in \mathrm{dom}([\alpha\!\restriction_l \to_\bot \beta])$, and then $[\alpha \rightharpoonup \beta](k) = (\mathrm{cl}(\alpha)(l), [\alpha\!\restriction_l \to_\bot \beta](m))$.

Theorem 4.4. *Let (D, α) and (E, β) be effective domains. Then $([D \rightharpoonup E], [\alpha \rightharpoonup \beta])$ is an effective domain.*

Proof. $\mathrm{dom}([\alpha \rightharpoonup \beta])$ is decidable: To decide whether $\langle l, m\rangle$ is in the domain of $[\alpha \rightharpoonup \beta]$ we first check if $l \in \mathrm{dom}(\mathrm{cl}(\alpha))$. If so we check if m is in the domain of $[\alpha\!\restriction_l \to_\bot \beta]$. This is decidable by Lemma 4.3 and the uniformity of the construction of the strict function space.

$\sqsubseteq$ is decidable since the order relations on the domains $\mathrm{cl}(D)$ and $[D\!\restriction_l \to_\bot E]$ are decidable uniformly in indices for (D, α), (E, β), and l. That the consistency relation on $[D \rightharpoonup E]_c$ is decidable and that $(f, g) \mapsto f \sqcup g$ is computable follows from the proof of Theorem 3.9. Finally, we note that an index for $\bot_{[D \rightharpoonup E]}$ can be obtained uniformly from indices for (D, α) and (E, β), and so we are done. □

Remark 4.3. As before, it actually follows from the proof of Theorem 4.4 that an index for $([D \rightharpoonup E], [\alpha \rightharpoonup \beta])$ can be obtained uniformly from indices for (D, α) and (E, β).

To be able to analyse the effective content of Propositions 3.12 and 3.13 we now introduce a notion of an effective partial continuous function.

Definition 4.3. Let (D, α) and (E, β) be effective domains and let $f : D \rightharpoonup E$ be a partial continuous function from D to E. $f : D \rightharpoonup E$ is called (α, β)-*effective* if there is an r.e. relation $R_f \subseteq \mathbb{N} \times \mathbb{N}$ such that if $m \in \mathrm{dom}(\alpha)$, $n \in \mathrm{dom}(\beta)$ and $\alpha(m) \in \mathrm{dom}(f)$, then

$$R_f(m, n) \iff \beta(n) \sqsubseteq_E f(\alpha(m)).$$

An r.e. index for R_f is called an *index* for $f : D \rightharpoonup E$.

If the numberings α and β are clear from the context we will economise on the notation and simply say that f is effective rather than (α, β)-effective. Note that we do not place any restrictions on the domain of definition of $f : D \rightharpoonup E$ (other than that it is a closed nonempty subset of D) for f to be effective. We merely require that there is an effective procedure which allows us to approximate $f(c)$ uniformly for each compact c which lies in the domain of f.

The following elementary lemma describes some basic but important properties of effective partial continuous functions.

Lemma 4.5. *Let (D, α), (E, β), and (F, γ) be effective domains, and suppose $f : D \rightharpoonup E$ and $g : E \rightharpoonup F$ are effective. Then*

1. *$g \circ f : D \to F$ is effective and an index for $g \circ f$ is obtained uniformly from indices for f and g.*

2. *If $x \in \mathrm{dom}(f)$ is computable in D then $f(x)$ is computable in E.*

Proof. Suppose $\alpha(m) \in \mathrm{dom}(g \circ f)$. Then $\gamma(n) \sqsubseteq_F (g \circ f)(\alpha(m))$ if and only if there is some $k \in \mathrm{dom}(\beta)$ such that $\beta(k) \sqsubseteq_E f(\alpha(m))$ and $\gamma(n) \sqsubseteq_F g(\beta(k))$. Since $f(\alpha(m)) \in \mathrm{dom}(g)$ we must have $\beta(k) \in \mathrm{dom}(g)$. It follows that $\gamma(n) \sqsubseteq_F (g \circ f)(\alpha(m))$ if and only if $(\exists k \in \mathrm{dom}(\beta))\big(R_f(m, k)$ and $R_g(k, n)\big)$ which is r.e. in R_f and R_g as required.

To prove 2, let $x \in \mathrm{dom}(f)$ be α-computable. Now, $\beta(n) \sqsubseteq_E f(x)$ if and only if there is some $m \in \mathrm{dom}(\alpha)$ such that $\beta(n) \sqsubseteq_E f(\alpha(m))$. Since $\mathrm{dom}(f)$ is downwards closed, $\alpha(m) \in \mathrm{dom}(f)$ for each $m \in \mathrm{dom}(\alpha)$ such that $\alpha(m) \sqsubseteq_D x$. Thus $\beta(n) \sqsubseteq_E f(x)$ if and only if $n \in \mathrm{dom}(\beta)$ and $(\exists m \in \mathrm{dom}(\alpha))\big(\alpha(m) \sqsubseteq_D x$ and $\beta(n) \sqsubseteq_E f(\alpha(m))\big)$. This is r.e. since x is computable and $f : D \rightharpoonup E$ is effective. □

Remark 4.4. If (D, α) is an effective domain then $\mathrm{id}_D : D \to D$ is effective. It follows by Lemma 4.5 that the class of effective domains with morphisms effective partial continuous functions form a category.

Note that now there are two notions of computability for partial continuous functions from (D, α) to (E, β). A partial continuous function $f : D \rightharpoonup E$ from D to E may either be effective in the sense of Definition 4.3, or $f : D \rightharpoonup E$ may be computable as an element of $([D \rightharpoonup E], [\alpha \rightharpoonup \beta])$. The next proposition relates these two notions of computability to each other.

Proposition 4.6. *Let (D, α) and (E, β) be effective domains and suppose $f : D \rightharpoonup E$ is a partial continuous function from D to E. Then $f : D \rightharpoonup E$ is computable as an element of $[D \rightharpoonup E]$ if and only if $f : D \rightharpoonup E$ is effective and $\mathrm{dom}(f)$ is computable.*

Proof. Suppose first that $f : D \rightharpoonup E$ is computable. Let $R_f \subseteq \mathbb{N} \times \mathbb{N}$ be the relation given by $R_f(m, n) \iff (\alpha(m) \searrow \beta(n)) : \downarrow \alpha(m) \to E$ is strict and $(\downarrow \alpha(m),\ (\alpha(m) \searrow \beta(n))) \sqsubseteq f$. Then R_f is r.e. since f is computable. Furthermore, if $m \in \mathrm{dom}(\alpha)$, $n \in \mathrm{dom}(\beta)$, and $\alpha(m) \in \mathrm{dom}(f)$, then $\beta(n) \sqsubseteq_E f(\alpha(m))$ if and only if $(\downarrow \alpha(m),\ (\alpha(m) \searrow \beta(n))) \sqsubseteq f$ if and only if $R_f(m, n)$. We conclude that $f : D \rightharpoonup E$ is effective. It is easy to show that $\mathrm{dom} : [D \rightharpoonup E] \to \mathrm{cl}(D)$ is effective and so $\mathrm{dom}(f)$ is computable in $\mathrm{cl}(D)$ by Lemma 4.5.

Conversely, suppose that $f : D \rightharpoonup E$ is effective and that $\mathrm{dom}(f)$ is computable in $\mathrm{cl}(D)$. Let $\langle k,\ \langle\langle l_1, m_1\rangle,\ \langle l_2, m_2\rangle,\ \dots,\ \langle l_n, m_n\rangle\rangle\rangle$ be an $[\alpha \rightharpoonup \beta]$-index for $g : D \rightharpoonup E$. Then $g \sqsubseteq f$ if and only if $\mathrm{cl}(\alpha)(k) \sqsubseteq \mathrm{dom}(f)$ and $m_i \sqsubseteq_E f(l_i)$ for each $1 \leq i \leq n$. This is r.e. since $f : D \rightharpoonup E$ is effective and $\mathrm{dom}(f)$ is computable. □

Let $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ be effective domain representations of the spaces X and Y and suppose that $f : X \to Y$ is a continuous function from X to Y. Then $f : X \to Y$ is called (D, E)-*effective* when there is an effective partial continuous function $\overline{f} : D \rightharpoonup E$ from D to E which represents f. A (D, E)-*index* for f is an index for the effective partial continuous function $\overline{f} : D \rightharpoonup E$.

It is easy to see that the identity on X is (D, D)-effective, and if $(F,\ F^R,\ \varphi)$ is an effective domain representation of Z and $g : Y \to Z$ is an (E, F)-effective continuous function, then $g \circ f : X \to Z$ is (D, F)-effective by Lemma 4.5. If the representations D, E and F are clear from the context we will drop the prefixes and simply say that f, g and $g \circ f$ are effective.

We now show that evaluation $(f, x) \mapsto f(x)$ is effective.

Proposition 4.7. *Let* (D, D^R, δ) *and* (E, E^R, ε) *be effective admissible domain representations of* X *and* Y. *Then* $\mathrm{eval} : [X \to_\omega Y] \times X \to Y$ *is effective.*

Proof. Let $S \subseteq [D \rightharpoonup E] \times D$ be the set $\downarrow\{f : D \rightharpoonup E;\ D^D \subseteq \mathrm{dom}(f)\} \times D^D$ and define $e : S \to E$ by $e(f, x) = \bigsqcup\{f(a);\ a \in \mathrm{approx}(x) \cap \mathrm{dom}(f)\}$. Before showing that (S, e) is an effective partial continuous function from $[D \rightharpoonup E] \times D$ to E we make the following two observations:

1. If $f \in [D \rightharpoonup E]^R$ then $D^D \subseteq \mathrm{dom}(f)$. Thus $[D \rightharpoonup E]^R \times D^R \subseteq S$.

2. If $(f, x) \in S$ and $x \in \mathrm{dom}(f)$ then $e(f, x) = f(x)$.

Thus, to show that evaluation is effective it is enough to show that (S, e) is an effective partial continuous function from $[D \rightharpoonup E] \times D$ to E.

The set S is a nonempty closed subset of $[D \rightharpoonup E] \times D$: S is clearly nonempty. To show that S is closed it is enough to show that $T = \downarrow\{f : D \rightharpoonup E;\ D^D \subseteq \mathrm{dom}(f)\}$ is a closed subset of $[D \rightharpoonup E]$. T is clearly downwards closed. To show that T is closed under suprema of directed sets let A be a directed subset of T and let $F = \bigsqcup A$. For each $f \in A$ we let $\overline{f} : D \rightharpoonup E$ be the partial continuous function from D to E given by $\mathrm{dom}(\overline{f}) = \mathrm{dom}(f) \cup D^D$ and $\overline{f}(x) = \bigsqcup\{f(a);\ a \in \mathrm{approx}(x) \cap \mathrm{dom}(f)\}$.

The function $\overline{f} : D \rightharpoonup E$ is well-defined for each $f \in A$: Choose some $g : D \rightharpoonup E$ such that $f \sqsubseteq g$ and $D^D \subseteq \mathrm{dom}(g)$. If $x \in \mathrm{dom}(f) \cup D^D$ then $x \in \mathrm{dom}(g)$. It follows that $f(a) \sqsubseteq_E g(a) \sqsubseteq_E g(x)$ for each $a \in \mathrm{approx}(x) \cap \mathrm{dom}(f)$. We conclude that $\{f(a);\ a \in \mathrm{approx}(x) \cap \mathrm{dom}(f)\}$ is consistent in E for each $x \in \mathrm{dom}(f) \cap D^D$ and $\overline{f} : D \rightharpoonup E$ is well-defined.

The function $\overline{f} : D \rightharpoonup E$ is a partial continuous function for each $f \in A$: $\overline{f}$ is clearly monotone, and $\overline{f}$ is strict since $\bot_D \in \mathrm{dom}(f)$ and f is strict. To show that $\overline{f}$ is continuous it is enough to observe that if B is a directed subset of $\mathrm{dom}(f) \cup D^D$ then

$$\overline{f}(\bigsqcup B) = \bigsqcup\{f(a);\ a \in \mathrm{approx}(\bigsqcup B) \cap \mathrm{dom}(f)\} =$$

$$\bigsqcup\{f(a);\ a \in \mathrm{approx}(b) \cap \mathrm{dom}(f) \text{ for some } b \in B\}.$$

The map $f \mapsto \overline{f}$ is monotone by construction. Let $C \subseteq [D \rightharpoonup E]$ be the set $\{\overline{f};\ f \in A\}$. Then C is directed and $G = \bigsqcup C \in T$ by construction. Since $f \sqsubseteq \overline{f}$ for each $f \in A$ we have $F \sqsubseteq G$. We conclude that $F \in T$ and so T is closed in $[D \rightharpoonup E]$.

The function $e : S \to E$ is well-defined: Let $f \in T$ and suppose $x \in D^D$. Let $g : D \rightharpoonup E$ be some partial continuous function from D to E such that $f \sqsubseteq g$ and $D^D \subseteq \mathrm{dom}(g)$. Since $f \sqsubseteq g$ it follows that $\{f(a);\ a \in \mathrm{approx}(x) \cap \mathrm{dom}(f)\}$ is bounded above by $g(x)$ in E. Now, $e(f, x)$ is well-defined since E is consistently complete.

(S, e) is a partial continuous function from D to E: Note that $e : S \to E$ is continuous if e is continuous in each argument. It is easy to see that e is monotone in each argument. To show that e is continuous, fix $x \in D^D$ and let A be a directed subset of T. Suppose $F = \bigsqcup A$ and let $B = \{\overline{f};\ f \in A\}$ and $G = \bigsqcup B$ as before. We claim that $\overline{F} = G$. Since the map $f \mapsto \overline{f}$ is monotone it follows immediately that $G \sqsubseteq \overline{F}$. Conversely, suppose that a is compact in D and $a \in \mathrm{dom}(\overline{F})$. Then $a \in \mathrm{dom}(F) \cup D^D$. If $a \in \mathrm{dom}(F)$ then $a \in \mathrm{dom}(f)$ for some $f \in A$ and so $a \in \mathrm{dom}(G)$. If $a \in D^D$ then $a \in \mathrm{dom}(G)$ since $D^D \subseteq \mathrm{dom}(G)$. We conclude that $\mathrm{dom}(G) = \mathrm{dom}(\overline{F})$. To show that $\overline{F} = G$ let a be compact in D and suppose $a \in \mathrm{dom}(F)$. Then $\overline{F}(a) = F(a) = \bigsqcup\{f(a);\ f \in A \text{ and } a \in \mathrm{dom}(f)\} = \bigsqcup\{\overline{f}(a);\ f \in A \text{ and } a \in \mathrm{dom}(f)\} = G(a)$. It follows that $\overline{F}$ and G agree on $\mathrm{dom}(F)$. Since $\overline{F} : D \rightharpoonup E$ is the least partial continuous extension of F to $\mathrm{dom}(F) \cup D^D$ it follows that $\overline{F} = G$. We now have $e(F, x) = \overline{F}(x) = G(x) = (\bigsqcup B)(x) = \bigsqcup\{\overline{f}(x);\ f \in A\} = \bigsqcup e[A \times \{x\}]$ as required.

Now, fix $f \in T$ and let C be a directed subset of D^D with least upper bound $x = \bigsqcup C$. Now $e(f, x) = \overline{f}(x) = \bigsqcup \overline{f}[C] = \bigsqcup e[\{f\} \times C]$ since $\overline{f} : D \rightharpoonup E$ is continuous. Thus, we conclude that $e : S \to E$ is continuous. Since e is strict it follows that e is a partial continuous function from $[D \rightharpoonup E] \times D$ to E.

The function $e : [D \rightharpoonup E] \times D \rightharpoonup E$ is effective: Let $(f, a) \in [D \rightharpoonup E] \times D$ be compact and suppose $(f, a) \in \mathrm{dom}(e)$. Choose step functions $(c_1 \searrow d_1)$, $(c_2 \searrow d_2)$, $\ldots$, $(c_n \searrow d_n)$ such that $f = (c_1 \searrow d_1) \sqcup (c_2 \searrow d_2) \sqcup \ldots \sqcup (c_n \searrow d_n)$ in $[\mathrm{dom}(f) \to_\perp E]$. Then $b \sqsubseteq_E e(f, a)$ if and only if $b \sqsubseteq_E \bigsqcup\{d_i \in E;\ c_i \sqsubseteq_D a\}$ which is clearly decidable since D and E are effective. This concludes the proof. □

Next, we consider type conversion in the category $\mathbf{PAdm}$. We will show that for a restricted class of effectively representable sequentially continuous functions $f : X \times Y \to Z$ in $\mathbf{PAdm}$, type conversion is effective and yields a new effective sequentially continuous function $f^* : X \to [Y \to_\omega Z]$.

We begin with some definitions:

Definition 4.4. Let (D, α), (E, β) and (F, γ) be effective domains and suppose $f : D \times E \rightharpoonup F$. f is called *right-computable* if f is effective and $\mathrm{dom}(f) = S \times T$ for some Scott-closed set $S \subseteq D$ and some computable

Scott-closed set $T \subseteq E$. Left-computability for a partial continuous function $f : D \times E \rightharpoonup F$ is defined analogously.

By Proposition 4.6, if $f : D \times E \rightharpoonup F$ and $\mathrm{dom}(f) = S \times T$ then $f \in [D \times E \rightharpoonup F]$ is computable if and only if f is both left- and right-computable.

Let $(D,\ D^R,\ \delta)$, $(E,\ E^R,\ \varepsilon)$, and $(F,\ F^R,\ \varphi)$ be effective domain representations of the spaces X, Y and Z and suppose that $f : X \times Y \to Z$ is sequentially continuous. We say that $f : X \times Y \to Z$ is *right-computable* if f is representable by some right-computable partial continuous function from $D \times E$ to F. Left-computability for sequentially continuous functions from $X \times Y$ to Z is defined analogously.

Proposition 4.8. *Let* $(D,\ D^R,\ \delta)$*,* $(E,\ E^R,\ \varepsilon)$*, and* $(F,\ F^R,\ \varphi)$ *be effective admissible domain representations of the spaces* X*,* Y *and* Z *and suppose* $f : X \times Y \to Z$ *is sequentially continuous. If* f *is right-computable then* $f^* : X \to [Y \to_\omega Z]$ *is effective.*

Proof. Let $\overline{f} : D \times E \rightharpoonup F$ be a right-computable representation of $f : X \times Y \to Z$. Let S be Scott-closed in D and let T be a computable Scott-closed subset of E such that $\mathrm{dom}(\overline{f}) = S \times T$. Since $\bot_D \notin D^R$ and $\bot_E \notin E^R$ we may assume that $\overline{f}(\bot_D, y) = \overline{f}(x, \bot_E) = \bot_F$ for each $x \in S$ and $y \in T$. (If not we may simply redefine $\overline{f} : D \times E \rightharpoonup F$ on the set $\{(x, y) \in S \times T;\ x = \bot_D \text{ or } y = \bot_E\}$.) Let $g : S \to [T \to F]$ be the continuous function given by $g(x)(y) = \overline{f}(x, y)$. If $x \in S$ then $g(x) : T \to F$ is strict since $\overline{f}(x, \bot_E) = \bot_F$ for each $x \in D$. Let $i_\bot : [T \to_\bot F] \to [E \rightharpoonup F]$ be the strict inclusion of $[T \to_\bot F]$ into $[E \rightharpoonup F]$. $i_\bot \circ g : S \to [E \rightharpoonup F]$ is strict since $\overline{f}(\bot_D, y) = \bot_F$ for each $y \in E$ and $i_\bot$ is strict. Since S is closed, $h = i_\bot \circ g$ is a partial continuous function from D to $[E \rightharpoonup F]$ and it is routine to prove that $h : D \rightharpoonup [E \rightharpoonup F]$ represents $f^* : X \to [Y \to_\omega Z]$.

To show that $h : D \rightharpoonup [E \rightharpoonup F]$ is effective, suppose that D, E, and F are effective with respect to the numberings $\alpha : \mathbb{N} \to D_c$, $\beta : \mathbb{N} \to E_c$, and $\gamma : \mathbb{N} \to F_c$. Let $\alpha(j) \in S$ and suppose $k = \langle l, m \rangle \in \mathbb{N}$ is in the domain of $[\beta \rightharpoonup \gamma]$ where $m = \langle \langle l_1, m_1 \rangle, \langle l_2, m_2 \rangle, \ldots, \langle l_n, m_n \rangle \rangle$. We would like to show that the relation $[\beta \rightharpoonup \gamma](k) \sqsubseteq_F h(\alpha(j))$ is r.e.

We have $[\beta \rightharpoonup \gamma](k) \sqsubseteq_F h(\alpha(j))$ if and only if either $[\beta \rightharpoonup \gamma](k) = \bot_{[E \rightharpoonup F]}$ or $g(\alpha(j)) \neq \bot_{[T \to_\bot F]}$, $\mathrm{cl}(\beta)(l) \subseteq T$ and $\gamma(m_i) \sqsubseteq_F f(\alpha(j), \beta(l_i))$ for each $1 \leq i \leq n$. To show that this is r.e. it is enough to show that the relation $g(\alpha(j)) \neq \bot_{[T \to_\bot F]}$ is r.e. since T is computable and f is effective.

The relation $g(\alpha(j)) \neq \bot_{[T \to_\bot F]}$ is r.e. since $g(\alpha(j)) \neq \bot_{[T \to_\bot F]}$ if and only if there is some compact element $b \in T$ such that $f(\alpha(j), b) \neq \bot_F$ if and only if there is a $p \in \mathrm{dom}(\beta)$ and $q \in \mathrm{dom}(\gamma)$ such that

$$\big(\beta(p) \in T,\ \gamma(q) \sqsubseteq_F f(\alpha(j), \beta(p)),\ \text{and}\ \gamma(q) \neq \bot_F\big).$$

We conclude that $h : D \rightharpoonup [E \rightharpoonup F]$ is $(\alpha, [\beta \rightharpoonup \gamma])$-effective. It follows immediately that $f^* : X \to [Y \to_\omega Z]$ is effective and so we are done. □

Remark 4.5. Note that it follows by the proof of Proposition 4.8 that an index for $f^* : X \to [Y \to_\omega Z]$ is obtained uniformly from an index for $\overline{f} : D \times E \rightharpoonup F$ and an index for T.

In many cases of interest, E^D is a computable Scott-closed subset of E. By the next proposition, this is enough to ensure that every effectively representable sequentially continuous function $f : X \times Y \to Z$ has a right-computable representation.

Proposition 4.9. *Let (D, D^R, δ), (E, E^R, ε), and (F, F^R, φ) be effective admissible domain representations of the spaces X, Y and Z and suppose that E^D is a computable Scott-closed subset of E. Then every effective sequentially continuous function from $X \times Y$ to Z has a right-computable representation.*

Proof. If $\overline{f} : D \times E \to F$ is an effective representation of $f : X \times Y \to Z$ then $D^D \times E^D \subseteq \mathrm{dom}(\overline{f})$ since $D^D \times E^D$ is the closure of $D^R \times E^R$. It follows that $\overline{f}\restriction_{D^D \times E^D} : D \times E \rightharpoonup F$ is a partial continuous function from $D \times E$ to F which represents f.

$\overline{f}\restriction_{D^D \times E^D}$ is effective since f is (indeed, if $a \in D$, $b \in E$, and $c \in F$ are compact and $(a, b) \in D^D \times E^D$ then $c \sqsubseteq_F \overline{f}\restriction_{D^D \times E^D}(a, b)$ if and only if $c \sqsubseteq_F \overline{f}(a, b)$ and $\overline{f}\restriction_{D^D \times E^D}$ is right-computable since E^D is a computable Scott-closed subset of E. □

Remark 4.6. We note that an index for $\overline{f}$ is by definition also an index for the right-computable partial continuous function $(D^D \times E^D, \overline{f}\restriction_{D^D \times E^D})$.

Propositions 4.8 and 4.9 together with Remarks 4.5 and 4.6 now immediately yield the following corollary:

Corollary 4.10. *Let (D, D^R, δ), (E, E^R, ε), and (F, F^R, φ) be effective admissible domain representations of the spaces X, Y and Z and suppose that E^D is a computable Scott-closed subset of E. Let $f : X \times Y \to Z$ be effective. Then $f^* : X \to [Y \to_\omega Z]$ is effective and an index for f^* is obtained uniformly from an index for f and an index for E^D.*

5 Conclusions

To be able to represent continuous functions between domain representable spaces by total continuous functions on domains, it is in general necessary to consider only dense domain representations. This is problematic from a computability theoretic point of view since there is no effective construction which allows us to pass from a non-dense domain representation of a topological space to a dense representation of the same space.

We argue that a viable alternative to working exclusively with densely representable spaces is to represent continuous functions by *partial* continuous functions, rather than total continuous functions on the domain level. This allows us to effectively circumvent the (sometimes exceedingly difficult) problem of finding dense domain representations of the spaces under consideration.

We have shown that the category of admissibly representable spaces with morphisms functions representable by some partial continuous function is Cartesian closed. We have also shown that the corresponding effective category is very close to being Cartesian closed. The only construction which is not completely effective is type conversion, which still preserves effectivity in many important cases.

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Paper III

A Domain Theoretic Approach to Effective Distribution Theory

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Abstract

We use effective domain theory to introduce a notion of effectivity for test functions and for distributions. We will show that the Fourier transform on $\mathscr{S}$ is effective, and that this results lifts to the space of tempered distributions. We also show that convolution is an effective operation on distributions. Finally, as an application of this result we show that it is possible to compute primitives in $\mathscr{D}'$.

Key Words: Computable analysis, effective domain theory, distribution theory.

1 Introduction

The theory of distributions has become an important conceptual tool in the theory of partial differential equations. There are at least two important reasons for the relative success of distribution theory. Every distribution is infinitely differentiable, something which frees the differential calculus of distributions from some of the problems which arise in the classical theory of partial differential equations because nondifferentiable functions exist. Another interesting and important feature of the theory of distributions is that it is possible to apply Fourier transform techniques to many problems where it cannot be done in a more classical setting. An excellent example of this is the proof (given by Malgrange and Ehrenpreis in the early fifties) that every partial differential equation with constant coefficients has a fundamental solution. That is, that there exists, for each nonconstant complex polynomial p in n variables, a distribution e such that $p(D)e = \delta$, where δ is the Dirac measure. Given the fundamental solution e one may compute a solution u to the more general nonhomogeneous equation $p(D)u = v$ by noticing that $u = e * v$ solves $p(D)u = v$.

A natural question is whether or not it would be possible to give some effective content to theorems like the one of Malgrange and Ehrenpreis. That is, if it is possible to *compute* the fundamental solution e given the complex polynomial p. To be able to make sense of such questions we need to be able to say what it means for a function taking a complex polynomial as input and giving a distribution as output to be computable by an algorithm. Since both the space of complex polynomials as well as the space of distributions are uncountable we cannot compute on them directly, but must resort to computing on approximations of the data we are interested in, rather than on the data itself. Questions like this one have been studied before by Washihara [25] using computability structures, by Weihrauch and Zhong in [27] and [28] in the context of Type Two Effectivity (TTE), and in the setting of constructive analysis by Ishihara and Yoshida in [15] and [29].

Here we will use effective domains to study computable processes in distribution theory. The domain theoretic approach has many similarities to TTE, but one of the main differences is that it gives us more flexibility to model computations directly on the approximations of the elements of the space we are interested in. In this way, one may argue that effective domain theory gives a general and uniform way to extend the ideas and methods of interval analysis to any space with an effective domain representation, where effective continuous functions on domains correspond directly to validated numerical algorithms in interval analysis. Having said this, we freely admit that we owe a substantial intellectual debt to Weihrauch and Zhong. Much of the theory developed in this paper is heavily inspired by the two papers [27] and [28].

2 Preliminaries

We will show that the theory of effective domains and effective domain representations can be used to introduce a notion of effectivity for test functions, and for distributions. Thus, we begin by recalling some definitions and notation from the theory of Scott-Ershov domains. This section should not be thought of as an introduction to the subject, but rather as a review of some of the relevant definitions and notational conventions. For a thorough treatment of domain theory we recommend the book [22] or the comprehensive survey given in [1].

2.1 Domain theory

A *Scott-Ershov domain* (or simply domain) is a bounded complete algebraic cpo. Let D be a domain. Then D_c denotes the set of compact elements in D. Given $x \in D$ we write $\mathrm{approx}(x)$ for the set $\{a \in D_c;\ a \sqsubseteq x\}$. Since D is algebraic, $\text{approx}(x)$ is directed and $\bigsqcup \text{approx}(x) = x$ for each $x \in D$. D is *countably based* if D_c is a countable set.

A *preordered conditional upper semilattice* (precusl) is a preordered set $(P, \sqsubseteq)$ with a distinguished least element $\bot$, which is consistently complete in the following sense: Whenever p and q are consistent in P there is some $r \in P$ such that $p, q, \sqsubseteq r$ and if $p, q \sqsubseteq s$ for some $s \in P$ then $r \sqsubseteq s$. r is called *a least upper bound* for p and q. It is clear that r need not be unique and so p and q may have many least upper bounds in P. P is a *conditional upper semilattice* (cusl) if $\sqsubseteq$ is a partial order on P. When P is a cusl then least upper bounds in P are unique whenever they exist.

Given a precusl P we write $\text{Idl}(P)$ for the set $\{I \subseteq P;\ I \text{ is an ideal}\}$. We order $\text{Idl}(P)$ by $I \sqsubseteq J \iff I \subseteq J$. Then $\text{Idl}(P) = (\text{Idl}(P), \sqsubseteq, \{\bot\})$ is a domain. The domain $\text{Idl}(P)$ is called the *ideal completion* of P. If D is a domain then D_c is a cusl and $D \cong \text{Idl}(D_c)$.

The *Scott-topology* on a domain D is generated by the collection of open sets $\uparrow a = \{x \in D;\ a \sqsubseteq x\}$ where a ranges over D_c. Let D and E be domains. A *partial continuous function*[1] from D to E is a pair (S, f) where S is a nonempty Scott-closed subset of D and $f : S \to E$ is a strict continuous function from S to E. We write $f : D \rightharpoonup E$ if $(\text{dom}(f), f)$ is a partial continuous function from D to E.

A precusl P is *computable* if there is a numbering $\alpha : \mathbb{N} \to P$ of P with respect to which the relations $\alpha(m) \sqsubseteq \alpha(n)$, "$\alpha(m)$ and $\alpha(n)$ are consistent", and $\alpha(k) = \alpha(m) \sqcup \alpha(n)$ are all recursive. The domain D is *effective* if the cusl D_c is computable. Thus in particular, if P is a computable precusl then $D = \text{Idl}(P)$ is an effective domain. When (D, α) and (E, β) are effective domains then $x \in D$ is *α-computable* (or simply computable) if the set $\{m \in \mathbb{N};\ \alpha(m) \sqsubseteq x\}$ is r.e. and if $f : D \rightharpoonup E$ is a partial continuous function then f is *(α, β)-effective* (or simply effective) if there is an r.e. relation R such that $R(m, n) \iff \beta(n) \sqsubseteq_E f(\alpha(m))$ whenever $\alpha(m) \in \text{dom}(f)$.

2.2 Representation theory

Let X be a topological space. A *domain representation* of X is a triple $(D,\ D^R,\ \delta)$ where D is a domain, D^R a nonempty subset of D, and $\delta : D^R \to X$ a continuous function from D^R onto X. When D is an effective domain then $(D,\ D^R,\ \delta)$ is an *effective domain representation* of X. We will assume throughout that $\bot \notin D^R$. For $r \in D$ and $x \in X$ we write $r \prec x$ and say that *r approximates x* if $x \in \delta[\uparrow r \cap D^R]$. Thus, $r \prec x$ if and only if there is some $s \in D^R$ such that $r \sqsubseteq s$ and $\delta(s) = x$. The domain representation $(D,\ D^R,\ \delta)$

[1]For an introduction to partial continuous functions on domains, we refer the reader to [7].

is *countably based* if D is countably based, and $(D,\ D^R,\ \delta)$ is *dense* if the set D^R is dense in D.

Let $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ be effective domain representations of X and Y respectively. We say that $x \in X$ is *D-computable* (or simply computable if the representation $(D,\ D^R,\ \delta)$ is clear from the context) if there is some computable $r \in D^R$ such that $\delta(r) = x$.

There is a corresponding notion of representability for functions from X to Y. A function $f : X \to Y$ is *(D, E)-representable* (or simply representable) if there is a partial continuous function $\overline{f} : D \rightharpoonup E$ such that $D^R \subseteq \mathrm{dom}(\overline{f})$, $\overline{f}[D^R] \subseteq E^R$ and $f(\delta(r)) = \varepsilon(\overline{f}(r))$ for each $r \in D^R$. Thus, $f : X \to Y$ is representable if there is a partial continuous function $\overline{f} : D \rightharpoonup E$ which makes the diagram in Figure 1 commute.

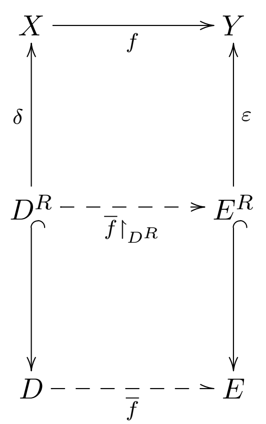


Figure 1.

If $\overline{f} : D \rightharpoonup E$ is effective we say that f is *(D, E)-effective* (or simply effective if the domain representations $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ are clear from the context).

The representation $(E,\ E^R,\ \varepsilon)$ of Y is called (ω-)*admissible*[2] if for each triple $(D,\ D^R,\ \delta)$ where D is a countably based domain, $D^R \subseteq D$, and $\delta : D^R \to Y$ is continuous, the function δ factors through ε via some partial continuous function $\overline{\delta} : D \rightharpoonup E$ as in Figure 2.

The following simple observation indicates why admissibility is interesting from a purely representation theoretic point of view.

[2]This is a reformulation of the definition of an ω-admissible domain representation found in [14]. It is equivalent to the definition in [14], as well as that of a ω-projecting equilogical space found in [2] and [16].

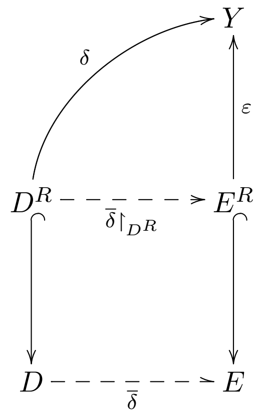


Figure 2.

Theorem 2.2.1. *Suppose $(D,\, D^R,\, \delta)$ is a countably based representation of X and $(E,\, E^R,\, \varepsilon)$ is an admissible representation of Y. Then every sequentially continuous function from X to Y is representable.*

(For a proof of Theorem 2.2.1 see [7].)

Theorem 2.2.1 can be used as a tool in constructing a representation of the space of sequentially continuous functions from X to Y over the domain $[D \rightharpoonup E]$ of partial continuous functions from D to E. For details, see [7].

To show that $(E,\, E^R,\, \varepsilon)$ is an admissible representation of the space Y it is sometimes useful to "prune" E to get a dense representation. To be somewhat more precise, we define E_c^D as the set $\{c \in E_c;\ c \sqsubseteq r \text{ for some } r \in E^R\}$ and let $E^D = \{\bigsqcup B;\ B \subseteq E_c^D \text{ is directed}\}$. Then $(E^D,\, \sqsubseteq,\, \bot)$ is a domain, and E^R is a dense subset of E^D. It follows that $(E^D,\, E^R,\, \varepsilon)$ is a dense domain representation of Y. The following elementary result can be found in [14]:

Theorem 2.2.2. *Let $(E,\, E^R,\, \varepsilon)$ be a domain representation of the space Y. Then $(E,\, E^R,\, \varepsilon)$ is admissible if and only if $(E^D,\, E^R,\, \varepsilon)$ is admissible.*

Thus, to show that $(E,\, E^R,\, \varepsilon)$ is admissible it is enough to show that the corresponding dense representation $(E^D,\, E^R,\, \varepsilon)$ is admissible.

2.3 Pseudobases and admissible pairs

Let X be a topological space. A *pseudobasis* for X is a collection $\mathscr{B}$ of subsets of X such that $X \in \mathscr{B}$, $B, C \in \mathscr{B}$ and $B \cap C \neq \emptyset \implies B \cap C \in \mathscr{B}$, and for each convergent sequence $(x_n)_n \longrightarrow x$ in X, and each open neighbourhood U of x, there is some $B \in \mathscr{B}$ such that $x \in B \subseteq U$ and $x_n \in B$ for almost all n. Pseudobases are related to admissibility in the following way:

Theorem 2.3.1. *Let X be a topological space. Then X has a countably based and admissible domain representation if and only if X is T_0 and has a countable pseudobasis.*

For a proof of Theorem 2.3.1 we refer the reader to the paper [14] by Hamrin. The notion of pseudobases is originally due to Schröder [19].

If $(D,\ D^R,\ \delta)$ is a countable based admissible domain representation of X, then the collection of all sets of the form $\{x \in X;\ a \prec x\}$, where a ranges over D_c, forms a pseudobasis on X.

Conversely, if $\mathscr{B}$ is a pseudobasis on X we may construct an admissible domain representation over the ideal completion of $(\mathscr{B}, \supseteq)$. More precisely, if $I \in \mathrm{Idl}(\mathscr{B}, \supseteq)$ we say that I *converges* to $x \in X$ (and write $I \longrightarrow x$) if $x \in B$ for each $B \in I$, and for each open neighbourhood U of x there is a $B \in I$ such that $B \subseteq U$. We may now define a domain representation of X over the ideal completion of $(\mathscr{B}, \supseteq)$ by setting $D = \mathrm{Idl}(\mathscr{B}, \supseteq)$, $D^R = \{I \in \mathrm{Idl}(\mathscr{B}, \supseteq);\ I$ is convergent$\}$, and $\delta(I) = x$ if and only if I converges to x. The function $\delta : D^R \to X$ is well-defined since X is T_0, and continuous and onto since $\mathscr{B}$ is a pseudobasis. That $(D,\ D^R,\ \delta)$ is admissible follows by Theorem 6.8 in [14]. The domain representation $(D,\ D^R,\ \delta)$ is called the *standard representation of X over $\mathscr{B}$.*

The problem with this construction is that in general, there is no numbering of $\mathscr{B}$ which makes the superset relation $\supseteq$ on $\mathscr{B}$ computable. It follows that the domain $\mathrm{Idl}(\mathscr{B}, \supseteq)$ is noneffective in general.

To remedy this we need to be able to "perturb" the cusl $(\mathscr{B}, \supseteq)$ enough to get a computable precusl $(B, \sqsubseteq)$ which behaves enough like the original for the construction above to go through with $(B, \sqsubseteq)$ in place of $(\mathscr{B}, \supseteq)$. To make this notion of perturbation precise we need the notion of an admissible pair.

Definition 2.3.1. Let D and E be domains and let $a : D \to E$ and $s : E \to D$ be continuous functions. Then (a, s) is an *admissible pair* for (D, E) if $a \circ s = \mathrm{id}_E$ and $\mathrm{id}_D \sqsubseteq s \circ a$.

If (a, s) is an admissible pair for (D, E) then $a \dashv s$ is a Galois correspondence and s is completely determined by a (and similarly, a is uniquely determined by s). To be more precise, we have $s(y) = \bigsqcup\{c \in D_c;\ a(c) \sqsubseteq y\}$ for all $y \in E$. The left adjoint of the admissible pair (a, s) is called an *admissible* function.

Admissible pairs are interesting since they allow us to "lift" an admissible domain representation over E along the admissible map a to construct a new admissible domain representation of the same space over D.

Theorem 2.3.2. *Let D and E be domains and suppose $(E,\ E^R,\ \varepsilon)$ is an admissible domain representation of X. Let (a, s) be an admissible pair for*

(D, E). *If* $s[E^R] \subseteq D^R \subseteq a^{-1}[E^R]$ *and* $\delta = (\varepsilon \circ a)\restriction_{D^R} : D^R \to X$ *then* $(D,\, D^R,\, \delta)$ *is an admissible domain representation of* X.

For a proof, we refer the reader to [6].

Informally, we think of the domain D as a computationally well-behaved perturbation of E. In the special case when $(E,\, E^R,\, \varepsilon)$ is constructed as the standard representation over a pseudobasis $\mathscr{B}$ on X it is often more convenient to work directly with the pseudobasis $\mathscr{B}$, rather than with the ideal completion of $\mathscr{B}$. To be able to do this we need a version of Theorem 2.3.2 for precusls.

Definition 2.3.2. If P and Q are precusls we say that $a : P \to Q$ is *pre-admissible* if a is a monotone function such that

1. For each $c \in Q$ there is some $b \in P$ such that $c \sqsubseteq a(b) \sqsubseteq c$.
2. If $a(b)$ and $a(c)$ are consistent in Q then b and c are consistent in P, and if d is a least upper bound of b and c in P then $a(d)$ is a least upper bound of $a(b)$ and $a(c)$ in Q.

As before, we think of the elements in P as computationally well-behaved representations of the elements in Q, and we think of P as a perturbed version of Q. Since every pre-admissible function from P to Q extends to an admissible function from $\mathrm{Idl}(P)$ to $\mathrm{Idl}(Q)$ we (more or less) immediately have the following result:

Theorem 2.3.3. *Let* X *be a* T_0 *space and let* $\mathscr{B}$ *be a countable pseudobasis on* X. *Let* $(B, \sqsubseteq)$ *be a precusl and suppose* $a : (B, \sqsubseteq) \to (\mathscr{B}, \supseteq)$ *is pre-admissible. Let* $D = \mathrm{Idl}(B, \sqsubseteq)$ *and assume that* $D^R \subseteq D$ *satisfies*

1. $a^{-1}[J] \in D^R$ *for each convergent ideal* $J \in \mathrm{Idl}(\mathscr{B}, \supseteq)$, *and*
2. $a[I]$ *is convergent for each* $I \in D^R$.

Define $\delta : D^R \to X$ *by* $\delta(I) = x$ *if and only if* $I \in D^R$ *and* $a[I] \longrightarrow x$ *in* $\mathrm{Idl}(\mathscr{B}, \supseteq)$. *Then* $(D,\, D^R,\, \delta)$ *is an admissible domain representation of* X.

For a proof of Theorem 2.3.3 see [6].

2.4 Almost standard domain representations and faithful extensions

The space $\mathscr{D}$ of smooth functions with compact support is constructed as the inductive limit of the spaces $(\mathscr{D}_k)_{k \geq 1}$, where $\mathscr{D}_k$ is the space of smooth functions with support contained in the interval $[-k, k]$. We will show that each of the spaces $\mathscr{D}_k$ has an effective and admissible domain representation $(D_k,\ D_k^R,\ \delta_k)$ which may be constructed from a pseudobasis on $\mathscr{D}_k$ as in the previous section. To construct an admissible domain representation of $\mathscr{D}$ it will be convenient to review a result from [6].

Definition 2.4.1. A domain representation $(D,\ D^R,\ \delta)$ of the T_0-space X is called *almost standard* if there is a pseudobasis $\mathscr{B}$ on X, and an admissible pair (a, s) for (D, E) such that $s[E^R] \subseteq D^R \subseteq a^{-1}[E^R]$ and $\delta = (\varepsilon \circ a)\restriction_{D^R} : D^R \to X$, where $(E,\ E^R,\ \varepsilon)$ is the standard representation of X over $\mathscr{B}$.

Thus in particular, if the domain representation $(D,\ D^R,\ \delta)$ is constructed using either Theorem 2.3.2 or Theorem 2.3.3 then $(D,\ D^R,\ \delta)$ is almost standard.

Definition 2.4.2. Let $(D,\ D^R,\ \delta)$ and $(E,\ E^R,\ \varepsilon)$ be domain representations of the spaces X and Y and suppose that $X \subseteq Y$. Let $e : D \to E$ be an embedding. Then e is a *faithful extension* if

1. $r \in D^R \iff e(r) \in E^R$ for each $r \in D$.
2. $e : D \to E$ represents the inclusion $X \hookrightarrow Y$.
3. If $c \neq \bot_E$ is compact in E and $c \in e[D]$, then $\{y \in Y;\ c \prec y\} \subseteq X$.
4. If c is compact in E, $\{y \in Y;\ c \prec y\}$ is nonempty, and $\{y \in Y;\ c \prec y\} \subseteq X$, then $c \in e[D]$.

Intuitively, if $e : D \to E$ is a faithful extension then e preserves both the domain theoretic structure on D (since e is an embedding) as well as the domain representation over D. In [6] we proved the following theorem:

Theorem 2.4.1. *Suppose that $(X_i)_{i\in I}$ is a countable directed system of T_1-spaces and let $((D_i)_{i\in I},\ (e_{i,j})_{i\leq j\in I})$ be a directed system of domains where*

1. *$(D_i,\ D_i^R,\ \delta_i)$ is an almost standard domain representation of X_i for each $i \in I$.*
2. *$e_{i,j}$ is a faithful extension for all $i \leq j \in I$.*

Then $(\varinjlim(D_i)_{i\in I},\ \bigcup_{i\in I} e_i[D_i^R],\ \bigcup_{i\in I}(\delta_i \circ p_i))$ is an almost standard domain representation of the inductive limit of $(X_i)_{i\in I}$, where (e_i, p_i) is the canonical embedding projection pair for $(D_i, \varinjlim(D_i)_{i\in I})$.

This Theorem will allow us to construct an effective and admissible domain representation of $\mathscr{D}$, the inductive limit of $(\mathscr{D}_k)_{k\geq 1}$.

2.5 Some standard domain representations

Before we go on to introduce domain representations of the standard test function spaces it will be useful to review the construction of the standard domain representations of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$.

To construct a domain representation of the real numbers, we let $B_{\mathbb{R}}$ be the set $\{[p,q] \subseteq \mathbb{R};\ p,q \in \mathbb{Q} \text{ and } p \leq q\} \cup \{\mathbb{R}\}$. We let $D_{\mathbb{R}}$ be the ideal completion of $(B_{\mathbb{R}}, \supseteq)$. Then $D_{\mathbb{R}}$ is an effective domain. Given $I \in D_{\mathbb{R}}$ and a real number r we say that I *represents* r if

1. $r \in S$ for each interval $S \in I$.
2. For each $n \in \mathbb{N}$ there is an interval $S \in I$ such that $\operatorname{diam}(S) < 2^{-n}$.

We let $D^R_{\mathbb{R}} \subseteq D_{\mathbb{R}}$ be the set of ideals in $D_{\mathbb{R}}$ which represent real numbers and define $\delta_{\mathbb{R}} : D^R_{\mathbb{R}} \to \mathbb{R}$ by $\delta_{\mathbb{R}}(I) = r$ if and only if I represents r. Then $(D_{\mathbb{R}},\ D^R_{\mathbb{R}},\ \delta_{\mathbb{R}})$ is an effective domain representation of $\mathbb{R}$, and it is easy to show that $(D_{\mathbb{R}},\ D^R_{\mathbb{R}},\ \delta_{\mathbb{R}})$ is admissible.

The standard representation of $\mathbb{C}$ is defined analogously. We let $B_{\mathbb{C}}$ be the set $\{[p,q] \times i[r,s] \subseteq \mathbb{C};\ p,q,r,s \in \mathbb{Q},\ p \leq q, \text{ and } r \leq s\} \cup \{\mathbb{C}\}$ of complex boxes with complex rational corners, and let $D_{\mathbb{C}}$ be the ideal completion of $(B_{\mathbb{C}}, \supseteq)$. It is clear that $D_{\mathbb{C}}$ is an effective domain. The set of representing ideals in $D_{\mathbb{C}}$ is defined exactly as in the case of $D_{\mathbb{R}}$. Thus, if $I \in D_{\mathbb{C}}$ and $z \in \mathbb{C}$ we say that I *represents* z if

1. $z \in T$ for each $T \in I$.
2. For each $n \in \mathbb{N}$ there is some $T \in I$ such that $\operatorname{diam}(T) < 2^{-n}$.

As before, we let $D^R_{\mathbb{C}}$ denote the set of ideals in $D_{\mathbb{C}}$ which represent complex numbers, and we define the function $\delta_{\mathbb{C}} : D^R_{\mathbb{C}} \to \mathbb{C}$ by $\delta_{\mathbb{C}}(I) = z$ if and only if I represents z. It is easy to show that $(D_{\mathbb{C}},\ D^R_{\mathbb{C}},\ \delta_{\mathbb{C}})$ is an effective and admissible domain representation of $\mathbb{C}$.

If X is a (real or complex) vector space and $(D_X,\ D^R_X,\ \delta_X)$ is an effective domain representation of X we say that X is an *effective vector space* if addition $(x,y) \mapsto x+y$ on X and scalar multiplication $(z,x) \mapsto z \cdot x$ are effective, and 0 is computable as an element of X. It is easy to show that addition and multiplication on the reals $\mathbb{R}$ and complex numbers $\mathbb{C}$ are effective, and so $\mathbb{R}$ and $\mathbb{C}$ are both effective vector spaces. (It is trivial to verify that 0 is computable as an element of either $\mathbb{R}$ or $\mathbb{C}$.)

Finally, to construct a domain representation of the natural numbers, we let $D_{\mathbb{N}}$ be the flat domain over $\mathbb{N}$. Thus $D_{\mathbb{N}} = \mathbb{N} \cup \{\bot\}$ where $\bot \sqsubseteq n$ for each

$n \in \mathbb{N}$, and $m \sqsubseteq n$ if and only if $m = n$ for all $m, n \in \mathbb{N}$. The domain $D_{\mathbb{N}}$ is clearly effective. The set of representing elements in $D_{\mathbb{N}}$ is simply $D_{\mathbb{N}}^R = \mathbb{N}$, and the representing map $\delta_{\mathbb{N}} : \mathscr{D}_{\mathbb{N}}^R \to \mathbb{N}$ is simply the identity on $\mathbb{N}$. It is clear that $(D_{\mathbb{N}},\ D_{\mathbb{N}}^R,\ \delta_{\mathbb{N}})$ is an effective admissible domain representation of $\mathbb{N}$.

3 Computability on spaces of test functions

We will first need to construct effective domain representations of the spaces $\mathscr{E}$, $\mathscr{D}$, and $\mathscr{S}$, which are the spaces most commonly used as test function spaces in distribution theory. $\mathscr{E}$ is the space of smooth (or infinitely differentiable) functions from $\mathbb{R}$ to $\mathbb{C}$, $\mathscr{D}$ is the space of compactly supported smooth functions from $\mathbb{R}$ to $\mathbb{C}$, and $\mathscr{S}$ is the space of rapidly decreasing smooth functions from $\mathbb{R}$ to $\mathbb{C}$.

3.1 An effective domain representation of $\mathscr{E}$

We denote by $\mathscr{E}$ the set of all smooth functions from $\mathbb{R}$ to $\mathbb{C}$, and for each $M \in \mathbb{N}$ we define a seminorm $\|\cdot\|_M : \mathscr{E} \to \mathbb{R}$ on $\mathscr{E}$ by

$$\|\varphi\|_M = \sup\{|\varphi^{(m)}(x)|;\ x \in [-M, M] \text{ and } m \leq M\}.$$

This family of seminorms generates a metrisable locally convex topology on $\mathscr{E}$, which will be denoted by $\tau_{\mathscr{E}}$, as follows: Let $U(m, n)$ be the set $\{\varphi \in \mathscr{E};\ \|\varphi\|_m < 2^{-n}\}$ and define $\mathscr{B}$ as the collection of all finite intersections of sets $U(m, n)$. Now, $\mathscr{B}$ is a local basis for $\tau_{\mathscr{E}}$.

To construct a domain representation of $\mathscr{E}$, let CRI denote the set of closed real intervals with rational endpoints and OCB the set of open complex boxes with complex rational corners. Now, given $S \in$ CRI and $T \in$ OCB, we let $B^{(n)}(S, T)$ be a notation for the set $\{\varphi \in \mathscr{E};\ \varphi^{(n)}[S] \subseteq T\}$. Finally, we define $B_{\mathscr{E}}$ as the collection of all finite sets $\{B^{(n_1)}(S_1, T_1), \ldots, B^{(n_k)}(S_k, T_k)\}$ such that

1. $S_1, \ldots, S_k \in$ CRI and $T_1, \ldots, T_k \in$ OCB.
2. If $n_i = n_j$ and $S_i \cap S_j \neq \emptyset$ then $T_i \cap T_j \neq \emptyset$.

