

# Effective equations for two-phase flow with trapping on the micro scale

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## EFFECTIVE EQUATIONS FOR TWO-PHASE FLOW WITH TRAPPING ON THE MICRO SCALE\*

C. J. VAN DUIJN<sup>†</sup>, A. MIKELIĆ<sup>‡</sup>, AND I. S. POP<sup>†</sup>

**Abstract.** In this paper we consider water-drive for recovering oil from a strongly heterogeneous porous column. The two-phase model uses Corey relative permeabilities and Brooks–Corey capillary pressure. The heterogeneities are perpendicular to the flow and have a periodic structure. This results in one-dimensional flow and a space periodic absolute permeability, reflecting alternating coarse and fine layers. Assuming many—or thin—layers, we use homogenization techniques to derive the effective transport equations. The form of these equations depends critically on the capillary number. The analysis is confirmed by numerical experiments.

**Key words.** homogenization, porous media flow, degenerate parabolic equations, Buckley–Leverett equation

**AMS subject classifications.** 35B27, 76M50, 76S05

**PII.** S0036139901385564

**1. Introduction.** A widely used technique for removing oil from reservoirs is water-drive. Water is pumped through injection wells into the reservoir, driving the oil to the production wells.

The presence of rock heterogeneities in the reservoir generally has an unfavorable effect on the recovery rate. For instance, when preferential paths (high permeability regions) exist from injection to production wells, much oil will be bypassed, and consequently the oil recovery rate will be small. Conversely, when rock heterogeneities occur perpendicular to the flow from injection to production wells (so-called cross-bedded or laminated structures), oil may be trapped at the interface between high and low permeability and become inaccessible to the flow, leading again to a reduction in recovery rate. This latter case was studied by Kortekaas [20], van Duijn, Molenaar, and de Neef [14], and more recently by van Lingen [22], who performed laboratory experiments using a porous column with periodically varying permeability; see Figure 1.1. In the same context, steady state solutions as well as an averaging procedure were considered by Dale and colleagues [10], [11].

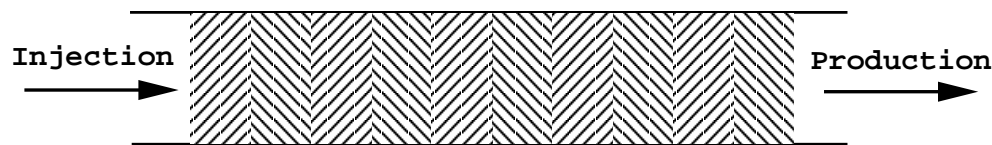


FIG. 1.1. Periodically varying porous medium with high (coarse) and low (fine) permeability layers.

The main purpose of this paper is to derive in a rational way the effective flow

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<sup>†</sup>Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands (C.J.v.Duijn@tue.nl, I.Pop@tue.nl).

<sup>‡</sup>Laboratoire d'Analyse Numérique, Université Lyon 1, 69622 Villeurbanne CEDEX, France (andro@lan.univ-lyon1.fr).

equations corresponding to Figure 1.1 when the ratio of the micro scale (periodicity length) to the macro scale (column length) is small.

To this end we consider a one-dimensional flow of two immiscible and incompressible phases (water being the wetting phase, oil the nonwetting phase) through a heterogeneous porous medium characterized by a constant porosity  $\Phi$  and a variable absolute permeability  $k = k(x)$ . The underlying equations describe the mass balance for the phases

$$(1.1) \quad \Phi \frac{\partial S_\alpha}{\partial t} + \frac{\partial q_\alpha}{\partial x} = 0 \quad (\alpha = o, w),$$

the momentum balance for the phases (Darcy law)

$$(1.2) \quad q_\alpha = -k(x) \frac{k_{r\alpha}(S_\alpha)}{\mu_\alpha} \frac{\partial p_\alpha}{\partial x},$$

and the complementary conditions

$$(1.3) \quad S_o + S_w = 1,$$

$$(1.4) \quad p_o - p_w = p_c(x, S_w).$$

Here  $S_\alpha, q_\alpha, k_{r\alpha}, \mu_\alpha$ , and  $p_\alpha$  denote, respectively, the saturation, specific discharge, relative permeability, viscosity, and pressure of phase  $\alpha$ . Throughout, we assume that the phase saturations are normalized, i.e.,  $0 \leq S_\alpha \leq 1$ . Condition (1.3) expresses the presence of only two phases. The phase pressures differ due to interface tension on the pore scale. This is expressed by (1.4), where  $p_c$  denotes the induced capillary pressure. In petroleum engineering it is usually described by the Leverett model (see Leverett [21] or Bear [2]):

$$(1.5) \quad p_c(x, S_w) = \sigma \sqrt{\frac{\Phi}{k(x)}} J(S_w),$$

where  $\sigma$  denotes the interfacial tension and  $J$  the Leverett function. The relative permeabilities  $k_{r\alpha} : [0, 1] \rightarrow [0, \infty)$  and the Leverett function  $J : (0, 1] \rightarrow [0, \infty)$  are assumed to be smooth generalizations of power law functions (see Corey [9] and Brooks and Corey [7]) satisfying the structural properties:

A<sub>1</sub>:  $k_{r\alpha}$  strictly increasing in  $[0, 1]$  with  $k_{r\alpha}(0) = 0$ ,

A<sub>2</sub>:  $J(0+) = \infty$ ,  $J(1) > 0$ , and  $J' < 0$  in  $(0, 1]$ ,

where the prime denotes differentiation.