$B_{\mathscr{E}}$ is clearly countable. We order $B_{\mathscr{E}}$ by $B \sqsubseteq C$ if and only if for each $B^{(n)}(S, T) \in B$ there are $B^{(n)}(S_1, T_1), \ldots, B^{(n)}(S_k, T_k) \in C$ such that $S \subseteq S_1 \cup \ldots \cup S_k$ and $T_i \subseteq T$ for each $1 \leq i \leq k$. Then $\sqsubseteq$ is a preorder on $B_{\mathscr{E}}$ with least element $\emptyset$. In fact, $B_{\mathscr{E}}$ is a computable precusl.

Lemma 3.1.1. *$B_{\mathscr{E}}$ is a computable precusl.*

Proof. To show that $(B_{\mathscr{E}}, \sqsubseteq)$ is a precusl, suppose that B and C are consistent in $B_{\mathscr{E}}$. Then $B \cup C \in B_{\mathscr{E}}$ and $B \cup C$ is clearly a least upper bound for B and C in $B_{\mathscr{E}}$.

It is clear that the set $B_{\mathscr{E}}$ and the order relation $\sqsubseteq$ on $B_{\mathscr{E}}$ are computable. Since $(B, C) \mapsto B \cup C$ is computable we conclude that $B_{\mathscr{E}}$ is a computable precusl as required. □

Informally, we think of $B \in B_{\mathscr{E}}$ as a notation for the set of all smooth functions φ such that $\varphi^{(n)}[S] \subseteq T$ for each $B^{(n)}(S, T) \in B$. Thus, we say that $\varphi \in \mathscr{E}$ *satisfies* $B \in B_{\mathscr{E}}$ if $\varphi^{(n)}[S] \subseteq T$ for each $B^{(n)}(S, T) \in B$. As a motivation for the definition of the preorder on $B_{\mathscr{E}}$ one may show that any φ which satisfies C also satisfies B whenever $B \sqsubseteq C$ in $B_{\mathscr{E}}$.

We may visualise an element of $B_{\mathscr{E}}$ as a collection of open boxes describing the behaviour of the approximated function, as well as a finite number of derivatives, on closed intervals.

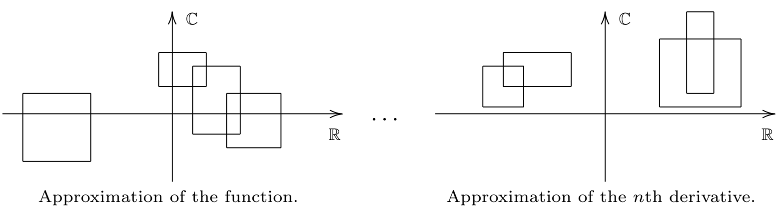


Figure 3. An element in $B_{\mathscr{E}}$.

We let $D_{\mathscr{E}}$ be the ideal completion of $B_{\mathscr{E}}$. Then $D_{\mathscr{E}}$ is an effective domain since $B_{\mathscr{E}}$ is a computable precusl. To define a domain representation of $\mathscr{E}$ over $D_{\mathscr{E}}$ we say that the ideal I in $D_{\mathscr{E}}$ *represents* $\varphi \in \mathscr{E}$ if and only if

1. φ satisfies every $B \in I$.

2. Given $x \in \mathbb{R}$, and $m, n \in \mathbb{N}$ there is a $B \in I$ and $B^{(m)}(S, T) \in B$ such that x is contained in the interior of S and the diameter of T is $< 2^{-n}$.

We let $D_{\mathscr{E}}^R$ be the set of representing elements $I \in D_{\mathscr{E}}$ and define $\delta_{\mathscr{E}} : D_{\mathscr{E}}^R \to \mathscr{E}$ by $\delta_{\mathscr{E}}(I) = \varphi$ if and only if I represents φ in $D_{\mathscr{E}}$. It is easy to see that $\delta_{\mathscr{E}}$ is well-defined. $\delta_{\mathscr{E}}$ is surjective since $\{B \in B_{\mathscr{E}};\ \varphi \text{ satisfies } B\}$ is in $D_{\mathscr{E}}^R$ for each $\varphi \in \mathscr{E}$. To show that $\delta_{\mathscr{E}}$ is a continuous function from $D_{\mathscr{E}}^R$ to $\mathscr{E}$ we will need the following elementary lemma.

The second requirement above may require some motivation. If I represents φ we require that I describes $\varphi^{(m)}$ arbitrarily well on an *open* neighbourhood of $x \in \mathbb{R}$. This strong approximation property is needed to ensure that we can compute (a representation of) $\varphi^{(m)}(x)$ from (representations of) φ and x. We note that it has another rather surprising consequence, apart from allowing us to compute evaluation. It forces $\delta_{\mathscr{E}} : D^R_{\mathscr{E}} \to \mathscr{E}$ to be a homeomorphism. (For details, see Appendix A.)

Lemma 3.1.2. *Suppose that $I \in D_{\mathscr{E}}$ represents $\varphi \in \mathscr{E}$. Then, for each basic open set $U(m,n)$ there is a $B \in I$ such that $\{\phi \in \mathscr{E};\ \phi$ satisfies $B\} \subseteq \varphi + U(m,n)$.*

Proof. For each $k \leq m$ we let C_k be the set $\{\mathrm{int}(S);\ S \in \mathrm{CRI},\ [-m,m] \cap \mathrm{int}(S) \neq \emptyset$ and $\{B^{(k)}(S,T)\} \in I$ for some $T \in \mathrm{OCB}$ with $\mathrm{diam}(T) < 2^{-n}\}$.

Since I represents φ it follows by 2 above that C_k is an open cover of $[-m,m]$. Since $[-m,m]$ is compact there are $\{B^{(k)}(S_1,T_1)\}, \ldots, \{B^{(k)}(S_l,T_l)\} \in I$ such that $[-m,m] \subseteq S_1 \cup \ldots \cup S_l$.

Let B_k be an upper bound for $\{B^{(k)}(S_1,T_1)\}, \ldots, \{B^{(k)}(S_l,T_l)\}$ in I and let B be an upper bound for $\{B_k;\ k \leq m\}$ in I. It follows immediately from the definition of B that $\{\phi \in \mathscr{E};\ \phi$ satisfies $B\} \subseteq \varphi + U(m,n)$, and so we are done. $\square$

Remark 3.1.1. We note that it follows from the proof of Lemma 3.1.2 that the finite set B satisfies the following comparatively stronger condition: For each $x \in [-m,m]$ and each $k \leq m$ there is some $B^{(k)}(S,T) \in B$ such that $x \in \mathrm{int}(S)$ and $\mathrm{diam}(T) < 2^{-n}$. This will be useful later.

Theorem 3.1.3. *$(D_{\mathscr{E}},\ D^R_{\mathscr{E}},\ \delta_{\mathscr{E}})$ is an effective domain representation of $\mathscr{E}$.*

Proof. We need to show that $\delta_{\mathscr{E}}$ is continuous. It is clearly enough to show that the set $\delta^{-1}_{\mathscr{E}}[U(m,n)]$ is open in $D^R_{\mathscr{E}}$ for all $m, n \in \mathbb{N}$. Let $I \in D^R_{\mathscr{E}}$ and suppose that I represents $\varphi \in U(m,n)$. Choose $k \in \mathbb{N}$ such that $\|\varphi\|_m + 2^{-k} < 2^{-n}$. Note that if $\|\psi\|_m < 2^{-k}$ then $\varphi + \psi \in U(m,n)$.

Since I represents φ there is some $B \in I$ such that $\{\phi \in \mathscr{E};\ \phi$ satisfies $B\} \subseteq \varphi + U(m,k) \subseteq U(m,n)$. Now, if $B \in J$ and J represents ψ we must have $\psi \in \{\phi \in \mathscr{E};\ \phi$ satisfies $B\} \subseteq U(m,n)$. It follows that $I \in \{J \in D^R_{\mathscr{E}};\ B \in J\} \subseteq \delta^{-1}_{\mathscr{E}}[U(m,n)]$. Since $\{J \in D^R_{\mathscr{E}};\ B \in J\}$ is open in $D^R_{\mathscr{E}}$ we conclude that $\delta_{\mathscr{E}}$ is continuous. $\square$

We would like to show that the representation $(D_{\mathscr{E}},\ D^R_{\mathscr{E}},\ \delta_{\mathscr{E}})$ is admissible. By Theorem 2.2.2 it is enough to show that the corresponding dense domain representation $(D^D_{\mathscr{E}},\ D^R_{\mathscr{E}},\ \delta_{\mathscr{E}})$ of $\mathscr{E}$ is admissible.

We let $B^D_{\mathscr{E}}$ be the set of all satisfiable elements B in $B_{\mathscr{E}}$. Thus, $B \in B^D_{\mathscr{E}}$ if and only if there is some $\varphi \in \mathscr{E}$ such that φ satisfies B. Then $D^D_{\mathscr{E}}$ is simply the ideal completion of $B^D_{\mathscr{E}}$. To show that $(D^D_{\mathscr{E}},\ D^R_{\mathscr{E}},\ \delta_{\mathscr{E}})$ is admissible we will use the theory of admissible pairs developed in [6]. Thus, let $\mathscr{B}_{\mathscr{E}}$ be the collection of all sets $\{\phi \in \mathscr{E};\ \phi$ satisfies $B\}$ where B ranges over $B^D_{\mathscr{E}}$, and define $a : (B^D_{\mathscr{E}}, \sqsubseteq) \to (\mathscr{B}_{\mathscr{E}}, \supseteq)$ by $a(B) = \{\phi \in \mathscr{E};\ \phi \text{ satisfies } B\}$. That $(D^D_{\mathscr{E}},\ D^R_{\mathscr{E}},\ \delta_{\mathscr{E}})$ is admissible follows by Lemmas 3.1.4 – 3.1.6.

Lemma 3.1.4. *$\mathscr{B}_{\mathscr{E}}$ is a pseudobasis on $\mathscr{E}$.*

Proof. It is clear that $\mathscr{E} \in \mathscr{B}_{\mathscr{E}}$, and that $\mathscr{B}_{\mathscr{E}}$ is closed under finite nonempty intersections. Suppose that $(\varphi_n)_n \longrightarrow \varphi$ in $\mathscr{E}$ and let U be an open neighbourhood of φ. Choose $m, n \in \mathbb{N}$ such that $\varphi + U(m, n) \subseteq U$. By Lemma 3.1.2 there is a $B \in D^D_{\mathscr{E}}$ such that $\varphi \in a(B) \subseteq \varphi + U(m, n) \subseteq U$.

The set $\{\varphi \in \mathscr{E};\ \varphi^{(m)}[S] \subseteq T\}$ is open in $\mathscr{E}$ for each $m \in \mathbb{N}$ and all $S \in \mathrm{CRI}$ and $T \in \mathrm{OCB}$. It follows that $a(B)$ (which can be written as a finite intersection of such sets) is open in $\mathscr{E}$. It follows that $\varphi_n \in a(B)$ for almost all n, and $\mathscr{B}_{\mathscr{E}}$ is a pseudobasis on $\mathscr{E}$. □

Lemma 3.1.5. $a : (B^D_{\mathscr{E}}, \sqsubseteq) \to (\mathscr{B}_{\mathscr{E}}, \supseteq)$ *is pre-admissible.*

Proof. We begin by noting that a is monotone, and a is surjective simply by the definition of $\mathscr{B}_{\mathscr{E}}$. To show that a preserves suprema, let $B, C \in B^D_{\mathscr{E}}$ and suppose that $a(B)$ and $a(C)$ are consistent in $\mathscr{B}_{\mathscr{E}}$. Then $a(B) \cap a(C)$ is nonempty, and there is some $\varphi \in \mathscr{E}$ which satisfies both B and C. Since $\varphi \in \mathscr{E}$ satisfies B and C if and only if φ satisfies $B \cup C$ it follows that $B \cup C$ is satisfiable, and $a(B \sqcup C) = a(B \cup C) = a(B) \cap a(C) = a(B) \sqcup a(C)$. We conclude that a preserves suprema, and so a is pre-admissible as required. □

Lemma 3.1.6. $(D^D_{\mathscr{E}},\ D^R_{\mathscr{E}},\ \delta_{\mathscr{E}})$ *is an admissible domain representation of* $\mathscr{E}$.

Proof. That $(D^D_{\mathscr{E}},\ D^R_{\mathscr{E}},\ \delta_{\mathscr{E}})$ is admissible follows by an application of Theorem 2.3.3. To prove this, we need to show that the domain representation of $\mathscr{E}$ over $D^D_{\mathscr{E}}$ is obtained by lifting the standard representation of $\mathscr{E}$ over the pseudobasis $\mathscr{B}_{\mathscr{E}}$ along a. More precisely, we need to show that

1. $a^{-1}[J] \in D^R_{\mathscr{E}}$ for each convergent ideal $J \subseteq \mathscr{B}_{\mathscr{E}}$.

2. $a[I]$ is convergent for each $I \in D^R_{\mathscr{E}}$.

3. $\delta_{\mathscr{E}}(I) = \varphi \iff a[I] \longrightarrow \varphi$ for every $I \in D^R_{\mathscr{E}}$.

To prove 1, let $J \in \mathrm{Idl}(\mathscr{B}_{\mathscr{E}})$ and suppose that $J \longrightarrow \varphi$. Let $I = a^{-1}[J] \subseteq B^D_{\mathscr{E}}$. It is clear that I is an ideal. We will show that I represents φ in $D_{\mathscr{E}}$.

Since $a(B) \in J$ for each $B \in I$ it follows that φ satisfies each $B \in I$. Choose $x \in \mathbb{R}$, and suppose $m, n \in \mathbb{N}$. There are $S \in \mathrm{CRI}$ and $T \in \mathrm{OCB}$ such that $\varphi^{(m)}[S] \subseteq T$, $x \in \mathrm{int}(S)$ and $\mathrm{diam}(T) < 2^{-n}$. Since the set $\{\phi \in \mathscr{E};\ \phi^{(m)}[S] \subseteq T\}$ is an open neighbourhood of φ in $\mathscr{E}$ and $J \longrightarrow \varphi$ in $\mathrm{Idl}(\mathscr{B}_{\mathscr{E}})$ there is some $B \in B^D_{\mathscr{E}}$ such that $a(B) \in J$ and $\phi^{(m)}[S] \subseteq T$

for each $\phi \in a(B)$. It follows that $C = B \cup \{B^{(m)}(S,T)\} \in B^D_{\mathscr{E}}$ and $a(C) = a(B) \in J$. We may thus conclude that $C \in I$ and I represents φ in $D_{\mathscr{E}}$ as required.

Next we need to show that $a[I]$ is convergent for each $I \in D^R_{\mathscr{E}}$: Thus, let $I \in D^R_{\mathscr{E}}$ and suppose that I represents φ in $D_{\mathscr{E}}$. Let $J = a[I]$. We need to show that $J \longrightarrow \varphi$ in $\mathrm{Idl}(\mathscr{B}_{\mathscr{E}})$.

If $a(B) \in J$ then $\varphi \in a(B)$ since I represents φ. Suppose that U is an open neighbourhood of φ in $\mathscr{E}$. Choose $m, n \in \mathbb{N}$ such that $\varphi + U(m,n) \subseteq U$. Since I represents φ in $D_{\mathscr{E}}$ there is some $B \in I$ such that $a(B) \subseteq \varphi + U(m,n) \subseteq U$ and so $J \longrightarrow \varphi$ as required.

Finally, we have to show that $\delta_{\mathscr{E}}(I) = \varphi$ if and only if $a[I] \longrightarrow \varphi$ for each $I \in D^R_{\mathscr{E}}$, but this follows immediately since I represents φ in $D_{\mathscr{E}}$ if and only if $a[I] \longrightarrow \varphi$ in $\mathrm{Idl}(\mathscr{B}_{\mathscr{E}})$ for $I \in D^R_{\mathscr{E}}$.

We may now apply Theorem 7 in [6] to conclude that the dense domain representation $(D^D_{\mathscr{E}}, D^R_{\mathscr{E}}, \delta_{\mathscr{E}})$ is admissible. □

By Theorem 2.2.2 the same is true of $(D_{\mathscr{E}}, D^R_{\mathscr{E}}, \delta_{\mathscr{E}})$, and so we have the following corollary.

Corollary 3.1.7. *$(D_{\mathscr{E}}, D^R_{\mathscr{E}}, \delta_{\mathscr{E}})$ is an admissible domain representation of $\mathscr{E}$.*

Next, we consider some of the standard operations on $\mathscr{E}$. We will show that addition, multiplication, evaluation, and differentiation are effective with respect to the representation $(D_{\mathscr{E}}, D^R_{\mathscr{E}}, \delta_{\mathscr{E}})$. To do this it will be convenient to define a normal form for the elements in $B_{\mathscr{E}}$. For $B = \{B^{(n_1)}(S_1,T_1), \ldots, B^{(n_k)}(S_k,T_k)\} \in B_{\mathscr{E}}$ we define $\mathrm{nf}(B)$, the *normal form* of B, by $B^{(n)}(S,T) \in \mathrm{nf}(B)$ if and only if there are $B^{(n)}(S_{i_1},T_{i_1}), \ldots, B^{(n)}(S_{i_l},T_{i_l}) \in B$ such that

1. $S = S_{i_1} \cap \ldots \cap S_{i_l}$ is nonempty.
2. $T = T_{i_1} \cap \ldots \cap T_{i_l}$.
3. If $B^{(n_i)}(S_i,T_i) \in B$ where $n = n_i$ and $S \subseteq S_i$ then $i = i_j$ for some $j \in \mathbb{N}$ such that $1 \leq j \leq l$.

It follows immediately that $\mathrm{nf}(B) \in B_{\mathscr{E}}$ and $B \sqsubseteq \mathrm{nf}(B) \sqsubseteq B$ in $B_{\mathscr{E}}$. Thus in particular, the function $\mathrm{nf} : B_{\mathscr{E}} \to B_{\mathscr{E}}$ is monotone. We say that $B \in B_{\mathscr{E}}$ *is on normal form* if $\mathrm{nf}(B) = B$.

Intuitively, we obtain the normal form of B by removing redundant information from B. To make this a little more precise, if B is the formal neighbourhood depicted in Figure 4 then $\mathrm{nf}(B)$ is the formal neighbourhood in Figure 5.

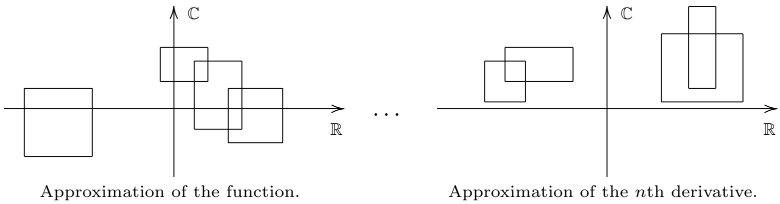


Figure 4. The formal neighbourhood B.

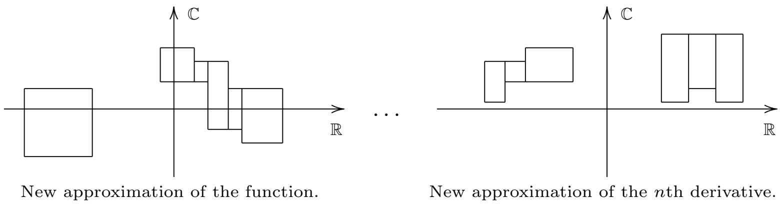


Figure 5. The normal form of B.

(It is clear that these pictures only give an intuitive idea of the relation between B and $\mathrm{nf}(B)$ since we have had to represent the complex plane as a one-dimensional object in order to be able to draw anything at all. However, these pictures do give an accurate description of what happens if we where to consider the real and imaginary parts separately.)

Proposition 3.1.8. *Evaluation* $(\varphi, r) \mapsto \varphi(r)$ *and differentiation* $(\varphi, m) \mapsto \varphi^{(m)}$ *are effective operations on* $\mathscr{E}$.

Proof. If $B = \{B^{(n_1)}(S_1, T_1), \ldots, B^{(n_k)}(S_k, T_k)\} \in B_{\mathscr{E}}$ is on normal form and $S \in B_{\mathbb{R}}$ we define $e(B, S)$ as the *greatest lower bound* of the finite set $\{\mathrm{cl}(T_i) \in B_{\mathbb{C}};\ n_i = 0, \text{and } S_i \cap S \neq \emptyset\}$ if $S \subseteq \bigcup\{S_i;\ n_i = 0\}$, and $e(B, S) = \bot$ otherwise. Then $e(B, S) \in B_{\mathbb{C}}$. We extend e to $B_{\mathscr{E}} \times B_{\mathbb{R}}$ by setting $e(B, S) = e(\mathrm{nf}(B), S)$ for arbitrary $B \in B_{\mathscr{E}}$.

It is an easy exercise in the definitions of the preorder on $B_{\mathscr{E}}$ and $\mathrm{nf} : B_{\mathscr{E}} \to B_{\mathscr{E}}$ to show that $e : B_{\mathscr{E}} \times B_{\mathbb{R}} \to B_{\mathbb{C}}$ is monotone, and so e extends to a continuous function $e : D_{\mathscr{E}} \times D_{\mathbb{R}} \to D_{\mathbb{C}}$.

To show that e represents evaluation, choose $I \in D^R_{\mathscr{E}}$ and $J \in D^R_{\mathbb{R}}$ and suppose that I represents the smooth function φ, and that J represents $r \in \mathbb{R}$. We note that $\varphi(r) \in C$ for each $C \in e(I, J)$ by the definition of $e : B_{\mathscr{E}} \times B_{\mathbb{R}} \to B_{\mathbb{C}}$. Thus, to show that $e(I, J)$ represents $\varphi(r)$ in $D_{\mathbb{C}}$ it is enough to show that for each $n \in \mathbb{N}$ there is some $C \in e(I, J)$ such that $\mathrm{diam}(C) < 2^{-n}$.

Thus, given $n \in \mathbb{N}$ we choose some $S \in J$ such that $r \in \mathrm{int}(S)$ and $\sup\{|\varphi(x) - \varphi(y)|;\ x, y \in S\} < 2^{-(n+1)}$. Choose $m \in \mathbb{N}$ such that $S \subseteq [-m, m]$ and $B \in I$ such that $\bigcap B \subseteq \varphi + U(m, n+2)$ as in Lemma

3.1.2. Since $S \subseteq [-m, m]$ we have $\bot_{\mathbb{C}} \sqsubset_{\mathbb{C}} e(B, S)$, and since $\varphi[S] \cap T_i \neq \emptyset$ and $\operatorname{diam}(T_i) < 2^{-(n+2)}$ whenever $B^{(0)}(S_i, T_i) \in B$ we must have $\operatorname{diam}(e(B, S)) < 2^{-(n+1)} + 2 \cdot 2^{-(n+2)} = 2^{-n}$. Since $e(B, S) \in e(I, J)$ it follows that $e(I, J)$ represents $\varphi(r)$ as required.

The function $e : D_{\mathscr{E}} \times D_{\mathbb{R}} \to D_{\mathbb{C}}$ is effective since the relation on $B_{\mathscr{E}} \times B_{\mathbb{R}} \times B_{\mathbb{C}}$ given by $e(B, S) = T$ is decidable.

To show that differentiation is effective, let $d : B_{\mathscr{E}} \times B_{\mathbb{N}} \to B_{\mathscr{E}}$ be the function given by $d(B, \bot_{\mathbb{N}}) = \emptyset$ and and if $m \in \mathbb{N}$ then $B^{(n)}(S, T) \in d(B, m)$ if and only if $B^{(m+n)}(S, T) \in B$. It follows immediately that d is a monotone function, and so d extends to a continuous function $d : D_{\mathscr{E}} \times D_{\mathbb{N}} \to D_{\mathscr{E}}$.

That d is effective follows immediately by the definition of d, and that d represents differentiation $(\varphi, m) \mapsto \varphi^{(m)}$ is an immediate consequence of the definition of the representing elements in $D_{\mathscr{E}}$. □

We are now in a position to give a general criterion for when a sequentially continuous function $f : X \to \mathscr{E}$ from some effectively representable space X into $\mathscr{E}$ is effective.

Theorem 3.1.9. *Let X be a topological space and let (D, D^R, δ) be an effective domain representation of X. Suppose that $f : X \to \mathscr{E}$ is sequentially continuous. Then f is effective if and only if the map $(x, r, m) \mapsto (f(x))^{(m)}(r)$ is effective.*

Proof. The direction from left to right follows immediately by Proposition 3.1.8 since effective representability is preserved under composition.

To prove the converse direction, suppose that the map $(x, r, m) \mapsto (f(x))^{(m)}(r)$ is effective and let $F : D \times D_{\mathbb{R}} \times D_{\mathbb{N}} \rightharpoonup D_{\mathbb{C}}$ be an effective partial continuous function which represents $(x, r, m) \mapsto (f(x))^{(m)}(r)$. Since the standard domain representations of $\mathbb{R}$ and $\mathbb{N}$ are dense it follows that the domain of F must be of the form $Q \times D_{\mathbb{R}} \times D_{\mathbb{N}}$ for some nonempty closed subset Q of D.

If T is the open complex box $(p, q) \times i(r, s)$ we write $c(T)$ for the closed complex box $[p+2^{-2}(q-p), q-2^{-2}(q-p)] \times i[r+2^{-2}(s-r), s-2^{-2}(s-r)]$. Note that $c(T) \in B_{\mathbb{C}}$ and $c(T) \subset T$ for each open box T in $\mathbb{C}$.

Now, given $m \in \mathbb{N}$ and $x \in Q$ we let $C_m(x) \subseteq B_{\mathscr{E}}$ be the set given by $B \in C_m(x)$ if and only if $B = \{B^{(m)}(S_1, T_1), \ldots, B^{(m)}(S_k, T_k)\}$ and $c(T_i) \sqsubseteq F(x, S_i, m)$ for each $1 \leq i \leq k$. We claim that $C_m(x)$ is directed for each $m \in \mathbb{N}$ and each $x \in Q$. Note that $C_m(x)$ is non-empty since we always have $\emptyset \in C_m(x)$. Let $B, C \in C_m(x)$ and suppose that $B^{(m)}(S_1, T_1) \in B$ and $B^{(m)}(S_2, T_2) \in C$. If $S_1 \cap S_2 \neq \emptyset$ then $S_1 \cap S_2 \in B_{\mathbb{R}}$ and $c(T_i) \sqsubseteq_{\mathbb{C}} F(x, S_i, m) \sqsubseteq_{\mathbb{C}} F(x, S_1 \cap S_2, m)$ for each $i = 1, 2$. Since $\bigcap F(x, S_1 \cap$

$S_2, m) \neq \emptyset$ and $\bigcap F(x, S_1 \cap S_2, m) \subseteq c(T_i)$ for each i, it follows that $T_1 \cap T_2 \supseteq c(T_1) \cap c(T_2) \neq \emptyset$. We conclude that $B \cup C \in B_{\mathscr{E}}$ and so $B \cup C \in C_m(x)$.

We now define $G_m : D \rightharpoonup D_{\mathscr{E}}$ by $\mathrm{dom}(G_m) = Q$ and $G_m(x) = \bigsqcup C_m(x)$ for each $x \in Q$ such that $x \neq \perp$, and $G_m(\perp) = \perp$. G_m is monotone by the definition of the set $C_m(x)$. To show that G_m is continuous suppose that A is a directed subset of Q. We need to show that $G_m(\bigsqcup A) \sqsubseteq \bigsqcup G_m[A]$. If $\bigsqcup A = \perp$ this follows immediately, so we may suppose that $\bigsqcup A \neq \perp$. If $B \sqsubseteq G_m(\bigsqcup A)$ then $B \sqsubseteq C$ for some $C \in B_{\mathscr{E}}$ such that $C = \{B^{(m)}(S_1, T_1), \ldots, B^{(m)}(S_k, T_k)\}$ and $c(T_i) \sqsubseteq F(\bigsqcup A, S_i, m)$ for each $1 \leq i \leq k$. Since $c(T_1), \ldots, c(T_k)$ are compact in $D_{\mathbb{C}}$ and F is continuous there is some $a \in A$ such that $c(T_i) \sqsubseteq F(a, S_i, m)$ for each $1 \leq i \leq k$. It follows that $B \sqsubseteq C \sqsubseteq G_m(a) \sqsubseteq \bigsqcup G_m[A]$. Since B was arbitrary we conclude that $\mathrm{approx}(G_m(\bigsqcup A)) \subseteq \mathrm{approx}(\bigsqcup G_m[A])$, and so $G_m(\bigsqcup A) \sqsubseteq \bigsqcup G_m[A]$ as required.

We note that if $B \in G_m(x)$ and $B^{(n)}(S, T) \in B$ then $m = n$. It follows that $G_{n_1}(x), G_{n_2}(x), \ldots, G_{n_k}(x)$ are consistent in $D_{\mathscr{E}}$, for all $n_1, n_2, \ldots, n_k \in \mathbb{N}$. Since $D_{\mathscr{E}}$ is consistently complete we can define $G : D \rightharpoonup D_{\mathscr{E}}$ by $\mathrm{dom}(G) = Q$ and $G(x) = \bigsqcup_{m \in \mathbb{N}} G_m(x)$ for each $x \in Q$. We claim that G is an effective partial continuous function which represents $f : X \to \mathscr{E}$.

G is a partial continuous function: G is clearly strict since each of the partial continuous functions G_m is strict, and G is monotone since G_m is monotone for each $m \in \mathbb{N}$. To show that G is continuous, let A be a directed subset of Q. We show that $G(\bigsqcup A) \sqsubseteq \bigsqcup G[A]$. Let $B = \{B^{(n_1)}(S_1, T_1), \ldots, B^{(n_k)}(S_k, T_k)\}$ and suppose that $B \sqsubseteq G(\bigsqcup A)$. Since $G(\bigsqcup A) = \bigsqcup_{m \in \mathbb{N}} G_m(\bigsqcup A)$ we have $B^{(n_i)}(S_i, T_i) \in C_i$ for some $C_i \in G_{n_i}(\bigsqcup A)$. Since G_{n_i} is continuous there is some $a_i \in A$ such that $C_i \in G_{n_i}(a_i)$. Let a be an upper bound for $a_1, \ldots, a_k$ in A. Then $\{B^{(n_i)}(S_i, T_i)\} \sqsubseteq C_i \sqsubseteq G_{n_i}(a) \sqsubseteq G(a)$ for each $1 \leq i \leq k$ and so $B \sqsubseteq G(a) \sqsubseteq \bigsqcup G[A]$. It follows that $\mathrm{approx}(G(\bigsqcup A)) \subseteq \mathrm{approx}(\bigsqcup G[A])$ and so $G(\bigsqcup A) \sqsubseteq \bigsqcup G[A]$ as required.

G is effective: To show that G is effective we need to show that the relation $B \sqsubseteq G(a)$ is r.e. uniformly in $a \in D_c \cap \mathrm{dom}(G)$. Thus, let a be some compact element in $\mathrm{dom}(G)$ and suppose that $B \in B_{\mathscr{E}}$. We note that $B \sqsubseteq G(a)$ if and only if $\{B^{(n)}(S, T)\} \sqsubseteq G_n(a)$ for each $B^{(n)}(S, T) \in B$. To check if $\{B^{(n)}(S, T)\} \sqsubseteq G_n(a)$ we simply search for some $C = \{B^{(n)}(S_1, T_1) \ldots, B^{(n)}(S_k, T_k)\} \in B_{\mathscr{E}}$ such that $\{B^{(n)}(S, T)\} \sqsubseteq C$ and $c(T_i) \sqsubseteq F(a, S_i, n)$ for each $1 \leq i \leq k$. This is r.e. since F is effective.

G represents f: Let $d \in D^R$ and suppose that $\delta(d) = x$. We need to show that $G(d)$ represents $f(x)$ in $D_{\mathscr{E}}$. Suppose first that $\{B^{(n)}(S, T)\} \in G(d)$. Since $G(d) = \bigsqcup_{m \in \mathbb{N}} G_m(d)$ we conclude that $\{B^{(n)}(S, T)\} \in G_n(d)$.

Choose $C = \{B^{(n)}(S_1, T_1), \dots, B^{(n)}(S_k, T_k)\} \in G_n(d)$ such that $\{B^{(n)}(S,T)\} \sqsubseteq C$ and $c(T_i) \sqsubseteq F(d, S_i, n)$ for each $1 \leq i \leq k$. Since F represents the map $(x, r, m) \mapsto (f(x))^{(m)}(r)$ we conclude that $(f(x))^{(n)}[S_i] \subseteq c(T_i) \subset T_i$ for each i. It follows that $f(x)$ satisfies C, and since $\{B^{(n)}(S,T)\} \sqsubseteq C$, $f(x)$ satisfies $\{B^{(n)}(S,T)\}$ as well. Since $B \in G(d)$ if and only if $\{B^{(n)}(S,T)\} \in G(d)$ for each $B^{(n)}(S,T) \in B$, we conclude that $f(x)$ satisfies every $B \in G(d)$.

Finally, let $r \in \mathbb{R}$, and let $m, n \in \mathbb{N}$. We would like to show that there is some $B \in G(d)$ such that $r \in \text{int}(S)$ and $\text{diam}(T) < 2^{-n}$ for some $B^{(m)}(S,T) \in B$. Let I be the ideal $\{S \in \text{CRI};\ r \in \text{int}(S)\}$. Then I represents r in $D_{\mathbb{R}}$. Choose T as some open complex interval with $\text{diam}(T) < 2^{-n}$ such that $(f(x))^{(m)}(r)$ lies in the interior of $c(T)$. It follows that $c(T) \sqsubseteq F(d, I, m)$. Since the map $I \mapsto F(d, I, m)$ is continuous there is some $S \in I$ such that $c(T) \sqsubseteq F(d, S, m)$. Since $c(T) \sqsubseteq F(d, S, m)$ we have $\{B^{(m)}(S,T)\} \in G_m(d) \subseteq G(d)$, and $G(d)$ represents $f(x)$ as required. □

Proposition 3.1.10. *The following operations on $\mathscr{E}$ are effective.*

1. *Scalar multiplication:* $(z, \varphi) \mapsto z \cdot \varphi$.
2. *Addition:* $(\varphi, \psi) \mapsto \varphi + \psi$.
3. *Multiplication:* $(\varphi, \psi) \mapsto \varphi \cdot \psi$.

Proof. This follows immediately by Proposition 3.1.8 and Theorem 3.1.9 since addition and multiplication on $\mathbb{C}$ are effective.

We prove 1. Let $\text{mul}_{\mathbb{C}} : D_{\mathbb{C}} \times D_{\mathbb{C}} \rightharpoonup D_{\mathbb{C}}$ be an effective representation of the map $(z, w) \mapsto z \cdot w$ from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$, and let $e : D_{\mathscr{E}} \times D_{\mathbb{R}} \rightharpoonup D_{\mathbb{C}}$ and $d : D_{\mathscr{E}} \times D_{\mathbb{N}} \rightharpoonup D_{\mathscr{E}}$ be effective partial continuous functions representing evaluation $(\varphi, r) \mapsto \varphi(r)$ and differentiation $(\varphi, m) \mapsto \varphi^{(m)}$. Let $s : D_{\mathbb{C}} \times D_{\mathscr{E}} \times D_{\mathbb{R}} \times D_{\mathbb{N}} \rightharpoonup D_{\mathbb{C}}$ be the partial continuous function given by $(I, J, K, m) \mapsto \text{mul}_{\mathbb{C}}(I, e(d(J, m), K))$. Then s represents the map given by $(z, \varphi, r, m) \mapsto z \cdot (\varphi^{(m)}(r)) = (z \cdot \varphi)^{(m)}(r)$. Since s is effective it follows immediately by Theorem 3.1.9 that scalar multiplication on $\mathscr{E}$ is effective. □

It is easy to show that the constant function which is zero everywhere is computable as an element of $\mathscr{E}$. It follows that $\mathscr{E}$ is an effective vector space.

In section 3.5 it will be useful to know that we can compute the supremum of $|\varphi(x)|$ over a given interval $[c, d]$. This result is well-known for continuous functions. For the sake of completeness we give a proof of this result for smooth functions φ.

Proposition 3.1.11. *The function* $(\varphi, c, d) \mapsto \sup\{|\varphi(x)|;\ c \leq x \leq d\}$ *is effective.*

(More precisely, we show that the function $s : \mathscr{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(\varphi, c, d) \mapsto \sup\{|\varphi(x)|;\ \min(c, d) \leq x \leq \max(c, d)\}$ is effective.)

Proof. Let $B \in B_{\mathscr{E}}$ and let $C, D \in B_{\mathbb{R}}$. We say that B is *complete* for (C, D) if the convex hull of $C \cup D$ is contained in $\bigcup\{S;\ B^{(0)}(S, T) \in B\}$. It is easy to see that if B_1 is complete for (C_1, D_1) and $(B_1, C_1, D_1) \sqsubseteq (B_2, C_2, D_2)$ in $B_{\mathscr{E}} \times B_{\mathbb{R}} \times B_{\mathbb{R}}$ then B_2 is complete for (C_2, D_2).

We write $\mathrm{abs}(z)$ for the absolute value of z. Let $B \in B_{\mathscr{E}}$ and $C, D \in B_{\mathbb{R}}$ and suppose that B is complete for (C, D). Let $u(B, C, D)$ be the real number given by $u(B, C, D) = \max\{(\sup \circ \mathrm{abs})[T];\ B^{(0)}(S, T) \in B \text{ and } S \cap H \neq \emptyset\}$, where H is the convex hull of $C \cup D$, and let $l(B, C, D)$ be the real number $l(B, C, D) = \max\{(\inf \circ \mathrm{abs})[T];\ B^{(0)}(S, T) \in B \text{ and } S \cap I(C, D) \neq \emptyset\}$, where

$$I(C, D) = \begin{cases} [\sup C,\ \inf D] & \text{if } \sup C < \inf D, \\ [\sup D,\ \inf C] & \text{if } \sup D < \inf C, \\ C \cup D & \text{if } C \cap D \neq \emptyset. \end{cases}$$

By going through the different cases it is easy to verify that $\sup\{|\varphi(x)|;\ c \leq x \leq d\} \in [l(B, C, D), u(B, C, D)]$ whenever $(B, C, D) \prec (\varphi, c, d)$. Let $s : D_{\mathscr{E}} \times D_{\mathbb{R}} \times D_{\mathbb{R}} \to D_{\mathbb{R}}$ be the continuous function given by $S \in s(B, C, D)$ if and only if B is complete for (C, D) and $[l(B, C, D), u(B, C, D)] \subseteq \mathrm{int}(S)$, and $s(B, C, D) = \bot$ otherwise. Then s is well-defined since $l(B_1, C_1, D_1) \leq l(B_2, C_2, D_2) \leq u(B_2, C_2, D_2) \leq u(B_1, C_1, D_1)$ whenever $(B_1, C_1, D_1) \sqsubseteq (B_2, C_2, D_2)$.

By considering the three cases when $c < d$, $c = d$, and $c > d$ separately, it is easy to show that s represents the function $(\varphi, c, d) \mapsto \sup\{|\varphi(x)|;\ c \leq x \leq d\}$. Finally, we note that the function s is effective since the relation $[l(B, C, D), u(B, C, D)] \subseteq \mathrm{int}(S)$ is semidecidable in B, C, D, and S. □

As a corollary of Proposition 3.1.11 we show that the seminorms $\varphi \mapsto \|\varphi\|_m$ are uniformly effective in m.

Corollary 3.1.12. *The function* $(\varphi, m) \mapsto \|\varphi\|_m$ *is effective.*

Proof. Fix $m \in \mathbb{N}$. Let $I_m = \{S \in B_{\mathbb{R}};\ -m \in S\}$, and $J_m = \{S \in B_{\mathbb{R}};\ m \in S\}$. Then I_m represents $-m$ and J_m represents m in $D_{\mathbb{R}}$. Let $d : D_{\mathscr{E}} \times D_{\mathbb{N}} \rightharpoonup D_{\mathscr{E}}$ be an effective representation of the map $(\varphi, m) \mapsto \varphi^{(m)}$, and let $s : D_{\mathscr{E}} \times D_{\mathbb{R}} \times D_{\mathbb{R}} \rightharpoonup D_{\mathbb{R}}$ be an effective representation of the map

$(\varphi, c, d) \mapsto \sup\{|\varphi(x)|;\ c \leq x \leq d\}$. Finally, let $f_m : D_{\mathbb{R}}^m \to D_{\mathbb{R}}$ be an effective representation of the map $(r_1, r_2, \ldots, r_m) \mapsto \max\{r_i;\ 1 \leq i \leq m\}$.

Now, define $\mathrm{sn_m} : D_{\mathscr{E}} \rightharpoonup D_{\mathbb{R}}$ as the partial continuous function given by $K \mapsto f_m(s(d(K,0), I_m, J_m),\ s(d(K,1), I_m, J_m),\ \ldots,\ s(d(K,m), I_m, J_m))$ if $m \in \mathbb{N}$, and $K \mapsto \perp$ otherwise. It is clear that sn_m is effective in m, and that $(K, m) \mapsto \mathrm{sn}_m(K)$ represents the seminorms $(\varphi, m) \mapsto \|\varphi\|_m$. □

In Section 3.5 we will be interested in the effective properties of convolution operators and the Fourier transform, and it will be useful to have some preliminary results about translation and reflection on $\mathscr{E}$. Thus, if φ is a smooth function from $\mathbb{R}$ to $\mathbb{C}$ we let $\mathrm{tr}_t(\varphi)$ and $\mathrm{rf}(\varphi)$ be the smooth functions given by $\mathrm{tr}_t(\varphi)(x) = \varphi(x - t)$, and $\mathrm{rf}(\varphi)(x) = \varphi(-x)$. It is clear that $\mathrm{tr}_t(\varphi)$ and $\mathrm{rf}(\varphi)$ are smooth functions, and the maps $(\varphi, t) \mapsto \mathrm{tr}_t(\varphi)$ and $\varphi \mapsto \mathrm{rf}(\varphi)$ are effective by the next proposition.

Proposition 3.1.13. *The functions $(\varphi, t) \mapsto \mathrm{tr}_t(\varphi)$ and $\varphi \mapsto \mathrm{rf}(\varphi)$ on $\mathscr{E}$ are effective.*

Proof. The map $(\varphi, t) \mapsto \mathrm{tr}_t(\varphi)$ is effective by Theorem 3.1.9 since $\mathrm{tr}_t(\varphi)^{(m)}(x) = \varphi^{(m)}(x - t)$. The map $\varphi \mapsto \mathrm{rf}(\varphi)$ is effective by the same theorem since $\mathrm{rf}(\varphi)^{(m)}(x) = (-1)^m \varphi^{(m)}(-x)$. □

3.2 An effective domain representation of $\mathscr{D}_k$

For each $k \in \mathbb{N}$ we let $\mathscr{D}_k$ denote the space of all $\varphi \in \mathscr{E}$ whose support lies in $[-k, k]$. The seminorms $\|\cdot\|_M : \mathscr{E} \to \mathbb{R}$ define a locally convex topology on $\mathscr{D}_k$ as follows: Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(m, n) = \mathscr{D}_k \cap U(m, n)$. Then $\mathscr{B}$ is a local basis for the topology τ_k on $\mathscr{D}_k$. We note that $\mathscr{D}_k$ is a subspace of $\mathscr{D}_l$ whenever $k \leq l$. We let $(\mathscr{D}, \tau_{\mathscr{D}})$ be the inductive limit of the spaces $\mathscr{D}_k$. Thus, $\mathscr{D} = \bigcup_{k \in \mathbb{N}} \mathscr{D}_k$ and $U \subseteq \mathscr{D}$ is open in $\mathscr{D}$ if and only if $U \cap \mathscr{D}_k$ is open in $\mathscr{D}_k$ for each k. With this topology $\mathscr{D}$ becomes a complete topological vector space. One can show that $\mathscr{D}$ is not locally countably based, and so in particular, $\mathscr{D}$ is not metrisable.

To construct an effective domain representation of $\mathscr{D}$ we will first construct effective domain representations of the individual spaces $\mathscr{D}_k$, and then apply Theorem 11 in [6] to construct an effective domain representation of $\mathscr{D}$ over the inductive limit of the domain representations of the individual spaces $\mathscr{D}_k$.

Fix $k \geq 1$. Since each $\varphi \in \mathscr{D}_k$ is a smooth continuous function supported in the compact interval $[-k, k]$ it is a reasonable assumption that a compact approximation of φ should contain both information about φ as an element of $\mathscr{E}$, as well as information about (i.e. bounds on) the support of φ. Thus, to construct a domain representation of $\mathscr{D}_k$ we let B_k be the set of all pairs (B, i)

where $B \in B_{\mathscr{E}}$ and $1 \leq i \leq k$, such that B is nonempty whenever $i < k$ and $0 \in T$ for each $B^{(n)}(S, T) \in B$ such that $S \setminus (-i, i) \neq \emptyset$.

We think of (B, i) as a notation for the collection of all smooth functions φ which satisfy B, with support contained in the interval $[-i, i]$. Thus, it is natural to order B_k by $(B, i) \sqsubseteq (C, j)$ if and only if $B \sqsubseteq C$ in $B_{\mathscr{E}}$ and $i \geq j$. It follows that $\sqsubseteq$ is a preorder on B_k with least element $(\emptyset, k)$. In fact,

Lemma 3.2.1. $(B_k, \sqsubseteq)$ *is a computable precusl.*

Proof. To show that $(B_k, \sqsubseteq)$ is a precusl it is enough to show that every consistent pair has a least upper bound in B_k, but this is clear since $(B \cup C, \min(i, j))$ is a least upper bound for (B, i) and (C, j) in B_k whenever (B, i) and (C, j) are consistent.

Since the order and consistency relations, as well as the partial map taking consistent pairs in B_k to a supremum (as above) are computable it follows that $(B_k, \sqsubseteq)$ is a computable precusl. □

We let D_k be the ideal completion of B_k. To define a domain representation of the space $\mathscr{D}_k$ over the effective domain D_k we say that a smooth function $\varphi \in \mathscr{D}_k$ *satisfies* $(B, i) \in B_k$ if $\varphi \in \mathscr{D}_i$ and φ satisfies B. If $I \in D_k$ then I *represents* $\varphi \in \mathscr{D}_k$ if

1. φ satisfies each pair $(B, i) \in I$.
2. For each $x \in \mathbb{R}$, and all $m, n \in \mathbb{N}$ there is a pair $(B, i) \in I$ such that $x \in \mathrm{int}(S)$ for some $B^{(m)}(S, T) \in B$ and $\mathrm{diam}(T) < 2^{-n}$.

We let D_k^R be the set $\{I \in D_k;\ I \text{ represents some } \varphi \in \mathscr{S}\}$ and define the function $\delta_k : D_k^R \to \mathscr{D}_k$ by $\delta_k(I) = \varphi$ if and only if I represents φ. It is easy to see that δ_k is well-defined and surjective.

Theorem 3.2.2. $(D_k,\ D_k^R,\ \delta_k)$ *is an effective domain representation of* $\mathscr{D}_k$.

Proof. We need to show that δ_k is continuous. Let U be some open neighbourhood of $\varphi \in \mathscr{D}_k$ and suppose that I represents φ in D_k. Choose m and n in $\mathbb{N}$ such that $\varphi + V(m, n) \subseteq U$. Since I represents φ it follows that $J = \{B \in B_{\mathscr{E}};\ (B, i) \in I \text{ for some } i\}$ represents φ in $D_{\mathscr{E}}$. It follows by Lemma 3.1.2 that there is some $B \in J$ such that $\{\phi \in \mathscr{E};\ \phi \text{ satisfies } B\} \subseteq \varphi + U(m, n)$. Choose $i \in \mathbb{N}$ such that $(B, i) \in I$. Then $\{\phi \in \mathscr{D}_i;\ \phi \text{ satisfies } B\} \subseteq (\varphi + U(m, n)) \cap \mathscr{D}_i = \varphi + V(m, n) \subseteq U$. It follows that $\{K \in D_k^R;\ (B, i) \in K\} \subseteq \delta_k^{-1}[U]$, and δ_k is continuous. □

To show that $(D_k,\ D_k^R,\ \delta_k)$ is an admissible domain representation of $\mathscr{D}_k$, we note that $(B, i) \in B_k$ is in the dense part of the representation

$(D_k,\ D_k^R,\ \delta_k)$ if and only if there is a $\varphi \in \mathscr{D}_k$ which satisfies (B,i). Thus, we let B_k^D be the set of all satisfiable pairs $(B,i) \in B_k$, and define D_k^D as the closure of B_k^D in D_k. Then D_k^D is a domain, and D_k^R is a dense subset of D_k^D by construction.

In view of Theorem 2.2.2, in order to show that the representation $(D_k,\ D_k^R,\ \delta_k)$ is admissible it is enough to show that the dense representation $(D_k^D,\ D_k^R,\ \delta_k)$ is admissible. As in the previous section, we employ Theorem 2.3.3 to the dense part of $(D_k,\ D_k^R,\ \delta_k)$. Thus, let $\mathscr{B}_k$ be the collection of all sets $\{\phi \in \mathscr{D}_i;\ \phi \text{ satisfies } B\}$ where (B,i) ranges over B_k^D. Then $\mathscr{D}_k \in \mathscr{B}_k$, and $\mathscr{B}_k$ is closed under finite nonempty intersections.

Lemma 3.2.3. *$\mathscr{B}_k$ is a pseudobasis on $\mathscr{D}_k$.*

Proof. To show that $\mathscr{B}_k$ is a pseudobasis on $\mathscr{D}_k$ suppose that $(\varphi_n)_n \longrightarrow \varphi$ in $\mathscr{D}_k$, and let U be an open neighbourhood of φ. Let m and n be natural numbers such that $\varphi + V(m,n) \subseteq U$, and let $I \in D_k^R$ represent φ in D_k. Let $J = \{B \in B_{\mathscr{E}};\ (B,i) \in I \text{ for some } i\}$. Then J represents φ in $D_{\mathscr{E}}$. By Lemma 3.1.2 there is some $B \in J$ such that $\{\phi \in \mathscr{E};\ \phi \text{ satisfies } B\} \subseteq \varphi + U(m,n)$. It follows that $\varphi \in \{\phi \in \mathscr{D}_k;\ \phi \text{ satisfies } B\} \subseteq \varphi + V(m,n) \subseteq U$, and since φ satisfies B, we have $(B,k) \in B_k^D$. Since the inclusion $\mathscr{D}_k \hookrightarrow \mathscr{E}$ is continuous we conclude that $\{\phi \in \mathscr{D}_k;\ \phi \text{ satisfies } B\}$ is open in $\mathscr{D}_k$. It follows that $\varphi_n \in \{\phi \in \mathscr{D}_k;\ \phi \text{ satisfies } B\}$ for almost all n, and so we are done. □

We let $a : (B_k^D, \sqsubseteq) \to (\mathscr{B}_k, \supseteq)$ be the monotone function which takes (B,i) to $\{\phi \in \mathscr{D}_i;\ \phi \text{ satisfies } B\}$. We would like to show that a is pre-admissible. It is easy to see that a must be monotone, and a is clearly surjective by the definition of $\mathscr{B}_k$.

Lemma 3.2.4. *$a : (B_k^D, \sqsubseteq) \to (\mathscr{B}_k, \supseteq)$ is pre-admissible.*

Proof. We need to show that a preserves suprema. Let (B,i) and (C,j) be in B_k^D, and suppose that $a(B,i)$ and $a(C,j)$ are consistent in $\mathscr{B}_k$. Choose a $\varphi \in \mathscr{D}_k$ which satisfies both (B,i) and (C,j). It follows that φ also satisfies $(B \cup C, \min(i,j)) = (B,i) \sqcup (C,j)$, and so in particular, we have $(B,i) \sqcup (C,j) \in B_k^D$. By the definition of a, we immediately get $a((B,i) \sqcup (C,j)) = a(B \cup C, \min(i,j)) = a(B,i) \cap a(C,j)$, and so a preserves suprema as required. □

Lemma 3.2.5. *$(D_k^D,\ D_k^R,\ \delta_k)$ is an admissible domain representation of $\mathscr{D}_k$.*

Proof. To apply Theorem 2.3.3 we need to show that the representation $(D_k^D,\ D_k^R,\ \delta_k)$ satisfies the following requirements:

1. $a^{-1}[J] \in D_k^R$ for each convergent ideal $J \subseteq \mathscr{B}_k$.

2. $a[I]$ is convergent for each $I \in D_k^R$.

3. $\delta_k(I) = \varphi \iff a[I] \longrightarrow \varphi$ for each $I \in D_k^R$.

To prove 1, let J be an ideal in $\mathscr{B}_k$ and suppose that J converges to φ. Let I be the ideal $a^{-1}[J] \subseteq B_k^D$. We will show that I represents φ in D_k.

Since J converges to φ we know that φ satisfies each pair $(B, i) \in I$. Choose $x \in \mathbb{R}$, and $m, n \in \mathbb{N}$. There are $S \in$ CRI and $T \in$ OCB such that $\varphi \in B^{(m)}(S,T)$, $x \in \mathrm{int}(S)$ and $\mathrm{diam}(T) < 2^{-n}$. Since $B^{(m)}(S,T) \cap \mathscr{D}_k$ is an open neighbourhood of φ in $\mathscr{D}_k$ and $J \longrightarrow \varphi$ there is some $(B,i) \in B_k^D$ such that $a(B,i) \in J$ and $a(B,i) \subseteq B^{(m)}(S,T)$. Let $C = B \cup \{B^{(m)}(S,T)\} \in B_{\mathscr{E}}$. Then $(C,i) \in B_k^D$ and $a(C,i) = a(B,i) \in J$. It follows that $(C,i) \in I$, and so I represents φ in D_k.

$a[I]$ is convergent for each $I \in D_k^R$: Choose I in D_k^R and suppose that I represents φ in D_k. Let $J = a[I]$. It is clearly enough to show that $J \longrightarrow \varphi$.

If $(B,i) \in I$ then $\varphi \in a(B,i)$ since I represents φ. Let U be an open neighbourhood of φ in $\mathscr{D}_k$, and choose $m, n \in \mathbb{N}$ such that $\varphi + V(m,n) \subseteq U$. Since I represents φ there is a pair $(B,i) \in I$ such that $\{\phi \in \mathscr{E};\ \phi$ satisfies $B\} \subseteq \varphi + U(m,n)$ in $\mathscr{E}$. It follows that $a(B,i) = \{\phi \in \mathscr{D}_i;\ \phi$ satisfies $B\} \subseteq \varphi + V(m,n) \subseteq U$ and so $J \longrightarrow \varphi$ as required.

Finally, to show that $\delta_k(I) = \varphi$ if and only if $a[I] \longrightarrow \varphi$ for each $I \in D_{\mathscr{E}}^R$ it is enough to note that $a[I] \longrightarrow \delta_k(I)$ for each $I \in D_k^R$. □

In view of Theorem 2.2.2 we may now immediately conclude that

Corollary 3.2.6. $(D_k,\ D_k^R,\ \delta_k)$ *is an admissible domain representation of* $\mathscr{D}_k$.

3.3 An effective domain representation of $\mathscr{D}$

To construct a domain representation of $\mathscr{D}$ we let $B_{\mathscr{D}}$ be the set of all pairs (B,i) where either $B \in B_{\mathscr{E}}$ is non-empty and $1 \leq i < \omega$ or $(B,i) = (\emptyset, \omega)$, and order $B_{\mathscr{D}}$ by $(B,i) \sqsubseteq (C,j)$ if and only if $B \sqsubseteq C$ and $i \geq j$. It follows that $(B_{\mathscr{D}}, \sqsubseteq)$ is a computable precusl. We let $D_{\mathscr{D}}$ be the ideal completion of $B_{\mathscr{D}}$. Then $D_{\mathscr{D}}$ is an effective domain.

As before, if $\varphi \in \mathscr{D}$ and $(B,i) \in B_{\mathscr{D}}$ we say that φ *satisfies* (B,i) if $\varphi \in \mathscr{D}_i$, and φ satisfies B, where $\mathscr{D}_\omega = \mathscr{D}$. To define a domain representation of $\mathscr{D}$ over $D_{\mathscr{D}}$ we say that $I \in D_{\mathscr{D}}$ *represents* $\varphi \in \mathscr{D}$ if

1. φ satisfies each pair $(B,i) \in I$.

2. For each $x \in \mathbb{R}$, and all $m, n \in \mathbb{N}$ there is a pair $(B,i) \in I$ such that $x \in \mathrm{int}(S)$ for some $B^{(m)}(S,T) \in B$ and $\mathrm{diam}(T) < 2^{-n}$.

We let $D_{\mathscr{D}}^R$ be the set of representing elements in $D_{\mathscr{D}}$, and define $\delta_{\mathscr{D}} : D_{\mathscr{D}}^R \to \mathscr{D}$ as the function taking I to φ if and only if I represents φ. To show that $(D_{\mathscr{D}},\, D_{\mathscr{D}}^R,\, \delta_{\mathscr{D}})$ is a domain representation of $\mathscr{D}$ we will show that $(D_{\mathscr{D}},\, D_{\mathscr{D}}^R,\, \delta_{\mathscr{D}})$ can be obtained as "the inductive limit" of the domain representations $(D_k,\, D_k^R,\, \delta_k)_k$.

If $1 \leq k \leq l$ we let $e_{k,l} : B_k \to B_l$ be the monotone function given by $(B, i) \mapsto (B, i)$ if B is nonempty, and $(\emptyset, k) \mapsto (\emptyset, l)$. It is clear that $e_{k,l}$ is monotone, and so $e_{k,l}$ extends to a continuous function $e_{k,l} : D_k \to D_l$. More is true however.

Lemma 3.3.1. *$e_{k,l} : D_k \to D_l$ is an embedding whenever $1 \leq k \leq l$.*

Proof. It is clear that $e_{k,l}$ is order-preserving and that $e_{k,l}$ takes compact elements to compact elements. To complete the proof it is enough to note that if (B, i) and (C, j) are compact in D_k and $e_{k,l}(B, i)$ and $e_{k,l}(C, j)$ are consistent in D_l, then (B, i) and (C, j) are consistent in D_k and $e_{k,l}((B, i) \sqcup (C, j)) = e_{k,l}(B, i) \sqcup e_{k,l}(C, j)$. It follows that $e_{k,l}$ is an embedding whenever $1 \leq k \leq l$. □

Note that $e_{k,k}$ is the identity on D_k for each k, and $e_{l,m} \circ e_{k,l} = e_{k,m}$ whenever $k \leq l \leq m$ in $\mathbb{N}$. It follows that $((D_k)_k,\, (e_{k,l})_{k \leq l})$ is a directed system (actually a chain) of domains.

Let $e_k : B_k \to B_{\mathscr{D}}$ be the monotone function given by $(B, i) \mapsto (B, i)$ if B is nonempty and $(\emptyset, k) \mapsto (\emptyset, \omega)$. Since e_k is monotone e_k extends to a continuous function $e_k : D_k \to D_{\mathscr{D}}$.

Lemma 3.3.2. *$e_k : D_k \to D_{\mathscr{D}}$ is an embedding for each k.*

Proof. The proof is identical to the proof of Lemma 3.3.1. □

Lemma 3.3.3. *The cone $(D_{\mathscr{D}},\, (e_k)_k)$ is an inductive limit of the directed system $((D_k)_k,\, (e_{k,l})_{k \leq l})$.*

Proof. Note that $(D_{\mathscr{D}},\, (e_k)_k)$ is a cone over $((D_k)_k,\, (e_{k,l})_{k \leq l})$ since $e_k = e_l \circ e_{k,l}$ whenever $k \leq l$.

To show that $(D_{\mathscr{D}},\, (e_k)_k)$ is an inductive limit of $((D_k)_k,\, (e_{k,l})_{k \leq l})$ it is enough to note that each compact element $(C, j) \in D_{\mathscr{D}}$ is equal to $e_k(B, i)$ for some k and some compact $(B, i) \in D_k$. If C is nonempty we have $(C, j) = e_j(C, j)$, and if $(C, j) = (\emptyset, \omega)$ we have $(C, j) = e_j(\emptyset, j)$. □

What is more interesting from our point of view is that each of the embeddings $e_{k,l} : D_k \to D_l$ restricts to a faithful extension $e_{k,l}^D : D_k^D \to D_l^D$ from D_k^D to D_l^D. It follows that the representation $(D_k^D,\, D_k^R,\, \delta_k)$ is preserved under $e_{k,l}^D$.

Lemma 3.3.4. *$e^D_{k,l} : D^D_k \to D^D_l$ is a faithful extension whenever $k \leq l$.*

Proof. It is clear that $e_{k,l}$ maps B^D_k into B^D_l. It follows that $e_{k,l}$ restricts to a continuous function $e^D_{k,l}$ from D^D_k to D^D_l. That $e^D_{k,l}$ is an embedding follows immediately since $e_{k,l}$ is an embedding.

Suppose that $I \in D^R_k$ represents φ in D_k. Then $e^D_{k,l}(I)$ represents φ in D_l since $e^D_{k,l}(I) = \{(B,i) \in B_l;\ (B,i) \sqsubseteq (C,j) \text{ for some } (C,j) \in I\}$. Conversely, if $e^D_{k,l}(I)$ represents φ in D_l, then φ satisfies each pair (B,i) in I. To show that I represents φ, choose $x \in \mathbb{R}$ and $m, n \in \mathbb{N}$. Since $e^D_{k,l}(I)$ represents φ there is a pair $(B,i) \in e^D_{k,l}(I)$ such that $x \in \operatorname{int}(S)$ and $\operatorname{diam}(T) < 2^{-n}$ for some $B^{(m)}(S,T) \in B$. Since $I \setminus \{(\emptyset, k)\}$ is cofinal in $e^D_{k,l}(I)$ there is some $(C,j) \in I$ such that $(B,i) \sqsubseteq (C,j)$ in B^D_l. It follows that $(B,j) \in I$, and hence, that I represents φ in D_k.

Finally, to show that $e^D_{k,l}$ is faithful we need to show that $(C,j) = e^D_{k,l}(B,i)$ for some $(B,i) \in D^D_k$ if and only if $\{\varphi \in \mathscr{D}_l;\ \varphi \text{ satisfies } (C,j)\} \subseteq \mathscr{D}_k$, for every compact (C,j) in D^D_l such that $(C,j) \neq \bot$. Thus, let (C,j) be a compact element in D^D_l, and suppose that $(C,j) \neq \bot$. If $(C,j) = e^D_{k,l}(B,i)$ for some $(B,i) \in D^D_k$ then $(B,i) \neq \bot$, $B \neq \emptyset$ and $1 \leq i \leq k$. It follows that $(C,j) = (B,i)$ and $\{\varphi \in \mathscr{D}_l;\ \varphi \text{ satisfies } (C,j)\} = \{\varphi \in \mathscr{D}_i;\ \varphi \text{ satisfies } B\} \subseteq \mathscr{D}_k$.

Conversely, suppose that $(C,j) \neq e^D_{k,l}(B,i)$ for each compact $(B,i) \in D^D_k$. Since $(C,j) \neq \bot$ we have $C \neq \emptyset$, and since $(C,j) \neq e^D_{k,l}(B,i)$ for each compact $(B,i) \in D^D_k$ we must have $k < j \leq l$. Choose some $\phi \in \mathscr{D}_j \setminus \mathscr{D}_k$ such that ϕ satisfies C. This is possible since $(C,j) \in B^D_l$ and $k < j$. Since ϕ satisfies (C,j) it follows that $\{\varphi \in \mathscr{D}_l;\ \varphi \text{ satisfies } (C,j)\}$ is not a subset of $\mathscr{D}_k$. □

It follows that $((D^D_k)_k, (e^D_{k,l})_{k \leq l})$ is a directed system of domains. Let $B^D_{\mathscr{D}}$ be the set of all satisfiable pairs $(B,i) \in B_{\mathscr{D}}$, and let $D^D_{\mathscr{D}}$ be the ideal completion of $B^D_{\mathscr{D}}$. Then $D^R_{\mathscr{D}} \subseteq D^D_{\mathscr{D}} \subseteq D_{\mathscr{D}}$, and $D^R_{\mathscr{D}}$ is dense in $D^D_{\mathscr{D}}$ by construction.

Since e_k maps B^D_k into $B^D_{\mathscr{D}}$, e_k restricts to an embedding $e^D_k : D^D_k \to D^D_{\mathscr{D}}$ of D^D_k into $D^D_{\mathscr{D}}$ as i Lemma 3.3.4.

Lemma 3.3.5. *The cone $(D^D_{\mathscr{D}}, (e^D_k)_k)$ is an inductive limit of $((D^D_k)_k, (e^D_{k,l})_{k \leq l})$.*

Proof. That $(D^D_{\mathscr{D}}, (e^D_k)_k)$ is a cone over the directed system $((D^D_k)_k, (e^D_{k,l})_{k \leq l})$ is immediate since we have $e^D_k = e^D_l \circ e^D_{k,l}$ for all $k \leq l$ in $\mathbb{N}$. The lemma follows since $J \in D^D_{\mathscr{D}}$ is compact if and only if J is equal to $e^D_k(B,i)$ for some k and some compact $(B,i) \in D^D_k$. □

Lemma 3.3.6. *J represents φ in $D_{\mathscr{D}}$ if and only if $e_k(I) = J$ for some $k \in \mathbb{N}$ and some $I \in D_k$ which represents φ in D_k.*

Proof. Suppose that J represents φ in $D^D_{\mathscr{D}}$, and choose k as the least natural number j such that $(C, j) \in J$ for some $C \in B_{\mathscr{E}}$. Let $I = (J \cap B_k) \cup \{(\emptyset, k)\}$. It follows that I is an ideal in B_k. We claim that I represents φ in D_k.

Since J represents φ in $D_{\mathscr{D}}$ it follows that φ satisfies each pair $(B, i) \in I$. Choose $x \in \mathbb{R}$ and $m, n \in \mathbb{N}$. Since J represents φ there is a pair $(B, i) \in J$ such that $x \in \mathrm{int}(S)$ for some $B^{(m)}(S, T) \in B$ with $\mathrm{diam}(T) < 2^{-n}$. Since $(C, k) \in J$ we must have $(B, k) \in J$, and since φ satisfies (B, k) we have $(B, k) \in B_k$. It follows that $(B, k) \in I$, and so I represents φ in D_k.

Note that if $(B, i) \in J$ then $(B, k) \in I$ as above. It follows that $I \setminus \{(\emptyset, k)\}$ is cofinal in J, and thus $e_k(I) = J$ as required.

Conversely, suppose that I represents φ in D_k, and let $e_k(I) = J$. Then $I \setminus \{(\emptyset, k)\}$ is cofinal in J, and so in particular, if $(B, i) \in J$ then φ satisfies (B, i). Since I represents φ in D_k and $I \setminus \{(\emptyset, k)\} \subseteq J$ it follows that J represents φ in $D_{\mathscr{D}}$, and so we are done. □

Lemma 3.3.7. *$(D^D_{\mathscr{D}}, D^R_{\mathscr{D}}, \delta_{\mathscr{D}})$ is an admissible domain representation of $\mathscr{D}$.*

Proof. By Lemma 3.3.6 we have $D^R_{\mathscr{D}} = \bigcup_k e_k[D^R_k]$ and $\delta_{\mathscr{D}}(e_k(I)) = \delta_k(I)$ for each k and each $I \in D^R_k$. Now the result follows by Lemmas 3.3.4 and 3.3.5 and an application of Theorem 4.11 in [6]. □

Since $(D^D_{\mathscr{D}}, D^R_{\mathscr{D}}, \delta_{\mathscr{D}})$ is the dense part of the representation $(D_{\mathscr{D}}, D^R_{\mathscr{D}}, \delta_{\mathscr{D}})$, we immediately conclude that

Corollary 3.3.8. *$(D_{\mathscr{D}}, D^R_{\mathscr{D}}, \delta_{\mathscr{D}})$ is an admissible domain representation of $\mathscr{D}$.*

We now go on to study some of the standard operations on the space $\mathscr{D}$. We will show that the inclusion maps $\mathscr{D}_k \hookrightarrow \mathscr{D} \hookrightarrow \mathscr{E}$ are effective, and that $\mathscr{D}$ is an effective vector space.

Proposition 3.3.9. *The inclusion $\mathscr{D}_k \hookrightarrow \mathscr{D}$ is effective for each k.*

Proof. This is clear since $e_k : D_k \to D_{\mathscr{D}}$ is effective and represents $\mathscr{D}_k \hookrightarrow \mathscr{D}$ by Lemma 3.3.6. □

The inclusion map from $\mathscr{D}$ into $\mathscr{E}$ is effectively representable by the (total) continuous function given by $(B, i) \mapsto B$. Thus, we have the following proposition:

Proposition 3.3.10. *The inclusion $\mathscr{D} \hookrightarrow \mathscr{E}$ is effective.*

Since the inclusion $\mathscr{D} \hookrightarrow \mathscr{E}$ is effective, we may (almost) immediately conclude that evaluation and differentiation on $\mathscr{D}$ are effective.

Proposition 3.3.11. *Evaluation* $(\varphi, r) \mapsto \varphi(r)$ *and differentiation* $(\varphi, m) \mapsto \varphi^{(m)}$ *are effective operations on* $\mathscr{D}$.

Proof. Evaluation on $\mathscr{D}$ is effective since the inclusion of $\mathscr{D}$ into $\mathscr{E}$ is effective, and evaluation on $\mathscr{E}$ is effective.

The proof that differentiation on $\mathscr{D}$ is effective is essentially the same as the proof that differentiation on $\mathscr{E}$ is effective, and so we leave it out. □

Moreover, since the inclusion of $\mathscr{D}$ into $\mathscr{E}$ is effective we may apply Theorem 3.1.9 to show that $\mathscr{D}$ is an effective vector space.

Proposition 3.3.12. *The following operations on* $\mathscr{D}$ *are effective.*

1. *Scalar multiplication:* $(z, \varphi) \mapsto z \cdot \varphi$.

2. *Multiplication by a smooth function:* $(\varphi, \psi) \mapsto \varphi \cdot \psi$ *from* $\mathscr{E} \times \mathscr{D}$ *to* $\mathscr{D}$.

3. *Addition:* $(\varphi, \psi) \mapsto \varphi + \psi$.

Proof. We prove 3. Since addition on $\mathscr{E}$ is effective it is enough to how that we can compute a bound on the support of $\varphi + \psi$ given bounds on the supports of the individual test functions φ and ψ, as in the proof of Proposition 3.4.10.

Hence, let $\mathrm{add}_{\mathscr{E}} : D_{\mathscr{E}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathscr{E}}$ be an effective partial continuous function which represents addition on $\mathscr{E}$. We note that if the supports of φ and ψ are contained in $[-k, k]$ and $[-l, l]$ respectively, then $\mathrm{supp}(\varphi + \psi) \subseteq [-\max(k, l), \max(k, l)]$. Thus, we define $\mathrm{add}_{\mathscr{D}} : D_{\mathscr{D}} \times D_{\mathscr{D}} \rightharpoonup D_{\mathscr{D}}$ as follows: $\mathrm{dom}(\mathrm{add}_{\mathscr{D}})$ is the closure of the set of all pairs of compact elements $((B_1, k_1),\ (B_2, k_2))$ such that $(B_1, B_2) \in \mathrm{dom}(\mathrm{add}_{\mathscr{E}})$, and if $((B_1, k_1),\ (B_2, k_2))$ is in $\mathrm{dom}(\mathrm{add}_{\mathscr{D}})$, then $\mathrm{add}_{\mathscr{D}}((B_1, k_1),\ (B_2, k_2)) =$

$$\{(C, j) \in B_{\mathscr{D}};\ C \sqsubseteq \mathrm{add}_{\mathscr{E}}(B_1, B_2) \text{ and } j \sqsubseteq \max(k_1, k_2)\}.$$

$\mathrm{add}_{\mathscr{D}}$ is clearly well-defined and it is easy to show that $\mathrm{add}_{\mathscr{D}}$ is monotone. It follows that $\mathrm{add}_{\mathscr{D}}$ extends to a partial continuous function from $D_{\mathscr{D}} \times D_{\mathscr{D}}$ to $D_{\mathscr{D}}$.

That $\mathrm{add}_{\mathscr{D}}$ represents addition on $\mathscr{D}$ is routine to show using the fact that $\mathrm{add}_{\mathscr{E}}$ represents addition on $\mathscr{E}$. □

The translation map takes elements in $\mathscr{D}$ to elements in $\mathscr{D}$, and the same is clearly true for reflection. Moreover,

Proposition 3.3.13. *The functions* $(\varphi, t) \mapsto \mathrm{tr}_t(\varphi)$ *and* $\varphi \mapsto \mathrm{rf}(\varphi)$ *on* $\mathscr{D}$ *are effective.*

Proof. Let $f : D_{\mathscr{D}} \times D_{\mathbb{R}} \rightharpoonup D_{\mathscr{E}}$ be an effective partial continuous function which represents the map $(\varphi, t) \mapsto \mathrm{tr}_t(\varphi)$ from $\mathscr{D} \times \mathbb{R}$ into $\mathscr{E}$. The partial continuous function f exists by Propositions 3.1.13 and 3.3.10.

Now, let φ be a smooth function with compact support contained in the interval $[-k, k]$. The support of $\mathrm{tr}_t(\varphi)$ must then be contained in the interval $[-k+t, k+t]$. Thus, let $g : D_{\mathscr{D}} \times D_{\mathbb{R}} \rightharpoonup D_{\mathscr{D}}$ be the partial continuous function given by $\mathrm{dom}(g) = \mathrm{dom}(f)$, and $(B, i) \in g(I, J)$ if and only if $B \in f(I, J)$ and there are $(C, j) \in I$, and $S \in J$ such that $i \geq j + \sup\{|x|;\ x \in S\}$.

It is clear that $g(I, J)$ is an ideal in $B_{\mathscr{D}}$ for each pair $(I, J) \in \mathrm{dom}(g)$, and g is clearly monotone. To show that g is continuous, suppose that $(B, i) \in g(I, J)$. Then, there are $(C, j) \in I$, and $S \in J$ such that $i \geq j + \sup\{|x|;\ x \in S\}$, and so it follows that $(B, i) \in g((C, j), S)$ and g is continuous. Finally, we note that g is strict since f is. It is clear from the definition of g that g represents the map $(\varphi, t) \mapsto \mathrm{tr}_t(\varphi)$ on $\mathscr{D}$.

To show that $\varphi \mapsto \mathrm{rf}(\varphi)$ is effective let $h : D_{\mathscr{D}} \rightharpoonup D_{\mathscr{E}}$ be an effective representation of the map $\varphi \mapsto \mathrm{rf}(\varphi)$ from $\mathscr{D}$ to $\mathscr{E}$. Let $k : D_{\mathscr{D}} \rightharpoonup D_{\mathscr{D}}$ be given by $\mathrm{dom}(k) = \mathrm{dom}(h)$ and $k(I) = \{(B, i) \in B_{\mathscr{D}};\ B \in h(C, i)$ for some $(C, i) \in I\}$. It is easy to show that k is continuous, and k is strict since h is. Finally, k represents the map $\varphi \mapsto \mathrm{rf}(\varphi)$ on $\mathscr{D}$ since $\mathrm{supp}(\varphi) \subseteq [-k, k] \implies \mathrm{supp}(\mathrm{rf}(\varphi)) \subseteq [-k, k]$. □

3.4 An effective domain representation of $\mathscr{S}$

We let $\mathscr{S}$ be the set of all $\varphi \in \mathscr{E}$ such that $\sup\{|x^k \varphi^{(m)}(x)|;\ x \in \mathbb{R}$ and $m \leq k\} < \infty$ for each $k \in \mathbb{N}$. In other words, $\varphi \in \mathscr{S}$ if φ is a smooth function from $\mathbb{R}$ to $\mathbb{C}$ such that $p \cdot \varphi^{(m)}$ is bounded on $\mathbb{R}$ for every polynomial p and every natural number m. These functions form a vector space sometimes known as the space of *rapidly decreasing functions*. Note that $p \cdot \varphi \in \mathscr{S}$ for each polynomial p and each $\varphi \in \mathscr{S}$. It follows that $\varphi \in L^1(\mathbb{R})$ for each $\varphi \in \mathscr{S}$. The countable family of norms given by

$$\|\varphi\|_{k,n} = \sup\{|x^k \varphi^{(m)}(x)|;\ x \in \mathbb{R} \text{ and } m \leq n\}$$

where $k, n \in \mathbb{N}$ defines a locally convex topology on $\mathscr{S}$ which we denote by $\tau_{\mathscr{S}}$. A local basis for $\tau_{\mathscr{S}}$ is given by the collection of all finite intersections of sets of the form $W(k, l) = \{\varphi \in \mathscr{S};\ \|\varphi\|_{k,k} < 2^{-l}\}$ where $k, l \in \mathbb{N}$.

Since $\mathscr{S} \subseteq \mathscr{E}$ it is a reasonable assumption that a compact approximation B of a rapidly decreasing smooth function φ should contain both information about φ as a smooth function from $\mathbb{R}$ to $\mathbb{C}$ as well as information about (that is, bounds on) $|x^k \varphi^{(m)}(x)|$. To make this idea more precise, let $R(k, m, n)$ be a notation for the set of all $\varphi \in \mathscr{S}$ such that $\sup\{|x^k \varphi^{(m)}(x)|;\ x \in \mathbb{R}\} < n$.

Let B_R be the collection of all finite sets $\{R(k_1, m_1, n_1), \ldots, R(k_j, m_j, n_j)\}$, and order B_R by $B \sqsubseteq C \iff$ for each $R(k, m, n_1) \in B$ there is some $R(k, m, n_2) \in C$ such that $n_1 \geq n_2$. It is clear that $\sqsubseteq$ is a preorder on B_R with least element $\emptyset$. Furthermore, if $B, C \in B_R$ then $B \cup C \in B_R$ and $B \cup C$ is a least upper bound for B and C in B_R. It follows that B_R is a computable precusl.

If φ is a rapidly decreasing smooth function and $B \in B_R$ we say that φ *satisfies* B if $\sup\{|x^k \varphi^{(m)}(x)|;\ x \in \mathbb{R}\} < n$ for each $R(k, m, n) \in B$. We note that every rapidly decreasing smooth function φ satisfies $\emptyset \in B_R$, and if $B \sqsubseteq C$ in B_R, then $\{\phi \in \mathscr{S};\ \phi \text{ satisfies } B\} \supseteq \{\phi \in \mathscr{S};\ \phi \text{ satisfies } C\}$.

Now, let D_R be the ideal completion of B_R and define $D_{\mathscr{S}}$ as the product $D_{\mathscr{E}} \times D_R$. Then $D_{\mathscr{S}}$ is an effective domain. To define a domain representation of $\mathscr{S}$ over $D_{\mathscr{S}}$ we say that $(I, J) \in D_{\mathscr{S}}$ *represents* $\varphi \in \mathscr{S}$ if

1. I represents φ in $D_{\mathscr{E}}$.
2. φ satisfies each $B \in J$.
3. Given k and m in $\mathbb{N}$ there is some $n \in \mathbb{N}$ and $B \in J$ such that $R(k, m, n) \in B$.

We let $D^R_{\mathscr{S}}$ be the set of representing elements in $D_{\mathscr{S}}$ and define $\delta_{\mathscr{S}} : D^R_{\mathscr{S}} \to \mathscr{S}$ by $\delta_{\mathscr{S}}(I, J) = \varphi$ if and only if (I, J) represents φ. To show that $(D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ is a domain representation of $\mathscr{S}$ it will be convenient to first prove the following lemma.

Lemma 3.4.1. *Suppose that $(I, J) \in D_{\mathscr{S}}$ represents $\varphi \in \mathscr{S}$. Then for each open set $W(k, l) \subseteq \mathscr{S}$ there are $B \in I$ and $C \in J$ such that*

$$\{\phi \in \mathscr{S};\ \phi \textit{ satisfies } B\} \cap \{\phi \in \mathscr{S};\ \phi \textit{ satisfies } C\} \subseteq \varphi + W(k, l).$$

Proof. Since (I, J) represents φ in $D_{\mathscr{S}}$ there are natural numbers $n_1, n_2, \ldots, n_k$ and $C_1, C_2, \ldots, C_k$ in J such that $R(k+1, i, n_i) \in C_i$ for each $1 \leq i \leq k$. Let $N = \max\{n_i;\ 1 \leq i \leq k\}$, and let C be an upper bound for $C_1, C_2, \ldots, C_k$ in J. By construction, if $\phi \in \mathscr{S}$ satisfies C then $|x^k \phi^{(m)}(x)| < N|x|^{-1}$ for each $x \neq 0$ and $m \leq k$.

Choose $M \in \mathbb{N}$ such that $N|x|^{-1} < 2^{-(l+2)}$ for $|x| \geq M$. Since the set $U \subseteq \mathscr{E}$ given by $U = \{\phi \in \mathscr{E};\ |x^k \phi^{(m)}(x)| < 2^{-(l+1)}$ for every $|x| \leq M$ and $m \leq k\}$ is open in $\mathscr{E}$ there are natural numbers m and n such that $\varphi + U(m, n) \subseteq \varphi + U$. Since I represents φ in $D_{\mathscr{E}}$ it follows by Lemma 3.1.2 that $\{\phi \in \mathscr{E};\ \phi \text{ satisfies } B\} \subseteq \varphi + U(m, n) \subseteq \varphi + U$ for some $B \in I$.

Now, suppose that $\phi \in \mathscr{S}$ satisfies both B and C. We need to show that the difference $\phi - \varphi$ is in $W(k, l)$. Since ϕ satisfies B we know that $\phi - \varphi$ is in

U, and so, that $|x^k(\phi - \varphi)^{(m)}(x)| < 2^{-(l+1)}$ for each $m \leq k$ on the interval $[-M, M]$. Since ϕ and φ both satisfy C we know that $|x^k(\phi - \varphi)^{(m)}(x)| \leq |x^k\phi^{(m)}(x)| + |x^k\varphi^{(m)}(x)| < 2^{-(l+2)} + 2^{-(l+2)} = 2^{-(l+1)}$ for each $m \leq k$ if $|x| \geq M$. We thus conclude that $\|\phi - \varphi\|_{k,k} \leq 2^{-(l+1)} < 2^{-l}$ as required. □

Theorem 3.4.2. *$(D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ is an effective domain representation of $\mathscr{S}$.*

Proof. That $D_{\mathscr{S}}$ is an effective domain is immediate from the definition of $D_{\mathscr{S}}$. To show that $(D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ is a domain representation of $\mathscr{S}$ we need to show that $\delta_{\mathscr{S}} : D^R_{\mathscr{S}} \to \mathscr{S}$ is well-defined, surjective and continuous.

The function $\delta_{\mathscr{S}}$ is clearly well-defined since I represents φ in $D_{\mathscr{E}}$ whenever (I, J) represents φ in $D_{\mathscr{S}}$. To show that $\delta_{\mathscr{S}}$ is surjective, choose some $\varphi \in \mathscr{S}$ and let $I = \{B \in B_{\mathscr{E}};\ \varphi \text{ satisfies } B\}$, and $J = \{B \in B_R;\ \varphi \text{ satisfies } B\}$. I and J are ideals and it follows immediately from the definition of the representing elements in $D_{\mathscr{S}}$ that (I, J) represents φ. Thus, $\delta_{\mathscr{S}}(I, J) = \varphi$ and $\delta_{\mathscr{S}}$ is surjective as required.

Finally, to show that $\delta_{\mathscr{S}}$ is continuous, let $U \subseteq \mathscr{S}$ be open in $\mathscr{S}$. Let $\varphi \in U$ and suppose that (I, J) represents φ.

Since $\varphi \in U$ there are natural numbers k and l such that $\varphi + W(k, l) \subseteq U$, and since (I, J) represents φ there are $B \in I$ and $C \in J$ such that $\{\phi \in \mathscr{S};\ \phi \text{ satisfies } B\} \cap \{\phi \in \mathscr{S};\ \phi \text{ satisfies } C\} \subseteq \varphi + W(k, l)$. It follows that $W = \{(K, L) \in D^R_{\mathscr{S}};\ B \in K \text{ and } C \in L\}$ is an open neighbourhood of (I, J) in $D^R_{\mathscr{S}}$ such that $W \subseteq \delta^{-1}_{\mathscr{S}}[U]$. We conclude that $\delta^{-1}_{\mathscr{S}}[U]$ is open in $D^R_{\mathscr{S}}$ and $\delta_{\mathscr{S}}$ is continuous. □

To show that the domain representation $(D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ is admissible we apply Theorem 2.3.2 to the dense part of the representation $(D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ as in the previous section.

A compact element (B, C) in $D_{\mathscr{S}}$ is in the dense part of $D_{\mathscr{S}}$ if and only if there is some $\varphi \in \mathscr{S}$ which satisfies both B and C. Thus, we let $B^D_{\mathscr{S}}$ be the set $B^D_{\mathscr{S}} = \{(B, C) \in B_{\mathscr{E}} \times B_R;$ there is a $\varphi \in \mathscr{S}$ which satisfies both C and $D\}$, and let $D^D_{\mathscr{S}}$ be the closure of $B^D_{\mathscr{S}}$ in $D_{\mathscr{S}}$. $D^D_{\mathscr{S}}$ is a domain, and $D^R_{\mathscr{S}}$ is dense in $D^D_{\mathscr{S}}$ by construction. To show that the domain representation $(D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ is admissible it is sufficient to show that the dense representation $(D^D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ is admissible by Theorem 2.2.2.

To be able to apply Theorem 2.3.2 to $(D^D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ we need to construct a pseudobasis $\mathscr{B}_{\mathscr{S}}$ on $\mathscr{S}$ and a pre-admissible function $a : B^D_{\mathscr{S}} \to \mathscr{B}_{\mathscr{S}}$ from $B^D_{\mathscr{S}}$ to $\mathscr{B}_{\mathscr{S}}$. To define a pseudobasis on $\mathscr{S}$, we let $\mathscr{B}_{\mathscr{S}}$ be the collection of all sets of the form $\{\phi \in \mathscr{S};\ \phi \text{ satisfies } B\} \cap \{\phi \in \mathscr{S};\ \phi \text{ satisfies } C\}$ where (B, C) ranges over $B^D_{\mathscr{S}}$. We have $\mathscr{S} = \{\phi \in \mathscr{S};\ \phi \text{ satisfies } \emptyset\} \cap \{\phi \in \mathscr{S};\ \phi$

satisfies $\emptyset\} \in \mathscr{B}_{\mathscr{S}}$, and it is clear that $\mathscr{B}_{\mathscr{S}}$ is closed under finite nonempty intersections.

Lemma 3.4.3. *$\mathscr{B}_{\mathscr{S}}$ is a pseudobasis on $\mathscr{S}$.*

Proof. Suppose that $(\varphi_n)_n \longrightarrow \varphi$ in $\mathscr{S}$ and let U be an open neighbourhood of φ in $\mathscr{S}$. Choose $k, l \in \mathbb{N}$ such that $\varphi + W(k,l) \subseteq U$. By Lemma 3.4.1 there is a pair $(B, C) \in B^D_{\mathscr{S}}$ such that $\varphi \in \{\phi \in \mathscr{S};\ \phi \text{ satisfies } B\} \cap \{\phi \in \mathscr{S};\ \phi \text{ satisfies } C\} \subseteq \varphi + W(k,l)$.

The set $\{\phi \in \mathscr{S};\ \phi \text{ satisfies } B\}$ is open in $\mathscr{S}$ since the inclusion $\mathscr{S} \hookrightarrow \mathscr{E}$ is continuous, and the set $\{\phi \in \mathscr{S};\ \phi \text{ satisfies } C\}$ is open in $\mathscr{S}$ since the set of all $\phi \in \mathscr{S}$ such that $\sup\{|x^k\phi^{(m)}(x)|;\ x \in \mathbb{R}\} < n$ is open in $\mathscr{S}$ for all natural numbers k, m, and n. It follows that $\varphi_n \in \{\phi \in \mathscr{S};\ \phi \text{ satisfies } B\} \cap \{\phi \in \mathscr{S};\ \phi \text{ satisfies } C\}$ for almost all n and we are done. □

Let $a : (B^D_{\mathscr{S}}, \sqsubseteq) \to (\mathscr{B}_{\mathscr{S}}, \supseteq)$ be the function given by $a(B, C) = \{\phi \in \mathscr{S};\ \phi \text{ satisfies } B\} \cap \{\phi \in \mathscr{S};\ \phi \text{ satisfies } C\}$. The function a is clearly monotone and a is surjective by construction.

Lemma 3.4.4. *$a : (B^D_{\mathscr{S}}, \sqsubseteq) \to (\mathscr{B}_{\mathscr{S}}, \supseteq)$ is pre-admissible.*

Proof. It is enough to show that a preserves suprema. Let (B_1, C_1), $(B_2, C_2) \in B^D_{\mathscr{S}}$ and suppose that $a(B_1, C_1)$ and $a(B_2, C_2)$ are consistent in $\mathscr{B}_{\mathscr{S}}$. Then $a(B_1, C_1) \cap a(B_2, C_2)$ is nonempty. It follows that there is a smooth function $\varphi \in \mathscr{S}$ which satisfies B_1, B_2, C_1, and C_2. Since φ satisfies both B_1 and B_2 we conclude that B_1 and B_2 are consistent in $B_{\mathscr{E}}$. Let $B = B_1 \sqcup B_2$ and let $C = C_1 \sqcup C_2$. Then φ satisfies both B and C, and so (B, C) is a least upper bound for (B_1, C_1) and (B_2, C_2) in $B^D_{\mathscr{S}}$. Since a smooth function $\phi \in \mathscr{S}$ satisfies B and C if and only if it satisfies B_1, B_2, C_1, and C_2, we conclude that $a(B, C) = a(B_1, C_1) \cap a(B_2, C_2)$ as required. □

Since $a : B^D_{\mathscr{S}} \to \mathscr{B}_{\mathscr{S}}$ is pre-admissible, a extends to an admissible function $a : D^D_{\mathscr{S}} \to \mathrm{Idl}(\mathscr{B}_{\mathscr{S}})$. Let $s : \mathrm{Idl}(\mathscr{B}_{\mathscr{S}}) \to D^D_{\mathscr{S}}$ be the right adjoint of a.

Lemma 3.4.5. *$(D^D_{\mathscr{S}},\ D^R_{\mathscr{S}},\ \delta_{\mathscr{S}})$ is an admissible domain representation of $\mathscr{S}$.*

Proof. That $(D^D_{\mathscr{S}},\ D^R_{\mathscr{S}},\ \delta_{\mathscr{S}})$ is admissible follows by an application of Theorem 2.3.2. We need to show that

1. $s(K) \in D^R_{\mathscr{S}}$ for each convergent ideal $K \subseteq \mathscr{B}_{\mathscr{S}}$.
2. $a(I, J)$ converges for each $(I, J) \in D^R_{\mathscr{S}}$.
3. $\delta_{\mathscr{S}}(I, J) = \varphi \iff a(I, J) \longrightarrow \varphi$ for each $(I, J) \in D^R_{\mathscr{S}}$.

To prove that $s(K) \in D^R_{\mathscr{S}}$ for each convergent ideal $K \subseteq \mathscr{B}_{\mathscr{S}}$, suppose that $K \subseteq \mathscr{B}_{\mathscr{S}}$ is an ideal and that $K \longrightarrow \varphi$. Let $s(K) = (I, J)$. It is clearly enough to show that (I, J) represents φ in $D_{\mathscr{S}}$.

By construction, we have $(I, J) = \bigsqcup\{(B, C) \in B^D_{\mathscr{S}};\ a(B, C) \in K\}$. Suppose that $B \in I$ and $C \in J$. Then φ satisfies B and C since $a(B, C) \in K$. Choose $x \in \mathbb{R}$, and $m, n \in \mathbb{N}$. We would like to show that there is some $B \in I$, and $B^{(m)}(S, T) \in B$ such that $x \in \mathrm{int}(S)$, and $\mathrm{diam}(T) < 2^{-n}$. To do this, choose $S \in$ CRI and $T \in$ OCB such that $\varphi^{(m)}[S] \subseteq T$, $x \in \mathrm{int}(S)$ and $\mathrm{diam}(T) < 2^{-n}$. Since $\{\phi \in \mathscr{S}; \phi^{(m)}[S] \subseteq T\}$ is an open neighbourhood of φ in $\mathscr{S}$ and $K \longrightarrow \varphi$ in $\mathrm{Idl}(\mathscr{B}_{\mathscr{S}})$ there is some $(B, C) \in B^D_{\mathscr{S}}$ such that $a(B, C) \in K$ and $a(B, C) \subseteq \{\phi \in \mathscr{S}; \phi^{(m)}[S] \subseteq T\}$. It follows that $(B \cup \{B^{(m)}(S, T)\}, C) \in B^D_{\mathscr{S}}$ and $a(B \cup \{B^{(m)}(S, T)\}, C) = a(B, C) \in K$. By the definition of (I, J) we have $B \cup \{B^{(m)}(S, T)\} \in I$.

Finally, let $k, m \in \mathbb{N}$ and choose $n \in \mathbb{N}$ such that $\sup\{|x^k \varphi^{(m)}(x)|;\ x \in \mathbb{R}\} < n$. Since $\{\phi \in \mathscr{S};\ \sup_{x\in\mathbb{R}} |x^k \phi^{(m)}(x)| < n\}$ is an open neighbourhood of φ in $\mathscr{S}$ there is a pair $(B, C) \in B^D_{\mathscr{S}}$ such that $a(B, C) \subseteq \{\phi \in \mathscr{S};\ \sup_{x\in\mathbb{R}} |x^k \phi^{(m)}(x)| < n\}$ and $a(B, C) \in K$. It follows that $C \cup \{R(k, m, n)\} \in J$ as above and so (I, J) represents φ as required.

To show that $a(I, J)$ is convergent for each $(I, J) \in D^R_{\mathscr{S}}$, let $(I, J) \in D^R_{\mathscr{S}}$, and suppose that (I, J) represents φ in $D_{\mathscr{S}}$. We would like to show that the ideal $a(I, J)$ converges to φ in $\mathrm{Idl}(\mathscr{B}_{\mathscr{S}})$. It is clear that $\varphi \in a(B, C)$ for each $B \in I$, and $C \in J$. Let U be an open neighbourhood of φ and choose k and l in $\mathbb{N}$ such that $\varphi + W(k, l) \subseteq U$. Since (I, J) represents φ there are $B \in I$ and $C \in J$ such that $a(B, C) \subseteq \varphi + W(k, l) \subseteq U$. Since $a(B, C) \in a(I, J)$ we see that $a(I, J)$ converges to φ in $\mathrm{Idl}(\mathscr{B}_{\mathscr{S}})$.

That $\delta_{\mathscr{S}}(I, J) = \varphi$ if and only if $a(I, J) \longrightarrow \varphi$ when $(I, J) \in D^R_{\mathscr{S}}$ follows immediately, since $a(I, J)$ converges to $\delta_{\mathscr{S}}(I, J)$ for each representing element $(I, J) \in D_{\mathscr{S}}$. □

The following corollary is now an immediate consequence of Theorem 2.2.2 and Lemma 3.4.5.

Corollary 3.4.6. *$(D_{\mathscr{S}}, D^R_{\mathscr{S}}, \delta_{\mathscr{S}})$ is an admissible domain representation of $\mathscr{S}$.*

It is clear that $\mathscr{D} \subseteq \mathscr{S}$, and it is easy to show that the inclusion $\mathscr{D} \hookrightarrow \mathscr{S}$ is continuous. More is true however.

Proposition 3.4.7. *The inclusion $\mathscr{D} \hookrightarrow \mathscr{S}$ is effective.*

Proof. If $B \in B_{\mathscr{E}}$ we say that B is *m-complete* on the interval $[-i, i]$, if for each x in the interval $[-i, i]$ there is some $B^{(m)}(S, T)$ such that $x \in \mathrm{int}(S)$. It is clear that if $(B, i) \sqsubseteq (C, j)$ in $B_{\mathscr{D}}$ and B is m-complete on $[-i, i]$ then C must also be m-complete on $[-j, j]$.

We define $b : B_{\mathscr{D}} \to D_R$ by $C \in b(B, i)$ if and only if $R(k, m, n) \in C$ implies that B is m-complete for $[-i, i]$, and for each $B^{(m)}(S, T) \in \mathrm{nf}(B)$ such that $S \cap [-i, i] \neq \emptyset$ we have $\sup\{|x|^k|z| \in \mathbb{R};\ x \in S \cap [-i, i]$ and $z \in T\} < n$. b is well-defined since $b(B, i)$ is an ideal for each $(B, i) \in B_{\mathscr{D}}$. To see that b is monotone, let $(B, i) \sqsubseteq (B', i')$ in $B_{\mathscr{D}}$. We may assume that B and B' are on normal form. Suppose that $C \in b(B, i)$. If $R(k, m, n) \in C$ then B is m-complete for $[-i, i]$. It follows that B' is m-complete for $[-i', i']$. Let $B^{(m)}(S', T') \in B'$, and suppose that $S \cap [-i', i'] \neq \emptyset$. Since B is m-complete for $[-i, i] \supseteq [-i', i']$, there are $B^{(m)}(S_1, T_1), B^{(m)}(S_2, T_2), \ldots, B^{(m)}(S_k, T_k) \in B$ such that $S' \cap [-i', i'] \subseteq (S_1 \cup S_2 \cup \ldots \cup S_k) \cap [-i, i]$, and $T' \subseteq T_j$ for each $1 \leq j \leq k$. It follows that $\sup\{|x|^k|z| \in \mathbb{R};\ x \in S' \cap [-i', i']$ and $z \in T'\} \leq \sup\{|x|^k|z| \in \mathbb{R};\ 1 \leq j \leq k, x \in S_j \cap [-i, i]$ and $z \in T_j\} < n$, and so we conclude that $C \in b(B', i')$.

Since b is monotone, b extends to a continuous function $b : D_{\mathscr{D}} \to D_R$. Suppose that $I \in D_{\mathscr{D}}$ represents $\varphi \in \mathscr{D}$. Let $k, m \in \mathbb{N}$. We would like to show that there is some $n \in \mathbb{N}$, and $C \in b(I)$ such that $R(k, m, n) \in C$. Since I represents φ, there is some $(B, i) \in I$ such that B is m-complete on the interval $[-i, i]$. Choose $n \in \mathbb{N}$ such that $\sup\{|x|^k|z| \in \mathbb{R};\ x \in S \cap [-i, i]$ and $z \in T\} < n$ for each $B^{(m)}(S, T) \in B$. It follows that $\{R(k, m, n)\} \in b(B, i) \subseteq b(I)$.

We now define $i : D_{\mathscr{D}} \to D_{\mathscr{S}}$ by $i(I) = (j(I), b(I))$, where $j : D_{\mathscr{D}} \to D_{\mathscr{E}}$ is the map given by $j(I) = \{B \in B_{\mathscr{E}};\ (B, i) \in I$ for some $i \in D_S\}$. The function i is clearly continuous, and effective since j and b are effective. That i represents $\mathscr{D} \hookrightarrow \mathscr{S}$ follows by the argument above. □

The same is true for the inclusion map from $\mathscr{S}$ into $\mathscr{E}$. Thus, we have the following proposition:

Proposition 3.4.8. *The inclusion $\mathscr{S} \hookrightarrow \mathscr{E}$ is effective.*

Proof. If (I, J) represents φ in $D_{\mathscr{S}}$, then I represents φ in $D_{\mathscr{E}}$. It follows that the inclusion $\mathscr{S} \hookrightarrow \mathscr{E}$ is effectively representable by the map $(I, J) \mapsto I$. □

Using the fact that the inclusion map from $\mathscr{S}$ into $\mathscr{E}$ is effective, It is now easy to show that evaluation as well as differentiation on $\mathscr{S}$ are effective.

Proposition 3.4.9. *Evaluation $(\varphi, r) \mapsto \varphi(r)$ and differentiation $(\varphi, m) \mapsto \varphi^{(m)}$ are effective operations on $\mathscr{S}$.*

Proof. That evaluation on $\mathscr{S}$ is effective follows immediately by Propositions 3.1.8 and 3.4.8.

That differentiation on $\mathscr{S}$ is effective is shown by reindexing, much as in the proof that differentiation on $\mathscr{E}$ is effective. □

We may also apply Theorem 3.1.9 to show that $\mathscr{S}$ is an effective vector space.

Proposition 3.4.10. *The following operations on $\mathscr{S}$ are effective.*

1. *Scalar multiplication:* $(z, \varphi) \mapsto z \cdot \varphi$.
2. *Multiplication:* $(\varphi, \psi) \mapsto \varphi \cdot \psi$.
3. *Addition:* $(\varphi, \psi) \mapsto \varphi + \psi$.

Proof. We start by considering scalar multiplication. If $T \in B_{\mathbb{C}}$ and $T \neq \perp$ we let $a(T)$ denote the least natural number a such that $\sup\{|z|;\ z \in T\} < a$. Now, given $T \in B_{\mathbb{C}}$ and $B \in B_R$ we define $\mathrm{smul}_R(T, B) \in B_R$ by $R(k, m, n) \in \mathrm{smul}_R(T, B)$ if $T \neq \perp$, and there is some $R(k, m, b) \in B$ such that $n = a(T) \cdot b$. It is clear that $\mathrm{smul}_R : B_{\mathbb{C}} \times B_R \to B_R$ is well-defined and monotone, and so smul_R extends to a continuous function $\mathrm{smul}_R : D_{\mathbb{C}} \times D_R \to D_R$.

Suppose that I represents z in $D_{\mathbb{C}}$, and that (J, K) represents φ in $D_{\mathscr{S}}$. If $A \in \mathrm{smul}_R(I, K)$ there are $T \in I$, and $B \in K$ such that $A \sqsubseteq \mathrm{smul}_R(T, B)$. It follows that $z \cdot \varphi$ satisfies A. Let $k, m \in \mathbb{N}$. Since (J, K) represents φ there is some $b \in \mathbb{N}$ and $B \in K$ such that $R(k, m, b) \in B$. Choose T in I such that $T \neq \perp$. Then $R(k, m, a(T) \cdot b) \in \mathrm{smul}_R(T, B) \in \mathrm{smul}_R(I, K)$.

Let $\mathrm{smul}_{\mathscr{E}} : D_{\mathbb{C}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathscr{E}}$ be an effective partial continuous function which represents scalar multiplication on $\mathscr{E}$. We define $\mathrm{smul}_{\mathscr{S}} : D_{\mathbb{C}} \times D_{\mathscr{S}} \rightharpoonup D_{\mathscr{S}}$ by $\mathrm{dom}(\mathrm{smul}_{\mathscr{S}})$ as the set $\{(I, J, K) \in D_{\mathbb{C}} \times D_{\mathscr{E}} \times D_R;\ (I, J) \in \mathrm{dom}(\mathrm{smul}_{\mathscr{E}})\}$ and $\mathrm{smul}_{\mathscr{S}}(I, J, K) = (\mathrm{smul}_{\mathscr{E}}(I, J), \mathrm{smul}_R(I, K))$ on $\mathrm{dom}(\mathrm{smul}_{\mathscr{S}})$. It follows that $\mathrm{smul}_{\mathscr{S}}$ is a partial continuous function, and $\mathrm{smul}_{\mathscr{S}}$ is effective since the relation $a(T) = n$ is decidable. That $\mathrm{smul}_{\mathscr{S}}$ represents scalar multiplication on $\mathscr{S}$ follows by the argument above.

To show that multiplication on $\mathscr{S}$ is effective, we note that $|x^k (\varphi\psi)^{(m)}(x)| \leq$

$$\sum_{i=1}^{m} \binom{m}{i} |x^k \varphi^{(i)}(x) \psi^{(m-i)}(x)| \leq \sum_{i=1}^{m} \binom{m}{i} |x^k \varphi^{(i)}(x)| |\psi^{(m-i)}(x)|.$$

Thus, if $B, C \in B_R$ we define $\mathrm{mul}_R(B, C) \in B_R$ by $R(k, m, n) \in \mathrm{mul}_R(B, C)$ if and only if there are $R(k, 0, b_0),\ R(k, 1, b_1),\ \ldots,\ R(k, m, b_m)$

in B, and $R(0,0,c_0)$, $R(0,1,c_1)$, ..., $R(0,m,c_m)$ in C such that

$$n = \sum_{i=0}^{m} \binom{m}{i} b_i c_{m-i}.$$

It is clear that $\mathrm{mul}_R(B,C) \in B_R$ for all $B, C \in B_R$, and that $\mathrm{mul}_R : B_R \times B_R \to B_R$ is monotone. It follows that mul_R extends to a continuous function from $D_R \times D_R$ to D_R.

Suppose that (I,J) and (K,L) represent φ and ψ in $\mathscr{S}$ respectively. If $A \in \mathrm{mul}_R(J,L)$ then $A \sqsubseteq \mathrm{mul}_R(B,C)$ for some $B \in J$ and $C \in L$. It follows that $\varphi \cdot \psi$ satisfies A. Choose k and m in $\mathbb{N}$. We would like to show that there is some $n \in \mathbb{N}$, and $B \in \mathrm{mul}_R(J,L)$ such that $R(k,m,n) \in B$. Since (I,J) represents φ there are $B_0, B_1, \ldots, B_m \in J$ and $b_0, b_1, \ldots, b_m \in \mathbb{N}$ such that $R(k,i,b_i) \in B_i$ for each $0 \le i \le m$. Let $B = B_0 \cup B_1 \cup \ldots \cup B_m$. Similarly, since (K,L) represents ψ there are $C_0, C_1, \ldots, C_m \in L$ and $c_0, c_1, \ldots, c_m \in \mathbb{N}$ such that $R(0,i,c_i) \in C_i$ for each $0 \le i \le m$. We let $C = C_0 \cup C_1 \cup \ldots \cup C_m$. By the choice of B and C there is a natural number n as above, such that $R(k,m,n) \in \mathrm{mul}_R(B,C) \in \mathrm{mul}_R(J,L)$.

Let $\mathrm{mul}_{\mathscr{E}} : D_{\mathscr{E}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathscr{E}}$ be an effective partial continuous function which represents multiplication on $\mathscr{E}$. We define $\mathrm{mul}_{\mathscr{S}} : D_{\mathscr{S}} \times D_{\mathscr{S}} \rightharpoonup D_{\mathscr{S}}$ by $\mathrm{dom}(\mathrm{mul}_{\mathscr{S}}) = \{(I,J,K,L) \in D_{\mathscr{E}} \times D_R \times D_{\mathscr{E}} \times D_R;\ (I,K) \in \mathrm{dom}(\mathrm{mul}_{\mathscr{E}})\}$, and $\mathrm{mul}_{\mathscr{S}}(I,J)(K,L) = (\mathrm{mul}_{\mathscr{E}}(I,K), \mathrm{mul}_R(J,L))$ on $\mathrm{dom}(\mathrm{mul}_{\mathscr{S}})$. It is clear that $\mathrm{mul}_{\mathscr{S}}$ is an effective partial continuous function, and that $\mathrm{mul}_{\mathscr{S}}$ represents multiplication on $\mathscr{S}$.

That addition on $\mathscr{S}$ is effective follows similarly, using that $|x^k(\varphi+\psi)^{(m)}(x))| \le |x^k\varphi^{(m)}(x)| + |x^k\psi^{(m)}(x)|$ for all $\varphi, \psi \in \mathscr{S}$. □

The norm $\|\varphi\|_{k,n} = \sup\{|x^k\varphi^{(m)}(x)|;\ x \in \mathbb{R} \text{ and } m \le n\}$ of φ is effective, uniformly in k and n, by the next proposition.

Proposition 3.4.11. *The norm* $(\varphi,k,n) \mapsto \|\varphi\|_{k,n}$ *is effective.*

Proof. Let k and n be natural numbers, and let $(B,C) \in B_{\mathscr{S}}$. We say that (B,C) is (k,n)-*complete* if there is a notation $B^{(m)}(S,T) \in B$ for each $m \le n$ such that $0 \in S$, and there are natural numbers $N_0, N_1, \ldots, N_n$ such that $R(k{+}1, m, N_m) \in C$ for each $m \le n$. Intuitively, if (B,C) is (k,n)-complete then C contains enough information to bound $\|\varphi\|_{k,n}$ from above for each $\varphi \in \mathscr{S}$ which satisfies C. In particular, we must have $|x^k\varphi^{(m)}(x)| \le Nx^{-1}$ for all $x \in \mathbb{R}$ and all $m \le n$ whenever φ satisfies C.

Let (B,C) be (k,n)-complete. We may assume that B is on normal form. (If not, we simply replace B by $\mathrm{nf}(B)$ in the right-hand side of all definitions.) Let N_m be the least natural number n such that $R(k,m,n) \in C$ and let $N =$

$\max\{N_m;\ m \le n\}$. To get an upper bound on the norm of any rapidly decreasing smooth function approximated by (B, C) we define $U(B) : \mathbb{R} \to \mathbb{R}$ by $U(B)(x) = \sup\{|x^k z| \in \mathbb{R};\ z \in T$ for some $B^{(m)}(S, T) \in B$ such that $x \in S$ and $m \le n\}$. We let $U(B, C) : \mathbb{R} \to \mathbb{R}$ be the map given by $U(B, C)(x) = \min\{U(B), Nx^{-1}\}$ if $x \neq 0$, and $U(B, C)(0) = U(B)(0)$. It is clear that $|x^k \varphi^{(m)}(x)| < U(B, C)(x)$ for each $m \le n$ whenever $\varphi \in \mathscr{S}$ satisfies B and C. Finally, we let $u(B, C) = \sup\{U(B, C)(x);\ x \in \mathbb{R}\}$. It follows that $\|\varphi\|_{k,n} < u(B, C)$ whenever $\varphi \in \mathscr{S}$ satisfies B and C.

To bound the norm of φ from below we let $L(B) : \mathbb{R} \to \mathbb{R}$ be the function given by $L(B)(x) = \inf\{|x^k z| \in \mathbb{R};\ z \in T$ for some $B^{(m)}(S, T) \in B$ such that $x \in S$, and $L(B)(x) = 0$ if there is no $B^{(m)}(S, T) \in B$ such that $x \in S$ and $m \le n$. We let $l(B) = \sup\{L(B)(x);\ x \in \mathbb{R}\}$. By the definition of $l(B)$ we have $l(B) < \|\varphi\|_{k,n}$ if $\varphi \in \mathscr{S}$ satisfies B and C.

We let $\mathrm{sn}_{k,n} : B^D_{\mathscr{S}} \to D_{\mathbb{R}}$ be the monotone function given by $\mathrm{sn}_{k,n}(B, C) = \{S \in B_{\mathbb{R}}; [l(B), u(B, C)] \subseteq S\}$, if (B, C) is (k, n)-complete, and $\mathrm{sn}_{k,n}(B, C) = \bot$ otherwise. It is easy to show that $\mathrm{sn}_{k,n}$ is well-defined and monotone, and so $\mathrm{sn}_{k,n}$ extends to a partial continuous function $\mathrm{sn}_{k,n} : D_{\mathscr{S}} \rightharpoonup D_{\mathbb{R}}$. Since the relation $[l(B), u(B, C)] \subseteq S$ is semidecidable for $(B, C) \in B_{\mathscr{S}}$ and $S \in B_{\mathbb{R}}$ it follows that $\mathrm{sn}_{k,n}$ is effective.

The proof that $\mathrm{sn}_{k,n}$ represents the map $\varphi \mapsto \|\varphi\|_{k,n}$ is routine, using the fact that the supremum of $|x^k \varphi^{(m)}(x)|$ is actually attained at some some $x \in \mathbb{R}$ whenever φ is a rapidly decreasing smooth function. □

3.5 Convolution and the Fourier transform on $\mathscr{S}$

Many interesting operators on spaces of smooth functions are defined in terms of integrals. Two examples, which are particularly important from our point of view, are convolution and the Fourier transform. In this section we will study effectivity for some integral operators on the spaces $\mathscr{E}$, $\mathscr{S}$, and $\mathscr{D}$. More precisely, we will show that

1. The definite integral $(\varphi, c, d) \mapsto \int_c^d \varphi$ from $\mathscr{E} \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$ is effective.

2. The definite integrals $\varphi \mapsto \int_{\mathbb{R}} \varphi$ from $\mathscr{S}$ to $\mathbb{C}$ and $\varphi \mapsto \int_{\mathbb{R}} |\varphi|$ from $\mathscr{S}$ to $\mathbb{R}$ are both effective.

3. The Fourier transform $\varphi \mapsto \hat{\varphi}$ on $\mathscr{S}$ is effective.

4. Convolution $(\varphi, \psi) \mapsto \varphi * \psi$ from $\mathscr{E} \times \mathscr{D}$ to $\mathscr{E}$ is effective.

Since each $\varphi \in \mathscr{E}$ is locally integrable, the integral $\int_c^d \varphi$ is well-defined. The function given by $(\varphi, c, d) \mapsto \int_c^d \varphi$ is effective by the following proposition.

Proposition 3.5.1. *The definite integral* $(\varphi, c, d) \mapsto \int_c^d \varphi$ *is effective.*

Proof. Since $\int_c^d \varphi = \int_c^d \mathrm{Re}(\varphi(x))dx + i\int_c^d \mathrm{Im}(\varphi(x))dx$ it is enough to show that the maps $(\varphi, c, d) \mapsto \int_c^d \mathrm{Re}(\varphi(x))dx$ and $(\varphi, c, d) \mapsto \int_c^d \mathrm{Im}(\varphi(x))dx$ are effective. We show that the integral of the real part of φ is effective. The proof that the integral of the imaginary part of φ is effective is entirely analogous.

For simplicity, we consider the case when $c < d$. The more general case is only slightly more technical. Let $B \in B_{\mathscr{E}}$, and let $C, D \in B_{\mathbb{R}}$. We say that B is *complete* on (C, D) if the convex hull of $C \cup D$ is contained in the set $\bigcup\{S;\ B^{(0)}(S,T) \in B$ for some $T\}$.

Suppose that $B \in B_{\mathscr{E}}$ is complete on (C, D). We may assume that B is on normal form. To compute upper and lower bounds on the definite integral of (the real part of) B over (C, D) we simply integrate along the upper and lower boundaries of the boxes $S \times \mathrm{Re}[T]$ where $B^{(0)}(S,T) \in B$, and S intersects the convex hull of $C \cup D$. To be more precise, let l_1, l_2, and l_3 be the rational numbers given by

$$\begin{aligned}
l_1(B,C,D) &= \mathrm{diam}(C)\cdot\min(\{0\}\cup\{\inf \mathrm{Re}[T];\ B^{(0)}(S,T)\in B \\
&\qquad \text{and } S\cap C \neq \emptyset\}). \\
l_2(B,C,D) &= \text{the sum of all } \mathrm{diam}(S\cap[\sup C, \inf D])\cdot \inf \mathrm{Re}[T] \\
&\qquad \text{where } B^{(0)}(S,T)\in B \text{ and } S\cap[\sup C,\inf D]\neq\emptyset. \\
l_3(B,C,D) &= \mathrm{diam}(D)\cdot\min(\{0\}\cup\{\inf \mathrm{Re}[T];\ B^{(0)}(S,T)\in B \\
&\qquad \text{and } S\cap D \neq \emptyset\}).
\end{aligned}$$

We let $l(B,C,D) = l_1(B,C,D) + l_2(B,C,D) + l_3(B,C,D)$. It follows immediately by the definition of l_1, l_2, and l_3 above that $l(B,C,D) \leq \int_c^d \mathrm{Re}(\varphi(x))dx$ whenever φ satisfies B, $c \in C$, and $d \in D$. We may bound the integral from above in the same way. Thus, let u_1, u_2, and u_3 be the rational numbers given by

$$\begin{aligned}
u_1(B,C,D) &= \mathrm{diam}(C)\cdot\max(\{0\}\cup\{\sup \mathrm{Re}[T];\ B^{(0)}(S,T)\in B \\
&\qquad \text{and } S\cap C \neq \emptyset\}). \\
u_2(B,C,D) &= \text{the sum of all } \mathrm{diam}(S\cap[\sup C, \inf D])\cdot \sup \mathrm{Re}[T] \\
&\qquad \text{where } B^{(0)}(S,T)\in B \text{ and } S\cap[\sup C,\inf D]\neq\emptyset. \\
u_3(B,C,D) &= \mathrm{diam}(D)\cdot\max(\{0\}\cup\{\inf \mathrm{Re}[T];\ B^{(0)}(S,T)\in B \\
&\qquad \text{and } S\cap D \neq \emptyset\}).
\end{aligned}$$

Let $u(B,C,D) = u_1(B,C,D) + u_2(B,C,D) + u_3(B,C,D)$. Then $\int_c^d \mathrm{Re}(\varphi(x))dx \leq u(B,C,D)$ whenever φ satisfies B, $c \in C$, and $d \in D$.

We define $\mathrm{int}(B,C,D)$ as the interval $[l(\mathrm{nf}(B),C,D),\ u(\mathrm{nf}(B),C,D)]$ if B is complete on (C,D), and $\mathrm{int}(B,C,D) = \perp$ otherwise. It is easy to verify that the function $\mathrm{int} : B_{\mathscr{E}} \times B_{\mathbb{R}} \times B_{\mathbb{R}} \to B_{\mathbb{R}}$ is monotone. It follows that int extends to a continuous function $\mathrm{int} : D_{\mathscr{E}} \times D_{\mathbb{R}} \times D_{\mathbb{R}} \to D_{\mathbb{R}}$. That int represents integration on $\mathscr{E}$ follows since every smooth function is Riemann integrable. □

As noted in section 3.4, if φ is a rapidly decreasing function then $\varphi \in L^1(\mathbb{R})$. It follows that the definite integrals $\int_{\mathbb{R}} \varphi$ and $\int_{\mathbb{R}} |\varphi|$ are well-defined.

Proposition 3.5.2. *Integration $\varphi \mapsto \int_{\mathbb{R}} \varphi$ from $\mathscr{S}$ to $\mathbb{C}$ is effective.*

Proof. As in the proof of Proposition 3.5.1 it is enough to show that the maps $\varphi \mapsto \int_{\mathbb{R}} \mathrm{Re}(\varphi(x))dx$, and $\varphi \mapsto \int_{\mathbb{R}} \mathrm{Im}(\varphi(x))dx$ are effective. Here, we will consider the integral of the real part of φ.

If $(B, C) \in B_{\mathscr{S}}$ we say that (B, C) is *complete* if there is a natural number $N \geq 1$ such that $[-N, N] \subseteq \bigcup\{S;\ B^{(0)}(S, T) \in B \text{ for some } T\}$, and some $k \geq 2$ and $n \in N$ such that $R(k, 0, n) \in C$. It is easy to show that if (B, C) is complete, and $(B, C) \sqsubseteq (B', C')$ in $B_{\mathscr{S}}$, then (B', C') must be complete as well.

Now, suppose that $(B, C) \in B_{\mathscr{S}}$ is complete. We may assume that B is on normal form. To define the integral of (B, C) we let $L(B, C)(x) \subseteq \mathbb{R}$ be the set

$$\{\inf \mathrm{Re}[T];\ B^{(0)}(S, T) \in B \text{ and } x \in S\} \cup$$

$$\{-n|x|^{-k};\ R(k, 0, n) \in C \text{ and } x \neq 0\},$$

and define a function $l(B, C) : \mathbb{R} \to \mathbb{R}$ by $l(B, C)(x) = \max L(B, C)(x)$. The function $l(B, C)$ is integrable since (B, C) is complete. We let $U(B, C)(x)$ be the set

$$\{\sup \mathrm{Re}[T];\ B^{(0)}(S, T) \in B \text{ and } x \in S\} \cup$$

$$\{n|x|^{-k};\ R(k, 0, n) \in C \text{ and } x \neq 0\},$$

and define $u(B, C) : \mathbb{R} \to \mathbb{R}$ by $u(B, C)(x) = \min U(B, C)(x)$. The function $u(B, C)$ is integrable as before.

Suppose that $\varphi \in \mathscr{S}$ satisfies (B, C), and let $l \in L(B, C)(x)$, and $u \in U(B, C)(x)$. Then $l \leq \mathrm{Re}(\varphi(x)) \leq u$. It follows that $l(B, C)(x) \leq \mathrm{Re}(\varphi(x)) \leq u(B, C)(x)$ for all $x \in \mathbb{R}$, and so $\int_{\mathbb{R}} l(B, C) \leq \int_{\mathbb{R}} \mathrm{Re}(\varphi(x))dx \leq \int_{\mathbb{R}} u(B, C)$ for each $\varphi \in \mathscr{S}$ which satisfies (B, C). We let $\mathrm{int} : B^D_{\mathscr{S}} \to B_{\mathbb{R}}$ be the function given by $(B, C) \mapsto [\int_{\mathbb{R}} l(B, C), \int_{\mathbb{R}} u(B, C)]$ if (B, C) is complete, and $(B, C) \mapsto \perp$ otherwise. The function $\mathrm{int} : B^D_{\mathscr{S}} \to B_{\mathbb{R}}$ is well-defined since $\int_{\mathbb{R}} l(B, C)$, and $\int_{\mathbb{R}} u(B, C)$ are rational, and $\int_{\mathbb{R}} l(B, C) \leq \int_{\mathbb{R}} \mathrm{Re}(\varphi(x))dx \leq \int_{\mathbb{R}} u(B, C)$ whenever (B, C) is a complete approximation of φ. Since int is monotone, int extends to a partial continuous function $\mathrm{int} : D_{\mathscr{S}} \rightharpoonup D_{\mathbb{R}}$.

To show that int represents the map $\varphi \mapsto \int_{\mathbb{R}} \mathrm{Re}(\varphi(x))dx$, suppose that (I, J) represent φ in $D_{\mathscr{S}}$. Let $B \in I$, and $C \in J$. Then $(B, C) \in \mathrm{dom}(\mathrm{int})$ and $\int_{\mathbb{R}} \mathrm{Re}(\varphi(x))dx \in \mathrm{int}(B, C)$ by construction. Let $M \in \mathbb{N}$, and choose $n \in \mathbb{N}$ and $C \in J$ such that $R(2, 0, n) \in C$. Choose $N \in \mathbb{N}$ such that $\int_{-\infty}^{-N} n|x|^{-2} + \int_{N}^{\infty} n|x|^{-2} < 2^{-(M+2)}$. Since I represents φ in $D_{\mathscr{E}}$ there is an element $B \in I$ such that the interval $[-N, N]$ is contained in the set

$$\bigcup\{S;\ B^{(0)}(S, T) \in B \text{ for some } T \text{ with } \mathrm{diam}(T) < 2^{-(M+2)}N^{-1}\}.$$

It follows that (B, C) is complete, and $\mathrm{diam}(\mathrm{int}(B, C)) =$

$$\left|\int_{\mathbb{R}} u(B, C) - \int_{\mathbb{R}} l(B, C)\right| < 2N(2^{-(M+2)}N^{-1}) + 2(2^{-(M+2)}) = 2^{-M}.$$

Since M was arbitrary we conclude that $\mathrm{int}(I, J)$ represents $\int_{\mathbb{R}} \mathrm{Re}(\varphi(x))dx$.

It is clear that we can compute the rational numbers $\int_{\mathbb{R}} l(B, C)$, and $\int_{\mathbb{R}} u(B, C)$ from the pair (B, C) whenever $(B, C) \in \mathrm{dom}(\mathrm{int})$ is complete. It follows that int is effective. □

Proposition 3.5.3. *The definite integral $\varphi \mapsto \int_{\mathbb{R}} |\varphi|$ from $\mathscr{S}$ to $\mathbb{R}$ is effective.*

Proof. If $\varphi \in \mathscr{S}$ then $|\varphi|$ is only piecewise $C^1(\mathbb{R})$ in general and so we cannot apply Proposition 3.5.2 to show that $\varphi \mapsto \int_{\mathbb{R}} |\varphi|$ is effective. However, the proof of Proposition 3.5.2 easily adapts to show that the integral $\varphi \mapsto \int_{\mathbb{R}} |\varphi|$ is effective. We leave the details to the reader. □

Remark 3.5.1. We note in passing that since the embedding $\mathscr{D} \hookrightarrow \mathscr{S}$ is effective we immediately know that the maps $\varphi \mapsto \int_{\mathbb{R}} \varphi$ from $\mathscr{D}$ to $\mathbb{C}$, and $\varphi \mapsto \int_{\mathbb{R}} |\varphi|$ from $\mathscr{D}$ to $\mathbb{R}$ are effective as well.

The most important integral operator on $\mathscr{S}$ for our purposes is arguably the Fourier transform, which is given by $\hat{\varphi}(t) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e_{-t}\varphi(x)dx$, where $e_t(x) = e^{itx}$. One can show that the Fourier transform is a continuous, linear, and one-to-one mapping of $\mathscr{S}$ onto $\mathscr{S}$. The inverse Fourier transform is also a continuous function on $\mathscr{S}$. It is given by $\varphi(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e_x\hat{\varphi}(t)dt$.

Since the exponential basis functions $(e_t)_t$ are eigenfunctions of the differential operator ∂_n, the Fourier transform takes partial differential equations to algebraic equations. This property naturally makes the Fourier transform, and its inverse, important tools in the PDE-toolbox, and it also makes them interesting from the point of view of computable analysis.

To show that the Fourier transform and the inverse Fourier transform on $\mathscr{S}$ are effective we will need the following lemma.

Lemma 3.5.4. *The function from $\mathscr{S} \times \mathbb{R}$ into $\mathscr{S}$ given by $(\varphi, t) \mapsto e_{-t} \cdot \varphi$ is effective, and the function from $\mathscr{S} \times \mathbb{N}$ into $\mathscr{S}$ given by $(\varphi, n) \mapsto x^n\varphi$ is effective.*

Proof. The function $t \mapsto e_{-t}$ from $\mathbb{R}$ to $\mathscr{E}$ is effective by Theorem 3.1.9. By a second application of the same theorem it follows that the function from $\mathscr{S} \times \mathbb{R}$ into $\mathscr{E}$ given by $(\varphi, t) \mapsto e_{-t} \cdot \varphi$ is effective. To show that $(\varphi, t) \mapsto e_{-t} \cdot \varphi$ from $\mathscr{S} \times \mathbb{R}$ into $\mathscr{S}$ is effective it is enough to show that we can compute bounds on $|x^k(e_{-t} \cdot \varphi)^{(m)}(x)|$ from t, and similar bounds on $|x^k\varphi^{(m)}(x)|$. Using Leibniz rule we have

$$|x^k(e_{-t} \cdot \varphi)^{(m)}(x)| \leq \sum_{s=0}^{m} \binom{m}{s} |x^k(-it)^{m-s}e_{-t}(x) \cdot \varphi^{(s)}(x)| =$$

$$\sum_{s=0}^{m} \binom{m}{s} |t|^{m-s}|x^k\varphi^{(s)}(x)| \leq m!(|t| + 1)^m \sum_{s=0}^{m} |x^k\varphi^{(s)}(x)|$$

since the absolute value of e_{-t} is 1. Thus, given $(B, C) \in B_{\mathscr{S}}$ and $S \in B_{\mathbb{R}}$ we let $g(B, C, S) \subseteq B_R$ be the set given by $C' \in g(B, C, S)$ if and only if $R(k, m, n) \in C' \implies$ there are $R(k, 0, n_0), R(k, 1, n_1), \ldots, R(k, m, n_m) \in C$ such that

$$m! \sup\{(|t| + 1)^m;\ t \in S\} \sum_{s=0}^{m} n_s \leq n.$$

It follows easily from the definition of g that $g(B, C, S)$ is an ideal in B_R for $(B, C) \in B_{\mathscr{S}}$ and $S \in B_{\mathbb{R}}$. Furthermore, if $(B, C) \sqsubseteq (B', C')$ in $B_{\mathscr{S}}$ and $S \sqsubseteq S'$ in $B_{\mathbb{R}}$, then $g(B, C, S) \subseteq g(B', C', S')$, and so g extends to a continuous function $g : D_{\mathscr{S}} \times D_{\mathbb{R}} \to D_R$.

The function g is clearly effective. To see that g gives the correct bounds on the product $e_t \cdot \varphi$, Let (I, J) represent φ in $D_{\mathscr{S}}$, and let K represents t in $D_{\mathbb{R}}$. Now, if $B \in I$, $C \in J$, and $S \in K$, then $e_t \cdot \varphi$ satisfies each $C' \in g(B, C, S)$. Moreover, if $k, m \in \mathbb{N}$ we may choose $n_0, n_1, \ldots, n_m \in \mathbb{N}$ and $C'' \in J$ such that $R(k, 0, n_0)$, $R(k, 1, n_1)$, $\ldots$, $R(k, m, n_m) \in C''$. Then $R(k, m, n) \in g(B, C, S)$ for each $n \in \mathbb{N}$ such that

$$m! \sup\{(|t| + 1)^m;\ t \in S\} \sum_{s=0}^{m} n_s \leq n.$$

Now, if $f : D_{\mathscr{S}} \times D_{\mathbb{R}} \rightharpoonup D_{\mathscr{E}}$ is an effective representation of the function $(\varphi, t) \mapsto e_{-t} \cdot \varphi$ from $\mathscr{S} \times \mathbb{R}$ to $\mathscr{E}$ we define $h : D_{\mathscr{S}} \times D_{\mathbb{R}} \rightharpoonup D_{\mathscr{S}}$ by $\mathrm{dom}(h) = \mathrm{dom}(f)$, and $h(I, J, K) = (f(I, J, K), g(I, J, K))$. It follows immediately that h is effective, and that h represents the function $(\varphi, t) \mapsto e_{-t} \cdot \varphi$ from $\mathscr{S} \times \mathbb{R}$ to $\mathscr{S}$.

That the function from $\mathscr{S} \times \mathbb{N}$ to $\mathscr{S}$ given by $(\varphi, n) \mapsto x^n\varphi$ is effective follows by a similar application of Leibniz rule. We leave the details to the reader. $\square$

Remark 3.5.2. Since the map $(\varphi, n) \mapsto (-i)^n\varphi$ is from $\mathscr{S} \times \mathbb{N}$ to $\mathscr{S}$ is effective it follows immediately by the previous lemma that the function given by $(\varphi, n) \mapsto (-ix)^n\varphi$ from $\mathscr{S} \times \mathbb{N}$ to $\mathscr{S}$ is effective.

We are now ready to show that the Fourier transform on $\mathscr{S}$, as well as its inverse, are effective.

Proposition 3.5.5. *The Fourier transform on $\mathscr{S}$ is effective.*

Proof. We begin by showing that the function $\varphi \mapsto \hat{\varphi}$ from $\mathscr{S}$ into $\mathscr{E}$ is effective. By Theorem 3.1.9 it is enough to show that $(\varphi, t, m) \mapsto \hat{\varphi}^{(m)}(t)$ is effective. We have

$$\hat{\varphi}^{(m)}(t) = \frac{d^m}{dt^m}\Big(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e_{-t}\varphi(x)dx\Big) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e_{-t}(-ix)^m\varphi(x)dx.$$

It now follows immediately by Proposition 3.5.2 and Lemma 3.5.4 that the function $\varphi \mapsto \hat{\varphi}$ from $\mathscr{S}$ to $\mathscr{E}$ is effective.

To show that the Fourier transform is an effective map on $\mathscr{S}$ we need to be able to compute bounds on $|t^k\hat{\varphi}^{(m)}(t)|$ for all natural numbers k and m. The following approximation will be useful

$$|t^k\hat{\varphi}^{(m)}(t)| = \Big|t^k\frac{d^m}{dt^m}\Big(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e_{-t}\varphi(x)dx\Big)\Big| =$$

$$\Big|t^k\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e_{-t}(-ix)^m\varphi(x)dx\Big| \leq \Big|\int_{\mathbb{R}} e_{-t}\frac{d^k}{dx^k}\Big(x^m\varphi(x)\Big)dx\Big| =$$

$$\Big|\int_{\mathbb{R}} e_{-t}\Big(\sum_{s=0}^{\min(k,m)}\binom{k}{s}\big(\frac{m!}{s!}x^{m-s}\varphi^{(k-s)}(x)\big)\Big)dx\Big| \leq$$

$$k!m!\sum_{s=0}^{\min(k,m)}\int_{\mathbb{R}}|e_{-t}x^{m-s}\varphi^{k-s}(x)|dx =$$

$$k!m!\sum_{s=0}^{\min(k,m)}\int_{\mathbb{R}}|x^{m-s}\varphi^{k-s}(x)|dx.$$

Note that $x^{m-s}\varphi^{(k-s)} \in \mathscr{S}$ for all k, m and $s \leq \min(k, m)$, and so in particular, if

$$B_{k,m}(\varphi) = k!m!\sum_{s=0}^{\min(k,m)}\int_{\mathbb{R}}|x^{m-s}\varphi^{k-s}(x)|dx,$$

then $0 \leq B_{k,m}(\varphi) < \infty$ and $B_{k,m}(\varphi)$ is well-defined.

The map $(\varphi, k, m) \mapsto B_{k,m}(\varphi)$ is effective by Proposition 3.5.3 and Lemma 3.5.4. Let $b : D_{\mathscr{S}} \times D_{\mathbb{N}} \times D_{\mathbb{N}} \rightharpoonup D_{\mathbb{R}}$ be an effective partial continuous representation of $(\varphi, k, m) \mapsto B_{k,m}(\varphi)$. Since the standard representation of $\mathbb{N}$ is dense, we must have $\mathrm{dom}(b) = Q \times D_{\mathbb{N}} \times D_{\mathbb{N}}$ for some nonempty closed subset $Q \subseteq D_{\mathscr{S}}$. Now, given a compact element $(B, C) \in Q$ we define $g(B, C) \subseteq B_R$ by $C' \in g(B, C)$ if and only if $R(k, m, n) \in C' \implies$ there is some $S \in b((B, C), k, m)$ such that $\sup S \leq n$. It follows immediately from the definition of g that $g(B, C)$ is an ideal in B_R. Since the function g is monotone we can extend g to a partial continuous function $g : D_{\mathscr{S}} \rightharpoonup D_R$ with $\mathrm{dom}(g) = Q$.

It is clear that g is effective, and $D^R_{\mathscr{S}} \subseteq \mathrm{dom}(g)$ since b represents the map $(\varphi, k, m) \mapsto B_{k,m}(\varphi)$. To show that g gives the correct bounds on $|t^k \hat{\varphi}^{(m)}(t)|$, let (I, J) represent φ in $D_{\mathscr{S}}$. By the definition of $B_{k,m}(\varphi)$ we know that $|t^k \hat{\varphi}^{(m)}(t)| \leq B_{k,m}(\varphi)$. It follows immediately that $|t^k \hat{\varphi}^{(m)}(t)| \leq \sup S$ for each closed interval S which contains $B_{k,m}(\varphi)$. Now, if $R(k, m, n) \in C$ for some $C \in g(I, J)$, then $n \geq \sup S$ for some interval S containing $B_{k,m}(\varphi)$ (since b represents $(\varphi, k, m) \mapsto B_{k,m}(\varphi)$). It follows that $\hat{\varphi}$ satisfies each $C \in g(I, J)$. Let $k, m \in \mathbb{N}$, and choose $B \in I$ and $C \in J$ for which there is some $S \in b((B, C), k, m)$ which is bounded above. (This is possible since b represents the map $(\varphi, k, m) \mapsto B_{k,m}(\varphi)$.) Let $n \geq \sup S$. Then $\{R(k, m, n)\} \in g(B, C)$ as required.

Now, let $f : D_{\mathscr{S}} \rightharpoonup D_{\mathscr{E}}$ be an effective representation of the Fourier transform as a function from $\mathscr{S}$ to $\mathscr{E}$. We define $h : D_{\mathscr{S}} \rightharpoonup D_{\mathscr{S}}$ by $\mathrm{dom}(h) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$, and $h(I, J) = (f(I, J), g(I, J))$ for each pair $(I, J) \in \mathrm{dom}(h)$. It follows immediately that h is an effective partial continuous function which represents the Fourier transform on $\mathscr{S}$. □

To show that the inverse Fourier transform is effective, we may simply modify the proof of Lemma 3.5.4 and Proposition 3.5.5. Alternatively, since the Fourier transform is periodic with period four it follows that the inverse Fourier transform is simply the Fourier transform composed with itself three times. In any case, we have following

Corollary 3.5.6. *The inverse Fourier transform on $\mathscr{S}$ is effective.*

Before we go on to study computability on the space of distributions, we need to prove one result about the convolution of two smooth functions.

If φ and ψ are two functions from $\mathbb{R}$ to $\mathbb{C}$, the *convolution* of φ and ψ is given by the integral

$$(\varphi * \psi)(t) = \int_{\mathbb{R}} \varphi(x)\psi(t - x)dx.$$

In general, we need some requirements on the functions φ and ψ to ensure that the right-hand side exists. For instance, it is easy to see that it is enough to assume that φ and ψ are continuous and that one of either φ or ψ has compact support. If φ and ψ are smooth functions then $\varphi*\psi$ is smooth and $(\varphi*\psi)^{(m)} = \varphi^{(m)} * \psi = \varphi * \psi^{(m)}$. It follows that the convolution operator $*$ maps $\mathscr{E} \times \mathscr{D}$ into $\mathscr{E}$.

Proposition 3.5.7. *Convolution* $(\varphi, \psi) \mapsto \varphi*\psi$ *is an effective map from* $\mathscr{E} \times \mathscr{D}$ *to* $\mathscr{E}$.

Proof. By Theorem 3.1.9 it is enough to show that the function given by $(\varphi, \psi, t, m) \mapsto (\varphi * \psi)^{(m)}(t) = \int_{\mathbb{R}} \varphi^{(m)}(x)\psi(t-x)dx$ is effective. The result now follows since differentiation on $\mathscr{E}$, translation and reflection on $\mathscr{D}$, multiplication from $\mathscr{E} \times \mathscr{D}$ to $\mathscr{D}$ as well as integration on $\mathscr{D}$ are effective. □

4 Computability on spaces of distributions

A *distribution* is a sequentially continuous linear functional on $\mathscr{D}$. As is customary, we denote the space of distributions by $\mathscr{D}'$. $\mathscr{D}'$ becomes a locally convex topological vector space when equipped with the weak*-topology. To be more precise, the topology $\tau_{\mathscr{D}'}$ on $\mathscr{D}'$ is the weakest topology on $\mathscr{D}'$ which makes the functionals $\{u \mapsto u(\varphi)\}_{\varphi \in \mathscr{D}}$ continuous. Thus, if $F(\varphi, U)$ = the set of distributions which map φ into U, then the collection of all $F(\varphi, U)$, where φ ranges over $\mathscr{D}$ and U ranges over a topological basis in $\mathbb{C}$, form a subbasis for the topology $\tau_{\mathscr{D}'}$ on $\mathscr{D}'$.

4.1 A domain representation of $\mathscr{D}'$

We will construct a domain representation of $\mathscr{D}'$ by restricting the standard representation of the space of sequentially continuous functions from $\mathscr{D}$ to $\mathbb{C}$ to $\mathscr{D}'$. Thus, let $[\mathscr{D} \to_\omega \mathbb{C}]$ denote the space of sequentially continuous functions from $\mathscr{D}$ to $\mathbb{C}$. If $V \subseteq \mathscr{D}$ and $W \subseteq \mathbb{C}$ we define $F(V, W) \subseteq [\mathscr{D} \to_\omega \mathbb{C}]$ as the set of all sequentially continuous functions $u : \mathscr{D} \to \mathbb{C}$ such that $u[V] \subseteq W$. A subbasis for the topology τ_ω on $[\mathscr{D} \to_\omega \mathbb{C}]$ is given by the collection of all sets $F(\{\varphi_n\}_n \cup \{\varphi\}, W)$ where $(\varphi_n)_n \longrightarrow \varphi$ in $\mathscr{D}$, and W is open in $\mathbb{C}$.

Since the domain representation of $\mathscr{D}$ defined in section 3.3 is countably based and the standard representation of $\mathbb{C}$ is admissible, it follows by Theorem 3.6 in [7] that every sequentially continuous function from $\mathscr{D}$ to $\mathbb{C}$ is representable by a partial continuous function from $D_{\mathscr{D}}$ to $D_{\mathbb{C}}$. Let $[D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]$ denote the domain of partial continuous functions from $D_{\mathscr{D}}$ to $D_{\mathbb{C}}$ as defined in [7], and let $[D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]^R$ denote the set of partial continuous functions which represent sequentially continuous functions. Finally, let $[\delta_{\mathscr{D}} \rightharpoonup \delta_{\mathbb{C}}] : [D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]^R \to [\mathscr{D} \to_\omega \mathbb{C}]$ be the function given by $[\delta_{\mathscr{D}} \rightharpoonup \delta_{\mathbb{C}}](f) = u$ if and only if f represents u. $([D \rightharpoonup D_{\mathbb{C}}], [D \rightharpoonup D_{\mathbb{C}}]^R, [\delta \rightharpoonup \delta_{\mathbb{C}}])$ is an admissible domain representation of $([X \to_\omega \mathbb{C}], \tau_\omega)$ by Theorem 3.12 in [7].

Now, since every distribution is sequentially continuous we may restrict the representation of $[\mathscr{D} \to_\omega \mathbb{C}]$ to get a domain representation of $\mathscr{D}'$. More precisely, let $D_{\mathscr{D}'}$ be the domain of partial continuous functions from $D_{\mathscr{D}}$ to $D_{\mathbb{C}}$ and let $D^R_{\mathscr{D}'}$ be the set of partial continuous functions $f : D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}$ which represent distributions. Finally, let $\delta_{\mathscr{D}'} : D^R_{\mathscr{D}'} \to \mathscr{D}'$ be the function given by $\delta_{\mathscr{D}'}(f) = u$ if and only if f represents u.

Theorem 4.1.1. *$(D_{\mathscr{D}'}, D^R_{\mathscr{D}'}, \delta_{\mathscr{D}'})$ is an effective domain representation of $\mathscr{D}'$.*

Proof. $D_{\mathscr{D}'}$ is an effective domain by Theorem 4.4 in [7] since $D_{\mathscr{D}}$ and $D_{\mathbb{C}}$ are effective. Since $\delta_{\mathscr{D}'} : D^R_{\mathscr{D}'} \to \mathscr{D}'$ is the restriction of the representing map $[\delta_{\mathscr{D}} \rightharpoonup \delta_{\mathbb{C}}] : [D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]^R \to [\mathscr{D} \to_\omega \mathbb{C}]$ to $D^R_{\mathscr{D}'}$, and the weak*-topology on $\mathscr{D}'$ is weaker than the subspace topology on $\mathscr{D}'$ with respect to $[\mathscr{D} \to_\omega \mathbb{C}]$, it follows that $\delta_{\mathscr{D}'}$ is continuous. $\delta_{\mathscr{D}'}$ is surjective since every sequentially continuous function from $\mathscr{D}$ to $\mathbb{C}$ is representable as a partial continuous function from $D_{\mathscr{D}}$ to $D_{\mathbb{C}}$. □

It follows by functional analysis that the domain representation of $\mathscr{D}'$ is admissible: If X is a topological vector space we denote the dual of X by X^*, and the weak*-topology on X^* by τ^*.

Lemma 4.1.2. *Let X be a topological vector space. If X is an F-space[3] then the inclusion of (X^*, τ^*) into $([X \to_\omega \mathbb{C}], \tau_\omega)$ is sequentially continuous.*

Proof. Let $(u_m)_m \longrightarrow u$ in (X^*, τ^*). We need to show that $(u_m)_m$ converges to u in $([X \to_\omega \mathbb{C}], \tau_\omega)$. By Lemma 3.11 in [7] it is enough to show that $(u_m(t_m))_m \longrightarrow u(t)$ in $\mathbb{C}$ for each convergent sequence $(t_n)_n \longrightarrow t$ in X. Thus, suppose that $(t_n)_n$ converges to t in X.

Suppose first that $(t_n) \longrightarrow 0$. The set $U(x) = \{u_n(x); n \in \mathbb{N}\}$ is bounded in $\mathbb{C}$ for each $x \in X$ since $(u_m)_m$ converges to u in X^*. Thus, by an application of the Banach-Steinhaus Theorem we may conclude that the family $(u_m)_m \subseteq X^*$ is equicontinuous.

Let d be a complete invariant metric on X, which is compatible with the topology on X. (d exists since X is an F-space.) As is customary, we write $B(0, r)$ for the open ball $\{x \in X;\ d(0, x) < r\}$ centered at 0 with radius r. Since the set $B(0, 1)$ is bounded in X there is some $M \in \mathbb{N}$ such that

[3]If X is a topological vector space then X is an *F-space* if the topology on X is induced by a complete invariant metric d. For details, c.f. [18].

$|u_m(x)| \leq M$ for each $m \in \mathbb{N}$ and each $x \in B(0,1)$. Since $2^n B(0, 2^{-n}) \subseteq B(0,1)$ we have $|u_m(x)| = 2^{-n}|u_m(2^n x)| \leq 2^{-n}M$ for each $m \in \mathbb{N}$ and each $x \in B(0, 2^{-n})$. Thus in particular, we must have $(u_m(t_m))_m \longrightarrow 0$ as required.

The general case when $(t_n)_n \longrightarrow t$ in X now follows by considering the sequence $(t - t_n)_n \longrightarrow 0$, since d is invariant. □

Lemma 4.1.3. *The inclusion of $\mathscr{D}'$ into $[\mathscr{D} \to_\omega \mathbb{C}]$ is sequentially continuous.*

Proof. Note that $\mathscr{D}$ is not an F-space so we cannot apply Lemma 4.1.2 directly. Thus, let $(u_m)_m \longrightarrow u$ in $\mathscr{D}'$. To show that $(u_m)_m \longrightarrow u$ in $[\mathscr{D} \to_\omega \mathbb{C}]$ it is enough to show that $(u_m(\varphi_m))_m \longrightarrow u(\varphi)$ for each convergent sequence $(\varphi_n)_n \longrightarrow \varphi$ in $\mathscr{D}$.

Suppose that $(\varphi_n)_n \longrightarrow \varphi$ in $\mathscr{D}$ and choose $k \in \mathbb{N}$ such that $\{\varphi_n\}_n \cup \{\varphi\} \subseteq \mathscr{D}_k$ and $(\varphi_n)_n \longrightarrow \varphi$ in $\mathscr{D}_k$. Since the inclusion $\mathscr{D}_k \hookrightarrow \mathscr{D}$ is continuous it follows that each distribution restricts to a continuous linear functional on $\mathscr{D}_k$. Thus in particular, we have $\{u_m\}_m \cup \{u\} \subseteq \mathscr{D}_k^*$ and $(u_m)_m \longrightarrow u$ in $\mathscr{D}_k^*$ by the definition of the weak*-topology on the dual space of $\mathscr{D}_k$. Since $\mathscr{D}_k$ is an F-space, we may apply Lemma 4.1.2 to conclude that $(u_m)_m \longrightarrow u$ in $[\mathscr{D}_k \to_\omega \mathbb{C}]$. By an application of Lemma 3.11 in [7] we see that $(u_m(\varphi_m))_m \longrightarrow u(\varphi)$ in $\mathbb{C}$. Since the convergent sequence $(\varphi_n)_n \longrightarrow \varphi$ was arbitrary we must have $(u_m)_m \longrightarrow u$ in $[\mathscr{D} \to_\omega \mathbb{C}]$ as required. □

We may now apply Lemma 4.1.3 to show that the representation $(D_{\mathscr{D}'}, D^R_{\mathscr{D}'}, \delta_{\mathscr{D}'})$ of $\mathscr{D}'$ is admissible.

Theorem 4.1.4. *$(D_{\mathscr{D}'}, D^R_{\mathscr{D}'}, \delta_{\mathscr{D}'})$ is an admissible domain representation of $\mathscr{D}'$.*

Proof. Let D be a countably based domain, let D^R be a nonempty subset of D and suppose that $\delta : D^R \to \mathscr{D}'$ is continuous. Let $i : \mathscr{D}' \to [\mathscr{D} \to_\omega \mathbb{C}]$ denote the inclusion of $\mathscr{D}'$ into $[\mathscr{D} \to_\omega \mathbb{C}]$. Since D is countably based and i is sequentially continuous it follows that $i \circ \delta : D^R \to [\mathscr{D} \to_\omega \mathbb{C}]$ is continuous.

Since the standard representation $([D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}], [D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]^R, [\delta_{\mathscr{D}} \rightharpoonup \delta_{\mathbb{C}}])$ of $([\mathscr{D} \to_\omega \mathbb{C}], \tau_\omega)$ is admissible there is a partial continuous function $\overline{\delta} : D \rightharpoonup [D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]$ such that

$$D^R \subseteq \mathrm{dom}(\overline{\delta}),\ \overline{\delta}[D^R] \subseteq E^R, \text{ and } (i \circ \delta)(r) = ([\delta_{\mathscr{D}} \rightharpoonup \delta_{\mathbb{C}}] \circ \overline{\delta})(r)$$

for each $r \in D^R$. Since δ maps D^R into $\mathscr{D}'$ it follows immediately that $\overline{\delta}$ maps D^R into $D^R_{\mathscr{D}'}$, and so δ factors though $\delta_{\mathscr{D}'}$ as required. □

Evaluation $(u, \varphi) \mapsto u(\varphi))$ is sequentially continuous by Lemma 4.1.3, and effective by Proposition 4.7 in [7].

Proposition 4.1.5. *Evaluation* $(u, \varphi) \mapsto u(\varphi)$ *on* $\mathscr{D}'$ *is effective.*

Proof. Let $e : [D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}] \times D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}$ be an effective partial continuous function which represents evaluation $[\mathscr{D} \to_{\omega} \mathbb{C}] \times \mathscr{D} \to \mathbb{C}$. e exists by Proposition 4.7 in [7]. Since $\delta_{\mathscr{D}'}$ is simply the restriction of $[\delta_{\mathscr{D}} \rightharpoonup \delta_{\mathbb{C}}]$ to $D^R_{\mathscr{D}'}$ we conclude that e represents evaluation from $\mathscr{D}' \times \mathscr{D}$ to $\mathbb{C}$. □

To show that type conversion from $[X \times \mathscr{D} \to \mathbb{C}]$ to $[X \to \mathscr{D}']$ preserves effectivity we need to show that the dense part of the representation $(D_{\mathscr{D}},\, D^R_{\mathscr{D}},\, \delta_{\mathscr{D}})$ is a computable element in the domain $\mathrm{cl}(D_{\mathscr{D}})$ of nonempty closed subsets of $D_{\mathscr{D}}$.

Lemma 4.1.6. $D^D_{\mathscr{D}}$ *is a computable element in* $\mathrm{cl}(D_{\mathscr{D}})$.

To prove Lemma 4.1.6 we will construct a countable dense set subset $\{f_{k,n}\}_{k,n\in\mathbb{N}}$ of $\mathscr{D}$ with the property that the relation "$f_{k,n}$ satisfies (B, i)" is semidecidable with respect to the natural numbering of $B_{\mathscr{D}}$. Since $(B, i) \in D^D_{\mathscr{D}}$ if and only if (B, i) is satisfiable if and only if there are natural numbers k and n such that $f_{k,n}$ satisfies (B, i), it follows by Proposition 4.2 in [7] that $D^D_{\mathscr{D}}$ is computable in $\mathrm{cl}(D_{\mathscr{D}})$.

Now, to define the countable dense set $\{f_{k,n}\}_{n\in\mathbb{N}} \subseteq \mathscr{D}$, we let

$$\sigma(x) = \begin{cases} m \cdot \exp((x^2-1)^{-1}) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where exp is the exponential function, and m is a constant such that $\int_{\mathbb{R}} \sigma = 1$. We let $\sigma_n(x) = 2^n \rho(2^n x)$. It is routine to show that σ_n is a smooth function for each n with support $\mathrm{supp}(\sigma_n) = [-2^{-n},\, 2^{-n}]$. The sequence $(\sigma_n)_n$ is sometimes called an *approximate identity* on $\mathbb{R}$. The following result can be found in any introductory text on distribution theory (c.f. [13]).

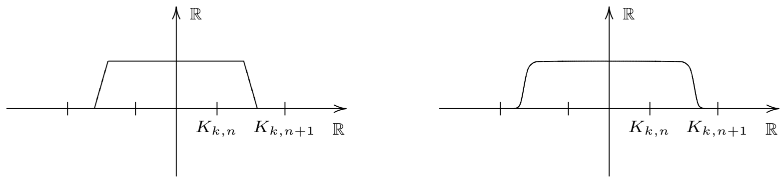
Fact 4.1.1 (Regularisation)**.** Let $f \in C^k(\mathbb{R})$ and suppose that f has compact support $K \subseteq \mathbb{R}$. Let $f_n = \sigma_n * f$ denote the convolution of σ_n and f. Then

1. $f_n \in \mathscr{D}$ and the support of f_n is contained in the 2^{-n}-neighborhood of K.
2. $(f_n^{(m)})_n$ converges uniformly to $f^{(m)}$ for each $m \leq k$.

For $A \subseteq \mathbb{R}$ we let $d(x,\, A) = \inf\{|x-a|;\ a \in A\}$. It follows from the triangle equality that $x \mapsto d(x,\, A)$ is continuous. Let $K_{k,n} = [-k + 2^{-n},\, k - 2^{-n}]$ and let

$$c_{k,n}(x) = \begin{cases} 1 - 2^{n+1} d(x,\, K_{k,n}) & \text{if } x \in K_{k,n+1} \\ 0 & \text{if } x \notin K_{k,n+1}. \end{cases}$$

The function $c_{k,n}$ is continuous and $c_{k,n}(x) = 1$ if $x \in K_{k,n}$. We let $\rho_{k,n} = \sigma_{n+2} * c_{k,n+1}$. It follows immediately from Fact 4.1.1 and the definition of $c_{k,n}$ that $\rho_{k,n} \in \mathscr{D}_k$, $0 \leq \rho_{k,n}(x) \leq 1$ for all x and $\rho_{k,n}(x) = 1$ for all $x \in K_{k,n}$. Thus, $\rho_{k,n}$ is a smooth approximation of $c_{k,n}$ with support contained in the interval $[-k, k]$.



The piecewise linear function $c_{k,n}$. The smooth approximation $\rho_{k,n}$ of $c_{k,n}$.

Figure 6. The functions $c_{k,n}$ and $\rho_{k,n}$.

Now, Let $\mathrm{P} \subseteq \mathscr{E}$ be the (countable) set of polynomials with complex rational coefficients and let $\mathrm{TP}_k = \{\rho_{k,n} \cdot p;\ n \in \mathbb{N} \text{ and } p \in P\} \subseteq \mathscr{D}_k$. Thus, TP_k is the set of smoothly truncated polynomials with complex rational coefficients.

Lemma 4.1.7. TP_k *is dense in* $\mathscr{D}_k$*.*

Proof. Let $\varphi \in \mathscr{D}_k$ and choose $M, n \in \mathbb{N}$. It is enough to show that there is a truncated polynomial $f \in \mathrm{TP}_k$ in the set $\varphi + U(M, n)$.

Suppose first that $\mathrm{supp}(\varphi) \subseteq (-k,\ k)$. Then $\mathrm{supp}(\varphi) \subseteq K_{k,n}$ for some n. Choose N so that the support of φ is contained in $K_{k,N}$. Then $\rho_{k,N} \cdot \varphi = \varphi$, and if $f = \rho_{k,N} \cdot p \in \mathrm{TP}_k$ then

$$\|\varphi - f\|_M = \|\rho_{k,N} \cdot (\varphi - p)\|_M =$$

$$\sup \Big| \sum_{i=0}^{M} \binom{M}{i} \rho_{k,N}^{(i)}(x) \cdot (\varphi - p)^{(M-i)}(x) \Big| \leq$$

$$\sup \Big(\sum_{i=0}^{M} \binom{M}{i} |\rho_{k,N}^{(i)}(x)| \cdot |(\varphi - p)^{(M-i)}(x)| \Big) \leq$$

$$\sum_{i=0}^{M} \binom{M}{i} \|\rho_{k,N}\|_M \|\varphi - p\|_M <$$

$$(M+1)! \, \|\rho_{k,N}\|_M \, \|\varphi - p\|_M \, .$$

By the Stone-Weierstrass Theorem there is a polynomial $p \in P$ such that $\|\varphi - p\|_M < (2^n \cdot (M+1)! \, \|\rho_{k,N}\|_M)^{-1}$. Choosing $f = \rho_{k,N} \cdot p$ we now get

$$\|\varphi - f\|_M < (M+1)! \, \|\rho_{k,N}\|_M \, \|\varphi - p\|_M < 2^{-n}.$$

For an arbitrary $\varphi \in \mathscr{D}_k$ we let $\varphi_n(x) = \varphi((1+2^{-n})x)$. Then $(\varphi_n^{(m)})_n$ converges uniformly to $\varphi^{(m)}$ for each m and $\operatorname{supp}(\varphi_n) \subseteq (-k,\, k)$ for each n. It follows that there is a natural number N such that $\|\varphi - \varphi_N\|_M < 2^{-(n+1)}$ and a truncated polynomial $f \in \mathrm{TP}_k$ such that $\|\varphi_N - f\|_M < 2^{-(n+1)}$ as above. Putting these together we get

$$\|\varphi - f\|_M \leq \|\varphi - \varphi_N\|_M + \|\varphi_N - f\|_M < 2^{-n},$$

and so we are done. □

Remark 4.1.1. If U is open in $\mathscr{D}$ then $U \cap \mathscr{D}_k$ is open in $\mathscr{D}_k$ since the inclusion $\mathscr{D}_k \hookrightarrow \mathscr{D}$ is continuous. Thus, if TP is the union of the sets TP_k for $k \geq 1$ then TP is a countable dense subset of $\mathscr{D}$.

Let (B,k) be a compact element in $D_{\mathscr{D}}$. Then either $(B,k) = \perp$, in which case $(B,k) \in D_{\mathscr{D}}^D$, or $(B,k) \neq \perp$, and then $(B,k) \in D_{\mathscr{D}}^D$ if and only if the set $\{\varphi \in \mathscr{D}_k;\ \varphi \text{ satisfies } B\}$ is nonempty. Since the set $\{\varphi \in \mathscr{D}_k;\ \varphi \text{ satisfies } B\}$ is open in $\mathscr{D}_k$ for each $B \in B_{\mathscr{E}}$ it follows that $\{\varphi \in \mathscr{D}_k;\ \varphi \text{ satisfies } B\}$ is nonempty if and only if there is a truncated polynomial $f \in \mathrm{TP}_k$ which satisfies B. We claim that this problem is semidecidable in B and k. To prove this, let $p : \mathbb{N} \to \mathscr{E}$ be an effective function mapping $\mathbb{N}$ onto the set P. (That is, let p be an effective enumeration of the set $\mathrm{P} \subseteq \mathscr{E}$.) The set TP_k is effectively enumerable in the same way by the following lemma.

Lemma 4.1.8. *The function $(k,n) \mapsto \rho_{k,n}$ from $\{k \in \mathbb{N};\ k \geq 1\} \times \mathbb{N}$ to $\mathscr{E}$ is effective.*

Proof. Since the function σ is computable as an element in $\mathscr{E}$ it follows by Theorem 3.1.9 that the approximate identity $n \mapsto \sigma_n$ is effective. Now,

$$\rho_{k,n}^{(m)}(t) = (\sigma_{n+2} * c_{k,n+1})^{(m)}(t) = (\sigma_{n+2}^{(m)} * c_{k,n+1})(t) =$$

$$\int_{\mathbb{R}} 2^{(m+1)n}\sigma^{(m)}(2^n(x-t))c_{k,n+1}(x)dx = \int_{K_{k,n+1}} 2^{(m+1)n}\sigma^{(m)}(2^n(x-t))dx$$

$$+ \int_{-k+2^{-(n+2)}}^{-k+2^{-(n+1)}} 2^{(m+1)n}\sigma^{(m)}(2^n(x-t))\Big(1 - 2^{n+2}(-k+2^{-(n+2)} - x)\Big)dx$$

$$+ \int_{k-2^{-(n+1)}}^{k-2^{-(n+2)}} 2^{(m+1)n}\sigma^{(m)}(2^n(x-t))\Big(1 - 2^{n+2}(k-2^{-(n+2)} - x)\Big)dx.$$

It follows by Proposition 3.5.1 that $(k,m,n,t) \mapsto \rho_{k,n}^{(m)}(t)$ is effective, and so $(k,n) \mapsto \rho_{k,n}$ is also effective by Theorem 3.1.9. □

Now, let $f_{k,\langle m,n\rangle} = \rho_{k,m} \cdot p_n$. Then $n \mapsto f_{k,n}$ is an effective enumeration of TP_k for each $k \geq 1$. We are now in a position to give a proof of Lemma 4.1.6:

Proof of Lemma 4.1.6. Let (B,k) be a compact element in $D_{\mathscr{D}}$. Then either $(B,k) = \bot$ and then $(B,k) \in D^D_{\mathscr{D}}$, or $(B,k) \neq \bot$ and then $(B,k) \in D^D_{\mathscr{D}}$ if and only if there is some $f \in \mathrm{TP}_k$ which satisfies B. Thus, if $(B,k) \neq \bot$ then $(B,k) \in D^D_{\mathscr{D}}$ if and only if $f_{k,n}$ satisfies B for some n. Let $F : D_{\mathbb{N}} \times D_{\mathbb{N}} \rightharpoonup D_{\mathscr{E}}$ be an effective representation of the enumeration $(k,n) \mapsto f_{k,n}$ of TP. We claim that $f_{k,n}$ satisfies B if and only if $B \in F(k,n)$.

Let $B = \{B^{(n_1)}(S_1, T_1),\ B^{(n_2)}(S_2, T_2),\ \ldots,\ B^{(n_j)}(S_j, T_j)\}$, and suppose that $f_{k,n}$ satisfies B. Since S_i is compact and T_i is open for each $1 \leq i \leq j$ there is a natural number M such that

$$\inf\{d(z,w);\ z \in f^{(n_i)}_{k,n}[S_i] \text{ and } w \in \mathbb{C} \setminus T_i\} > 2^{-M}$$

for each $1 \leq i \leq j$. It is clear that we may choose M such that $M \geq n_i$ and $M \geq \sup\{|x|;\ x \in S_i\}$ for each $1 \leq i \leq j$. By Lemma 3.1.2, and the remark immediately following it, there is a $C \in F(k,n)$ such that for any $x \in [-M, M]$ and any $m \leq M$ there is some $B^{(m)}(S,T) \in C$ such that $x \in \mathrm{int}(S)$ and $\mathrm{diam}(T) < 2^{-M}$. It follows immediately that $B \sqsubseteq C$ in $B_{\mathscr{E}}$. Since $F(k,n)$ is downwards closed, we must have $B \in F(k,n)$ as required.

For the converse direction it is enough to note that $f_{k,n}$ satisfies each $B \in F(k,n)$ since $F(k,n)$ represents $f_{k,n}$ in $D_{\mathscr{E}}$.

Thus, to decide if $(B,k) \in D^D_{\mathscr{D}}$ we simply check if $(B,k) = \bot$ in $D_{\mathscr{D}}$. If not, we search for a natural number n such that $B \in F(k,n)$. This is semidecidable (with respect to the natural numbering of $B_{\mathscr{E}}$) since F is effective. It follows that $D^D_{\mathscr{D}}$ is a computable element in $\mathrm{cl}(D_{\mathscr{D}})$ by Proposition 4.2 in [7]. □

We may now show that type conversion from $[X \times \mathscr{D} \to \mathbb{C}]$ to $[X \to \mathscr{D}']$ preserves effectivity. If X is a topological space and $f : X \times \mathscr{D} \to \mathbb{C}$ is sequentially continuous and linear in the second argument, then $\varphi \mapsto f(x,\varphi)$ defines a distribution for each $x \in X$. We write $f^* : X \to \mathscr{D}'$ for the map which takes $x \in X$ to the distribution given by $\varphi \mapsto f(x,\varphi)$. f^* is called the *transpose* of f.

Proposition 4.1.9. *Let X be a topological space and suppose that $(D,\ D^R,\ \delta)$ is an effective domain representation of X. Let $f : X \times \mathscr{D} \to \mathbb{C}$ be sequentially continuous and linear in the second argument. Then $f : X \times \mathscr{D} \to \mathbb{C}$ is effective if and only if $f^* : X \to \mathscr{D}'$ is effective.*

Proof. Suppose that $f : X \times \mathscr{D} \to \mathbb{C}$ is effective. Then $f^* : X \to [\mathscr{D} \to_\omega \mathbb{C}]$ is effective by Corollary 4.10 in [7] since the dense part of $D_{\mathscr{D}}$ is computable in $\mathrm{cl}(D_{\mathscr{D}})$. Since $\varphi \mapsto f(x, \varphi)$ is a distribution for each $x \in X$, f^* maps X into $\mathscr{D}'$. It follows that any representation of f^* takes representing elements in D^R to representing elements in $D^R_{\mathscr{D}'}$, and hence, that f^* is effective with respect to the representations $(D,\, D^R,\, \delta)$ and $(D_{\mathscr{D}},\, D^R_{\mathscr{D}},\, \delta_{\mathscr{D}'})$.

Conversely, if $f^* : X \to \mathscr{D}'$ is effective then $f : X \times \mathscr{D} \to \mathbb{C}$ is effective since $f(x, \varphi) = f^*(x)(\varphi)$ for all $x \in X$ and $\varphi \in \mathscr{D}$. □

Given a function φ from $\mathbb{R}$ to $\mathbb{C}$ we define u_φ as the distribution given by $u_\varphi(\psi) = \int_{\mathbb{R}} \varphi\psi$. This definition makes sense whenever φ is locally integrable since ψ has compact support. In particular, if $\varphi \in \mathscr{E}$ then φ is locally integrable, and so $\varphi \mapsto u_\varphi$ defines an embedding of the space $\mathscr{E}$ into the space of distributions $\mathscr{D}'$.

Proposition 4.1.10. *The embedding $\varphi \mapsto u_\varphi$ from $\mathscr{E}$ to $\mathscr{D}'$ is effective.*

Proof. By Proposition 4.1.9 it is enough to show that $(\varphi, \psi) \mapsto \int_{\mathbb{R}} \varphi\psi$ is effective, but this is clear from Propositions 3.3.12 and 3.5.2. □

The corresponding results for $\mathscr{S}$ and $\mathscr{D}$ follow immediately from Proposition 4.1.10 since the inclusions $\mathscr{S} \hookrightarrow \mathscr{E}$ and $\mathscr{D} \hookrightarrow \mathscr{E}$ are effective.

We may also embed the real numbers into $\mathscr{D}'$ as follows: If $x \in \mathbb{R}$ we let δ_x be the distribution given by $\delta_x(\varphi) = \varphi(x)$. δ_x is known as the *Dirac measure* at x. The Dirac measure at x is effective in x by the next Proposition.

Proposition 4.1.11. *The embedding $x \mapsto \delta_x$ is effective.*

Proof. As before, it is enough to show that $(x, \varphi) \mapsto \delta_x(\varphi)$ is effective, but this is immediate since evaluation from $\mathscr{D} \times \mathbb{R}$ to $\mathbb{C}$ is effective. □

Many of the standard operations on the space $\mathscr{D}'$ are defined as extensions of the corresponding operations on $\mathscr{D}$. More precisely, if $f : \mathscr{D} \to \mathscr{D}$ is a continuous function on $\mathscr{D}$ such that $\varphi \mapsto u(f(\varphi))$ defines a distribution for each $u \in \mathscr{D}'$ we may sometimes "lift" f to get a continuous function $F : \mathscr{D}' \to \mathscr{D}'$ given by $F(u)(\varphi) = u(f(\varphi))$ for each $u \in \mathscr{D}'$ and each $\varphi \in \mathscr{D}$. The canonical example here is differentiation on $\mathscr{D}'$: If u is a distribution we define u' by $u'(\varphi) = u(-\varphi')$. (Note that it follows immediately by the definition that we have $u'_\varphi = u_{\varphi'}$ for each $\varphi \in \mathscr{E}$.) Another example is multiplication by a smooth function defined by $(\varphi \cdot u)(\psi) = u(\varphi \cdot \psi)$. By Proposition 4.1.9 such liftings preserve effectivity.

Corollary 4.1.12. *Suppose that $f : \mathscr{D} \to \mathscr{D}$ lifts to a continuous function $F : \mathscr{D}' \to \mathscr{D}'$. If $f : \mathscr{D} \to \mathscr{D}$ is effective then so is $F : \mathscr{D}' \to \mathscr{D}'$.*

Proof. If $f : \mathscr{D} \to \mathscr{D}$ is effective then so is $\mathrm{ev} \circ (\mathrm{id}_{\mathscr{D}'} \times f) : \mathscr{D}' \times \mathscr{D} \to \mathbb{C}$. Since $F = (\mathrm{ev} \circ (\mathrm{id}_{\mathscr{D}'} \times f))^*$ it follows that F is effective. □

We may now apply Proposition 4.1.9 and Corollary 4.1.12 to show that $\mathscr{D}'$ is an effective vector space.

Corollary 4.1.13. *The following operations on $\mathscr{D}'$ are effective.*

1. *Differentiation:* $(u, m) \mapsto u^{(m)}$.
2. *Scalar multiplication:* $(z, u) \mapsto z \cdot u$.
3. *Addition:* $(u, v) \mapsto u + v$.
4. *Multiplication by a smooth function* $(\varphi, u) \mapsto \varphi \cdot u$.

Proof. This follows immediately by Proposition 4.1.9 and Corollary 4.1.12 since addition and multiplication on $\mathbb{C}$ are effective. □

4.2 Representing the space of distributions with compact support

Let u be a distribution and let U be an open subset of $\mathbb{R}$. If $u(\varphi) = 0$ for each $\varphi \in \mathscr{D}$ such that $\operatorname{supp}(\varphi) \subseteq U$, we say that u *vanishes* on U. Let V be the union of all open sets U such that u vanishes on U. Then u vanishes on V. The complement of V is called the *support of* u. We denote the support of u by $\operatorname{supp}(u)$.

If $u \in \mathscr{D}'$ has compact support then u extends in a unique way to a continuous linear functional on $\mathscr{E}$. This extension defines a vector space isomorphism from the space of compactly supported distributions to the dual space of $\mathscr{E}$. We may (and will) therefore identify the space of distributions with compact support with the dual space of $\mathscr{E}$, which we denote by $\mathscr{E}'$. The topology on $\mathscr{E}'$ is the weak*-topology induced by $\mathscr{E}$.

To construct a domain representation of $\mathscr{E}'$ we simply copy the construction carried out for $\mathscr{D}'$ in the previous section. Since most of the proofs will be identical to the proofs of the corresponding results given in section 4.1, we will leave them out.

Thus, let $D_{\mathscr{E}'}$ be the effective domain $[D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}]$ and let $D^R_{\mathscr{E}'} \subseteq D_{\mathscr{E}'}$ be the set of partial continuous functions from $D_{\mathscr{E}}$ to $D_{\mathbb{C}}$ which represent some distribution with compact support. Define $\delta_{\mathscr{E}'} : D_{\mathscr{E}'} \to \mathscr{E}'$ by $\delta_{\mathscr{E}'}(f) = u$ if and only if f represents u. As before, it follows immediately that

Theorem 4.2.1. *$(D_{\mathscr{E}'}, D^R_{\mathscr{E}'}, \delta_{\mathscr{E}'})$ is an effective domain representation of $\mathscr{E}'$.*

That $(D_{\mathscr{E}'}, D^R_{\mathscr{E}'}, \delta_{\mathscr{E}'})$ is an admissible follows by an application of Lemma 4.1.2.

Theorem 4.2.2. *$(D_{\mathscr{E}'}, D^R_{\mathscr{E}'}, \delta_{\mathscr{E}'})$ is an admissible domain representation of $\mathscr{E}'$.*

Proof. The inclusion of $\mathscr{E}'$ into $[\mathscr{E} \to_\omega \mathbb{C}]$ is sequentially continuous since $\mathscr{E}$ is an F-space. The rest of the proof is identical to the proof of Theorem 4.1.4. □

As in the case of $\mathscr{D}'$ it follows by domain theory that evaluation from $\mathscr{E}' \times \mathscr{E}$ to $\mathbb{C}$ is effective.

Proposition 4.2.3. *Evaluation $(u, \varphi) \mapsto u(\varphi)$ on $\mathscr{E}'$ is effective.*

Proof. The proof is essentially the same as the proof of Proposition 4.1.5 and so we leave it out. □

To show that type conversion from $[X \times \mathscr{E} \to \mathbb{C}]$ to $[X \to \mathscr{E}']$ preserves effectivity we need to show that the dense part of the domain representation of $\mathscr{E}$ is computable as a Scott-closed subset of $D_{\mathscr{E}}$. This is Lemma 4.2.4.

Lemma 4.2.4. *$D^D_{\mathscr{E}}$ is a computable element in* $\mathrm{cl}(D_{\mathscr{E}})$.

Proof. Since $\mathscr{D}$ is dense in $\mathscr{E}$, and the inclusion $\mathscr{D} \hookrightarrow \mathscr{E}$ is continuous, it follows that TP is dense in $\mathscr{E}$.

Thus, if B is compact in $D_{\mathscr{E}}$ then $B \in D^D_{\mathscr{E}}$ if and only if B is satisfiable. Since the set $\{\varphi \in \mathscr{E};\ \varphi \text{ satisfies } B\}$ is open in $\mathscr{E}$ it follows that $B \in D^D_{\mathscr{E}}$ if and only if there are natural numbers k and n such that $f_{k,n}$ satisfies B. That this is semidecidable with respect to the natural numbering of $B_{\mathscr{E}}$ follows as in the proof of Lemma 4.1.6. □

As before, it follows that type conversion from $[X \times \mathscr{E} \to \mathbb{C}]$ to $[X \to \mathscr{E}']$ preserves effectivity.

Proposition 4.2.5. *Let X be a topological space and suppose that $(D,\ D^R,\ \delta)$ is an effective domain representation of X. Let $f : X \times \mathscr{E} \to \mathbb{C}$ be sequentially continuous and linear in the second argument. Then $f : X \times \mathscr{E} \to \mathbb{C}$ is effective if and only if $f^* : X \to \mathscr{E}'$ is effective.*

We may also show that "lifting" continuous functions from $[\mathscr{E} \to \mathscr{E}]$ to $[\mathscr{E}' \to \mathscr{E}']$ preserves effectivity.

Corollary 4.2.6. *Suppose that $f : \mathscr{E} \to \mathscr{E}$ lifts to a continuous function $F : \mathscr{E}' \to \mathscr{E}'$. If $f : \mathscr{E} \to \mathscr{E}$ is effective then so is $F : \mathscr{E}' \to \mathscr{E}'$.*

As an immediate consequence of Proposition 4.2.5 and Corollary 4.2.6 we conclude that $\mathscr{E}'$ is an effective vector space. In fact,

Corollary 4.2.7. *The following operations on $\mathscr{E}'$ are effective.*

1. *Differentiation:* $(u, m) \mapsto u^{(m)}$.

2. *Scalar multiplication:* $(z, u) \mapsto z \cdot u$.

3. *Addition:* $(u, v) \mapsto u + v$.

In particular, it follows that $\mathscr{E}'$ is an effective vector space since the distribution $\varphi \mapsto 0$ is computable as an element of $\mathscr{E}'$.

4.3 Representing the space of tempered distributions

A distribution u is called *tempered* if u extends to a continuous linear functional on $\mathscr{S}$. This extension defines a vector space isomorphism between the dual space of $\mathscr{S}$ on the one hand, and the space of tempered distributions on the other. It is customary to identify the space of tempered distributions with the dual of $\mathscr{S}$, which we denote by $\mathscr{S}'$. If we equip $\mathscr{S}'$ with the weak*-topology induced by $\mathscr{S}$ then $\mathscr{S}'$ becomes a locally convex topological vector space.

To construct a domain representation of the space $\mathscr{S}'$ we simply mimic the construction in section 4.1. Thus, we let $D_{\mathscr{S}'}$ be the effective domain $[D_{\mathscr{S}} \rightharpoonup D_{\mathbb{C}}]$ and define $D^R_{\mathscr{S}'} \subseteq D_{\mathscr{S}'}$ by $f \in D^R_{\mathscr{S}'}$ if f represents a tempered distribution. Finally, we let $\delta_{\mathscr{S}'} : D^R_{\mathscr{S}'} \to \mathscr{S}'$ be the function given by $\delta_{\mathscr{S}'}(f) = u$ if and only if f represents the tempered distribution u.

Theorem 4.3.1. *$(D_{\mathscr{S}'}, D^R_{\mathscr{S}'}, \delta_{\mathscr{S}'})$ is an effective domain representation of $\mathscr{S}'$.*

Proof. The proof is identical to the proof of Theorem 4.1.1 □

We may also show that the representation $(D_{\mathscr{S}'}, D^R_{\mathscr{S}'}, \delta_{\mathscr{S}'})$ of $\mathscr{S}'$ is admissible.

Theorem 4.3.2. *$(D_{\mathscr{S}'}, D^R_{\mathscr{S}'}, \delta_{\mathscr{S}'})$ is an admissible domain representation of $\mathscr{S}'$.*

Proof. The proof is identical to the proof of Theorem 4.2.2 since $\mathscr{S}$ is an F-space. □

It is also immediate that evaluation on $\mathscr{S}'$ is effective.

Proposition 4.3.3. *Evaluation $(u, \varphi) \mapsto u(\varphi)$ on $\mathscr{S}'$ is effective.*

As before, we will need a technical lemma to ensure that type conversion from $[X \times \mathscr{S} \to \mathbb{C}]$ to $[X \to \mathscr{S}']$ preserves effectivity.

Lemma 4.3.4. *$D^D_{\mathscr{S}}$ is a computable element in* $\mathrm{cl}(D_{\mathscr{S}})$.

Proof. Since TP is dense in $\mathscr{D}$, $\mathscr{D}$ is dense in $\mathscr{S}$, and the inclusion $\mathscr{D} \hookrightarrow \mathscr{S}$ is continuous, it follows that TP is dense in $\mathscr{S}$.

Now, if (B, C) is compact in $D_{\mathscr{S}}$ then $\{\varphi \in \mathscr{S};\ \varphi \text{ satisfies } B\} \cap \{\varphi \in \mathscr{S};\ \varphi \text{ satisfies } C\}$ is open in $\mathscr{S}$. It follows that $(B, C) \in D^D_{\mathscr{S}}$ if and only if there is some $f \in \mathrm{TP}$ which satisfies both B and C. Thus, to show that $D^D_{\mathscr{S}}$ is computable in $\mathrm{cl}(D_{\mathscr{S}})$ it is enough to show that the relation "There are natural numbers k and n such that $f_{k,n}$ satisfies both B and C" is semidecidable (with respect to the natural numbering of $B_{\mathscr{S}}$).

Let $(B, C) \in B_{\mathscr{S}}$. To check if $f_{k,n}$ satisfies both B and C we first check if $f_{k,n}$ satisfies B. That this is semidecidable with respect to the natural numbering of $B_{\mathscr{S}}$ follows as in the proof of Lemma 4.1.6. If $f_{k,n}$ satisfies B we check if $f_{k,n}$ also satisfies C as follows: Suppose that $C = \{R(k_1, m_1, n_1), \ldots, R(k_j, m_j, n_j)\}$. For each $1 \leq i \leq j$ we need to check if

$$|x^{k_i} f^{(m_i)}_{k,n}(x)| < n_i \text{ for all } x.$$

Since $f_{k,n}$ has compact support contained in the interval $[-k, k]$, this is true if and only if

$$|x^{k_i} f^{(m_i)}_{k,n}(x)| < n_i \text{ for all } |x| \leq k.$$

This is semidecidable since the map $(k_i, k, m_i, n) \mapsto \sup\{|x^{k_i} f^{(m_i)}_{k,n}(x)|;\ |x| \leq k\}$ is effective by Propositions 3.1.8, 3.1.10, and 3.1.11. □

We may now show that type conversion preserves effectivity

Proposition 4.3.5. *Let X be a topological space and suppose that $(D,\ D^R,\ \delta)$ is an effective domain representation of X. Let $f : X \times \mathscr{S} \to \mathbb{C}$ be sequentially continuous and linear in the second argument. Then $f : X \times \mathscr{S} \to \mathbb{C}$ is effective if and only if $f^* : X \to \mathscr{S}'$ is effective.*

Proof. The proof is the same as the proof of Proposition 4.1.9. □

It follows immediately that lifting preserves effectivity, and that the standard operations on $\mathscr{S}'$ are effective. In particular, since $\varphi \mapsto 0$ is computable, $\mathscr{S}'$ is an effective vector space.

Corollary 4.3.6. *Suppose that $f : \mathscr{S} \to \mathscr{S}$ lifts to a continuous function $F : \mathscr{S}' \to \mathscr{S}'$. If $f : \mathscr{S} \to \mathscr{S}$ is effective then so is $F : \mathscr{S}' \to \mathscr{S}'$.*

Corollary 4.3.7. *The following operations on $\mathscr{S}'$ are effective.*

1. *Differentiation:* $(u, m) \mapsto u^{(m)}$.

2. *Scalar multiplication:* $(z, u) \mapsto z \cdot u$.

3. Addition: $(u, v) \mapsto u + v$.

The proofs of Corollaries 4.3.6 and 4.3.7 are identical to the proofs of the corresponding results for $\mathscr{D}'$ and so we leave them out.

The Fourier transform and the inverse Fourier transform on $\mathscr{S}$ lift to continuous, linear, one-to-one maps from $\mathscr{S}'$ onto $\mathscr{S}'$ as in Corollary 4.3.6. Thus,

Corollary 4.3.8. *The Fourier transform and the inverse Fourier transform on $\mathscr{S}'$ are effective.*

Since the inclusion map from $\mathscr{S}$ to $\mathscr{E}$ is continuous, it follows that every distribution with compact support is a tempered distribution. Thus, $\mathscr{E}' \subseteq \mathscr{S}'$, and the inclusion map from $\mathscr{E}'$ into $\mathscr{S}'$ is continuous. Moreover

Proposition 4.3.9. *The inclusion $\mathscr{E}' \hookrightarrow \mathscr{S}'$ is effective.*

Proof. Let i be the inclusion of $\mathscr{S}$ into $\mathscr{E}$ and let $j : \mathscr{E}' \times \mathscr{S} \to \mathbb{C}$ be the effective sequentially continuous function given by $(u, \varphi) \mapsto u(i(\varphi))$. Since j^* is the inclusion map $\mathscr{E}' \hookrightarrow \mathscr{S}'$ it follows that the inclusion of $\mathscr{E}'$ into $\mathscr{S}'$ is effective. □

We may also prove that the inclusion map from $\mathscr{S}'$ into $\mathscr{D}'$ is effective.

Proposition 4.3.10. *The inclusion $\mathscr{S}' \hookrightarrow \mathscr{D}'$ is effective.*

Proof. The proof is identical to the proof of Proposition 4.3.9. □

5 Convolution and fundamental solutions

Let $p(x)$ be a complex polynomial, and let D denote the standard differential operator d/dx. If e is a distribution then e is a *fundamental solution* of the operator $p(D)$ if e satisfies $p(D)e = \delta$ where δ is the Dirac measure at 0. By a Theorem by Malgrange and Ehrenpreis, such fundamental solutions always exist.

Fundamental solutions are interesting for the following reason: If the distribution e satisfies $p(D)e = \delta$, and the convolution $u = e * v$ is defined, then u solves the linear differential equation $p(D)u = v$, since $p(D)(e * v) = (p(D)e) * v = \delta * v = v$. In this section we will show that convolution is an effective operation on the space of distributions, and, as an application of this result, we will show that the solution operator corresponding to the differential operator D^m is effective in m.

5.1 The convolution of a distribution and a smooth function

If u is a distribution and φ is a function we define the convolution of u with φ as the function given by $(u * \varphi)(t) = u(\mathrm{rf}(\mathrm{tr}_t(\varphi)))$ whenever the right-hand side of the equation makes sense. We note that for this to be the case, it is enough to assume that φ is smooth, and that either u or φ has compact support. If $u * \varphi$ is well-defined we have $(u * \varphi)^{(m)} = u^{(m)} * \varphi = u * \varphi^{(m)}$ as before.

The following two propositions show that convolution of a distribution and a smooth function is an effective operation. The first takes care of the case when the support of the function is compact, and the second takes care of the case when the support of the distribution is compact.

Proposition 5.1.1. *Convolution* $(u, \varphi) \mapsto u*\varphi$ *is an effective map from* $\mathscr{D}' \times \mathscr{D}$ *to* $\mathscr{E}$.

Proof. Since $(u * \varphi)^{(m)}(t) = (u * \varphi^{(m)})(t) = u(\mathrm{rf}(\mathrm{tr}_t(\varphi^{(m)})))$, the result now follows by an application of Theorem 3.1.9. □

Proposition 5.1.2. *Convolution* $(u, \varphi) \mapsto u*\varphi$ *is an effective map from* $\mathscr{E}' \times \mathscr{E}$ *to* $\mathscr{E}$.

Proof. By a similar application of Theorem 3.1.9. □

Suppose that φ is a smooth function with compact support contained in the interval $[-k, k]$. Then $\mathrm{supp}(x \mapsto \varphi(t - x))$ is contained in $t + [-k, k]$. Thus, if u is a distribution with compact support contained in the interval $[-l, l]$, then $(t + [-k, k]) \cap [-l, l] = \emptyset$ whenever $|t| > k + l$, and so we must have $\mathrm{supp}(u * \varphi) \subseteq [-(k + l), (k + l)]$. It follows that convolution $(u, \varphi) \mapsto u * \varphi$ maps $\mathscr{E}' \times \mathscr{D}$ into $\mathscr{D}$.

Proposition 5.1.3. *Convolution* $(u, \varphi) \mapsto u*\varphi$ *is an effective map from* $\mathscr{E}' \times \mathscr{D}$ *to* $\mathscr{D}$.

Proof. Suppose that $f \in D_{\mathscr{E}'}$ represents $u \in \mathscr{E}'$. Let C be a bounded (as a subset of $\mathbb{C}$) approximation of 0 in $B_{\mathbb{C}}$, let $B \neq \perp$ approximate the constant function $x \mapsto 0$ in $B_{\mathscr{E}}$, and suppose that $C \sqsubseteq f(B)$ in $B_{\mathbb{C}}$. Let K be the least natural number k such that $k \geq \sup\{|x|; x \in S\}$ for each $B^{(m)}(S, T) \in B$. We claim that the support of u must be contained in the interval $[-K, K]$. Thus, let $\varphi \in \mathscr{E}$ and suppose that $\mathrm{supp}(\varphi) \subseteq \mathbb{R} \setminus [-K, K]$. Then $c\varphi(x) = 0$ for each $c \in \mathbb{C}$, and each $x \in [-K, K]$. It follows that $c\varphi$ satisfies B for each $c \in \mathbb{C}$. Since f represents u we conclude that $cu(\varphi) = u(c\varphi) \in C$ for each $c \in \mathbb{C}$. Since the set $C \subseteq \mathbb{C}$ is bounded it follows that $u(\varphi) = 0$ as required.

Now, for each $k \geq 1$, let $B_k \in B_{\mathscr{E}}$ be the compact approximation of the constant function $x \mapsto 0$ given by

$$B_k = \{B^{(m)}([-k,k],\, [-2^{-k},2^{-k}] \times i[-2^{-k},2^{-k}]);\ 0 \leq m \leq k\}$$

and let $C = [-1,1] \times i[-1,1]$. Finally, define $s : D_{\mathscr{E}} \to D_S$ by $s(f) =$ the least natural number k such that $B_k \in \mathrm{dom}(f)$ and $C \sqsubseteq f(B_k)$, and $s(f) = \bot$ if no such k exists.

The function $s : D_{\mathscr{E}} \to D_S$ is continuous: It is enough to show that s is monotone since every monotone function into D_S is continuous. Thus, let f and g be partial continuous functions from $D_{\mathscr{E}}$ to $D_{\mathbb{C}}$ and suppose that $f \sqsubseteq g$. If $B_k \in \mathrm{dom}(f)$ and $C \sqsubseteq f(B_k)$ then $B_k \in \mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $C \sqsubseteq f(B_k) \sqsubseteq g(B_k)$. It follows that $s(f) \sqsubseteq s(g)$ in D_S and so s is monotone.

The set $\{B_k;\ k \geq 1\}$ is a directed subset of $D_{\mathscr{E}}$ and $I = \bigsqcup\{B_k;\ k \geq 1\}$ represents the constant function $x \mapsto 0$ in $D_{\mathscr{E}}$. Thus, if $f \in D_{\mathscr{E}'}$ represents u then $I \in \mathrm{dom}(f)$ and $C \sqsubseteq f(I)$ in $D_{\mathbb{C}}$. It follows that $C \sqsubseteq f(B_k)$ for some k and so $k \sqsubseteq s(f)$ in D_S.

Now, to compute the convolution of a compactly supported distribution and a compactly supported smooth function, let $c : D_{\mathscr{E}'} \times D_{\mathscr{D}} \rightharpoonup D_{\mathscr{E}}$ be an effective representation of the map $(u,\varphi) \mapsto u * \varphi$ from $\mathscr{E}' \times \mathscr{D}$ to $\mathscr{E}$, and let $a : D_S \times D_S \to D_S$ be the monotone function given by $a(k,l) = k + l$, and $a(k,\bot) = a(\bot,l) = a(\bot,\bot) = \bot$. We let $t : D_{\mathscr{E}'} \times D_{\mathscr{D}} \to D_S$ be the continuous function given by

$$t(f,I) = \bigsqcup\{a(k,l);\ k \sqsubseteq s(f) \text{ and } (B,l) \in I \text{ for some } B \in B_{\mathscr{E}}\}.$$

It is clear that t is continuous, and if f represents u in $D_{\mathscr{E}'}$ and I represents φ in $D_{\mathscr{D}}$ then $t(f,I) \neq \bot$ and $\mathrm{supp}(u*\varphi)$ is contained in $[-t(f,I), t(f,I)]$. We let $d : D_{\mathscr{E}'} \times D_{\mathscr{D}} \rightharpoonup D_{\mathscr{D}}$ be the partial continuous function given by $\mathrm{dom}(d) = \mathrm{dom}(c)$, and $d(f,J) = \{(B,k) \in B_{\mathscr{D}};\ B \in c(f,J), \text{ and } k \geq t(f,J)\}$. It follows immediately that d is an effective partial continuous representation of the map $(u,\varphi) \mapsto u * \varphi$. □

As a simple application of Proposition 5.1.1, let $(\sigma_n)_n$ be the approximate identity from Section 4.1, and let u be a distribution. Since $(\sigma_n)_n \longrightarrow \delta$ in $\mathscr{D}'$, and convolution is sequentially continuous we have $(u * \sigma_n)_n \longrightarrow u * \delta = u$. Since the sequence $(\sigma_n)_n$ is effective it follows by Proposition 5.1.1 that the sequence $(u * \sigma_n)_n$ is effective in u. Thus, it follows that the space $\mathscr{E}$ is *effectively dense* in $\mathscr{D}'$ in the following precise sense: Given a (representation of a) distribution u we can construct a (representation of a) sequence in $\mathscr{E}$ which converges to u in $\mathscr{D}'$. Furthermore, if u is a computable element of $\mathscr{D}'$ we can actually *compute* an effective sequence in $\mathscr{E}$ which converges to u in $\mathscr{D}'$.

5.2 The convolution of two distributions

If u and v are distributions we define $u * v$ as the continuous linear functional which takes $\varphi \in \mathscr{D}$ to $(u * (v * \mathrm{rf}(\varphi))(0) \in \mathbb{C}$. $u * v$ is well-defined if either u or v has compact support, and $u * v$ is a distribution. The convolution $(u, v) \mapsto u * v$ is effective by the next proposition.

Proposition 5.2.1. *Convolution* $(u, v) \mapsto u * v$ *from* $\mathscr{D}' \times \mathscr{E}'$ *to* $\mathscr{D}'$ *is effective.*

Proof. By Proposition 4.1.9 it is enough to show that the map $(u, v, \varphi) \mapsto (u * v)(\varphi)$ from $\mathscr{D}' \times \mathscr{E}' \times \mathscr{D}$ to $\mathbb{C}$ is effective, but this follows immediately by Propositions 5.1.1 and 5.1.3 since reflection on $\mathscr{D}$ and evaluation on $\mathscr{E}$ are effective. ☐

5.3 Computing primitives in $\mathscr{D}'$ using convolution

It is also possible to give more symmetric requirements on the distributions u and v to ensure that the convolution $u * v$ of u and v exists. An interesting example from the point of view of applications is when the supports of u and v are contained in the half-open interval $[-k, \infty)$ for some natural number k.

We let $\mathscr{E}_+$ be the collection of smooth functions with support contained in $[-k, \infty)$ for some $k \in \mathbb{N}$. The topology on $\mathscr{E}_+$ is simply the subspace topology with respect to $\mathscr{E}$. To construct a domain representation of $\mathscr{E}_+$, we let $D_{\mathscr{E}_+}$ be the domain $D_{\mathscr{E}} \times D_S$, and if $(I, k) \in D_{\mathscr{E}_+}$ we say that (I, k) *represents* $\varphi \in \mathscr{E}_+$ if and only if I represents φ in $D_{\mathscr{E}}$, and $\mathrm{supp}(\varphi) \subseteq [-k, \infty)$. We let $D^R_{\mathscr{E}_+}$ be the set of all representing elements in $D_{\mathscr{E}_+}$, and define $\delta_{\mathscr{E}_+} : D^R_{\mathscr{E}_+} \to \mathscr{E}_+$ as the function given by $\delta_{\mathscr{E}_+}(I, k) = \varphi$ if and only if (I, k) represents φ.

Proposition 5.3.1. $(D_{\mathscr{E}_+}, D^R_{\mathscr{E}_+}, \delta_{\mathscr{E}_+})$ *is an effective domain representation of* $\mathscr{E}_+$.

Proof. The domain $D_{\mathscr{E}_+}$ is clearly effective, and the function $\delta_{\mathscr{E}_+}$ is surjective by definition, and continuous since $\delta_{\mathscr{E}_+} = \delta_{\mathscr{E}} \circ f$, where $f : D_{\mathscr{E}_+} \to D_{\mathscr{E}}$ is the left projection restricted to $D^R_{\mathscr{E}_+}$. ☐

Similarly, we let $\mathscr{D}'_+$ be the space of distributions supported in the half-open interval $[-k, \infty)$ for some $k \in \mathbb{N}$. The topology on $\mathscr{D}'_+$ is the subspace topology with respect to $\mathscr{D}'$ as before. We let $D_{\mathscr{D}'_+}$ be the domain $D_{\mathscr{D}'} \times D_S$, and if $(I, k) \in D_{\mathscr{D}'_+}$ we say that (I, k) *represents* the distribution $u \in \mathscr{D}'_+$ if and only if I represents u in $D_{\mathscr{D}'}$, and $\mathrm{supp}(u) \subseteq [-k, \infty)$. We let $D^R_{\mathscr{D}'_+}$ be the set of all pairs $(I, k) \in D_{\mathscr{D}'_+}$ which represent some element in $\mathscr{D}'_+$ and define $\delta_{\mathscr{D}'_+} : D^R_{\mathscr{D}'_+} \to \mathscr{D}'_+$ by $\delta_{\mathscr{D}'_+}(I, k) = u$ if and only if (I, k) represents u.

Proposition 5.3.2. *$(D_{\mathscr{D}'_+}, D^R_{\mathscr{D}'_+}, \delta_{\mathscr{D}'_+})$ is an effective domain representation of $\mathscr{D}'_+$.*

Proof. $D_{\mathscr{D}'_+}$ is clearly an effective domain. The function $\delta_{\mathscr{D}'_+}$ is continuous and surjective since it factors through $\delta_{\mathscr{D}'}$ via the left projection from $D_{\mathscr{D}'_+}$ to $D_{\mathscr{D}'}$, restricted to $D^R_{\mathscr{D}'_+}$. □

Dually, we define $\mathscr{E}_-$ as the space of smooth functions supported in the half-open interval $(-\infty, k]$ for some $k \in \mathbb{N}$, and $\mathscr{D}'_-$ as the space of distributions supported in $(-\infty, k]$ for some $k \in \mathbb{N}$. It is clear that we may construct effective domain representations of these spaces in exactly the same way as above. Thus, we let $(D_{\mathscr{E}_-}, D^R_{\mathscr{E}_-}, \delta_{\mathscr{E}_-})$ be an effective domain representation of $\mathscr{E}_-$, and let $(D_{\mathscr{D}'_-}, D^R_{\mathscr{D}'_-}, \delta_{\mathscr{D}'_-})$ be an effective domain representation of $\mathscr{D}'_-$.

Now, suppose that u is a distribution with support in the half-open interval $[-k, \infty)$, and let σ_k be a smooth function in $\mathscr{E}_+$ such that $0 \leq \sigma_k(x) \leq 1$ for each x, and $\sigma_k(x) = 1$ on an open neighbourhood of $[-k, \infty)$. Then $u = \sigma_k u$ is in $\mathscr{D}'$. That is, if $\varphi \in \mathscr{D}$ then $u(\varphi) = (\sigma_k u)(\varphi) = u(\sigma_k \varphi)$. Since $\sigma_k \varphi \in \mathscr{D}$ for each $\varphi \in \mathscr{E}_-$ it follows that we can extend u to $\mathscr{E}_-$ by setting $u(\varphi) = (\sigma_k u)(\varphi) = u(\sigma_k \varphi)$ on $\mathscr{E}_-$. It is easy to show that this definition is independent of the cut-off function σ_k. Furthermore, defined in this way, evaluation from $\mathscr{D}'_+ \times \mathscr{E}_-$ to $\mathbb{C}$ is effective.

Proposition 5.3.3. *Evaluation from $\mathscr{D}'_+ \times \mathscr{E}_-$ to $\mathbb{C}$ is effective.*

Note that it is not enough to show that we can compute (a representation of) σ_n from n since the function from D_S to $D_{\mathscr{E}}$ given by $n \mapsto$ a representation of σ_n would not be continuous. In fact, it is easy to see that it would not even be monotone. It turns out that the solution is to, in a sense, "evaluate u over the collection of all smooth functions $\sigma_n \varphi$ such that $n \geq k$ simultaneously".

Proof of Proposition 5.3.3. Let $(\sigma_n)_n$ be an effective sequence of smooth functions such that $0 \leq \sigma_n(x) \leq 1$ for all x, $\sigma_n(x) = 1$ on $[-(n+1), \infty)$, and $\sigma_n(x) = 0$ on $(-\infty, -(n+2)]$. (The effective sequence $(\sigma_n)_n$ may be constructed as in Lemma 4.1.8.) Let $s : D_{\mathbb{N}} \to D_{\mathscr{E}}$ be an effective continuous representation of the map $n \mapsto \sigma_n$. (We may assume that s is total since $\mathbb{N}$ is dense in $\mathscr{D}_{\mathbb{N}}$.)

Let Q be the closure of $D^R_{\mathscr{D}'_+}$ in $D_{\mathscr{D}'_+}$, and let R be the closure of $D^R_{\mathscr{E}_-}$ in $D_{\mathscr{E}_-}$. Let $m : D_{\mathscr{E}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathscr{E}}$ be an effective representation of the map $(\varphi, \psi) \mapsto \varphi \cdot \psi$ on $\mathscr{E}$. Then $t : D_{\mathbb{N}} \times D_{\mathscr{E}_-} \rightharpoonup D_{\mathscr{D}}$ given by $\mathrm{dom}(t) = D_{\mathbb{N}} \times R$, and $t(n, (J, l)) = \{(B, k) \in B_{\mathscr{D}};\ B \sqsubseteq m(s(n), J), \text{ and } k \geq \max(l, n)\}$ on $\mathrm{dom}(t)$, is an effective representation of the map $(n, \varphi) \mapsto \sigma_n \cdot \varphi$ from $\mathbb{N} \times \mathscr{E}_-$

to $\mathscr{D}$. Now, if $(f,k) \in Q$ and $(C,l) \in R$ are compact we let $E(f,k,C,l)$ denote the set of all $S \in B_{\mathbb{C}}$ such that

$$S \sqsubseteq f(t(m,B,n)) \text{ for some } m \sqsubseteq k,\ n \sqsubseteq l,$$

$$\text{and } B \sqsubseteq C \text{ such that } (m,B,n) \in \operatorname{dom}(f \circ t).$$

We claim that $E(f,k,C,l)$ is a nonempty and consistent for all compact $(f,k) \in Q$ and $(C,l) \in R$. To prove this, let $(f,k) \in Q$ and $(C,l) \in R$ be compact. Then $\perp \in E(f,k,C,l)$ and so $E(f,k,C,l)$ is nonempty. Suppose that $S_1, S_2 \in E(f,k,C,l)$. To show that S_1 and S_2 are consistent in $B_{\mathbb{C}}$ it is enough to show that $S_1 \cap S_2$ is nonempty. Since $(f,k) \in Q$ there is a partial continuous function $g \in D_{\mathscr{D}'}$, and a distribution $u \in \mathscr{D}'_+$ such that $f \sqsubseteq g$ in $D_{\mathscr{D}'}$, g represents u in $D_{\mathscr{D}'}$, and $\operatorname{supp}(u) \subseteq [-k,\infty)$. Similarly, since $(C,l) \in R$ there is an ideal $J \in D_{\mathscr{E}}$ and a smooth function $\phi \in \mathscr{E}_-$ such $C \in J$, J represents ϕ in $D_{\mathscr{E}}$, and $\operatorname{supp}(\phi) \subseteq (-\infty, l]$. For each $i \in \{1,2\}$ there are B_i in $B_{\mathscr{E}}$, $m_i \sqsubseteq k$ and $n_i \sqsubseteq l$ in D_S such that $(m_i,B_i,n_i) \in \operatorname{dom}(f \circ t)$, and $S_i \sqsubseteq f(t(m_i,B_i,n_i))$ in $D_{\mathbb{C}}$. It follows that

$$S_i \sqsubseteq f(t(m_i,B_i,n_i)) \sqsubseteq g(t(m_i,B_i,n_i)) \sqsubseteq g(t(k,J,l))$$

in $D^R_{\mathbb{C}}$. Since J represents ϕ in $D_{\mathscr{E}}$ it follows that (J,l) represents ϕ in $D_{\mathscr{E}_-}$. Since t represents the map $(n,\varphi) \mapsto \sigma_n \cdot \varphi$ and g represents u in $D_{\mathscr{D}'}$ we conclude that $g(t(k,J,l))$ represents $u(\sigma_k\phi)$ in $D_{\mathbb{C}}$. Since $S_i \sqsubseteq g(t(k,J,l))$ for each $i \in \{1,2\}$ we have $u(\phi) = u(\sigma_k\phi) \in S_1 \cap S_2$, and $S_1 \cap S_2$ is nonempty as required.

It follows easily from the definition of E above that $E(f,k,B,l) \subseteq E(g,m,C,n)$ whenever $(f,k) \sqsubseteq (g,m)$ in $D_{\mathscr{D}'_+}$ and $(B,l) \sqsubseteq (C,n)$ in $D_{\mathscr{E}_-}$. We conclude that there is a (unique) partial continuous function $e : D_{\mathscr{D}'_+} \times D_{\mathscr{E}_-} \rightharpoonup D_{\mathbb{C}}$ such that $\operatorname{dom}(e) = R \times Q$ and $e(f,k,C,l) = \bigsqcup E(f,k,C,l)$ for all compact $(f,k) \in R$ and $(C,l) \in Q$. We would like to show that e represents evaluation from $\mathscr{D}'_+ \times \mathscr{E}_-$ to $\mathbb{C}$. Thus, let (g,k) represent u in $D_{\mathscr{D}'_+}$ and let (J,l) represent ϕ in $D_{\mathscr{E}_-}$. If $S \sqsubseteq e(g,k,J,l)$ then $S \sqsubseteq e(f,m,C,n)$ for some compact $(f,m) \sqsubseteq (g,k)$ and $(C,n) \sqsubseteq (J,l)$. It follows that there are $S_1, S_2, \ldots, S_p \in E(f,m,C,n)$ such that $S \sqsubseteq S_1 \sqcup S_2 \sqcup \ldots \sqcup S_p$. Each S_i contains $u(\phi)$ as above, and so we must have $u(\phi) \in S_1 \cap S_2 \cap \ldots \cap S_p \subseteq S$. Let $N \in \mathbb{N}$, and choose $S \in B_{\mathbb{C}}$ such that $u(\phi) \in \operatorname{int}(S)$ and $\operatorname{diam}(S) < 2^{-N}$. Since $g(t(k,J,l))$ represents $u(\phi)$ in $D_{\mathbb{C}}$ we have $S \sqsubseteq g(t(k,J,l))$. It follows that there are compact $(f,k) \sqsubseteq (g,k)$ and $(C,l) \sqsubseteq (J,l)$ such that $(k,C,l) \in \operatorname{dom}(f \circ t)$ and $S \sqsubseteq f(t(k,C,l))$. By the definition of E we have $S \in E(f,k,C,l)$, and so $S \sqsubseteq e(f,k,C,l) \sqsubseteq e(g,k,J,l)$. We conclude that $e(g,k,J,l)$ represents $u(\phi)$ in $D_{\mathbb{C}}$. Since (g,k) and (J,l) where arbitrary, we are done. $\square$

Now, let u be a distribution with support contained in the interval $[-k, \infty)$, and let φ be a smooth function with compact support contained in the interval $[-l, \infty)$. Then $\operatorname{supp}(x \mapsto \varphi(t-x)) \subseteq t+(-\infty, l]$. Thus, if $t < -(k+l)$, then $(t + (-\infty, l]) \cap [-k, \infty) = \emptyset$. It follows that the support of $u * \varphi$ is contained in $[-(k + l), \infty)$.

Proposition 5.3.4. *Convolution* $(u, \varphi) \mapsto u*\varphi$ *from* $\mathscr{D}'_+ \times \mathscr{E}_+$ *to* $\mathscr{E}_+$ *is effective.*

Proof. It is easy to show that differentiation and translation on $\mathscr{E}_+$, as well as reflection from $\mathscr{E}_+$ to $\mathscr{E}_-$ are effective. Now, since $(u * \varphi)^{(m)}(t) = u(\mathrm{rf}(\mathrm{tr}_t(\varphi^{(m)})))$ it follows that $(u, \varphi, m, t) \mapsto (u * \varphi)^{(m)}(t)$ is an effective map from $\mathscr{D}'_+ \times \mathscr{E}_+ \times \mathbb{N} \times \mathbb{R}$ to $\mathbb{C}$. Thus, by Theorem 3.1.9 we know that the map $(u, \varphi) \mapsto u * \varphi$ from $\mathscr{D}_+ \times \mathscr{E}_+$ to $\mathscr{E}$ is effective. To show that convolution is effective as a map from $\mathscr{D}_+ \times \mathscr{E}_+$ into $\mathscr{E}_+$ it is enough to show that it is possible to compute a bound on the support of $u * \varphi$, given bounds on the support of u and the support of φ. However, this is clearly the case since $\operatorname{supp}(u) \subseteq [-k, \infty)$ and $\operatorname{supp}(\varphi) \subseteq [-l, \infty)$ implies that $\operatorname{supp}(u * \varphi) \subseteq [-(k + l), \infty)$, and the function $b : D_S \times D_S \to D_S$ given by $b(k, l) = k + l$, and $b(\bot, \bot) = b(k, \bot) = b(\bot, l) = \bot$ is continuous. □

Proposition 5.3.5. *Convolution is an effective operation on* $\mathscr{D}'_+$.

Proof. It is easy to show that the inclusion map from $\mathscr{D}$ into $\mathscr{E}_+$ is effective. Since $(u * v)(\varphi) = (u * (v * \mathrm{rf}(\varphi)))(0)$ the result follows by (two successive applications of) the previous proposition. □

By considering the dual case when $u \in \mathscr{D}'_-$ and $\varphi \in \mathscr{E}_-$ we immediately have

Proposition 5.3.6. *Convolution* $(u, \varphi) \mapsto u*\varphi$ *from* $\mathscr{D}'_- \times \mathscr{E}_-$ *to* $\mathscr{E}_-$ *is effective.*

We may also prove a version of Proposition 5.3.5 for $\mathscr{D}'_-$.

Proposition 5.3.7. *Convolution is an effective operation on* $\mathscr{D}'_-$.

Since the proofs of Propositions 5.3.6 and 5.3.7 are entirely analogous to those of Propositions 5.3.4 and 5.3.5 we may safely leave them out.

A primitive to the distribution v is any solution to the differential equation $u' = v$. Let D denote the differential operator d/dx on $\mathscr{D}'$. It is well-known that the Heaviside function H given by $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$ is a fundamental solution to D. That is, $H' = \delta$ in $\mathscr{D}'$. It follows that $H * v$ is a solution to the differential equation $u' = v$ whenever the convolution $H * v$ is defined. By Proposition 5.3.5 we know that $H * v$ exists if v lies in $\mathscr{D}'_+$. On the other hand, by a simple computation one can show that $(1 - H)$ is also a fundamental solution to D. Since every distribution v can be decomposed as a sum $v = v_+ + v_-$ where $v_+ \in \mathscr{D}'_+$ and $v_- \in \mathscr{D}'_-$, we can compute a solution to the differential equation $u' = v$ by first decomposing v as $v = v_+ + v_-$, and then computing $u = H * v_+ + (1 - H) * v_-$. Since both H and $(1 - H)$ are fundamental solutions to the differential operator D one now has $Du = D(H * v_+ + (1 - H) * v_-) = H' * v_+ + (1 - H)' * v_- = \delta * v_+ + \delta * v_- = v_+ + v_- = v$ as required.

To show that the map $v \mapsto H * v_+ + (1 - H) * v_-$ is effective we need the following simple lemma.

Lemma 5.3.8. *The Heaviside function H is a computable element of $\mathscr{D}'_+$.*

Proof. Since the dense part of the representation $(D_{\mathscr{D}}, D^R_{\mathscr{D}}, \delta_{\mathscr{D}})$ is a computable element in $\mathrm{cl}(D_{\mathscr{D}})$ by Lemma 4.1.6, it follows by Proposition 4.6 in [7] that it is enough to show that the map $\varphi \mapsto H(\varphi) = \int_0^\infty \varphi$ is effective. This is easy to show using the same techniques as in section 3.5. We leave the details to the reader. □

We note that it follows immediately from Lemma 5.3.8 that $(1 - H)$ is a computable element of $\mathscr{D}'_-$.

Proposition 5.3.9. *The differential operator D has an effective solution operator.*

(That is, there is an effective function $s : \mathscr{D}' \to \mathscr{D}'$ such that $s(u)' = u$ for each distribution u.)

Proof. Let σ be a computable element of $\mathscr{E}$ such that $0 \leq \sigma(x) \leq 1$ for each x, $\sigma(x) = 1$ for each $x \geq 1$, and $\sigma(x) = 0$ for each $x \leq -1$. (It is easy to construct σ using the same methods as in Lemma 4.1.8. Again, we feel that we may safely leave the details to the reader.) Now, $u_+ = \sigma u$ is in $\mathscr{D}'_+$ for each $u \in \mathscr{D}'$, and the map $u \mapsto u_+$ from $\mathscr{D}'$ to $\mathscr{D}'_+$ is effective since σ is computable and the map $(\sigma, u) \mapsto \sigma \cdot u$ is effective. Similarly, if we let $u_- = (1-\sigma)u$ then $u_- \in \mathscr{D}'_-$ for each $u \in \mathscr{D}'$, and it is clear that the function $u \mapsto u_-$ from $\mathscr{D}'$ to $\mathscr{D}'_-$ is effective as well. Now, $u = u_+ + u_-$ where $u_+ \in \mathscr{D}'_+$ and $u_- \in \mathscr{D}'_-$, and if $s(u) = H * u_+ + (1-H) * u_-$ then $s(u)' = u$ as above, and $s : \mathscr{D} \to \mathscr{D}$ is effective by Propositions 5.3.5 and 5.3.7. □

Since the map $(u, m) \mapsto s^m(u)$ is effective it follows immediately from the previous proposition that D^m has an effective solution operator which can be computed uniformly in m.

Proposition 5.3.10. *The differential operator D^m has an effective solution operator which can be computed uniformly in m.*

A Appendices

A.1 Some topological properties of the domain representations introduced in this paper

We have already show that (most of) the domain representations introduced in this paper are admissible, but there are other purely topological characterisations of domain representations which may be interesting from a representation theoretic point of view. Here, we consider some of the domain representations introduced above in the light of some purely topological characterisations of domain representations.

Definition A.1.1. Let $(D,\, D^R,\, \delta)$ be a domain representation of the space X.

1. $(D,\, D^R,\, \delta)$ is *upwards closed* if D^R is upwards closed in D, and if $r \sqsubseteq s$ in D^R and $\delta(r) = x$ then $\delta(s) = x$.

2. $(D,\, D^R,\, \delta)$ is *dense* if D^R is dense in D.

3. $(D,\, D^R,\, \delta)$ is a *retract representation* if δ has a continuous right inverse.

4. $(D,\, D^R,\, \delta)$ is a *quotient representation* if δ is a quotient map.

5. $(D,\, D^R,\, \delta)$ is *open* if the map δ is open.

To help us characterise the domain representations introduced in this paper, we will need the following two theorems.

Theorem A.1.1. *Let $(D,\, D^R,\, \delta)$ be an admissible domain representation of the topological space X. Then δ is a quotient map if and only if X is a sequential space.*

Theorem A.1.2. *Let $(D,\, D^R,\, \delta)$ be an admissible domain representation of the space X. Then $(D,\, D^R,\, \delta)$ is a retract representation of X if and only if X is countably based.*

The first theorem has been proved in the context of TTE by Schröder in [19], as well as for equilogical spaces by Bauer in [2]. The second result is a slight strengthening of a Theorem due to Blanck. For a proof of the second theorem we refer the reader to [8].

Theorem A.1.3. *The topological properties of the domain representations of the spaces $\mathscr{E}$, $\mathscr{D}$, $\mathscr{S}$, $\mathscr{D}'$, $\mathscr{E}'$, and $\mathscr{S}'$ are summarised in Figure 7.*

	$\mathscr{E}$	$\mathscr{D}$	$\mathscr{S}$	$\mathscr{D}'$	$\mathscr{E}'$	$\mathscr{S}'$
Upwards closed				×	×	×
Dense						
Admissible	×	×	×	×	×	×
Quotient	×	×	×			
Retract	×		×			
Open	×		×			
δ is a bijection	×					
δ is a homeomorphism	×					

Figure 7. A cross in the column of X means that the domain representation $(D,\, D^R,\, \delta)$ of X has the corresponding property.

Proof. To show that the representations of the spaces $\mathscr{E}$, $\mathscr{D}$, and $\mathscr{S}$ are not upwards closed, let $B_n = \{B^{(m)}([-n, n], (2^{-n}, 2^{-n}) \times i(2^{-n}, 2^{-n}));\ m \leq n\} \in B_{\mathscr{E}}$ and $I = \bigsqcup_n B_n$. Then I represents the constant function $x \mapsto 0$ in $\mathscr{E}$. Let C be the compact element $\{B^{(0)}([-1, 1], (0, 1) \times i(0, 1))\}$ in $B_{\mathscr{E}}$. Then I and C are consistent, but $I \sqcup C$ is not in $D^R_{\mathscr{E}}$ since C does not approximate $x \mapsto 0$. It follows immediately that the representations of $\mathscr{E}$, $\mathscr{D}$, and $\mathscr{S}$ are not upwards closed.

On the other hand, since the standard representation of $\mathbb{C}$ is upwards closed, it follows that the representations of $\mathscr{D}'$, $\mathscr{E}'$, and $\mathscr{S}'$ are upwards closed.

It is easy to show that the representations of $\mathscr{E}$, $\mathscr{D}$, and $\mathscr{S}$ are not dense. To see that the representation of $\mathscr{D}'$ is not dense, let (B, i) be a compact approximation of the constant function $x \mapsto 0$ in $D_{\mathscr{D}}$ and choose $T \in \mathscr{B}_{\mathbb{C}}$ such that $0 \notin T$. Let $f : D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}$ be any compact element in $D_{\mathscr{D}'}$ such that $(B, i) \in \mathrm{dom}(f)$ and $T \sqsubseteq f(B, i)$. If $f \prec u$ for some distribution u, then $u(\varphi) \subseteq T$ for each $\varphi \in \mathscr{D}$ which satisfies (B, i). Thus in particular, we have $u(0) \in T$ which is impossible since u is linear. Since $\uparrow f \cap D^R_{\mathscr{D}'} = \emptyset$ we conclude that the representation of $\mathscr{D}'$ is not dense. That the domain representations of $\mathscr{E}'$ and $\mathscr{S}'$ are not dense is proved similarly.

The only sequential spaces on the list are the metrisable spaces $\mathscr{E}$ and $\mathscr{S}$, and the inductive limit $\mathscr{D}$ (c.f. Dudley [9]). It follows that the domain representations of $\mathscr{E}$, $\mathscr{D}$ and $\mathscr{S}$ are the only quotient representations of the spaces on the list. The space $\mathscr{D}$ is not countably based (in fact, it is not even locally countably based), and the spaces $\mathscr{D}'$, $\mathscr{E}'$, and $\mathscr{S}'$ cannot be countably based as they are not even sequential. Hence, we conclude that the only retract representations on the list are the domain representations of the spaces $\mathscr{E}$ and $\mathscr{S}$.

It is easy to show that $\delta_{\mathscr{E}}$ is a bijection using the same methods as in the proof of Lemma 4.1.6. Since $\delta_{\mathscr{E}}$ is a quotient map it follows that $\delta_{\mathscr{E}}$ is a homeomorphism from $D^R_{\mathscr{E}}$ to $\mathscr{E}$.

Finally, to show that $\delta_{\mathscr{S}}$ is open it is enough to note that the set $\delta_{\mathscr{S}}(\uparrow(B,C) \cap D^R_{\mathscr{S}}) = \{\varphi \in \mathscr{S};\ \varphi \text{ satisfies } B\} \cap \{\varphi \in \mathscr{S};\ \varphi \text{ satisfies } C\}$ is open in $\mathscr{S}$ for each compact element (B,C) in $D_{\mathscr{S}}$. □

Since the spaces $\mathscr{D}'$, $\mathscr{E}'$, and $\mathscr{S}'$ are not sequential, Theorems A.1.1 and A.1.2 give very general restrictions on any admissible domain representation of one of these spaces. In particular, if $(D,\ D^R,\ \delta)$ is an admissible domain representation of either $\mathscr{D}'$, $\mathscr{E}'$, or $\mathscr{S}'$, then $(D,\ D^R,\ \delta)$ is neither a quotient nor a retract representation.

Index of notation

We try to adhere to the following notational conventions: The letters $B, C, \ldots$ are used for elements in either $B_{\mathscr{E}}$ or $B_{\mathscr{S}}$. Letters $S, T, \ldots$ denote either real intervals or complex boxes. Finally, the letters $I, J, \ldots$ are used to denote ideals.

$D_{\mathbb{R}}$	The standard representation of $\mathbb{R}$.	9
D_R	The ideal completion of B_R.	29
$D_{\mathscr{S}}$	The standard representation of $\mathscr{S}$.	29
$\mathscr{E}_-$	The space of smooth functions supported in $(-\infty, k]$ for some k.	59
$\mathscr{E}_+$	The space of smooth functions supported in $[-k, \infty)$ for some k.	58
$\{f_{k,n}\}_{k,n\in\mathbb{N}}$	The canonical enumeration of TP.	48
$\mathrm{nf}(B)$	The normal form of $B \in B_{\mathscr{E}}$.	14
OCB	The set of open complex boxes with complex rational corners.	10
$\mathrm{rf}(\varphi)$	The reflection of $\varphi \in \mathscr{E}$.	20
$R(k, m, n)$	A notation for the set of all $\varphi \in \mathscr{S}$ such that $\sup_{x\in\mathbb{R}} \lvert x^k \varphi^{(m)}(x)\rvert < n$.	28
$\lVert\varphi\rVert_M$	The M-norm of $\varphi \in \mathscr{E}$.	10
$\lVert\varphi\rVert_{k,n}$	The (k, n)-norm of $\varphi \in \mathscr{S}$.	28
$[X \to_\omega Y]$	The space of sequentially continuous functions from X to Y.	43
TP	The set of smoothly truncated complex rational polynomials in $\mathscr{D}$.	48
TP_k	The set of smoothly truncated complex rational polynomials in $\mathscr{D}_k$.	47
$\mathrm{tr}_t(\varphi)$	The translation of $\varphi \in \mathscr{E}$.	20
f^*	The transpose of f.	49
$U(m, n)$	A basic open set in $\mathscr{E}$.	10
u_φ	The distribution $\psi \mapsto \int_{\mathbb{R}} \varphi\psi$.	50
$V(m, n)$	A basic open set in $\mathscr{D}_k$.	20
$W(k, l)$	A basic open set in $\mathscr{S}$.	28

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Paper IV

EFFECTIVE STRUCTURE THEOREMS FOR SPACES OF COMPACTLY SUPPORTED DISTRIBUTIONS

FREDRIK DAHLGREN

Abstract

In this paper we investigate the effective content of the structure theorem for the space $\mathscr{E}'$ of distributions with compact support. We show that each of the four characterisations of the space of distributions with compact support given in the structure theorem (below) gives rise to an effective domain representation of $\mathscr{E}'$. To show that the structure theorem has effective content, we study the reducibility properties of these four domain representations.

We also employ the same method to prove an effective version of a similar structure theorem for the space of distributions supported at the origin.

Key Words: Computable analysis, effective domain theory, distribution theory.

1 Introduction

Generalised functions are interesting since they extend the available classical methods for solving partial differential equations. In this context, the theory of distributions over the space $\mathscr{D}$ of test functions (which are smooth functions with compact support) has been particularly successful.

The space of distributions over $\mathscr{D}$ has two subspaces which are of particular interest from the point of view of applications: The space of tempered distributions, which is mapped bijectively onto itself by the Fourier transform, and the space of distributions with compact support which is very well-behaved under convolution. It turns out that the space of distributions with compact support has many other interesting purely structural properties. In fact, there are at least four essentially different ways to characterise the class of distributions with compact support. (Here, we restrict ourselves to the one-dimensional case.)

Theorem 1.1. *Let u be a continuous linear functional on $\mathscr{D}$. Then the following are equivalent*

(UE) *u extends (in a unique way) to a linear continuous functional on $\mathscr{E}$.*

(CS) *The support of u is a compact subset of $\mathbb{R}$.*

(FO) *u has finite order.*

(WC) *There are compactly supported continuous functions $g_0, g_2, \ldots, g_n$ such that $u = \sum_{k=0}^{n} g_k^{(k)}$.*

The abbreviation (UE) stands for "Unique Extension". (CS) stands for "Compact Support" and (FO) stands for "Finite Order". The last abbreviation (WC) is meant to stand for "Weak derivatives of Continuous functions". The equivalence $(\mathrm{CS}) \iff (\mathrm{WC})$ is usually known as the structure theorem for the space of distributions with compact support. (The derivatives in (WC) should of course be understood in a distribution sense.)

In [9] we employed the equivalence $(\mathrm{UE}) \iff (\mathrm{CS})$ to construct an admissible domain representation of the space $\mathscr{E}'$ of compactly supported distributions. As we shall see shortly, each of the different characterisations of the space of compactly supported distributions given in Theorem 1.1 can be used to construct an effective domain representation of $\mathscr{E}'$, and each representation yields a different notion of approximation for the space of compactly supported distributions.

One way of studying how these four essentially different characterisations of the class of distributions with compact support are related to each other from a computability theoretic point of view would be to investigate the reducibility properties of the corresponding domain representations. Intuitively, if $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ are two different domain representation of the same topological space X, we say that D is continuously reducible to E if we can translate representations of an element $x \in X$ in D to representations of x in E in a continuous way. If D and E are effective domains, and the translation can be performed by an effective continuous function we say that D is effectively reducible to E. If D is effectively reducible to E we can approximate each element x in X arbitrarily well in E, by first approximating x in D and then translating the result to E. If we have effective translations in both directions it does not matter from the point of view of effective calculability if we work with the domain representation D or the domain representation E. Each element $x \in X$ may be approximated to any desired degree of accuracy in any one of the representations by simply approximating x in the other representation and then translating the result back to the original representation.

2 Preliminaries from domain theory

We start with a brief recollection of some important definitions from the theories of Scott-Ershov domains and domain representations. For a more in depth introduction to domain theory we recommend [1] and [20].

A *Scott-Ershov domain* (or simply domain) is a bounded complete algebraic dcpo. Let D be a domain. Then D_c denotes the set of compact elements in D. Given $x \in D$ we write $\mathrm{approx}(x)$ for the set $\{a \in D_c;\ a \sqsubseteq x\}$. Since D is algebraic, $\mathrm{approx}(x)$ is directed and $\bigsqcup \mathrm{approx}(x) = x$ for each $x \in D$.

A *preordered conditional upper semilattice* (precusl) is a preordered set $(P, \sqsubseteq)$ with a distinguished least element $\perp$, which is *consistently complete* in the following sense: Whenever p and q are consistent in P there is some $r \in P$ such that $p, q, \sqsubseteq r$ and if $p, q \sqsubseteq s$ for some $s \in P$ then $r \sqsubseteq s$. r is called *a least upper bound* for p and q. It is clear that r need not be unique and so p and q may have many least upper bounds in P. The precusl P is a *conditional upper semilattice* (cusl) if $\sqsubseteq$ is a partial order on P. When P is a cusl then least upper bounds in P are unique whenever they exist. For a precusl P we let $\mathrm{Idl}(P)$ be the set $\{I \subseteq P;\ I \text{ is an ideal}\}$. We order $\mathrm{Idl}(P)$ by $I \sqsubseteq J \iff I \subseteq J$. Then $\mathrm{Idl}(P) = (\mathrm{Idl}(P), \sqsubseteq, \{\perp\})$ is a domain. $\mathrm{Idl}(P)$ is called the *ideal completion* of P. If D is a domain then D_c is a cusl and $D \cong \mathrm{Idl}(D_c)$.

Let D and E be domains. A *partial continuous function*[1] from D to E is a pair (S, f) where S is a nonempty closed subset of D and $f : S \to E$ is a strict[2] Scott-continuous function from S to E. We write $f : D \rightharpoonup E$ if $(\mathrm{dom}(f), f)$ is a partial continuous function from D to E.

If P is a precusl then P is *computable* if there is a numbering $\alpha : \mathbb{N} \to P$ of P with respect to which the relations $\alpha(m) \sqsubseteq \alpha(n)$, "$\alpha(m)$ and $\alpha(n)$ are consistent", $\alpha(k) = \alpha(m) \sqcup \alpha(n)$ are all recursive. The domain D is *effective* if the cusl D_c is computable. Thus in particular, if P is a computable precusl then $D = \mathrm{Idl}(P)$ is an effective domain. When (D, α) and (E, β) are effective domains then $x \in D$ is α*-computable* (or simply computable) if the set $\{m \in \mathbb{N};\ \alpha(m) \sqsubseteq x\}$ is r.e. and if $f : D \rightharpoonup E$ is a partial continuous function then f is (α, β)*-effective* (or simply effective) if there is an r.e. relation R such that $R(m, n) \iff \beta(n) \sqsubseteq_E f(\alpha(m))$ whenever $\alpha(m) \in \mathrm{dom}(f)$.

[1] For an introduction to partial continuous functions on domains, we refer the reader to [8].

[2] $f : S \to E$ is *strict* if $f(\perp) = \perp$.

2.1 Representation theory

There are many ways to introduce a notion of computability on uncountable spaces. Here, we will use effective domain representations over algebraic domains as developed by Stoltenberg-Hansen, Tucker, Blanck and Hamrin in [3], [4], [6], [12], and [20].

Let X be a topological space. A *domain representation* of X is a triple $D = (D,\, D^R,\, \delta)$ where D is a domain, D^R a nonempty subset of D, and $\delta : D^R \to X$ a continuous function from D^R onto X. We assume throughout that $\bot \notin D^R$. Given $r \in D$ and $x \in X$ we write $r \prec x$ and say that r *approximates* x if $x \in \delta[\uparrow r \cap D^R]$. That is, $r \prec x$ if and only if there is some $s \in D^R$ such that $r \sqsubseteq s$ and $\delta(s) = x$. The *dense part* of the domain representation $(D,\, D^R,\, \delta)$ is the set $D^D = \mathrm{cl}(D^R) = \mathrm{cl}(\{a \in D_c;\; a \prec x$ for some $x \in X\}$. The domain representation $(D,\, D^R,\, \delta)$ is an *effective* representation of X if the domain D is effective.

Let $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ be effective domain representations of the spaces X and Y. We say that $x \in X$ is *D-computable* (or simply computable when the representation $(D,\, D^R,\, \delta)$ is clear from the context) if there is a computable element $r \in D^R$ such that $\delta(r) = x$.

There is a corresponding notion of representability for functions from X to Y. A function $f : X \to Y$ is (D, E)-*representable* (or simply representable) if $f \circ \delta$ factors through ε via some partial continuous function $\overline{f} : D \rightharpoonup E$ as in Figure 1.

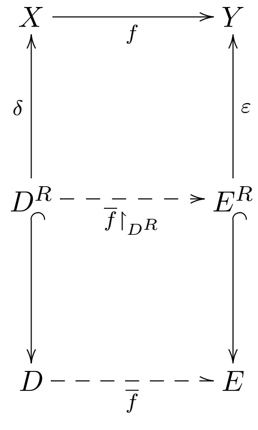


Figure 1.

That is, if there is a partial continuous function $\overline{f} : D \rightharpoonup E$ such that $D^R \subseteq \mathrm{dom}(\overline{f})$, $\overline{f}[D^R] \subseteq E^R$ and $f(\delta(r)) = \varepsilon(\overline{f}(r))$ for each $r \in D^R$. If $\overline{f} : D \rightharpoonup E$ is effective we say that f is (D, E)-*effective* (or simply effective if the domain representations $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ are clear from the context).

Definition 2.1. Let $(E,\, E^R,\, \varepsilon)$ be a domain representation of the space Y. The representation $(E,\, E^R,\, \varepsilon)$ is called (ω-)*admissible*[3] if for each triple $(D,\, D^R,\, \delta)$ where D is a countably based domain, $D^R \subseteq D$, and $\delta : D^R \to Y$ is continuous, there is a partial continuous function $\overline{\delta} : D \rightharpoonup E$ such that $D^R \subseteq \mathrm{dom}(\overline{\delta})$, $\overline{\delta}[D^R] \subseteq E^R$, and $\delta(x) = \varepsilon(\overline{\delta}(x))$ for each $x \in D^R$. Or put slightly differently, there is a $\overline{\delta} : D \rightharpoonup E$ which makes the top triangle in the following diagram commute.

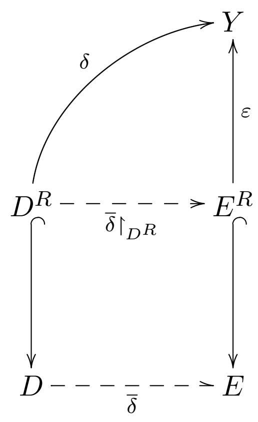


Figure 2.

If $(D,\, D^R,\, \delta)$ is a countably based representation of X and $(E,\, E^R,\, \varepsilon)$ is an admissible representation of Y, then every sequentially continuous function from X to Y is representable as a partial continuous function $\overline{f} : D \rightharpoonup E$. This property can be used to construct a domain representation of the space of sequentially continuous functions from X to Y over the domain $[D \rightharpoonup E]$ of partial continuous functions from D to E. For details, we refer the reader to [8].

In what follows we will be studying translation properties of different domain representations of the same space. These translations are usually known as reductions in the literature (c.f. [5]). Here we will always consider partial reductions since we are working over a category of domains with partial continuous functions.

[3]This is a reformulation of the definition of an ω-admissible domain representation found in [12]. It is equivalent to the definition in [12], as well as that of a ω-projecting equilogical space found in [2] and [14].

Definition 2.2. If $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ are two domain representations of the same space Y, we say that D is *continuously reducible* to E (written $D \leq_{\mathrm{c}} E$) if there is a partial continuous function $r : D \rightharpoonup E$ such that $D^R \subseteq \mathrm{dom}(r)$, and $\delta(x) = \varepsilon(r(x))$ for each $x \in D^R$. We write $D \equiv_{\mathrm{c}} E$ if $D \leq_{\mathrm{c}} E \leq_{\mathrm{c}} D$ and say that D and E are *continuously equivalent*. If D and E are effective domains, and $D \leq_{\mathrm{c}} E$ via some effective partial continuous function $r : D \rightharpoonup E$, we say that D is *effectively reducible* to E (written $D \leq_{\mathrm{e}} E$). If $D \leq_{\mathrm{e}} E \leq_{\mathrm{e}} D$ we say that D and E are *effectively equivalent* and write $D \equiv_{\mathrm{e}} E$.

As an immediate consequence of Definition 2.2 we note that every domain representation of Y is continuously reducible to E if E is admissible. (In fact, one can show that the property of being the $\leq_{\mathrm{c}}$-largest representation of a space Y is equivalent to being admissible.)

Note that in order for D to be effectively reducible to E we require that the reduction (or witness) $r : D \rightharpoonup E$ is a continuous function. If we relax this condition slightly we get a weaker notion of reducibility which still has enough effective properties to be interesting.

Definition 2.3. Let $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ be two domain representations of the same topological space Y.

Let $(a_n)_n$ be an infinite sequence of compact elements in D. We say that $(a_n)_n$ *δ-converges* to $y \in Y$ (written $(a_n)_n \longrightarrow_\delta y$) if $A = \{a_n;\ n \in \mathbb{N}\}$ is directed, $\bigsqcup A \in D^R$, and $\delta(\bigsqcup A) = y$.

If the domains D and E are effective, we say that D is *Turing reducible* to E (written $D \leq_{\mathrm{T}} E$) if there is a Turing machine M which given a sequence $(a_n)_n$ of compact elements from D such that $(a_n)_n \longrightarrow_\delta y \in Y$ on the input tape[4], M computes forever and writes a sequence $(b_n)_n$ of compact elements from E such that $(b_n)_n \longrightarrow_\varepsilon y$ on the one-way output tape[5]. The representations D and E are *Turing-equivalent* (written $D \equiv_{\mathrm{T}} E$) if $D \leq_{\mathrm{T}} E \leq_{\mathrm{T}} D$.

It is perhaps not immediately obvious from the definition above that Turing reducibility is a weaker notion than effective reducibility. That this is so follows by the next lemma.

Lemma 2.4. *Let $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ be effective domain representations of the topological spaces X and Y. If $f : X \to Y$ is effective with respect*

[4]We assume that the sequence $(a_n)_n$ is coded on the input-tape in some way using the numbering of the compact elements of D. The details of exactly how this is done need not concern us here.

[5]We require that the output tape is write-only, which means that the Turing machine may not erase anything written on the this tape. This is standard and is done to ensure that the result written on the output tape after any finite amount of time represents an initial segment of the sequence $(b_n)_n$

to the representations D and E then there is a Turing machine M which given a sequence $(a_n)_n$ of compact elements from D such that $(a_n)_n \longrightarrow_\delta x \in X$ on the input tape, M computes forever and writes a sequence $(b_n)_n$ of compact elements from E such that $(b_n)_n \longrightarrow_\varepsilon f(x)$ on the one-way output tape.

Proof. Suppose that $f : X \to Y$ be effective via $\overline{f} : D \rightharpoonup E$, and let $(a_n)_n$ be a sequence of compact elements in D such that $(a_n)_n \longrightarrow x \in X$. Since $(a_n)_n \longrightarrow x$ we have $a_n \prec x$ for all n. It follows that $a_n \in \mathrm{dom}(\overline{f})$ for each n. By simply going through the compact elements in E one by one and checking if they are less than some $\overline{f}(a_n)$ we can compute a new sequence $(b_n)_n$ in E such that $(b_n)_n \longrightarrow_\varepsilon f(x)$, and it is clear that this can be done by a Turing machine. □

Corollary 2.5. *Let $(D,\, D^R,\, \delta)$ and $(E,\, E^R,\, \varepsilon)$ be two effective domain representations of the same space X. If $D \leq_{\mathrm{e}} E$ then $D \leq_{\mathrm{T}} E$.*

2.2 Some standard representations

To be able to construct domain representations of the space of distributions with compact support which correspond to the four different characterisations of the space given in the structure theorem, we need to be able to represent the real and complex numbers, the space $\mathscr{E}$ of smooth functions from the reals to the complex numbers, and the space $\mathscr{D}$ of smooth functions with compact support. We will also need to be able to represent the space of distributions over $\mathscr{D}$. In this section we will give a brief review of how this may be done. We will not give any proofs and very little motivation, but will simply give the definitions. For details and proofs we refer the interested reader to [9].

Let $D_{\mathbb{N}}$ be the flat domain over $\mathbb{N}$. To construct a domain representation of $\mathbb{N}$ over $D_{\mathbb{N}}$ we simply let $D_{\mathbb{N}}^R = \mathbb{N}$, and define $\delta_{\mathbb{N}} : D_{\mathbb{N}}^R \to \mathbb{N}$ as the identity on $\mathbb{N}$. Let $(D,\, D^R,\, \delta)$ be an effective domain representation of the space X. An *effective sequence* in X is a sequence $(x_n)_n$ such that the map $n \mapsto x_n$ is $(D_{\mathbb{N}}, D)$-effective.

We denote the standard algebraic closed interval representation of the real numbers by $(D_{\mathbb{R}},\, D_{\mathbb{R}}^R,\, \delta_{\mathbb{R}})$. Since $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$ we may construct a domain representation of $\mathbb{C}$ over $D_{\mathbb{R}}^2$. We denote this representation of $\mathbb{C}$ by $(D_{\mathbb{C}},\, D_{\mathbb{C}}^R,\, \delta_{\mathbb{C}})$.

The domain representation of $\mathscr{E}$ is constructed as the ideal completion of an appropriately chosen collection $B_{\mathscr{E}}$ of step functions:

Definition 2.6. Given a closed interval $S \subseteq \mathbb{R}$ with rational endpoints and an open box $T \subseteq \mathbb{C}$ with complex rational corners we let $B^{(m)}(S, T)$ be a notation for the set of all smooth functions φ from $\mathbb{R}$ to $\mathbb{C}$ such that

$\varphi^{(m)}[S] \subseteq T$. We write $B_{\mathscr{E}}$ for the collection of all finite sets $\{B^{(m_1)}(S_1, T_1), B^{(m_2)}(S_2, T_2), \ldots, B^{(m_n)}(S_n, T_n)\}$ where each $S_k \subseteq \mathbb{R}$ is a closed interval with rational endpoints and each $T_k \subseteq \mathbb{C}$ is an open box with complex rational corners. If $\varphi \in \mathscr{E}$ we say that φ *satisfies* $B \in B_{\mathscr{E}}$ if $\varphi^{(m)}[S] \subseteq T$ for each $B^{(m)}(S, T) \in B$.

An element in $B_{\mathscr{E}}$ may be thought of as a "generalised step function". It may be visualised as a family of boxes with complex rational endpoints enclosing the graph of the function, the first derivative of the function, and so on in finitely many steps.

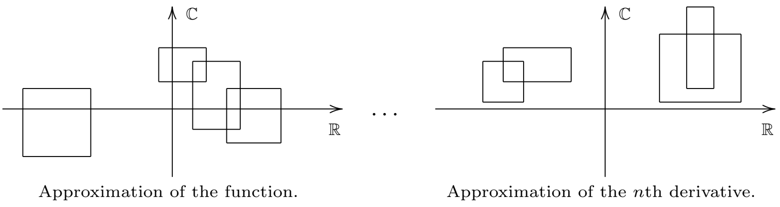


Figure 3. A generalised step function in $B_{\mathscr{E}}$.

We can define a preorder on $B_{\mathscr{E}}$ as follows: Given two elements $B, C \in B_{\mathscr{E}}$ we let $B \sqsubseteq C$ if and only if for each $B^{(m)}(S, T) \in B$ there are notations $B^{(m)}(S_1, T_1), B^{(m)}(S_2, T_2), \ldots, B^{(m)}(S_n, T_n) \in C$ such that $S \subseteq \bigcup_{k=1}^{n} S_k$ and $\bigcup_{k=1}^{n} T_k \subseteq T$.

Definition 2.7. $D_{\mathscr{E}}$ is the ideal completion of $B_{\mathscr{E}}$. If $I \in D_{\mathscr{E}}$ we say that I *represents* $\varphi \in \mathscr{E}$ if $I = \{B \in B_{\mathscr{E}};\ \varphi \text{ satisfies } B\}$. The set of representing elements in $D_{\mathscr{E}}$ is denoted by $D_{\mathscr{E}}^R$. The map $\delta_{\mathscr{E}} : D_{\mathscr{E}}^R \to \mathscr{E}$ is given by $\{B \in B_{\mathscr{E}};\ \varphi \text{ satisfies } B\} \mapsto \varphi$.

The original definition of the domain representation $(D_{\mathscr{E}}, D_{\mathscr{E}}^R, \delta_{\mathscr{E}})$ of $\mathscr{E}$ given in [9] was formulated slightly differently, but it is easily shown to be equivalent to the definition given here. In [9] we showed that

Theorem 2.8. *$(D_{\mathscr{E}}, D_{\mathscr{E}}^R, \delta_{\mathscr{E}})$ is an effective and admissible domain representation of $\mathscr{E}$, and $\delta_{\mathscr{E}}$ is a homeomorphism.*

Standard operations such as evaluation, addition, multiplication, and scalar multiplication on $\mathscr{E}$ are all effective with respect to the representation $D_{\mathscr{E}}$. For details, see [9].

To construct a domain representation of the space $\mathscr{D}$ of smooth functions with compact support we simply restrict the representation $D_{\mathscr{E}}$ to $\mathscr{D}$ and add information about (i.e. bounds on) the support.

Definition 2.9. D_S is the domain $\{\bot \sqsubseteq \ldots \sqsubseteq 2 \sqsubseteq 1\}$, and $D_{\mathscr{D}} = D_{\mathscr{E}} \times D_S$. We say that $(I, k) \in D_{\mathscr{D}}$ *represents* $\varphi \in \mathscr{D}$ if I represents φ in $D_{\mathscr{E}}$ and $\text{supp}(\varphi) \subseteq [-k, k]$. The collection of representing elements in $D_{\mathscr{D}}$ is denoted by $D^R_{\mathscr{D}}$. Finally, we let $\delta_{\mathscr{D}} : D^R_{\mathscr{D}} \to \mathscr{D}$ be the function given by $\delta_{\mathscr{D}}(I, k) = \varphi$ if and only if (I, k) represents φ.

It is easy to show that $\delta_{\mathscr{D}}$ is well-defined and surjective. Since it is also continuous we have the following Theorem (which was proved in [9]).

Theorem 2.10. *$(D_{\mathscr{D}}, D^R_{\mathscr{D}}, \delta_{\mathscr{D}})$ is an effective and admissible domain representation of $\mathscr{D}$.*

Since the representations $D_{\mathscr{D}}$ and $D_{\mathbb{C}}$ are admissible it follows that every continuous linear functional on $\mathscr{D}$ lifts to a partial continuous function from $D_{\mathscr{D}}$ to $D_{\mathbb{C}}$. We may use this to construct a domain representation of the space $\mathscr{D}'$ of distributions over the domain $[D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]$ of partial continuous functions from $D_{\mathscr{D}}$ to $D_{\mathbb{C}}$.

Definition 2.11. Let $D_{\mathscr{D}'} = [D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}]$, and let $D^R_{\mathscr{D}'}$ be the collection of partial continuous functions which represent distributions (in the sense of section 2.1). We define $\delta_{\mathscr{D}'} : D^R_{\mathscr{D}'} \to \mathscr{D}'$ as the map given by $\delta_{\mathscr{D}'}(f) = u$ if and only if f represents u.

Using the theory of partial continuous functions on domains as developed in [8], one may now show that

Theorem 2.12. *$(D_{\mathscr{D}'}, D^R_{\mathscr{D}'}, \delta_{\mathscr{D}'})$ is an effective and admissible domain representation of $\mathscr{D}'$.*

In [9] we also showed that evaluation from $\mathscr{D}' \times \mathscr{D}$ to $\mathbb{C}$ and type conversion from $[X \times \mathscr{D} \to \mathbb{C}]$ to $[X \to \mathscr{D}']$ are effective with respect to the representation $D_{\mathscr{D}'}$ of $\mathscr{D}'$.

3 Domain representations of the space of distributions with compact support

Let u be a distribution and let V be an open subset of $\mathbb{R}$. We say that u *vanishes on* V if $u(\varphi) = 0$ for each test function φ with support contained in V. Let W be the union of all open subsets of $\mathbb{R}$ on which u vanishes. The *support* of u is the complement of W. Here, we will be interested in the space of distributions with compact support.

As we have seen, there are at least four essentially different ways to characterise the class of distributions with compact support. As we shall see shortly, each characterisation $(\star)$ gives rise to a different domain representation $D_\star$ of the space of compactly supported distributions. To study how these characterisations are related from a computability theoretic standpoint we will study the reducibility properties of the corresponding domain representations.

3.1 Representing compactly supported distributions as continuous linear functionals on $\mathscr{E}$

Let u be a distribution with compact support $K \subseteq \mathbb{R}$ and let σ be a smooth function which is equal to 1 on some open neighbourhood of K. Then $\sigma u = u$, and since $\sigma u(\varphi) = u(\sigma\varphi)$ is well-defined for each $\varphi \in \mathscr{E}$ it follows that u extends to a continuous linear functional $\overline{u}$ on $\mathscr{E}$. Since $\mathscr{D}$ is dense in $\mathscr{E}$ this extension is unique.

Conversely, if v is a continuous linear functional on $\mathscr{E}$, then $v{\restriction_{\mathscr{D}}}$ is a distribution, and the support of $v{\restriction_{\mathscr{D}}}$ is a compact set. This yields a characterisation of the distributions with compact support as the class of distributions which extend to continuous linear functionals on $\mathscr{E}$.

Theorem 3.1. *Let u be a distribution. Then u extends to a continuous linear functional on $\mathscr{E}$ if and only if u has compact support.*

It follows that we may identify the dual space of $\mathscr{E}$ with the space of distributions with compact support.

Now, since the domain representations $D_{\mathscr{E}}$ and $D_{\mathbb{C}}$ are admissible it follows by Corollary 3.8 in [8] that every compactly supported distribution (viewed as a continuous linear functional from $\mathscr{E}$ to $\mathbb{C}$) is representable by a partial continuous function from $D_{\mathscr{E}}$ to $D_{\mathbb{C}}$. It follows that we may mimic the construction of the standard representation of $\mathscr{D}'$ to construct a domain representation of $\mathscr{E}'$.

Definition 3.2. We let D_{UE} be the domain $[D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}]$ of partial continuous functions from $D_{\mathscr{E}}$ to $D_{\mathbb{C}}$. The totality D^R_{UE} on D_{UE} is given by $f \in D^R_{\mathrm{UE}}$ if and only if f represents a distribution with compact support (in the sense of section 2.1). We let $\delta_{\mathrm{UE}}(f) = u$ if and only if f represents u.

Theorem 3.3. $(D_{\mathrm{UE}},\, D^R_{\mathrm{UE}},\, \delta_{\mathrm{UE}})$ *is an effective admissible domain representation of $\mathscr{E}'$.*

(For a proof of Theorem 3.3 we refer the reader to [9].)

Now, if $(D,\, D^R,\, \delta)$ is an effective domain representation of $\mathscr{E}'$ we say that *evaluation (from $\mathscr{E}' \times \mathscr{E}$ to $\mathbb{C}$) is D-effective* if the evaluation map $(u, \varphi) \mapsto$

$u(\varphi)$ is $(D \times D_{\mathscr{E}},\, D_{\mathbb{C}})$-effective. We say that *type conversion preserves D-effectivity* (or that *D-effectivity is preserved under type conversion*) if for every topological space X, every effective domain representation $(D_X,\, D_X^R,\, \delta_X)$ of X, and every $(D_X \times D_{\mathscr{E}},\, D_{\mathbb{C}})$-effective function $f : X \times \mathscr{E} \to \mathbb{C}$ which is continuous and linear in the second argument, the transpose $f^* : X \to \mathscr{E}'$ of f is (D_X, D)-effective.

In [9], we showed that evaluation is D_{UE}-effective, and that type conversion preserves D_{UE}-effectivity.

Proposition 3.4. *Evaluation from $\mathscr{E}' \times \mathscr{E}$ to $\mathbb{C}$ is D_{UE}-effective.*

Proposition 3.5. *Type conversion preserves D_{UE}-effectivity.*

(For proofs of Propositions 3.4 and 3.5 see [9].)

Proposition 3.4 may be strengthened by noting that the domain representation D_{UE} is the $\leq_{\mathrm{e}}$-greatest effective representation D of $\mathscr{E}'$ with a D-effective evaluation map.

Proposition 3.6. *Let D be an effective domain representation of $\mathscr{E}'$. Evaluation $(u, \varphi) \mapsto u(\varphi)$ is D-effective if and only if $D \leq_{\mathrm{e}} D_{\mathrm{UE}}$.*

Proof. Let D be an effective domain representation of $\mathscr{E}'$ and suppose that evaluation is D-effective. Let $\mathrm{ev} : D \times D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}$ be an effective partial continuous representation of the evaluation map $(u, \varphi) \mapsto u(\varphi)$. Since D_{UE}-effectivity is preserved under type conversion it follows that there is an effective partial continuous function $\mathrm{ev}^* : D \rightharpoonup D_{\mathrm{UE}}$ which represents the transpose of the evaluation map $(u, \varphi) \mapsto u(\varphi)$. The result follows since the transpose of the evaluation map is the identity on $\mathscr{E}'$. □

We may strengthen Proposition 3.5 in a similar way by noting that D_{UE} is the $\leq_{\mathrm{e}}$-least effective representation of $\mathscr{E}'$ with effective type conversion.

Proposition 3.7. *Let D be an effective domain representation of $\mathscr{E}'$. Then type conversion preserves D-effectivity if and only if $D_{\mathrm{UE}} \leq_{\mathrm{e}} D$.*

Proof. Let D be an effective domain representation of $\mathscr{E}'$, and suppose that type conversion preserves D-effectivity. Let $\mathrm{ev} : D_{\mathrm{UE}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}$ be an effective partial continuous representation of the evaluation map $(u, \varphi) \mapsto u(\varphi)$. Since type conversion preserves D-effectivity it follows that there is an effective partial continuous function $\mathrm{ev}^* : D_{\mathrm{UE}} \rightharpoonup D$ which represents the transpose of $(u, \varphi) \mapsto u(\varphi)$. Since this is simply the identity on $\mathscr{E}'$ we conclude that $D_{\mathrm{UE}} \leq_{\mathrm{e}} D$ via ev^*. □

By combining Propositions 3.6 and 3.7 we immediately get the following corollary.

Corollary 3.8. *Let D be an effective domain representation of $\mathscr{E}'$. Then $D \equiv_{\mathrm{e}} D_{\mathrm{UE}}$ if and only if evaluation is D-effective and type conversion preserves D-effectivity.*

3.2 Representing compactly supported distributions as continuous linear functionals on $\mathscr{D}$ with bounded support

In view of the Heine-Borel property we know that $u \in \mathscr{D}'$ has compact support if and only if the support of u is a bounded subset of $\mathbb{R}$. Thus, it makes sense to represent an element $u \in \mathscr{E}'$ as a pair (f, k) where f is an representation of u in $D_{\mathscr{D}'}$, and k is a bound on the support of u. We will now make this idea more precise.

Definition 3.9. Let D_S be the domain $\{\bot \sqsubseteq \ldots \sqsubseteq 2 \sqsubseteq 1\}$ and let D_{CS} be the domain $D_{\mathscr{D}'} \times D_S$. We define the totality on D_{CS} by $(f, k) \in D^R_{\mathrm{CS}}$ if and only if $f \in D^R_{\mathscr{D}'}$ and $k \in \mathbb{N}$ is a bound on the support of $\delta_{\mathscr{D}'}(f)$. Finally, we let $\delta_{\mathrm{CS}} : D^R_{\mathrm{CS}} \to \mathscr{E}'$ be the function given by $\delta_{\mathrm{CS}}(f, k) = u$ if and only if $\delta_{\mathscr{D}'}(f) = u$.

Since a lower bound on the support of a distribution u yields more information about u, it is natural to use D_S rather than $D_{\mathbb{N}}$ to represents the bound on $\mathrm{supp}(u)$. More may be said on this point however, and we will return to this question at the end of this section.

We need to show that $(D_{\mathrm{CS}}, D^R_{\mathrm{CS}}, \delta_{\mathrm{CS}})$ is an effective domain representation of $\mathscr{E}'$. It is clear that the domain D_{CS} is effective, and that δ_{CS} is well-defined and surjective, but we need to prove that δ_{CS} is continuous.

Theorem 3.10. $(D_{\mathrm{CS}}, D^R_{\mathrm{CS}}, \delta_{\mathrm{CS}})$ *is an effective domain representation of $\mathscr{E}'$.*

Proof. To show that δ_{CS} is continuous it is enough to show that δ_{CS} is sequentially continuous since D^R_{CS} is a countably based space.

Thus, suppose that $(f_n, k_n)_n \longrightarrow (f, k)$ in D^R_{CS}, and let $u_n = \delta_{\mathrm{CS}}(f_n, k_n)$ and $u = \delta_{\mathrm{CS}}(f, k)$. We would like to prove that $(u_n)_n \longrightarrow u$ in $\mathscr{E}'$. That amounts to showing that $(u_n(\varphi))_n \longrightarrow u(\varphi)$ for each $\varphi \in \mathscr{E}$.

Since $(f_n)_n \longrightarrow f$ in $D^R_{\mathscr{D}'}$ we conclude that $(u_n)_n \longrightarrow u$ in $\mathscr{D}'$. It follows that $(u_n(\varphi))_n \longrightarrow u(\varphi)$ for each $\varphi \in \mathscr{D}$. Since $(k_n)_n \longrightarrow k$ in D_S we must have $k_n \leq k$ for all n greater than some $N \in \mathbb{N}$. Let $K = \max\{k_n;\ n \in \mathbb{N}\}$ and choose $\sigma \in \mathscr{D}$ such that $0 \leq \sigma(x) \leq 1$ for all x, $\sigma(x) = 1$ on $[-(K+1), (K+1)]$, and $\sigma(x) = 0$ outside $[-(K+2), (K+2)]$. Since σ is equal to 1 on an open neighbourhood of the support of each u_n and u, we have $(u_n(\varphi))_n = (u_n(\sigma\varphi))_n \longrightarrow u(\sigma\varphi) = u(\varphi)$ for each $\varphi \in \mathscr{E}$ as required. $\square$

To show that D_{CS} is effectively reducible to D_{UE} it is enough to show that evaluation is D_{CS}-effective. The following is is an effective version of the classical proof that every compactly supported distribution extends to a continuous linear functional on $\mathscr{E}$.

Proposition 3.11. *Evaluation from $\mathscr{E}' \times \mathscr{E}$ to $\mathbb{C}$ is D_{CS}-effective.*

Proof. Let $(\sigma_n)_n$ be an effective sequence of smooth functions such that $0 \leq \sigma_n(x) \leq 1$ for all x, $\sigma_n(x) = 1$ on $[-(n+1), (n+1)]$, and $\sigma_n(x) = 0$ on $[-(n+2), (n+2)]$. (The effective sequence $(\sigma_n)_n$ may be constructed as in Lemma 4.1.8 in [9].) Let $s : D_{\mathbb{N}} \to D_{\mathscr{E}}$ be an effective continuous representation of the map $n \mapsto \sigma_n$. (We may assume that s is total since $\mathbb{N}$ is dense in $D_{\mathbb{N}}$.)

Let Q be the closure of D^R_{CS} in D_{CS}, and let R be the closure of $D^R_{\mathscr{E}}$ in $D_{\mathscr{E}}$. Let $m : D_{\mathscr{E}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathscr{E}}$ be an effective representation of the map $(\varphi, \psi) \mapsto \varphi\psi$ on $\mathscr{E}$. Then $t : D_{\mathbb{N}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathscr{D}}$ given by $\mathrm{dom}(t) = D_{\mathbb{N}} \times R$, and $t(n, J) = \{(B, k) \in B_{\mathscr{D}};\ B \sqsubseteq m(s(n), J), \text{ and } k \geq n\}$ on $\mathrm{dom}(t)$, is an effective representation of the map $(n, \varphi) \mapsto \sigma_n\varphi$ from $\mathbb{N} \times \mathscr{E}$ to $\mathscr{D}$.

Now, if $(f, k) \in Q$ and $C \in R$ are compact we let $\mathrm{EV}(f, k, C)$ denote the set of all $S \in B_{\mathbb{C}}$ such that

$$S \sqsubseteq f(t(n, B)) \text{ for some } n \sqsubseteq k \text{ and some}$$

$$B \sqsubseteq C \text{ such that } (n, B) \in \mathrm{dom}(f \circ t).$$

We claim that $\mathrm{EV}(f, k, C)$ is nonempty and consistent for all compact $(f, k) \in Q$ and $C \in R$. To prove this, let $(f, k) \in Q$ and $C \in R$ be compact. Then $\perp \in \mathrm{EV}(f, k, C)$ and so $\mathrm{EV}(f, k, C)$ is nonempty. To show that $\mathrm{EV}(f, k, C)$ is consistent it is enough to show that $\{S_1, S_2\}$ is consistent for all $S_1, S_2 \in \mathrm{EV}(f, k, C)$. Thus, suppose that $S_1, S_2 \in \mathrm{EV}(f, k, C)$. To show that S_1 and S_2 are consistent in $B_{\mathbb{C}}$ it is enough to show that $S_1 \cap S_2$ is nonempty. Since $(f, k) \in Q$ there is a partial continuous function $g \in D_{\mathscr{D}'}$, and a distribution $u \in \mathscr{E}'$ such that $f \sqsubseteq g$ in $D_{\mathscr{D}'}$, g represents u in $D_{\mathscr{D}'}$, and $\mathrm{supp}(u) \subseteq [-k, k]$. Similarly, since $C \in R$ there is an ideal $J \in D_{\mathscr{E}}$ and a smooth function $\phi \in \mathscr{E}$ such $C \in J$, and J represents ϕ in $D_{\mathscr{E}}$. Now, for $i \in \{1, 2\}$, since $S_i \in \mathrm{EV}(f, k, C)$ there is a $n_i \sqsubseteq k$ in D_S and a $B_i \sqsubseteq C$ in $B_{\mathscr{E}}$ such that $(n_i, B_i) \in \mathrm{dom}(f \circ t)$, and $S_i \sqsubseteq f(t(n_i, B_i))$ in $D_{\mathbb{C}}$. It follows that

$$S_i \sqsubseteq f(t(n_i, B_i)) \sqsubseteq g(t(n_i, B_i)) \sqsubseteq g(t(k, J))$$

in $D_{\mathbb{C}}$. Since t represents the map $(n, \varphi) \mapsto \sigma_n\varphi$ and g represents u in $D_{\mathscr{D}'}$ we conclude that $g(t(k, J))$ represents $u(\sigma_k\phi)$ in $D_{\mathbb{C}}$. Since $S_i \sqsubseteq g(t(k, J))$ for each $i \in \{1, 2\}$ we have $u(\phi) = u(\sigma_k \phi) \in S_1 \cap S_2$, and $S_1 \cap S_2$ is nonempty as required.

It follows immediately from the definition of EV that $\mathrm{EV}(f, k, B) \subseteq \mathrm{EV}(g, m, C)$ whenever $(f, k) \sqsubseteq (g, m)$ in D_{CS} and $B \sqsubseteq C$ in $D_{\mathscr{E}}$. We conclude that there is a (unique) partial continuous function $\mathrm{ev} : D_{\mathrm{CS}} \times D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}$ such that $\mathrm{dom}(\mathrm{ev}) = R \times Q$ and $\mathrm{ev}(f, k, C) = \bigsqcup \mathrm{EV}(f, k, C)$ for all compact $(f, k) \in R$ and $C \in Q$. We would like to show that ev represents evaluation from $\mathscr{E}' \times \mathscr{E}$ to $\mathbb{C}$. Thus, let (g, m) represent u in D_{CS} and let J represent ϕ in $D_{\mathscr{E}}$. If $S \sqsubseteq \mathrm{ev}(g, m, J)$ then $S \sqsubseteq \mathrm{ev}(f, k, C)$ for some compact $(f, k) \sqsubseteq (g, m)$ and $C \sqsubseteq J$. It follows that there are $S_1, S_2, \ldots, S_n \in \mathrm{EV}(f, k, C)$ such that $S \sqsubseteq S_1 \sqcup S_2 \sqcup \ldots \sqcup S_n$. Each S_i contains $u(\phi)$ as above, and so we must have $u(\phi) \in S_1 \cap S_2 \cap \ldots \cap S_n \subseteq S$. Let $N \in \mathbb{N}$, and choose $S \in B_{\mathbb{C}}$ such that $u(\phi) \in S^\circ$ and $\mathrm{diam}(S) < 2^{-N}$. Since $g(t(k, J))$ represents $u(\phi)$ in $D_{\mathbb{C}}$ we have $S \sqsubseteq g(t(k, J))$. It follows that there are compact $(f, k) \sqsubseteq (g, m)$ and $C \sqsubseteq J$ such that $(k, C) \in \mathrm{dom}(f \circ t)$ and $S \sqsubseteq f(t(k, C))$. By the definition of EV we have $S \in \mathrm{EV}(f, k, C)$, and so $S \sqsubseteq \mathrm{ev}(f, k, C) \sqsubseteq \mathrm{ev}(g, m, J)$. We conclude that $\mathrm{ev}(g, m, J)$ represents $u(\phi)$ in $D_{\mathbb{C}}$. Since (g, m) and J where arbitrary, we are done. □

As an immediate consequence of Proposition 3.11 we have

Corollary 3.12. *D_{CS} is effectively reducible to D_{UE}.*

What is more interesting is that the converse is also true. The original idea behind the following proof is due to Weihrauch and Zhong [25].

Proposition 3.13. *D_{UE} is effectively reducible to D_{CS}.*

Proof. Let $f : D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}$ be a partial continuous function representing the compactly supported distribution u, and suppose that $C \sqsubseteq f(B)$ where B is some nontrivial (i.e. nonempty) approximation of 0 in $B_{\mathscr{E}}$, and C is bounded (as a subset of $\mathbb{C}$). Let K be the least natural number k such that $k \geq \sup\{|x|; x \in S\}$ for each $B^{(m)}(S, T) \in B$. We claim that $\mathrm{supp}(u)$ is contained in $[-K, K]$. Thus, let $\varphi \in \mathscr{E}$ and suppose that $\mathrm{supp}(\varphi) \subseteq \mathbb{R} \setminus [-K, K]$. Then $c\varphi = 0$ on the interval $[-K, K]$ and so $B \prec c\varphi$ for each $c \in \mathbb{C}$. It follows that $C \prec u(c\varphi) = cu(\varphi)$ for each $c \in \mathbb{C}$, and hence, that $cu(\varphi) \in C$ for each $c \in \mathbb{C}$. Since C is bounded we conclude that $u(\varphi) = 0$.

For each $k \geq 1$, we let $B_k \in B_{\mathscr{E}}$ be the compact approximation of the constant function 0 given by $B_k = \{B^{(m)}([-k, k], [-2^{-k}, 2^{-k}] \times i[-2^{-k}, 2^{-k}]); 0 \leq m \leq k\}$ and let $C = [-1, 1] \times i[-1, 1]$. Finally, we let $s : D_{\mathrm{UE}} \to D_S$ be the function $s(f) =$ the least natural number k such that $B_k \in \mathrm{dom}(f)$ and $C \sqsubseteq f(B_k)$, and $s(f) = \bot$ if no such k exists.

The function $s : D_{\mathscr{E}} \to D_S$ is continuous: It is enough to show that s is monotone since every monotone function into D_S is continuous. Thus, let f and g be partial continuous functions from $D_{\mathscr{E}}$ to $D_{\mathbb{C}}$ and suppose that $f \sqsubseteq g$. If $B_k \in \mathrm{dom}(f)$ and $C \sqsubseteq f(B_k)$ then $B_k \in \mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $C \sqsubseteq f(B_k) \sqsubseteq g(B_k)$. It follows that $s(f) \sqsubseteq s(g)$ in D_S and so s is monotone. The function s is effective since the relation "$B_k \in \mathrm{dom}(f)$ and $C \sqsubseteq f(B_k)$" is decidable for compact elements f in D_{UE}.

The set $\{B_k\}_k$ is a directed subset of $D_{\mathscr{E}}$ and $I = \bigsqcup_k B_k$ represents the constant function 0 in $D_{\mathscr{E}}$. Thus, if $f \in D^R_{\mathrm{UE}}$ represents u then $I \in \mathrm{dom}(f)$ and $C \sqsubseteq f(I)$ in $D_{\mathbb{C}}$. It follows that $C \sqsubseteq f(B_k)$ for some k and so $k \sqsubseteq s(f)$ in D_S. We conclude that $s(f)$ is a bound on the support of $\delta_{\mathrm{UE}}(f)$ for each $f \in D^R_{\mathrm{UE}}$.

Let $i : D_{\mathrm{UE}} \rightharpoonup D_{\mathscr{D}'}$ be an effective representation of the inclusion map $\mathscr{E}' \hookrightarrow \mathscr{D}'$. (The partial continuous function i exists by Propositions 4.2.9 and 4.3.8 in [9].) Let $r : D_{\mathrm{UE}} \rightharpoonup D_{\mathrm{CS}}$ be the partial continuous function given by $\mathrm{dom}(r) = \mathrm{dom}(i)$ and $r(f) = (i(f), s(f))$ for each $f \in \mathrm{dom}(r)$. Then r is continuous since i and s are continuous, and effective since i and s are effective. That r represents the identity on $\mathscr{E}'$ follows by the argument above. □

Corollary 3.14. *The representations D_{UE} and D_{CS} are effectively equivalent.*

It is natural to ask if it would be possible to construct the domain representation D_{CS} over $D_{\mathscr{D}'} \times D_{\mathbb{N}}$ rather than $D_{\mathscr{D}'} \times D_S$. This would give us a domain representation of $\mathscr{E}'$ (in the same way as above), but it would not be equivalent to the admissible representation D_{UE}. The reason is that there is no $(D_{\mathrm{UE}}, D_{\mathbb{N}})$-effective function taking u to a bound on the support of u.

Lemma 3.15. *There is no partial continuous function $s : D_{\mathrm{UE}} \rightharpoonup D_{\mathbb{N}}$ such that $D^R_{\mathrm{UE}} \subseteq \mathrm{dom}(s)$ and $\mathrm{supp}(u) \subseteq [-s(f), s(f)]$ whenever f represents u in D_{UE}.*

Proof. Suppose for a contradiction that $s : D_{\mathrm{UE}} \rightharpoonup D_{\mathbb{N}}$ is a partial continuous function such that $D^R_{\mathrm{UE}} \subseteq \mathrm{dom}(s)$ and $\mathrm{supp}(u) \subseteq [-s(f), s(f)]$ whenever f represents u in D_{UE}. Let δ_k be the compactly supported distribution given by $\delta(\varphi) = \varphi(k)$, and let $u_n = 2^{-n}\delta_{k+1}$. Then $(u_n)_n \longrightarrow 0$ in $\mathscr{E}'$. Since D_{UE} is admissible it follows that every convergent sequence in $\mathscr{E}'$ lifts[6] to a convergent sequence in D_{UE}. That is, there is a sequence $(f_n)_n$ and an element $f \in D^R_{\mathrm{UE}}$ such that $\delta_{\mathrm{UE}}(f_n) = u_n$, $\delta_{\mathrm{UE}}(f) = 0$, and $(f_n)_n \longrightarrow f$ in D_{UE}.

[6]For a proof of this, see [12].

Since $\mathrm{supp}(u_n) \not\subseteq [-k,k]$ we must have $s(f_n) > k$ for each n. Since s is continuous it follows that $s(f) > k$. We conclude that for each natural number k there is a representation f of the distribution 0 such that $s(f) > k$.

Let $J = \{T \in D_{\mathbb{C}};\ T \text{ is compact and } 0 \in T\}$. Then J is the largest representation of 0 in $D_{\mathbb{C}}$. Let $g : D_{\mathscr{E}} \to D_{\mathbb{C}}$ be the strict continuous function given by $g(I) = J$ if $I \neq \bot$, and $g(\bot) = \bot$. It follows immediately that g represents 0 in D_{UE}, and since J is the largest representation of 0 in $D_{\mathbb{C}}$, g is the largest representation of 0 in D_{UE}. Suppose that $s(g) = k$, and choose a representation f of 0 in D_{UE} such that $s(f) = n > k$. Since $f \sqsubseteq g$ we must have $s(f) = s(g)$, and so $k = s(g) = s(f) = n > k$ which of course is a contradiction. □

Thus, if we want a representation of $\mathscr{E}'$ which is equivalent to the admissible representation D_{UE} it is not possible to work over $D_{\mathscr{D}'} \times D_{\mathbb{N}}$.

3.3 Representing compactly supported distributions as distributions with finite order

If u is a distribution and $K \subseteq \mathbb{R}$ is compact there are constants C and m (which may depend on K) such that $|u(\varphi)| \leq C \|\varphi\|_m$ for all $\varphi \in \mathscr{D}$ with $\mathrm{supp}(\varphi) \subseteq K$. (That this is so follows since u restricts to a continuous functional on the space of test functions with compact support in K.) If u has compact support this local result may be turned into a corresponding global result.

Theorem 3.16. *Let u be a distribution. Then u has compact support if and only if there are constants C and m such that $|u(\varphi)| \leq C \|\varphi\|_m$ for all $\varphi \in \mathscr{D}$.*

Proof. We prove the implication from right to left. (The direction from left to right is standard.) Suppose that there are constants C and m such that $|u(\varphi)| \leq C \|\varphi\|_m$ for all $\varphi \in \mathscr{D}$. We claim that the support of u must be contained in $[-m,m]$. Thus, suppose that $\varphi \in \mathscr{D}$ and that $\mathrm{supp}(\varphi)$ does not intersect $[-m,m]$. Then $|u(\varphi)| \leq C \|\varphi\|_m = 0$, and $u(\varphi) = 0$ as required. □

If $u \in \mathscr{D}'$ we say that u has *finite order* if there are constants C and m such that $|u(\varphi)| \leq C \|\varphi\|_m$ holds for all $\varphi \in \mathscr{D}$. The *order* of u in this case is the least natural number m for which there is a corresponding C such that $|u(\varphi)| \leq C \|\varphi\|_m$ is true for all $\varphi \in \mathscr{D}$. We may thus reformulate Theorem 3.16 as follows: A distribution has compact support if and only if it has finite order.

This characterisation of the class of compactly supported distributions suggests a new way to represent the space $\mathscr{E}'$. To approximate an element u in $\mathscr{E}'$ we simply approximate u in the sense of the representation $D_{\mathscr{D}'}$ of $\mathscr{D}'$, and supply bounds on the order of u. More precisely, we let D_O be the domain $D_S \times D_S$. (We will use D_O to approximate the constants C and m above. Since tighter bounds give more information about the behaviour of the distribution, it makes sense to approximate the order of the distribution using the domain D_O rather than $D_{\mathbb{N}} \times D_{\mathbb{N}}$.)

Definition 3.17. Let D_{FO} be the domain $D_{\mathscr{D}'} \times D_O$. If $(f, k, m) \in D_{\mathrm{FO}}$ we say that (f, k, m) *represents* the compactly supported distribution u if f represents u in $D_{\mathscr{D}'}$, $k, m \neq \bot$, and $|u(\varphi)| \leq k \|\varphi\|_m$ for all $\varphi \in \mathscr{D}$. We denote the set of representing elements in D_{FO} by D^R_{FO}, and define $\delta_{\mathrm{FO}} : D^R_{\mathrm{FO}} \to \mathscr{E}'$ by $\delta_{\mathrm{FO}}(f, k, m) = u$ if and only if (f, k, m) represents u.

It is clear that D_{FO} is effective, and that δ_{FO} is well-defined and surjective. To show that $(D_{\mathrm{FO}}, D^R_{\mathrm{FO}}, \delta_{\mathrm{FO}})$ is an effective domain representation of $\mathscr{E}'$ it is enough to show that δ_{FO} is continuous.

Theorem 3.18. $(D_{\mathrm{FO}}, D^R_{\mathrm{FO}}, \delta_{\mathrm{FO}})$ *is an effective domain representation of* $\mathscr{E}'$.

Proof. Suppose that (f, k, m) in D^R_{FO} represents u. Then $\mathrm{supp}(u)$ must be contained in the interval $[-m, m]$ as in the proof of Proposition 3.16. It follows that we may use the same technique as in the proof of Proposition 3.10 to show that δ_{FO} is continuous. We leave the details in the safe hands of the reader. □

We have already seen how to compute a bound on the support of a distribution, given information about (i.e. bounds on) the order of the distribution. This suggests that it is possible to effectively translate approximations in D_{FO} to approximations in D_{CS}. That this translation is continuous and effective follows by the next proposition.

Proposition 3.19. $D_{\mathrm{FO}} \leq_{\mathrm{e}} D_{\mathrm{CS}}$.

Proof. The function $r : D_{\mathrm{FO}} \to D_{\mathrm{CS}}$ given by $(f, k, m) \mapsto (f, m)$ is continuous, effective, and represents the identity on $\mathscr{E}'$. □

That it is possible to compute the extra information contained in an approximation in D_{FO} from an approximation in D_{CS} is less obvious. To prove this, we will need the following lemma.

Lemma 3.20. *There is an effective sequence* $(\sigma_n)_n$ *and a total recursive function* $\beta : \mathbb{N} \to \mathbb{N}$ *such that*

1. $0 \leq \sigma_n(x) \leq 1$ *for all* x, $\sigma_n(x) = 1$ *on a neighbourhood of* $[-n, n]$, *and* $\sigma_n(x) = 0$ *outside the interval* $[-(n+1), (n+1)]$ *for each* n.

2. *The function* β *is increasing.*

3. $\|\sigma_n\|_k \leq \beta(k)$ *for all natural numbers* n *and* k*.*

(The fact that $(\sigma_n)_n$ is effective is not important here. We will actually only use this to show that the function β is recursive.)

Proof. Let σ be a computable element of $\mathscr{D}$ such that $0 \leq \sigma(x) \leq 1$, $\operatorname{supp}(\sigma) \subseteq [-2^{-3}, 2^{-3}]$, and $\int_{\mathbb{R}} \sigma = 1$. (The function σ may be constructed as in Lemma 4.1.8 in [9].) Given a subset A of $\mathbb{R}$ we let $d(x, A) = \inf\{d(x, a); a \in A\}$. Let $A_n = [-(n+2^{-2}), (n+2^{-2})]$, and let $B_n = [-(n+2^{-1}), (n+2^{-1})]$. For each $n \in \mathbb{N}$ we let c_n be the piecewise linear function given by

$$c_n(x) = \begin{cases} 1 - 2^2 d(x,\, A_n) & \text{if } x \in B_n \\ 0 & \text{if } x \notin B_n \end{cases}$$

By the definition of c_n we have $c_n(x) = 1$ on A_n, and $c_n(x) = 0$ outside B_n. Let σ_n be the smooth function $\sigma_n(x) = (\sigma * c_n)(x) = \int_{\mathbb{R}} \sigma(t) c_n(x - t) dt$. Then $0 \leq \sigma_n(x) \leq 1$ for all x, and by computing the integral it is easy to show that $\sigma_n(x) = 1$ for all $x \in [-(n + 2^{-3}), (n + 2^{-3})]$, and $\sigma_n(x) = 0$ for all x outside the interval $[-(n + 2^{-1} + 2^{-3}), (n + 2^{-1} + 2^{-3})]$. Moreover, the function $n \mapsto \sigma_n$ is $(D_{\mathbb{N}}, D_{\mathscr{D}})$-effective. This follows since integration $(\varphi, c, d) \mapsto \int_c^d \varphi$ is an effective map from $\mathscr{E} \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$. (For details, see [9].)

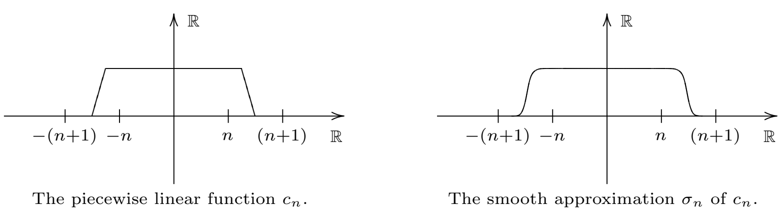


Figure 4. The functions c_n and σ_n.

Since the inclusion map $\mathscr{D} \hookrightarrow \mathscr{E}$ is effective, and the map $(\varphi, m) \mapsto \|\varphi\|_m$ is effective with respect to the representations $D_{\mathscr{E}}$, $D_{\mathbb{N}}$, and $D_{\mathbb{R}}$, we conclude that $(\|\sigma_n\|_{n+1})_n$ is an effective sequence in $\mathbb{R}$. Let $b : \mathbb{N} \to \mathbb{R}$ be a (total) effective continuous representation of the map $n \mapsto \|\sigma_n\|_{n+1}$. The set $\{(S, n) \in D_{\mathbb{R}} \times \mathbb{N};\; S \text{ is compact and } S \sqsubseteq b(n)\}$ is semidecidable since b is effective. To compute $\beta(n)$ we list the compact approximations $S_1, S_2, S_3, \ldots$ of $b(n)$. If $S_k = [r, s]$ is the first bounded interval which occurs in this list, we choose $\beta(n)$ as the least natural number N such that $N \geq \max\{s, \beta(0), \beta(1), \ldots, \beta(n-1)\}$. It is clear that β is total and recursive, and β is increasing by construction.

It is clear that $\|\sigma_k\|_{k+1} \leq \beta(k)$ for each k. To show that β satisfies 3 above it is enough to show that $\|\sigma_n\|_k \leq \|\sigma_k\|_{k+1}$ for all $n, k \in \mathbb{N}$. Note that

$\|\sigma_n\|_k \geq 1$ for all n and all $k \geq 1$, and $\|\sigma_n\|_{k+1} = \|\sigma_k\|_{k+1}$ for all $n \leq k$ since the only difference between σ_n and σ_k is the size of the interval where the function is identically equal to 1. Suppose first that $n \geq k$. Then $\|\sigma_n\|_k = 1$ since $\sigma_n(x) = 1$ on $[-n, n]$. It follows that $\|\sigma_n\|_k = 1 \leq \|\sigma_k\|_{k+1}$. On the other hand, if $n < k$ we have $\|\sigma_n\|_k \leq \|\sigma_n\|_{k+1} = \|\sigma_k\|_{k+1}$, and so we are done. □

To show that D_{CS} is effectively reducible to D_{FO} we will employ a technique similar to the one used in the proof of Proposition 3.13.

Proposition 3.21. *D_{CS} is effectively reducible to D_{FO}.*

Proof. As in the proof of Proposition 3.13 we let $B_k \in B_{\mathscr{E}}$ be the compact approximation of the constant function 0 given by $B_k = \{B^{(m)}([-k,k], [-2^{-k}, 2^{-k}] \times i[-2^{-k}, 2^{-k}]);\, 0 \leq m \leq k\}$ for $k \geq 1$, and define $C \in D_{\mathbb{C}}$ by $C = [-1, 1] \times i[1, 1]$.

Now, let $(f, m) \in D_{\mathrm{CS}}$ and suppose that (f, m) approximates the compactly supported distribution u. Suppose further that $(B_k, n) \in \mathrm{dom}(f)$ for some $k \in \mathbb{N}$ and $n \geq m+1$, and that $C \sqsubseteq f(B_k, n)$ in $D_{\mathbb{C}}$. If $\varphi \in \mathscr{D}$ satisfies $\|\varphi\|_k < 2^{-k}$ and $\mathrm{supp}(\varphi) \subseteq [-n, n]$, then $(B_k, n) \prec \varphi$. It follows that $C \prec u(\varphi)$ in $D_{\mathbb{C}}$, and $|u(\varphi)| \leq \sup\{|z|;\ z \in C\} < 2$. Since $\left\|(2^{k+1}\|\varphi\|_k)^{-1}\varphi\right\|_k < 2^{-k}$ for all nonzero $\varphi \in \mathscr{D}$ we have

$$|u(\varphi)| = (2^{k+1}\|\varphi\|_k)\left|u\big((2^{k+1}\|\varphi\|_k)^{-1}\varphi\big)\right| < 2^{(k+2)}\|\varphi\|_k$$

for all $\varphi \in \mathscr{D}$ with support contained in the interval $[-n, n]$. Since the support of u is contained in the interval $[-n, n]$ we may use this approximation to compute a global bound on $|u(\varphi)|$.

Let $(\sigma_n)_n$ be the effective sequence defined in Lemma 3.20. Since $n \geq m+1$ we have $\mathrm{supp}(\sigma_m) \subseteq [-n, n]$, and since $\mathrm{supp}(u) \subseteq [-m, m]$ we have $u(\varphi) = (\sigma_m u)(\varphi) = u(\sigma_m \varphi)$ for each $\varphi \in \mathscr{D}$. Let $\beta : \mathbb{N} \to \mathbb{N}$ be the total recursive function defined in Lemma 3.20. Then

$$|u(\varphi)| = |u(\sigma_m\varphi)| < 2^{(k+2)}\|\sigma_m\varphi\|_k \leq$$

$$2^{(k+2)}\|\sigma_m\|_k\|\varphi\|_k \leq 2^{(k+2)}\beta(k)\|\varphi\|_k$$

for each $\varphi \in \mathscr{D}$. Thus, given $(f, m) \in D_{\mathrm{CS}}$ we let $o(f, m) = (2^{K+2}\beta(K), K)$ where K is the least $k \in \mathbb{N}$ such that $(B_k, n) \in \mathrm{dom}(f)$ for some $n \geq m+1$, and $C \sqsubseteq f(B_k, n)$. Finally, if no such k exists we let $o(f, m) = \bot$.

The function $o : D_{\mathrm{CS}} \to D_{\mathrm{O}}$ is continuous: It is enough to show that o is monotone since every monotone function into D_{O} is continuous. Thus, let $(f, m) \sqsubseteq (g, n)$ in D_{CS}, and suppose that $o(f, m) = (2^{K+2}\beta(K), K)$. It follows by the definition of o above that $(B_K, N) \in \mathrm{dom}(f)$ for some $N \geq m+1$, and $C \sqsubseteq f(B_K, N)$. Since $(f, m) \sqsubseteq (g, n)$ in D_{CS} we immediately have $(B_K, N) \in \mathrm{dom}(g)$, $N \geq m+1 \geq n+1$, and $C \sqsubseteq f(B_K, N) \sqsubseteq g(B_K, N)$. Since the function $k \mapsto 2^{k+2}\beta(k)$ is increasing we conclude that $o(f, m) \sqsubseteq o(g, n)$ in D_{O}. We also note that o is effective since the problem "$(B_k, n) \in \mathrm{dom}(f)$, $n \geq m+1$, and $C \sqsubseteq f(B_k, n)$" is decidable for compact elements $(B_k, n) \in D_{\mathscr{D}}$ and $(f, m) \in D_{\mathrm{CS}}$.

Suppose now that (f, m) represents u in D_{CS}. Then $f : D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}$ represents $u : \mathscr{D} \to \mathbb{C}$, and so in particular, it follows that $D_{\mathscr{D}}^{D} \subseteq \mathrm{dom}(f)$ and $(B_k, n) \in \mathrm{dom}(f)$ for all n and $k \geq 1$. Let $I = \bigsqcup_{k \geq 1}(B_k, m+1)$. I is well-defined since $(B_1, m+1) \sqsubseteq (B_2, m+1) \sqsubseteq (B_3, m+1) \sqsubseteq \ldots$ in $B_{\mathscr{D}}$, and it is easy to see that I represents the constant function 0 in $D_{\mathscr{D}}$. Since $I \in D_{\mathscr{D}}^{R}$ we have $I \in \mathrm{dom}(f)$, and $f(I)$ represents $u(0) = 0$ in $D_{\mathbb{C}}$. Since $f(I)$ represents 0 we have $C \sqsubseteq f(I)$ in $D_{\mathbb{C}}$. By the continuity of f there is some $k \geq 1$ such that $C \sqsubseteq f(B_k, m+1)$. Hence we conclude that $(2^{k+2}\beta(k), k) \sqsubseteq o(f, m)$ in D_{O}.

Conversely, by the definition of o above it is clear that if (f, m) represents u in D_{CS} and $o(f, m) = (k, n)$ then $|u(\varphi)| \leq k \|\varphi\|_n$ for all $\varphi \in \mathscr{D}$. Thus, if $r : D_{\mathrm{CS}} \to D_{\mathrm{FO}}$ is given by $\mathrm{dom}(r) = D_{\mathrm{CS}}$ and $r(f, m) = (f, k, n)$, where $o(f, m) = (k, n)$, then r represents the identity on $\mathscr{E}'$. r is clearly effective since o is effective, and so $D_{\mathrm{CS}} \leq_{\mathrm{e}} D_{\mathrm{FO}}$ as required. □

In view of Corollary 3.14 we immediately have the following corollary to Propositions 3.19 and 3.21.

Corollary 3.22. *The representations D_{UE}, D_{CS}, and D_{FO} are all effectively equivalent.*

One may ask if it would be possible to construct D_{FO} over the effective domain $D_{\mathscr{D}'} \times D_{\mathbb{N}} \times D_{\mathbb{N}}$ rather than $D_{\mathscr{D}'} \times D_S \times D_S$. As in the case of D_{CS}, this would give us a new domain representation of $\mathscr{E}'$ (defined in exactly the same way as above). However, since we can compute an upper bound on the support of $u \in \mathscr{E}'$ from a bound on the order of u, it is fairly straightforward to show (using Lemma 3.15) that the resulting domain representation would not be equivalent to D_{UE}. In fact, it would not even be equivalent to the alternative representation D_{CS} defined over $D_{\mathscr{D}'} \times D_{\mathbb{N}}$ which was discussed at the end of the previous section.

3.4 Representing compactly supported distributions as finite sums of weak derivatives of continuous functions

By Theorem 1.1 we know that every compactly supported distribution u can be written as a finite sum $u = \sum_{k=0}^{n} g_k^{(k)}$ where $g_0, g_2, \ldots, g_n$ are continuous functions from $\mathbb{R}$ to $\mathbb{C}$ with compact support. This result is sometimes known as the structure theorem for the space of distributions with compact support. Since $g_0, g_2, \ldots, g_n$ are only continuous and not smooth in general, the derivatives above must be understood in the weak distributional sense, and so the identity above actually reads

$$u(\varphi) = \sum_{k=0}^{n} (-1)^k \int_{\mathbb{R}} g_k(x)\varphi^{(k)}(x)dx$$

for each $\varphi \in \mathscr{D}$. As before, we may use this characterisation to construct a domain representation of $\mathscr{E}'$.

We denote the space of continuous functions from $\mathbb{R}$ to $\mathbb{C}$ by $C(\mathbb{R})$, and write $C_c(\mathbb{R})$ for the space of continuous functions from $\mathbb{R}$ to $\mathbb{C}$ with compact support. The topology on $C(\mathbb{R})$ is given by the convergence relation $(g_n)_n \longrightarrow g$ if and only if $(g_n(x_n))_n \longrightarrow g(x)$ for each convergent sequence $(x_n)_n \longrightarrow x$ in $\mathbb{R}$, and the topology on $C_c(\mathbb{R})$ is given by the norm $\|g\| = \sup\{|g(x)|;\ x \in \mathbb{R}\}$.

We let $D_{C(\mathbb{R})}$ be the standard representation of $C(\mathbb{R})$ over the domain $[D_{\mathbb{R}} \rightharpoonup D_{\mathbb{C}}]$ of partial continuous functions from $D_{\mathbb{R}}$ to $D_{\mathbb{C}}$. More precisely, $f : D_{\mathbb{R}} \rightharpoonup D_{\mathbb{C}}$ is in $D^R_{C(\mathbb{R})}$ if and only if f represents some function $g : \mathbb{R} \to \mathbb{C}$, and if $f \in D^R_{C(\mathbb{R})}$ then $\delta_{C(\mathbb{R})}(f) = g$ if and only if f represents g.

Theorem 3.23. $(D_{C(\mathbb{R})}, D^R_{C(\mathbb{R})}, \delta_{C(\mathbb{R})})$ *is an effective admissible domain representation of* $C(\mathbb{R})$.

Proof. This follows by a direct application of Theorem 3.11 in [8]. □

Let $D_{C_c(\mathbb{R})}$ be the domain $D_{C(\mathbb{R})} \times D_{\mathrm{S}}$. We say that $(f, k) \in D_{C_c(\mathbb{R})}$ *represents* $g \in C_c(\mathbb{R})$ if f represents g in $D_{C(\mathbb{R})}$ and the support of g is contained in the interval $[-k, k]$. We let $D^R_{C_c(\mathbb{R})}$ denote the set of representing elements in $D_{C_c(\mathbb{R})}$ and define $\delta_{C_c(\mathbb{R})} : D^R_{C_c(\mathbb{R})} \to C_c(\mathbb{R})$ by $\delta_{C_c(\mathbb{R})}(f, k) = g$ if and only if (f, k) represents g.

Theorem 3.24. $(D_{C_c(\mathbb{R})}, D^R_{C_c(\mathbb{R})}, \delta_{C_c(\mathbb{R})})$ *is an effective domain representation of* $C_c(\mathbb{R})$.

Proof. Since it is clear that $\delta_{C_c(\mathbb{R})}$ is well-defined and surjective, it remains to show that it is continuous. As usual, since $D^R_{C_c(\mathbb{R})}$ is countably based, it is enough to prove sequential continuity. To this aim, suppose that $(f_n, k_n)_n \longrightarrow (f, k)$ in $D^R_{C_c(\mathbb{R})}$, and let $g_n = \delta_{C_c(\mathbb{R})}(f_n, k_n)$ and $g = \delta_{C_c(\mathbb{R})}(f, k)$. Since $(f_n)_n \longrightarrow f$ in $D_{C(\mathbb{R})}$ we must have

$$(g_n)_n = (\delta_{C_c(\mathbb{R})}(f_n, k_n))_n = (\delta_{C(\mathbb{R})}(f_n))_n \longrightarrow \delta_{C(\mathbb{R})}(f) =$$

$$\delta_{C_c(\mathbb{R})}(f, k) = g.$$

in $C(\mathbb{R})$. (Thus in particular, we know that $(g_n)_n \longrightarrow g$ pointwise.)

To show that $(g_n) \longrightarrow g$ in $C_c(\mathbb{R})$, we note that $k_n \leq k$ for almost all n since $(k_n)_n \longrightarrow k$ in D_S. Let $K = \max\{k_n;\ n \in \mathbb{N}\}$. Since $(g_n) \longrightarrow g$ pointwise it follows that $(g_n) \longrightarrow g$ uniformly on $[-K, K]$. We thus conclude that $(g_n) \longrightarrow g$ in $C_c(\mathbb{R})$ as required. □

We may now define a domain representation of $\mathscr{E}'$ which encodes the information implicit in the structure theorem for the class of distributions with compact support.

Definition 3.25. D_{WC} is the domain given by $D_{\mathrm{WC}} = \coprod_{n\geq 1} D^n_{C_c(\mathbb{R})}$. The element $r = (r_1, r_2, \ldots, r_n) \in D_{\mathrm{WC}}$ *represents* the compactly supported distribution $u = \sum_{k=1}^n g_k^{(k)}$ if r_k represents g_k in $D_{C_c(\mathbb{R})}$ for each $1 \leq k \leq n$. We write D^R_{WC} for the set of representing elements in D_{WC}, and define $\delta_{\mathrm{WC}} : D^R_{\mathrm{WC}} \to \mathscr{E}'$ by $\delta_{\mathrm{WC}}(r) = u$ if and only if r represents u.

Theorem 3.26. $(D_{\mathrm{WC}}, D^R_{\mathrm{WC}}, \delta_{\mathrm{WC}})$ *is an effective domain representation of* $\mathscr{E}'$.

Proof. It is clear that D_{WC} is effective, and that δ_{WC} is well-defined and onto. We will show that δ_{WC} is sequentially continuous. Suppose that $(r_n)_n \longrightarrow r$ in D_{WC}, and let $u_n = \delta_{\mathrm{WC}}(r_n)$ and $u = \delta_{\mathrm{WC}}(r)$. To show that $(u_n)_n \longrightarrow u$ in $\mathscr{E}'$ we need to show that $u_n(\varphi) \longrightarrow u(\varphi)$ for each $\varphi \in \mathscr{E}$.

Thus, let $\varphi \in \mathscr{E}$. Suppose that $r \in D^m_{C_c(\mathbb{R})}$, and that $r = (s_1, s_2, \ldots, s_m)$. We may assume that r_n is in $D^m_{C_c(\mathbb{R})}$ for all n. It follows that each r_n is on the form $r_n = (s_{1,n}, s_{2,n}, \ldots, s_{m,n})$. Let $g_{k,n}$ be the continuous function represented by $s_{k,n}$, and let g_k be the continuous function represented by s_k. Since $(r_n)_n \longrightarrow r$ in D_{WC} we know that $(s_{k,n})_n \longrightarrow s_k$ in $D_{C_c(\mathbb{R})}$ for each $1 \leq k \leq m$. It follows that $(g_{k,n})_n \longrightarrow g_k$ uniformly for each $1 \leq k \leq m$. This means that $(G_{k,n})_n = ((-1)^k g_{k,n}\varphi^{(k)})_n$ converges uniformly to $G_k = (-1)^k g_k \varphi^{(k)}$ for each $1 \leq k \leq m$. This in turn means that $(u_n(\varphi))_n = (\sum_{k=1}^m \int_{\mathbb{R}} G_{k,n})_n \longrightarrow \sum_{k=1}^m \int_{\mathbb{R}} G_k = u(\varphi)$ as required. □

To show that evaluation is D_{WC}-effective (and in turn, that $D_{\mathrm{WD}} \leq_{\mathrm{e}} D_{\mathrm{UE}}$), we will need the following two lemmas.

Lemma 3.27. *Multiplication $(g, \varphi) \mapsto g\varphi$ from $C_c(\mathbb{R}) \times \mathscr{E}$ into $C_c(\mathbb{R})$ is effective.*

Proof. We begin by showing that multiplication from $C_c(\mathbb{R}) \times \mathscr{E}$ into $C(\mathbb{R})$ is effective. Since type conversion from $[C_c(\mathbb{R}) \times \mathscr{E} \times \mathbb{R} \to \mathbb{C}]$ to $[C_c(\mathbb{R}) \times \mathscr{E} \to C(\mathbb{R})]$ preserves effectivity[7] it is enough to show that the function $(g, \varphi, r) \mapsto g(r)\varphi(r)$ is effective. This is clear since evaluation from $\mathscr{E} \times \mathbb{R}$ into $\mathbb{C}$ is effective, the inclusion $C_c(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ is effective, evaluation from $C(\mathbb{R}) \times \mathbb{R}$ into $\mathbb{C}$ is effective, and multiplication on $\mathbb{C}$ is effective.

Now, to show that multiplication from $C_c(\mathbb{R}) \times \mathscr{E}$ into $C_c(\mathbb{R})$ is effective it is enough to show that we can compute a bound on the support of $g\varphi$ from a bound on the support of g. This is simple since $\operatorname{supp}(g\varphi) \subseteq \operatorname{supp}(g)$. □

Lemma 3.28. *Integration $g \mapsto \int_{\mathbb{R}} g$ is an effective map from $C_c(\mathbb{R})$ to $\mathbb{C}$.*

It is well-known that integration $(g, k) \mapsto \int_{-k}^{k} g$ is effective with respect to the representations $D_{C(\mathbb{R})}$, $D_{\mathbb{N}}$ and $D_{\mathbb{C}}$. However, since we are using a different order relation on the natural numbers (to represent the support of g), we need to be slightly more careful.

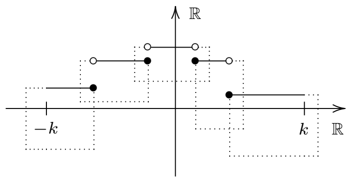
Proof. Since $\int_{\mathbb{R}} g = \int_{\mathbb{R}} \operatorname{Re}(g) + \int_{\mathbb{R}} \operatorname{Im}(g)$ it is enough to show that $g \mapsto \int_{\mathbb{R}} \operatorname{Re}(g)$ and $g \mapsto \int_{\mathbb{R}} \operatorname{Im}(g)$ are effective. We will consider the integral of the real part of g. The proof for the integral of the imaginary part is completely analogous.

Let $f \in D_{C(\mathbb{R})}$ be compact, and suppose that $f = (S, (S_1 \searrow T_1) \sqcup (S_2 \searrow T_2) \sqcup \ldots \sqcup (S_n \searrow T_n))$. Let $k \in D_S$. We say that (f, k) is *complete* if $k \neq \bot$ and $[-k, k] \subseteq \bigcup\{S_i;\ T_i \text{ is bounded}\}$. It is easy to verify that the definition of completeness is independent of the representation of f. It is also easy to see that if $(f_1, k_1) \sqsubseteq (f_2, k_2)$ in $D_{C_c(\mathbb{R})}$ and (f_1, k_1) is complete then so is (f_2, k_2).

If (f, k) is complete and $x \in [-k, k]$ we define $U(f, k)(x)$ as the set $U(f, k)(x) = \{(\sup \circ \operatorname{Re})[T_i];\ T_i \text{ is bounded and } x \in S_i\}$. The set $U(f, k)(x)$ contains upper bounds of the real part of the value at x of every continuous function approximated by (f, k). We let $u(f, k) : [-k, k] \to \mathbb{R}$ be the function $u(f, k)(x) = \min U(f, k)(x)$. The function $u(f, k)$ is well-defined since (f, k) is complete. It follows immediately by the definition of $u(f, k)$ that any function $g \in C_c(\mathbb{R})$ approximated by (f, k) satisfies $\operatorname{Re}(g(x)) \leq u(f, k)(x)$ for all $x \in [-k, k]$.

Similarly, we define $L(f, k)(x)$ as the set $\{(\inf \circ \operatorname{Re})[T_i];\ T_i \text{ is bounded and } x \in S_i\}$ for $x \in [-k, k]$. We let $l(f, k) : [-k, k] \to \mathbb{R}$ be the function

[7] For a proof of this, see Corollary 4.10 in [9].



The real part of the approximation (f, k), and the corresponding function $u(f, k)$.

Figure 5.

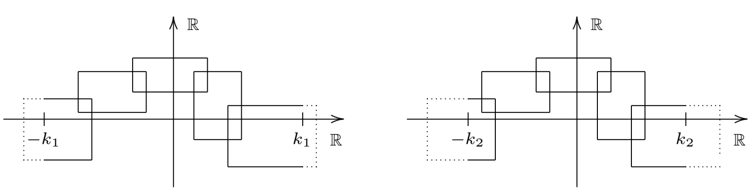
given by $l(f,k)(x) = \max L(f,k)(x)$ for each $x \in [-k,k]$. The function l is well-defined since (f,k) is complete. Furthermore, if $g \in C_c(\mathbb{R})$ is approximated by (f,k) then g satisfies $l(f,k)(x) \le \mathrm{Re}(g(x))$ for all $x \in [-k,k]$.

Now, let R denote the dense part of the domain $D_{C_c(\mathbb{R})}$, and let (f,k) be a compact element in R. If (f,k) is complete we define $\mathrm{int}(f,k) \in D_{\mathbb{R}}$ as the rational interval

$$\mathrm{int}(f,k) = \Big[\int_{-k}^{k} l(f,k)(x)dx,\ \int_{-k}^{k} u(f,k)(x)dx\Big].$$

If (f,k) is not complete we set $\mathrm{int}(f,k) = \bot$. Note that $\mathrm{int}(f,k)$ is well-defined since $\int_{-k}^{k} l(f,k)(x)dx \le \int_{-k}^{k} u(f,k)(x)dx$ as $l(f,k)(x) \le u(f,k)(x)$ for all $x \in [-k,k]$. Moreover, since $u(f,k)$ and $l(f,k)$ are piecewise constant functions taking values in $\mathbb{Q}$ it follows that $\int_{-k}^{k} l(f,k)(x)dx$ and $\int_{-k}^{k} u(f,k)(x)dx$ are both rational numbers. We conclude that $\mathrm{int}(f,k)$ is (a compact element) in $D_{\mathbb{R}}$ for each compact $(f,k) \in R$.

To show that int is monotone, suppose that $(f_1,k_1) \sqsubseteq (f_2,k_2)$ are compact in R. If (f_1,k_1) is not complete we have $\mathrm{int}(f_1,k_1) = \bot \sqsubseteq \mathrm{int}(f_2,k_2)$. Suppose that (f_1,k_1) is complete. Then (f_2,k_2) is complete as well. Since (f_1,k_1) and (f_2,k_2) are in R we must have $T \prec 0$ for each $(S \searrow T) \sqsubseteq f_1$ such that $S \cap ([-k_1,k_1] \setminus [-k_2,k_2]) \neq \emptyset$. (Note that this is not true in general.)



The real part of the approximation (f_1, k_1). The real part of the approximation (f_1, k_2).

Figure 6.

It follows that $\mathrm{int}(f_1,k_1) \sqsubseteq \mathrm{int}(f_1,k_2) \sqsubseteq \mathrm{int}(f_2,k_2)$ as required. Let $\mathrm{int} : D_{C_c(\mathbb{R})} \rightharpoonup D_{\mathbb{R}}$ be the unique continuous extension of int to R. Then int is an effective partial continuous function, and $D^R_{C_c(\mathbb{R})} \subseteq R = \mathrm{dom}(\mathrm{int})$ by the definition of R. That int represents integration on $C_c(\mathbb{R})$ follows easily since every continuous function is Riemann integrable. $\square$

We can now show that evaluation from $\mathscr{E}' \times \mathscr{E}$ into $\mathbb{C}$ is D_{WC}-effective. We may then immediately conclude that $D_{\mathrm{WC}} \leq_{\mathrm{e}} D_{\mathrm{UE}}$ by Proposition 3.6.

Proposition 3.29. *Evaluation* $(u, \varphi) \mapsto u(\varphi)$ *on* $\mathscr{E}$ *is* D_{WC}*-effective.*

Proof. It is enough to show that there is an effective partial continuous function $\mathrm{ev}_n : D^n_{C_c(\mathbb{R})} \times D_{\mathscr{E}} \rightharpoonup \mathscr{D}_{\mathbb{C}}$ which represents evaluation in the sense that ev_n takes each representation of the pair (u, φ) in $D^n_{C_c(\mathbb{R})} \times D_{\mathscr{E}}$ to a representation of $u(\varphi)$ in $D_{\mathbb{C}}$.

Fix $n \geq 1$, and let $\mathrm{sum}_n : D^n_{\mathbb{C}} \rightharpoonup D_{\mathbb{C}}$ be an effective representation of the map $(z_1, z_2, \ldots, z_n) \mapsto \sum_{i=1}^n z_i$. Let $d : D_{C_c(\mathbb{R})} \times D_{\mathbb{N}} \times D_{\mathscr{E}} \rightharpoonup D_{C_c(\mathbb{R})}$ be an effective representation of the map given by $(g, m, \varphi) \mapsto (-1)^m g\varphi^{(m)}$. The partial continuous function d exists by Lemma 3.27 since differentiation $(\varphi, m) \mapsto \varphi^{(m)}$ on $\mathscr{E}$ is effective. (For a proof of this, see [9].) Finally, let $\mathrm{int} : D_{C_c(\mathbb{R})} \rightharpoonup D_{\mathbb{C}}$ be the effective representation of the integration map from Lemma 3.28.

Let R denote the closure of the set $D^R_{\mathrm{WC}} \cap D^n_{C_c(\mathbb{R})}$ in $D^n_{C_c(\mathbb{R})}$ and let Q be the closure of $D^R_{\mathscr{E}}$ in $D_{\mathscr{E}}$. For $1 \leq k \leq n$ we let $s_k : D^n_{C_c(\mathbb{R})} \times D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}$ be the effective partial continuous function given by $\mathrm{dom}(s_k) = R \times Q$, and $s_k((r_1, r_2, \ldots, r_n), J) = \mathrm{int}(d(r_k, k, J))$ on $R \times Q$. Let $r_1, r_2, \ldots, r_n \in D^R_{C_c(\mathbb{R})}$ represent $g_1, g_2, \ldots, g_n \in C_c(\mathbb{R})$, and suppose that $J \in D^R_{\mathscr{E}}$ represents $\varphi \in \mathscr{E}$. Then $s_k((r_1, r_2, \ldots, r_n), J)$ represents $(-1)^k \int g_k \varphi^{(k)}$ in $D_{\mathbb{C}}$.

We may now define $\mathrm{ev}_n : (D_{C_c(\mathbb{R})} \times D_{\mathbb{N}})^n \times D_{\mathscr{E}} \rightharpoonup D_{\mathbb{C}}$ as the effective partial continuous function such that $\mathrm{dom}(\mathrm{ev}_n) = R \times Q$ and

$$\mathrm{ev}_n(r, J) = \mathrm{sum}_n(s_1(r, J),\ s_2(r, J),\ \ldots,\ s_n(r, J))$$

on $\mathrm{dom}(\mathrm{ev}_n)$. It is clear that ev_n is an effective partial continuous function, and ev_n represents evaluation by the definition of sum_n. $\square$

In light of Proposition 3.6 we immediately have the following corollary.

Corollary 3.30. D_{WC} *is effectively reducible to* D_{UE}*.*

It is natural to ask if the converse result holds as well. Interestingly, this is not the case. To prove this, we will show that there is a partial continuous function $o : D_{\mathrm{WC}} \rightharpoonup D_{\mathbb{N}}$ such that $o(r)$ is an upper bound on the order of $\delta_{\mathrm{WC}}(r)$ for each representation $r \in D^R_{\mathrm{WC}}$ of a distribution with support in the interval $[-1, 1]$. The result follows since (as we will see shortly) there cannot be a corresponding partial continuous function $o : D_{\mathrm{UE}} \rightharpoonup D_{\mathbb{N}}$ with the same property.

Lemma 3.31. *There is a partial continuous function $o : D_{\mathrm{WC}} \rightharpoonup D_{\mathbb{N}}$ such that $o(r)$ is an upper bound on the order of $\delta_{\mathrm{WC}}(r)$ for each $r \in D^R_{\mathrm{WC}}$ which represents a distribution with support contained in* $[-1, 1]$.

Proof. Suppose that $(r_1, r_2, \ldots, r_n)$ represents u in D_{WC}. Then $u = \sum_{k=1}^n g_k^{(k)}$ for some $g_1, g_2, \ldots, g_n \in D_{C_c(\mathbb{R})}$. If the support of u is contained in $[-1, 1]$ it follows that the order of u is bounded above by n. Let $o : D_{\mathrm{WC}} \to D_{\mathbb{N}}$ be the function given by $o(r_1, r_2, \ldots, r_n) = n$, and $o(\bot) = \bot$. It is easy to see that o is monotone, and so o is continuous. □

To prove the corresponding negative result for D_{UE} we will need the following lemma

Lemma 3.32. *The order of $\delta^{(m)}$ is m.*

Proof. It is clear that the order of $\delta^{(m)}$ is $\leq m$. Suppose that the order of $\delta^{(m)}$ is equal to $k < m$. Choose C such that $|\delta^{(m)}(\varphi)| \leq C \|\varphi\|_k$ holds for all $\varphi \in \mathscr{E}$. Choose $\sigma \in \mathscr{D}$ such that $0 \leq \sigma(x) \leq 1$ for all x, $\sigma(0) = 1$, and $\int \sigma = 1$. Let $\sigma_n = 2^n \sigma(2^n x)$. Then $\int \sigma_n = 1$ for all n and $\sigma_n(0) = 2^n$.

Now, let $\sigma_{0,n} = \sigma_n$, and given $\sigma_{i,n}$ for some $i < m$ we let $\sigma_{i+1,n}$ be the function $\sigma_{i+1,n}(x) = \int_{-\infty}^x \sigma_{i,n}(t)dt$. Since $\sigma_{i,n}$ is positive for each $i < m$ we have

$$\sup\{|\sigma_{i+1,n}(x)|;\ x \in [-k, k]\} = \sigma_{i+1,n}(k) = \int_{-\infty}^k \sigma_{i,n}(t)dt.$$

For $i = 0$ this is $\leq 2k$ since $\sigma_{0,n}(x) \leq 1$ for all x. For $i = 1$ this is $\leq (2k)^2$, and in the general case for $0 \leq i < m$, we have $\sup\{|\sigma_{i+1,n}(x)|; x \in [-k, k]\} \leq (2k)^{i+1}$.

It follows that $\|\sigma_{m,n}\|_k$ is bounded as n tends to infinity, but this is a contradiction since $|\delta^{(m)}(\sigma_{m,n})| = \sigma_n(0) = 2^n$, which tends to infinity as $n \longrightarrow \infty$. □

Lemma 3.33. *There is no partial continuous function $o : D_{\mathrm{UE}} \rightharpoonup D_{\mathbb{N}}$ such that $o(f)$ is an upper bound on the order of u for each representation $f \in D^R$ of a distribution u with support in the interval* $[-1, 1]$.

Proof. Suppose for a contradiction that $o : D_{\mathrm{UE}} \rightharpoonup D_{\mathbb{N}}$ exists. Fix $m \in \mathbb{N}$ and let $u_n = 2^{-n}\delta^{(m)}$. Then $u_n(\varphi) = 2^{-n}(-1)^{(m)}\varphi(0)^{(m)}$ and so $(u_n)_n \longrightarrow 0$ in $\mathscr{E}'$. Since D_{UE} is admissible, each convergent sequence in $\mathscr{E}'$ lifts[8] to a convergent sequence in D_{UE}. Choose f_n and f in D^R_{UE} such that $\delta_{\mathrm{UE}}(f_n) =$

[8]For a proof that every convergent sequence in $\mathscr{E}'$ lifts to a convergent sequence in D_{UE}, c.f. [12].

u_n, $\delta_{\text{UE}}(f) = u$, and $(f_n)_n \longrightarrow f$ in D_{UE}. It follows that $(o(f_n))_n \longrightarrow o(f)$ in $D_{\mathbb{N}}$, and so $o(f) \geq m$ by the characterisation of o. Since m was arbitrary it follows that for each m there is a representation $f \in D^R_{\text{UE}}$ of 0 such that $o(f) \geq m$.

Let J denote the ideal $\{T \in D_{\mathbb{C}};\ T \text{ is compact and } 0 \in T\}$ in $D_{\mathbb{C}}$. It is easy to see that J is the $\sqsubseteq$-largest representation of 0 in $D_{\mathbb{C}}$. Let $g : D_{\mathscr{E}} \to D_{\mathbb{C}}$ be the strict continuous function given by $g(I) = J$ for each $I \neq \bot$. Then g represents 0 in D_{UE}. In fact g is the $\sqsubseteq$-largest representation of 0 in D_{UE}. Now, if $o(g) = m$ there is a representation f of 0 in D_{UE} such that $o(f) \geq m + 1$. But then $o(g) = m + 1$ since $f \sqsubseteq g$ and o is monotone. This is our desired contradiction. □

Corollary 3.34. *D_{UE} is not continuously reducible to D_{WC}.*

3.5 The representation D_{UE} is Turing-reducible to D_{WC}

Even though D_{UE} is not effectively reducible to D_{WC} it may still be possible to compute a representation of u in D_{WC} from a representation of u in D_{UE} if the representation D_{UE} is Turing-reducible to D_{WC}. We will now show that this is indeed the case. To prove this, we will need an effective version of the following result for the Fourier transform on $\mathscr{E}'$.

Theorem 3.35. *Let u be a compactly supported distribution. Then $\hat{u}$ is a slowly increasing smooth function.*

Here $\hat{u}$, and in some places $\mathcal{F}(u)$, denotes the Fourier transform of u. For a proof of Theorem 3.35 we refer the reader to [10].

The class $\text{SI}(\mathbb{R})$ of *slowly increasing* smooth functions is the class of C^∞ functions $\varphi : \mathbb{R} \to \mathbb{C}$ with the property that for each k, there is a polynomial p_k such that $|\varphi^{(k)}(x)| \leq p_k(x)$ for all $x \in \mathbb{R}$.

Let $P(k, m, n)$ be a notation for the set of all $\varphi \in \text{SI}(\mathbb{R})$ such that $\varphi^{(k)}(x) \leq m(1 + x^2)^n$ for all x. We let B_{PB} be the collection of all finite subsets of the form $\{P(k_1, m_1, n_1), P(k_2, m_2, n_2), \ldots, P(k_s, m_s, n_s)\}$ with the order relation given by $B_1 \sqsubseteq B_2$ if and only if for every $P(k_1, m_1, n_1) \in B_1$ there is some $P(k_2, m_2, n_2) \in B_2$ such that $k_1 = k_2$, $m_1 \geq m_2$ and $n_1 \geq n_2$. Intuitively, we think of the set $\{P(k_1, m_1, n_1), P(k_2, m_2, n_2), \ldots, P(k_s, m_s, n_s)\}$ in B_{PB} as (a representation of) the set of all $\varphi \in \text{SI}(\mathbb{R})$ such that $|\varphi^{(k_i)}(x)| \leq m_i(1 + x^2)^{n_i}$ for all x and all $1 \leq i \leq s$. Since smaller values for m_i and n_i represent tighter bounds, the order relation on B_{PB} may be interpreted as an information ordering.

Definition 3.36. Let D_{PB} be the ideal completion of B_{PB} and let $D_{\mathrm{SI}(\mathbb{R})}$ be the effective domain $D_{\mathscr{E}} \times D_{\mathrm{PB}}$. If $(I, J) \in D_{\mathrm{SI}(\mathbb{R})}$ we say that (I, J) *represents* $\varphi \in \mathrm{SI}(\mathbb{R})$ if I represents φ in $D_{\mathscr{E}}$, and given any k there are m and n in $\mathbb{N}$ such that $P(k, m, n) \in B$ for some B in J. As usual we denote the set of representing elements in $D_{\mathrm{SI}(\mathbb{R})}$ by $D^R_{\mathrm{SI}(\mathbb{R})}$ and we define $\delta_{\mathrm{SI}(\mathbb{R})} : D^R_{\mathrm{SI}(\mathbb{R})} \to \mathrm{SI}(\mathbb{R})$ by $\delta_{\mathrm{SI}(\mathbb{R})}(I, J) = \varphi$ if and only if (I, J) represents φ.

It is clear that $\delta_{\mathrm{SI}(\mathbb{R})}$ is well-defined and surjective. It is continuous (with respect to the subspace topology on $\mathrm{SI}(\mathbb{R})$ as a subspace of $\mathscr{E}$) since $\delta_{\mathrm{SI}(\mathbb{R})}(I, J) = \delta_{\mathscr{E}}(I)$ for each pair $(I, J) \in D^R_{\mathrm{SI}(\mathbb{R})}$. Thus, we have the following theorem:

Theorem 3.37. *$(D_{\mathrm{SI}(\mathbb{R})}, D^R_{\mathrm{SI}(\mathbb{R})}, \delta_{\mathrm{SI}(\mathbb{R})})$ is an effective domain representation of* $\mathrm{SI}(\mathbb{R})$.

We would like to show that the function $u \mapsto \hat{u}$ is effective with respect to the representations D_{UE} and $D_{\mathrm{SI}(\mathbb{R})}$. This follows from the fact that the Fourier transform of a compactly supported distribution u is equal to u evaluated at $(2\pi)^{-\frac{1}{2}} e^{-itx}$. Throughout, we write D_x for the differential operator d/dx.

Proposition 3.38. *The Fourier transform $u \mapsto \hat{u}$ is $(D_{\mathrm{UE}}, D_{\mathrm{SI}(\mathbb{R})})$-effective.*

Proof. Since D_{UE} is effectively equivalent to D_{FO}, it is enough to show that the Fourier transform is $(D_{\mathrm{FO}}, D_{\mathrm{SI}(\mathbb{R})})$-effective.

Let e_t be the smooth function from $\mathbb{R}$ to $\mathbb{C}$ given by $e_t(x) = e^{itx}$. Then $\hat{u}(t) = (2\pi)^{-\frac{1}{2}} u(e_{-t})$ and we have $\hat{u}(t)^{(k)} = (2\pi)^{-\frac{1}{2}} u(D^k_t e_{-t}) = (-i)^k (2\pi)^{-\frac{1}{2}} u(x^k e_{-t})$. (For proofs, see [10].) Since evaluation is effective with respect to D_{FO} it follows that the map given by $(u, k, t) \mapsto \hat{u}(t)^{(k)}$ is $(D_{\mathrm{FO}} \times D_{\mathbb{N}} \times D_{\mathbb{R}}, D_{\mathbb{C}})$-effective. Thus, by Theorem 3.1.9 in [9] we conclude that $u \mapsto \hat{u}$ is $(D_{\mathrm{FO}}, D_{\mathscr{E}})$-effective.

To show that the Fourier transform from $\mathscr{E}'$ to $\mathrm{SI}(\mathbb{R})$ is $(D_{\mathrm{FO}}, D_{\mathrm{SI}(\mathbb{R})})$-effective, it is clearly enough to show that there is an effective partial continuous function $\mathrm{pb} : D_{\mathrm{FO}} \rightharpoonup D_{\mathrm{PB}}$ such that for each representing element r in D_{FO}:

1. $r \in \mathrm{dom}(\mathrm{pb})$.
2. If r represents u and $P(k, m, n) \in B$ for some $B \in \mathrm{pb}(r)$ then $|\hat{u}(x)^{(k)}| \leq m(1 + x^2)^n$ for all x.
3. For each $k \in \mathbb{N}$ there are natural numbers m and n such that $P(k, m, n) \in B$ for some $B \in \mathrm{pb}(r)$.

Now, given a compactly supported distribution u such that $|u(\varphi)| \leq M \|\varphi\|_N$ for all $\varphi \in \mathscr{E}$, we have

$$|\hat{u}(t)^{(k)}| = |(2\pi)^{-\frac{1}{2}} u(D_t^k e_{-t})| = |(-i)^k (2\pi)^{-\frac{1}{2}} u(x^k e_{-t})| \leq$$

$$M \left\| x^k e_{-t} \right\|_N \leq M \sum_{n=0}^{N} \sup \left\{ |D_x^n (x^k e_{-t})|;\ |x| \leq N \right\} \leq$$

$$M \sum_{n=0}^{N} \sup \left\{ \Big| \sum_{m=0}^{\max(k,n)} \binom{n}{m} \frac{k!}{(k-m)!} x^{k-m} (-it)^{n-m} e_{-t}) \Big|;\ |x| \leq N \right\} \leq$$

$$M \sum_{n=0}^{N} \sum_{m=0}^{\min(n,k)} \sup \left\{ \Big| \frac{n^m}{m!} \frac{k!}{(k-m)!} x^{k-m} (-it)^{n-m} e_{-t} \Big|;\ |x| \leq N \right\} \leq$$

$$M \sum_{n=0}^{N} \sum_{m=0}^{\min(n,k)} \sup \left\{ \Big| N^k \frac{k^m}{m!} x^{k-m} (-it)^{n-m} e_{-t} \Big|;\ |x| \leq N \right\}$$

since n over m is bounded above by $n^k/m!$ for all $m, n \in \mathbb{N}$. This in turn is less than or equal to

$$M \sum_{n=0}^{N} \sum_{m=0}^{\min(n,k)} \sup \left\{ |N^k k^k x^{k-m} (-it)^{n-m} e_{-t}|;\ |x| \leq N \right\} \leq$$

$$M N^k k^k \sum_{n=0}^{N} \sum_{m=0}^{\min(n,k)} \sup \left\{ |x^{k-m} t^{n-m}|;\ |x| \leq N \right\}$$

Since x^{k-m} is always less than N^k on $[-N, N]$, this is less than or equal to

$$M N^k k^k \sum_{n=0}^{N} \sum_{m=0}^{\min(n,k)} N^k |t^{n-m}| \leq M N^{2k} k^k \sum_{n=0}^{N} \sum_{m=0}^{\min(n,k)} (1+t^2)^n \leq$$

$$M N^{2k} k^k \sum_{n=0}^{N} (k+1)(1+t^2)^n \leq M(N+1)^{2k+1}(k+1)^{k+1}(1+t^2)^N.$$

Thus, given (M, N) in D_{O} we define $b(M, N)$ as the ideal generated by (i.e. the down-set of) the set $\{P(k,\ M(N+1)^{2k+1}(k+1)^{k+1},\ N);\ k \in \mathbb{N}\} \subseteq D_{\mathrm{PB}}$ if both M and N are natural numbers, and $b(M, \perp) = b(\perp, M) = b(\perp, \perp) = \perp$. It is easy to see that $b : D_{\mathrm{O}} \to D_{\mathrm{PB}}$ is monotone, and so b is continuous. Furthermore, if we define $\mathrm{pb} : D_{\mathrm{FO}} \to D_{\mathrm{PB}}$ by $\mathrm{pb}(f, M, N) = b(M, N)$, then pb is a total continuous function from D_{FO} to D_{PB} satisfying the conditions 1 – 3 above. □

If $f : \mathbb{R} \to \mathbb{C}$ we write $\text{refl}(f)$ for the function given by $\text{refl}(f)(x) = f(-x)$. This definition immediately lifts to $\mathscr{D}'$: If u is a distribution we define $\text{refl}(u)$ as the distribution given by $\text{refl}(u)(\varphi) = u(\text{refl}(\varphi))$. Then $\text{refl}(u)$ is well-defined since $\text{refl}(\varphi) \in \mathscr{D}$ for each $\varphi \in \mathscr{D}$, and it is easy to show that $\text{refl}(u)$ is linear and continuous. In what follows, we write $\langle u, \varphi \rangle$ for $u(\varphi)$, and when it is convenient, we will denote the Fourier transform of $u \in \mathscr{E}'$ by $\mathcal{F}(u)$.

If $f \in \text{SI}(\mathbb{R})$ then $(1 + x^2)^{-n} f$ is in $L^1(\mathbb{R})$ if we choose $n \in \mathbb{N}$ large enough. It follows that $g = \mathcal{F}((1 + x^2)^{-n} f)$ is a continuous function[9]. Now, if $f(t) = \text{refl}(\hat{u})$ for some compactly supported distribution u we have

$$\langle (1 - D_x^2)^n g, \varphi \rangle = \langle (1 - D_x^2)^n \mathcal{F}((1 + x^2)^{-n} f), \varphi \rangle =$$

$$\langle \hat{f}, \varphi \rangle = \langle \text{refl}(\hat{u}), \hat{\varphi} \rangle = \langle \hat{u}, \text{refl}(\hat{\varphi}) \rangle = \langle \hat{u}, \mathcal{F}^{-1}(\varphi) \rangle = \langle u, \varphi \rangle$$

for each $\varphi \in \mathscr{D}$. It follows that $u = (1 - D_x^2)^n g = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g^{(2k)}$. We note that g does not have compact support in general. This is easily remedied however. Choose $\sigma \in \mathscr{D}$ such that $0 \le \sigma(x) \le 1$ for all x, and $\sigma(x) = 1$ on some open neighbourhood of the support of u. Then,

$$\langle u, \varphi \rangle = \langle u, \sigma\varphi \rangle = \langle (1 - D_x^2)^n g, \sigma\varphi \rangle = \langle\, g, (1 - D_x^2)^n (\sigma\varphi) \rangle =$$

$$\Big\langle\, g, \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sigma^{(2(n-k))} \varphi^{(2k)} \Big\rangle = \sum_{k=0}^{n} \Big\langle (-1)^k \binom{n}{k} g\sigma^{(2(n-k))}, \varphi^{(2k)} \Big\rangle.$$

Thus, if $g_{2k} = (-1)^k \binom{n}{k} g\sigma^{(2(n-k))}$ and $g_{2k+1} = 0$ for each $0 \le k \le n$, then g_k is a continuous function for each k, and the support of g_k is contained in the support of σ. Furthermore, we have

$$\langle u, \varphi \rangle = \sum_{k=0}^{2n} (-1)^k \langle\, g_k, \varphi^{(k)} \rangle = \Big\langle \sum_{k=0}^{2n} g_k^{(k)}, \varphi \Big\rangle$$

and $u = \sum_{k=0}^{2n} g_k^{(k)}$ as required.

By Corollary 3.34 we know that this algorithm does not translate to a partial continuous reduction from D_{UE} to D_{WC}. In fact, it is easy to see that we will run into problems if we want to compute n above. Since n depends on the order of u we know that n may decrease as the accuracy of the approximations of u in D_{UE} increase. This means that we will never be able to fix n or the functions g_k.

To show that D_{UE} is Turing-reducible to D_{WC} we will give an informal description of how a Turing machine would translate an infinite sequence of

[9] See [10].

compact approximations of u in D_{UE} into and infinite sequence of compact approximations of u in D_{WC}. Since we will use information about the support of u, as well as the order of u, to compute $g_0, g_2, \ldots, g_{2n}$, it will be convenient to work with D_{FO} rather than D_{UE}. (This is only to make the algorithm slightly less verbose. From a computability theoretic point of view it does not matter which of the two representations D_{UE} and D_{FO} we work with since they are effectively equivalent.)

Proposition 3.39. *D_{FO} is Turing-reducible to D_{WC}.*

To prove Proposition 3.39, we will need the following two Lemmas:

Lemma 3.40. *Reflection is an effective operation on* $\mathrm{SI}(\mathbb{R})$.

Proof. This follows immediately since reflection on $\mathscr{E}$ is effective, and any polynomial bound on $\varphi \in \mathrm{SI}(\mathbb{R})$ is also a bound on $\mathrm{refl}(\varphi)$. □

Let $B(\mathbb{R})$ be the subspace of $C(\mathbb{R})$ of all continuous functions $g : \mathbb{R} \to \mathbb{C}$ for which there are natural numbers s and t such that $|g(x)| \leq s(1 + x^2)^{-(t+2)}$ for all x. Note that every function $g \in B(\mathbb{R})$ is integrable. Let $(D_{B(\mathbb{R})}, D^R_{B(\mathbb{R})}, \delta_{B(\mathbb{R})})$ be the restriction of $D_{C(\mathbb{R})}$ to $B(\mathbb{R})$.

Lemma 3.41. *The Fourier transform $g \mapsto \hat{g}$ is an effective map from $B(\mathbb{R})$ to $C(\mathbb{R})$.*

Proof. Since type conversion from $[B(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}]$ to $[B(\mathbb{R}) \to C(\mathbb{R})]$ preserves effectivity[10], it is enough to show that the map

$$(g, t) \mapsto \hat{g}(t) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e_{-t}\, g(x)dx$$

is effective. Since $|e_{-t}| \leq 1$ for all $x, t \in \mathbb{R}$, this may be carried out using the same techniques as in Proposition 3.4.2 in [9]. □

Proof of Proposition 3.39. Let $(f_n, k_n, m_n)_n$ be a sequence of compact elements in D_{FO} and suppose that $(f_n, k_n, m_n)_n$ δ_{FO}-converges to $u \in \mathscr{E}'$. Let $(f, k, m) = \bigsqcup_n (f_n, k_n, m_n)_n$ and let p denote the least n such that $k_n, m_n \neq \bot$. (p exists since (f, k, m) is in D^R_{FO}.) Since $(f_p, k_p, m_p) \prec u$ we know that the support of u is contained in the interval $[-m_p, m_p]$. (For a proof of this, note that $m \leq m_p$ and then apply the same method as in the proof of Theorem 3.16.) From m_p we may compute an infinite sequence of compact elements $(a_n)_n$ in $D_{\mathscr{D}}$ such that $(a_n)_n \longrightarrow_{\delta_{\mathscr{D}}} \sigma$, where $\sigma \in \mathscr{D}$ satisfies $0 \leq \sigma(x) \leq 1$ for all x, $\sigma(x) = 1$ on $[-(m_p + 1), (m_p + 1)]$, and $\sigma(x) = 0$ outside $[-(m_p + 2), (m_p + 2)]$. This may be done as in Lemma 4.1.8 in [9].

[10] See [8].

From the sequence $(f_n, k_n, m_n)_n$ we may compute a sequence $(B_n, C_n)_n$ of compact elements in $D_{\mathrm{SI}(\mathbb{R})}$ such that $(B_n, C_n)_n \longrightarrow_{\delta_{\mathrm{SI}(\mathbb{R})}} f = \mathrm{refl}(\hat{u})$. This follows by Proposition 3.38 and Lemma 3.40. Let q be the least index n such that $C_n \neq \perp$ and $P(0, s, t) \in C_n$ for some $s, t \in \mathbb{N}$. It follows that $|f(x)| \leq s(1 + x^2)^t$ for all x, and so $(1 + x^2)^{-(t+2)}|f(x)| \leq s(1 + x^2)^{-2}$ for all x. We conclude that $g = (1 + x^2)^{-(t+2)} f$ is in $B(\mathbb{R})$. We let $N = t + 2$. Since multiplication on $C(\mathbb{R})$ is effective we can compute a sequence $(b_n)_n$ of compact elements in $D_{B(\mathbb{R})}$ such that $(b_n)_n \longrightarrow_{\delta_{B(\mathbb{R})}} (1 + x^2)^{-N} f$. Since the Fourier transform from $B(\mathbb{R})$ to $C(\mathbb{R})$ is effective by Lemma 3.41 we may now compute an infinite sequence $(c_n)_n$ of compact elements in $D_{C(\mathbb{R})}$ such that $(c_n)_n \longrightarrow_{\delta_{C(\mathbb{R})}} g = \mathcal{F}((1 + x^2)^{-N} f)$.

Finally, from $(c_n)_n$ we can compute sequences $(d_{0,n})_n$, $(d_{2,n})_n$, $\ldots$, $(d_{2N,n})_n$ of compact elements in $D_{C(\mathbb{R})}$ such that each of the sequences $(d_{2k,n})_n$ $\delta_{C(\mathbb{R})}$-converges to $g_{2k} = (-1)^k \binom{t+2}{k} g\sigma^{(2(t+2-k))}$, and $(d_{2k+1,n})_n$ $\delta_{C(\mathbb{R})}$-converges to 0. Since $u = \sum_{k=0}^{2N} g_k^{(k)}$ as above, it follows that $(e_n)_n$ given by $e_n = (d_{0,n}, d_{1,n}, \ldots, d_{2N,n})$ is a sequence of compact elements in D_{WC} which δ_{WC}-converges to u as required. □

To summarise, we have shown that each of the four different characterisations of the class of distributions with compact support given in Theorem 1.1 gives rise to an effective domain representation of $\mathscr{E}'$. Three of these, D_{UE}, D_{CS}, and D_{FO} are effectively equivalent, whereas the fourth representation D_{WC} is only Turing-equivalent (but not effectively equivalent) to the others. Thus, we have the following picture:

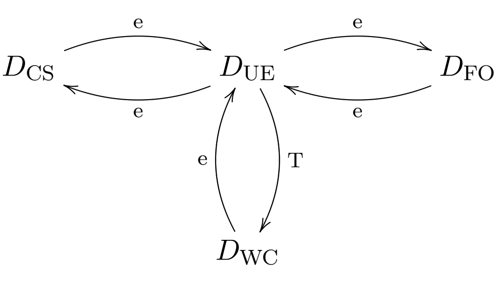


Figure 7. The reductions between the four domain representations of $\mathscr{E}'$.

The arrows marked “e” are effective reductions, and the arrow “T” denotes the Turing-reduction from D_{UE} to D_{WC}. Note that effective reductions compose to yield new effective reductions, whereas the composition of an effective reduction and a Turing-reduction yields a new Turing-reduction.

4 Domain representations of the space of distributions supported at the origin

The class $\mathscr{D}_0'$ of distributions with support in the singleton set $\{0\}$ forms another interesting subspace of the space of distributions. One can show that any distribution in $\mathscr{D}_0'$ can be written as a finite sum $\sum_{k=0}^{n} c_k \delta^{(k)}$. That is, we have the following structure theorem for the space of distribution with support in $\{0\}$.

Theorem 4.1. *Let u be a distribution. Then the following are equivalent:*

(SZ) The support of u is contained in the singleton set $\{0\}$.

(WD) There are complex numbers $c_1, c_2, \ldots, c_n$ such that $u = \sum_{k=0}^{n} c_k \delta^{(k)}$.

The abbreviation (SZ) is meant to stand for "Supported at Zero", and (WD) stands for "Weak derivatives of Dirac's δ-function".

The representation of u as a finite sum of weak derivatives of the distribution δ is unique in the following sense: Suppose that $\sum_{k=0}^{m} c_k \delta^{(k)} = \sum_{k=0}^{n} d_k \delta^{(k)}$ where $m \leq n$. Then $c_k = d_k$ for each $0 \leq k \leq m$ and $d_k = 0$ for each $k > m$. This fact will be used implicitly throughout this section.

Theorem 4.1 suggests two different ways in which one may choose to represent the elements in $\mathscr{D}_0'$. Since $\mathscr{D}_0'$ is a subspace of $\mathscr{D}'$ we may simply restrict the domain representation of $\mathscr{D}'$ to get an admissible and effective domain representation of $\mathscr{D}_0'$. Alternatively, we could take the characterisation (WD) as our starting point and represent the elements in $\mathscr{D}_0'$ as finite sums of weak derivatives of δ. In this section we will study the reducibility properties of the two domain representations arising from these two different ways of characterising the elements of the space $\mathscr{D}_0'$.

4.1 Representing $\mathscr{D}_0'$ as a subspace of $\mathscr{D}'$

Since $D_{\mathscr{D}'}$ is an effective admissible domain representation of $\mathscr{D}'$ we may restrict $D_{\mathscr{D}'}$ to $\mathscr{D}_0'$ to get an effective admissible domain representation of $\mathscr{D}_0'$.

Definition 4.2. Let D_{SZ} be the domain $D_{\mathscr{D}'}$ and let D_{SZ}^R be the set of all $f \in D_{\mathscr{D}'}^R$ which represent elements in $\mathscr{D}_0'$. Finally, we let $\delta_{\mathrm{SZ}} = \delta_{\mathscr{D}'}\restriction_{D_{\mathrm{SZ}}^R} : D_{\mathrm{SZ}}^R \to \mathscr{D}_0'$.

Since $D_{\mathrm{SZ}} = D_{\mathscr{D}'}$ and δ_{SZ} is the restriction of $\delta_{\mathscr{D}'}$ to D_{SZ}^R it follows immediately that $(D_{\mathrm{SZ}},\, D_{\mathrm{SZ}}^R,\, \delta_{\mathrm{SZ}})$ is an effective domain representation of $\mathscr{D}_0'$.

Theorem 4.3. *The domain representation D_{SZ} is admissible.*

Proof. Let D be a countably based domain, let $D^R \subseteq D$, and suppose that $f : D^R \to \mathscr{D}'_0$ is continuous. Since the inclusion $\mathscr{D}'_0 \hookrightarrow \mathscr{D}'$ is continuous and the representation $D_{\mathscr{D}'}$ is admissible we know that f factors through $\delta_{\mathscr{D}'}$ via some partial continuous function $\overline{f} : D \rightharpoonup D_{\mathscr{D}'}$.

By assumption f maps D^R into $\mathscr{D}'_0$, and so we must have $\overline{f}(x) \in D^R_{\mathrm{SZ}}$ for each $x \in D^R$. It follows that f factors through δ_{SZ} via $\overline{f}$. Since $(D,\, D^R,\, f)$ was arbitrary D_{SZ} is admissible. □

Let $(D,\, D^R,\, \delta)$ be an effective domain representation of $\mathscr{D}'_0$. We say that *evaluation (from $\mathscr{D}'_0 \times \mathscr{D}$ to $\mathbb{C}$) is D-effective* if the evaluation map $(u, \varphi) \mapsto u(\varphi)$ is $(D \times D_{\mathscr{D}},\, D_{\mathbb{C}})$-effective. We say that *type conversion preserves D-effectivity* (or alternatively, that *D-effectivity is preserved under type conversion*) if for every topological space X, every effective domain representation $(D_X,\, D^R_X,\, \delta_X)$ of X, and every $(D_X \times D_{\mathscr{D}},\, D_{\mathbb{C}})$-effective function $f : X \times \mathscr{E} \to \mathbb{C}$ such that $\varphi \mapsto f(x, \varphi)$ is in $\mathscr{D}'_0$ for each $x \in X$, the transpose $f^* : X \to \mathscr{D}'_0$ of f is (D_X, D)-effective.

As was the case for the admissible representation D_{UE} in the previous section, we may give alternative characterisations of the relations "$\ldots \leq_{\mathrm{e}} D_{\mathrm{SZ}}$" and "$D_{\mathrm{SZ}} \leq_{\mathrm{e}} \ldots$" in terms of evaluation and type conversion. We begin by showing that evaluation is D_{SZ}-effective, and that type conversion preserves D_{SZ}-effectivity.

Proposition 4.4. *Evaluation $(u, \varphi) \mapsto u(\varphi)$ is D_{SZ}-effective.*

Proof. This follows immediately since evaluation from $\mathscr{D}' \times \mathscr{D}$ to $\mathbb{C}$ is effective. □

Proposition 4.5. *Type conversion preserves D_{SZ}-effectivity.*

Proof. Let X be a topological space and let $(D_X,\, D^R_X,\, \delta_X)$ be an effective domain representation of X. Let $f : X \times \mathscr{D} \to \mathbb{C}$ be an effective map from $D \times \mathscr{D}$ to $\mathbb{C}$, and suppose that $\varphi \mapsto f(x, \varphi)$ is in $\mathscr{D}'_0$ for each $x \in X$. Let $\overline{f} : D_X \times D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}$ be an effective representation of f.

By an application of Proposition 4.1.9 in [9] there is an effective partial continuous function $g : D_X \rightharpoonup D_{\mathscr{D}'}$ which represents the transpose $f^* : X \to \mathscr{D}'$ of f. Since g represents the transpose of f we must have $g(x) \in D^R_{\mathrm{SZ}}$ for each $x \in D^R_X$. It follows that $g : D_X \rightharpoonup D_{\mathrm{SZ}}$ represents $f^* : X \to \mathscr{D}'_0$. □

We now go on to the promised characterisation of the the class of representations which are effectively reducible to D_{SZ}.

Proposition 4.6. *Let D be an effective domain representation of $\mathscr{D}'_0$. Then $(u,\varphi) \mapsto u(\varphi)$ is D-effective if and only if $D \leq_{\mathrm{e}} D_{\mathrm{SZ}}$.*

Proof. The proof is similar to the proof of Proposition 3.6, and so we leave it out. □

As before, we have $D_{\mathrm{SZ}} \leq_{\mathrm{e}} D$ precisely when D-effectivity is preserved under type conversion.

Proposition 4.7. *Let D be an effective domain representation of $\mathscr{D}'_0$. Then D-effectivity is preserved under type conversion if and only if $D_{\mathrm{SZ}} \leq_{\mathrm{e}} D$.*

Proof. The proof is identical to the proof of Proposition 3.7, □

Thus, we have the following characterisation of the class of effective domain representations of $\mathscr{D}'_0$ which are effectively equivalent to the representation D_{SZ}.

Corollary 4.8. *Let D be an effective domain representation of $\mathscr{D}'_0$. Then $D \equiv_{\mathrm{e}} D_{\mathrm{SZ}}$ if and only if evaluation is D-effective and type conversion preserves D-effectivity.*

The support of each element in $\mathscr{D}'_0$ is contained in the interval $[-1,1]$. It follows that the map $i : D_{\mathrm{SZ}} \to D_{\mathrm{CS}}$ given by $f \mapsto (f,1)$ is an effective representation of the inclusion map $\mathscr{D}'_0 \hookrightarrow \mathscr{E}'$.

Proposition 4.9. *The inclusion map $\mathscr{D}'_0 \hookrightarrow \mathscr{E}'$ is $(D_{\mathrm{SZ}}, D_{\mathrm{CS}})$-effective.*

Since evaluation from $\mathscr{E}' \times \mathscr{E}$ to $\mathbb{C}$ is D_{CS}-effective we immediately have the following corollary.

Corollary 4.10. *The evaluation map from $\mathscr{D}'_0 \times \mathscr{E}$ to $\mathbb{C}$ is D_{SZ}-effective.*

4.2 Representing elements in $\mathscr{D}'_0$ as finite sums of weak derivatives of Dirac's delta-function

By the structure theorem for the class of distributions supported at 0 we know that every element in $\mathscr{D}'_0$ can be written as a finite sum of weak derivatives of $\delta \in \mathscr{D}'$, where δ is the Dirac distribution given by $\delta(\varphi) = \varphi(0)$. To show that this equivalence has effective content, we need to construct a domain representation of $\mathscr{D}'_0$ which encodes the information implicit in the structure theorem, and then show that this representation is effectively equivalent (or Turing-equivalent) to D_{SZ}.

Definition 4.11. Let D_{WD} be the set of all finite sequences $(J_0, J_1, \ldots, J_n)$ where $J_k \in D_{\mathbb{C}}$ for each k together with a distinguished element $\perp$. We define an information order on D_{WD} as the transitive closure $\sqsubseteq$ of the relation R on D_{WD} given by

1. $(I_0, I_1, \ldots, I_n)\, R\, (J_0, J_1, \ldots, J_n)$ if $I_k \sqsubseteq J_k$ in $D_{\mathbb{C}}$ for each $0 \leq k \leq n$.

2. $(I_0, I_1, \ldots, I_n)\, R\, (J_0, J_1, \ldots, J_m)$ if $m < n$, $I_k \sqsubseteq J_k$ for all $0 \leq k \leq m$, and $I_k \prec 0$ for all $m < k \leq n$.

3. $\perp \sqsubseteq (J_0, J_1, \ldots, J_n)$ for all $n \in \mathbb{N}$ and $J_0, J_1, \ldots, J_n \in D_{\mathbb{C}}$.

Intuitively, we think of a finite sequence $(J_0, J_1, \ldots, J_n)$ as an approximation of $u = \sum_{k=0}^{n} c_k \delta^{(k)}$ if $J_k \prec c_k$ for each k. Note that, if $m < n$ and $J_k \prec 0$ for each $m < k \leq n$, then $(J_1, J_2, \ldots, J_m)$ contains more information than $(J_1, J_2, \ldots, J_n)$. This is part of the motivation for the second line in the definition of $\sqsubseteq$ above. A more pragmatic motivation for including the second line in the definition of the order relation on D_{WD} is to counteract the argument used to show that D_{UE} cannot be continuously reducible to D_{WC}. More precisely, if $(I_0, I_1, \ldots, I_n) \sqsubseteq (J_0, J_1, \ldots, J_m)$ always implied that $m = n$, then $o : D_{\mathrm{WD}} \to D_{\mathbb{N}}$ given by $o(I_0, I_1, \ldots, I_n) = n$ is an effective continuous function and $o(f)$ is an upper bound on the order of $\delta_{\mathrm{WD}}(f)$ for each $f \in D_{\mathrm{WD}}^R$. Since there cannot be such a function from D_{SZ} (for exactly the same reasons as in the case of D_{UE} above), we could never expect to be able to reduce D_{SZ} to D_{WD} in this case.

Proposition 4.12. $(D_{\mathrm{WD}}, \sqsubseteq)$ *is an effective domain.*

Proof. This follows since D_{WD} is the ideal completion of the cusl $(B_{\mathrm{WD}}, \sqsubseteq)$ where $B_{\mathrm{WD}} = \{(S_0, S_1, \ldots, S_n);\ n \in \mathbb{N}$ and $S_0, S_1, \ldots, S_n$ are compact $\} \cup \{\perp\}$, and $\sqsubseteq$ is the order relation defined above restricted to B_{WD}. □

We may now define a domain representation of $\mathscr{D}_0'$ over D_{WD} as follows:

Definition 4.13. Let D_{WD}^R be the set of all tuples $(J_0, J_1, \ldots, J_n) \in D_{\mathrm{WD}}$ such that $J_k \in D_{\mathbb{C}}^R$ for each $0 \leq k \leq n$, and let $\delta_{\mathrm{WD}} : D_{\mathrm{WD}}^R \to \mathscr{D}_0'$ be the function given by $\delta_{\mathrm{WD}}(J_0, J_1, \ldots, J_n) = \sum_{k=0}^{n} c_k \delta^{(k)}$ where J_k represents c_k for each $0 \leq k \leq n$.

Theorem 4.14. $(D_{\mathrm{WD}}, D_{\mathrm{WD}}^R, \delta_{\mathrm{WD}})$ *is an effective domain representation of* $\mathscr{D}_0'$.

Proof. As usual it is enough to show that δ_{WD} is sequentially continuous. Suppose that $(r_n)_n \longrightarrow r$ in D^R_{WD}, where $r_n = (J_{0,n}, J_{1,n}, \ldots, J_{m_n,n})$ and $r = (J_0, J_1, \ldots, J_m)$. Let $\perp_m = (\perp, \perp, \ldots, \perp)$ ($m+1$ times). Since $r \in D^{m+1}_{\mathbb{C}}$ we have $\perp_m \sqsubseteq r$, and since $(r_n)_n$ converges to r we must have $\perp_m \sqsubseteq r_n$ for almost all n. It follows that $m_n \leq m$ for almost all n. Since $m_n \leq m$ for almost all n we may assume that $m_n \leq m$ for all n. We let $c_{k,n} = \delta_{\mathbb{C}}(J_{k,n})$ if $1 \leq k \leq m_n$, and let $c_{k,n} = 0$ for each $m_n < k \leq m$. Finally, we let $c_i = \delta_{\mathbb{C}}(J_i)$. We would like to show that $(c_{k,n})_n \longrightarrow c_k$ for each $0 \leq k \leq m$.

Fix $k \leq m$. Now, either $k \leq m_n$ for almost all n or $m_n < k$ for infinitely many n. Suppose that $k \leq m_n$ for almost all n. In this case we may safely assume that $k \leq m_n$ for all natural numbers n. Since $(r_n)_n \longrightarrow r$ in D_{WD} we must have $(J_{k,n})_n \longrightarrow J_k$. Since $\delta_{\mathbb{C}}$ is continuous we immediately have $(c_{k,n}) \longrightarrow c_k$.

Suppose now that $m_n < k$ for infinitely many n. Let $S \in J_k$ and let $a = (\perp, \ldots, \perp, S, \perp, \ldots, \perp) \in D^{m+1}_{\mathbb{C}}$ (with S in the k:th position). Then $a \sqsubseteq r$, and so $a \sqsubseteq r_n$ for almost all n. It follows that $a \sqsubseteq r_n$ for some n such that $m_n < k$. We conclude that $S \prec 0$, and hence, that $c_k = 0$. Now, either $k \leq m_n$ for infinitely many n, or $m_n < k$ for almost all n. Suppose that $k \leq m_n$ for infinitely many n. Let $j : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function which enumerates the set $\{n \in \mathbb{N};\ k \leq m_n\}$ and let $(s_n)_n$ be the subsequence of $(r_n)_n$ given by $s_n = r_{j(n)}$. Then $(s_n)_n \longrightarrow r$, and $(J_{k,j(n)})_n \longrightarrow J_k$ as before. It follows that $(c_{k,j(n)})_n \longrightarrow c_k = 0$. Since $c_{k,n} = 0$ whenever $k > m_n$ we have $(c_{k,n})_n \longrightarrow c_k$. Finally, if $m_n < k$ for almost all n we have $c_{k,n} = 0$ for almost all n, and so $(c_{k,n})_n \longrightarrow 0 = c_k$. It follows that $(c_{k,n})_n \longrightarrow c_k$ for each $0 \leq k \leq m$.

Let $u_n = \delta_{\mathrm{WD}}(r_n)$, and let $u = \delta_{\mathrm{WD}}(u)$. To show that $(u_n)_n \longrightarrow u$ in $\mathscr{E}'$ it is enough to show that $u_n(\varphi) \longrightarrow u(\varphi)$ for each test function φ. Thus, suppose that $\varphi \in \mathscr{D}$. Then

$$(u_n(\varphi))_n = \Big(\sum_{k=0}^{m_n} (-1)^k c_{k,n} \varphi^{(k)}(0)\Big)_n =$$

$$\Big(\sum_{k=0}^{m} (-1)^k c_{k,n} \varphi^{(k)}(0)\Big)_n \longrightarrow \sum_{k=0}^{m} (-1)^k c_k \varphi^{(k)}(0) = u(\varphi).$$

Since $\varphi \in \mathscr{D}$ was arbitrary we conclude that $(u_n)_n \longrightarrow u$ in $\mathscr{D}'_0$. It follows immediately that $\delta_{\mathrm{WD}} : D_{\mathrm{WD}} \to \mathscr{D}_0$ is continuous. □

To show that D_{WD} is effectively reducible to D_{SZ} it is enough to show that evaluation from $\mathscr{D}'_0 \times \mathscr{D}$ to $\mathbb{C}$ is effective with respect to the representation

D_{WD}. This essentially follows since the evaluation map $\varphi \mapsto \varphi(0)$ is effective with respect to $D_{\mathscr{D}}$.

Proposition 4.15. *Evaluation from $\mathscr{D}'_0 \times \mathscr{D}$ to $\mathbb{C}$ is effective with respect to the representation D_{WD}.*

Proof. Let $\mathrm{ev}_k : D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}$ be an effective partial continuous representation of the map $\varphi \mapsto (-1)^k \varphi^{(k)}(0)$, and let $a, m : D_{\mathbb{C}} \times D_{\mathbb{C}} \mapsto D_{\mathbb{C}}$ be effective partial continuous functions representing addition and multiplication on $\mathbb{C}$. We write $a_n : D_{\mathbb{C}}^n \rightharpoonup D_{\mathbb{C}}$ for the partial continuous function $(J_0, J_2, \ldots, J_n) \mapsto a(\ldots a(a(J_0, J_1),\ J_2), \ldots,\ J_n)$. It is clear that a_n is effective (uniformly in n), and that a_n represents the function $(c_0, c_2, \ldots, c_n) \mapsto \sum_{k=0}^{n} c_k$ on $\mathbb{C}$.

Now, let $c = (S_0, S_2, \ldots, S_n)$ be compact in D_{WD} and let $d = (B, k)$ be a compact element in $D_{\mathscr{D}}^D$, the dense part of $D_{\mathscr{D}}$. We define $\mathrm{ev}(c, d)$ as

$$\mathrm{ev}(c, d) = a_n(m(S_0, \mathrm{ev}_0(B, k)),$$

$$m(S_1, \mathrm{ev}_1(B, k)),\ \ldots,\ m(S_n, \mathrm{ev}_n(B, k))).$$

It is routine to show that ev is monotone, using the fact that $a_n(S_0, S_2, \ldots, S_n) \sqsubseteq a_k(S_0, S_2, \ldots, S_k)$ whenever $k \leq n$ and $S_i \prec 0$ for each $k < i \leq n$. Let $\mathrm{ev} : D_{\mathrm{WD}} \times D_{\mathscr{D}} \rightharpoonup D_{\mathbb{C}}$ denote the continuous extension of ev to $D_{\mathrm{WD}} \times D_{\mathscr{D}}^D$. The function ev is clearly effective. Suppose that $r = (J_0, J_2, \ldots, J_n)$ is a representation of $u = \sum_{k=0}^{n} c_k \delta^{(k)}$ in D_{WD}, and let L be a representation of φ in $D_{\mathscr{D}}$. Then $\mathrm{ev}_k(L)$ represents $(-1)^k \varphi^{(k)}(0)$ in $D_{\mathbb{C}}$ and $\mathrm{ev}(r, L)$ represents

$$\sum_{k=0}^{n} c_k (-1)^k \varphi^{(k)}(0) = \left\langle \sum_{k=0}^{n} c_k \delta^{(k)},\ \varphi \right\rangle = \langle u, \varphi \rangle.$$

It follows that ev represents the evaluation map $(u, \varphi) \mapsto u(\varphi)$ from $\mathscr{D}'_0 \times \mathscr{D}$ to $\mathbb{C}$. □

By the characterisation of the relation "$\ldots \leq_{\mathrm{e}} D_{\mathrm{SZ}}$" above, we now have the following corollary.

Corollary 4.16. *D_{WD} is effectively reducible to D_{SZ}.*

To show that D_{SZ} is effectively reducible to D_{WD} we need to show that we can compute (representations of) the constants $c_0, c_1, \ldots, c_n$ from (a representation of) $u = \sum_{k=0}^{n} c_k \delta^{(k)}$ (in D_{SZ}). To compute $c_0, c_1, \ldots, c_n$ we will use the following lemma:

Lemma 4.17. *There is an effective sequence $(\sigma_k)_k$ of smooth functions such that $\sigma_k^{(k)}(x) = 1$ on an open neighbourhood of 0 for each k.*

Proof. There is a computable element $\sigma_0 \in \mathscr{E}$ such that $\text{supp}(\sigma_0) \subseteq [-1, 1]$ and $\sigma_0(x) = 1$ on a neighbourhood of 0. σ_0 may be constructed as in Lemma 4.1.8 in [9]. Now, given $\sigma_k \in \mathscr{E}$ such that $\text{supp}(\sigma_k) \subseteq [-1, \infty)$ and $\sigma_k^{(k)}(x) = 1$ on some open neighbourhood U of 0, we let $\sigma_{k+1}(x) = \int_{-1}^{x} \sigma_k(t)dt$. Then $\text{supp}(\sigma_{k+1}) \subseteq [-1, \infty)$, and $\sigma_{k+1}^{(k+1)}(x) = \sigma_k^{(k)}(x) = 1$ on U as required. The map $k \mapsto \sigma_k$ is effective since σ_0 is computable and integration on $\mathscr{E}$ is effective by Proposition 3.4.1 in [9]. □

Now, given the sequence $(\sigma_k)_k$ we may compute (representations of) the constants $c_0, c_1, \ldots, c_n$ from (a representation of) $u = \sum_{k=0}^{n} c_k \delta^{(k)}$ by noting that for each natural number $m \leq n$ we have

$$u(\sigma_m) = \Big\langle \sum_{k=0}^{n} c_k \delta^{(k)}, \sigma_m \Big\rangle = \sum_{k=0}^{n} (-1)^k c_k \sigma_m^{(k)}(0) =$$

$$\sum_{k=0}^{m-1} (-1)^k c_k \sigma_m^{(k)}(0) + (-1)^m c_m \sigma_m^{(m)}(0) + \sum_{k=m+1}^{n} (-1)^k c_k \sigma_m^{(k)}(0) =$$

$$\sum_{k=0}^{m-1} (-1)^k c_k \sigma_m^{(k)}(0) + (-1)^m c_m + 0.$$

In the last identity we used the fact that σ_0 is constant on an open neighbourhood of 0. It follows that $\sigma_m^{(k)}(0) = 0$ for each $k > m$. By rearranging the terms we get

$$c_m = (-1)^m \big(u(\sigma_m) - \sum_{k=0}^{m-1} (-1)^k c_k \sigma_m^{(k)}(0) \big)$$

for each distribution $u = \sum_{k=0}^{n} c_k \delta^{(k)}$ and each natural number $m \leq n$.

The constants $c_0, c_1, \ldots, c_n$ correspond (in some informal way) to the compactly supported continuous functions $g_0, g_1, \ldots, g_n$ in the structure theorem for the space of compactly supported distributions. Now, one of the key reasons why D_{UE} was only Turing-reducible and not effectively reducible to D_{WC} was that *each* of the functions $g_0, g_1, \ldots, g_n$ depended explicitly on n, and so if the upper bound n changed, each function g_m changed as well. We note that the value of c_m *does not* depend on n in this way. This independence of n will allow us to construct an effective reduction from D_{SZ} to D_{WD}.

Proposition 4.18. *D_{SZ} is effectively reducible to D_{WD}.*

Proof. For each k, we let $c_k : D_{\text{SZ}} \rightharpoonup D_{\mathbb{C}}$ be an effective representation of the map $u \mapsto (-1)^k \big(u(\sigma_k) - \sum_{j=0}^{k-1} (-1)^j c_j \sigma_k^{(j)}(0) \big)$. The partial function c_k

exists since the sequence $(\sigma_k)_k$ is effective and evaluation from $\mathscr{D}'_0 \times \mathscr{E}$ to $\mathbb{C}$ is effective with respect to the representation D_{SZ} of $\mathscr{D}'_0$.

Since the inclusion map $\mathscr{D}'_0 \hookrightarrow \mathscr{E}'$ is $(D_{\mathrm{SZ}}, D_{\mathrm{CS}})$-effective, and D_{CS} is effectively equivalent in D_{FO} there is an effective partial continuous function $o : D_{\mathrm{SZ}} \rightharpoonup D_S$ such that $D^R_{\mathrm{SZ}} \subseteq \mathrm{dom}(o)$ and $o(f)$ is an upper bound on the order of $\delta_{\mathrm{SZ}}(f)$ for each $f \in D^R_{\mathrm{SZ}}$. Now, let $f \in D^R_{\mathrm{SZ}}$ and suppose that $\delta_{\mathrm{SZ}}(f) = \sum_{k=0}^n c_k \delta^{(k)}$. Then c_k must be equal to 0 for each $k > o(f)$ since the order of u is bounded above by $o(f)$. (This follows by the same argument as in Lemma 3.32.)

We are now ready to define a partial continuous function r reducing D_{SZ} to D_{WD}. Let $\mathrm{dom}(r) = D^D_{\mathrm{SZ}}$, and for each $f \in \mathrm{dom}(r)$ such that $o(f) = n \neq \bot$ we let $r(f) = (c_0(f), c_1(f), \dots, c_n(f))$. Finally, if $f \in \mathrm{dom}(r)$ and $o(f) = \bot$ we let $r(f) = \bot$.

To show that r is monotone, suppose that $f \sqsubseteq g$ in D^R_{SZ}. If $o(f) = \bot$ it is clear that $r(f) \sqsubseteq r(g)$ so suppose that $o(f) = m$ for some $m \in \mathbb{N}$. Then $o(g) = k \leq m$ for some $k \in \mathbb{N}$. It is clear that $c_j(f) \sqsubseteq c_j(g)$ for each $0 \leq j \leq k$. Thus, to show that $r(f) \sqsubseteq r(g)$ it is enough to show that $c_j(f) \prec 0$ for each $k < j \leq m$. Since g lies in the dense part of D_{SZ} there is some distribution $u = \sum_{j=0}^n c_j \delta^{(j)}$ in $\mathscr{D}'_0$ such that $g \prec u$. It follows that the order of u is bounded above by k, and so $c_j = 0$ for each $k < j \leq n$. Since $f \prec u$ we have $c_j(f) \prec 0$ for each $k < j \leq m$ as required.

To show that r is continuous, let A be a directed subset of D^D_{SZ}. If $o(\bigsqcup A) = \bot$ we have $o(f) = \bot$ for each $f \in A$ and $r(\bigsqcup A) = \bigsqcup r[A]$ as required. Thus, we may assume that $o(\bigsqcup A) = n$ for some $n \in \mathbb{N}$. Since the set $\{f \in A;\ o(f) = o(\bigsqcup A)\}$ is cofinal in A, we may assume that $o(f) = n$ for each $f \in A$. We now have

$$r(\bigsqcup A) = (c_0(\bigsqcup A), c_1(\bigsqcup A), \dots, c_n(\bigsqcup A)) =$$

$$(\bigsqcup c_0[A], \bigsqcup c_1[A], \dots, \bigsqcup c_n[A]) = \bigsqcup r[A],$$

and r is continuous. We also note that r is effective since o is effective and c_k is effective uniformly in k.

To show that r represents the identity on $\mathscr{D}'_0$, let f represent the distribution $u = \sum_{k=0}^n c_k \delta^{(k)}$. in D_{SZ}. Since $f \in D^R_{\mathrm{SZ}}$ we must have $o(f) \neq \bot$. Suppose first that $o(f) = m \leq n$. Since m is an upper bound on the order of u we know that $c_k = 0$ for each $m < k \leq n$. Since $o(f) \neq \bot$ we have $r(f) = (c_0(f), c_1(f), \dots, c_m(f))$. By the definition of the function $c_k : D_{\mathrm{SZ}} \rightharpoonup D_{\mathbb{C}}$ we know that $c_k(f)$ represents $c_k \in \mathbb{C}$ for each $0 \leq k \leq m$. We conclude that $r(f)$ represents $\sum_{k=0}^m c_k \delta^{(k)} = \sum_{k=0}^n c_k \delta^{(k)} = u$. If $m \geq n$ we must have $\delta_{\mathbb{C}}(c_k(f)) = 0$ for each $n < k \leq m$ since the order of u is bounded above by n. It now follows as before that $r(f)$ represents u in D_{WD}. □

Thus, we immediately have the following corollary:

Corollary 4.19. *The representations D_{SZ} and D_{WD} are effectively equivalent.*

5 Concluding remarks

In this paper we have studied the effective content of the structure theorem for the space of distributions with compact support. This was done as follows:

1. To be able to study the effective content of the structure theorem for $\mathscr{E}'$ we coded the information implicit in each of the (classically equivalent) characterisations $(\star)$ in the structure theorem as an effective domain representation $D_\star$ of $\mathscr{E}'$.
2. We could then try to determine the effective content of the structure theorem by studying the reducibility properties of these four different domain representations. Intuitively, if $D_\star$ is effectively reducible (or Turing-reducible) to $D_\dagger$, we conclude that the corresponding implication $(\star) \implies (\dagger)$ is effective.

The domain representations D_{UE}, D_{CS}, and D_{FO} are all effectively equivalent. It follows that the equivalence $(\mathrm{UE}) \iff (\mathrm{CS}) \iff (\mathrm{FO})$ is effective. The representation D_{WC} is not effectively equivalent to the others since D_{UE} is not effectively reducible to D_{WC}. However, we can still translate a sequence of compact elements in D_{UE} which converges to a representation of $u \in \mathscr{E}'$ into a sequence of compact elements in D_{WC} which converges to a representation of the same distribution u. This translation is not Scott-continuous (it is not even extensional since two different sequences converging to the same representing element in D_{UE} may be sent to different sequences which converge to *different* representing elements in D_{WC}), but it is still computable since the translation can be carried out in a uniform way by a Turing machine. It follows that we can "compute a representation of u in D_{WC} from a representation of u in D_{UE}" and vice versa. (That is, given the ability to approximate a representation of u in D_{UE} to an arbitrary degree of precision, we can approximate a representation of u in D_{WC} to an arbitrary degree of precision as well, and vice versa.) We thus conclude that the equivalence $(\mathrm{UE}) \iff (\mathrm{WC})$ has effective content as well.

For the space of distributions with support contained in the singleton set $\{0\}$ the situation is simpler. The two representations D_{SZ} and D_{WD} corresponding to the two characterisations (SZ) and (WD) of the space of distributions supported at 0 are effectively equivalent. It follows that the equivalence $(\mathrm{SZ}) \iff (\mathrm{SZ})$ is effective as well. We note that one important reason why D_{SZ} was effective reducible to D_{WD} (as opposed to only being Turing-reducible to D_{WD}) was the fact that each distribution u with support in $\{0\}$ has a *unique* representation as a sum of weak derivatives of Dirac's δ-function. In particular, the constants $c_0, c_1, \ldots, c_n$ in the representation $u = \sum_{k=0}^{n} c_k \delta^{(k)}$ are independent of the order of the distribution u. This was not the case in the structure theorem for the space of distributions with compact support.

Another important insight in the second part of the paper was the realisation that we could not construct the domain representation D_{WD} over the disjoint sum of the domains $(D_{\mathbb{C}}^{m})_m$ if we wanted D_{SZ} to be effectively reducible to D_{WD}. We note that the same ideas could be employed in the construction of the domain representation D_{WC} in Section 3.4. It is easy to show that the domain representation of $\mathscr{E}'$ constructed in this way would be effectively reducible to D_{UE}, but it is still unknown if it would be effectively equivalent to to the representation D_{UE}.

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