Here we explicitly assume  $J(1) > 0$ . Physically this means that a certain pressure, the capillary entry pressure given by  $p_c(x, 1)$ , has to be exerted on the oil before it can enter a fully water-saturated medium.

Equation (1.1) and condition (1.3) imply that the total specific discharge  $q := q_o + q_w$  is constant in space. Throughout this paper we consider the discharge to be constant in time as well. With  $q > 0$  given, (1.1), (1.2) and conditions (1.3), (1.4) can be combined into a single transport equation for a single saturation. Since we are primarily interested in the oil flow, we use the oil saturation for that purpose. In doing so, it is convenient to redefine  $k_{rw}$ ,  $p_c$ , and  $J$  in terms of  $S_o$ . Setting

$$u = S_o \quad (S_w = 1 - u),$$

we now write

$$\begin{aligned}k_{rw}(u) &:= k_{rw}(1-u), \\ p_c(x, u) &:= p_c(x, 1-u), \\ J(u) &:= J(1-u).\end{aligned}$$

In terms of  $u$ , assumptions  $A_{1-2}$  become

$$\tilde{A}_1 : \begin{cases} k_{rw} \text{ strictly decreasing in } [0, 1] \text{ with } k_{rw}(1) = 0, \\ k_{ro} \text{ strictly increasing in } [0, 1] \text{ with } k_{ro}(0) = 0, \end{cases}$$

$$\tilde{A}_2 : \quad J(1-) = \infty, \quad J(0) > 0, \quad \text{and } J' > 0 \text{ in } [0, 1].$$

*Remark 1.1.* In most cases of practical interest the blow-up of  $J$  and  $J'$  near  $u = 1$  is balanced by the behavior of  $k_{rw}$  near that point, in the sense that  $k_{rw}(u)J'(u) \rightarrow 0$  as  $u \rightarrow 1$ . The consequence of this behavior and its possible failure is investigated by van Duijn and Floris [13]. Though important for the well-posedness of the mathematical formulation, no additional assumptions are required for the purpose of this paper.

Applying the scalings

$$(1.6) \quad x := \frac{x}{L_x}, \quad t := \frac{t q}{\Phi L_x}, \quad \text{and } k := \frac{k}{K},$$

where  $L_x$  is a characteristic macroscopic length scale and  $K$  a characteristic permeability value, we find for the oil saturation the balance equation

$$(1.7a) \quad \frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where

$$(1.7b) \quad F = f(u) - N_c k(x) \lambda(u) \frac{\partial}{\partial x} p_c(x, u).$$

Here

$$(1.8) \quad f(u) = \frac{k_{ro}(u)}{k_{ro}(u) + M k_{rw}(u)}$$

denotes the oil fractional flow function, and

$$(1.9) \quad \lambda(u) = k_{rw}(u) f(u), \quad p_c(x, u) = \frac{J(u)}{\sqrt{k(x)}}.$$

The two dimensionless numbers involved are the capillary number  $N_c$  and the viscosity ration  $M$ . They are given by

$$(1.10) \quad N_c = \frac{\sigma \sqrt{K \Phi}}{\mu_w q L_x} \quad \text{and} \quad M = \frac{\mu_o}{\mu_w}.$$

*Remark 1.2.* (i) Assumptions  $\tilde{A}_{1-2}$  imply

$$f(0) = 0, f(1) = 1, \quad \text{and} \quad f \text{ strictly increasing in } [0, 1],$$

$$\lambda(0) = \lambda(1) = 0 \quad \text{and} \quad \lambda(u) > 0 \text{ for } 0 < u < 1.$$

(ii) Depending on the specific application, the value of the capillary number may vary considerably. For instance, adding surfactants or polymers may substantially alter  $\sigma$  or  $\mu_w$ . Likewise, the flow rate  $q$  can have different values. Therefore we investigate in section 2 the consequences of having a moderate and a small value for  $N_c$ .

(iii) Petroleum engineers define the capillary number (1.10) in the reciprocal way, i.e.,  $N_c = \frac{\mu_w q L_\sigma}{\sigma \sqrt{K\Phi}}$ . Here we do not adopt this convention, because we want to emphasize the direct proportionality between the capillary number and the interface tension  $\sigma$ .

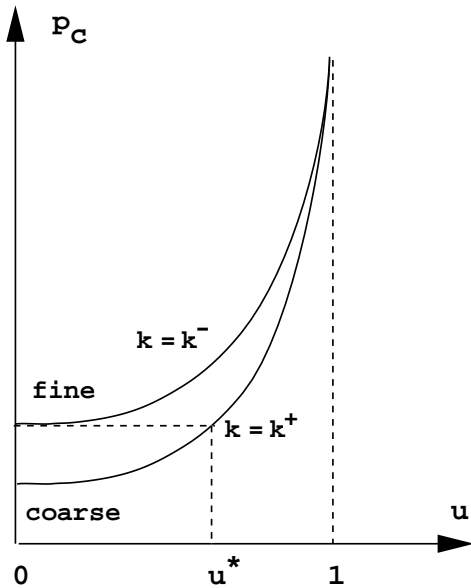


FIG. 1.2. Dimensionless capillary pressure in terms of oil saturation: the top (bottom) curve reflects fine (coarse) material.

Two typical capillary pressures  $p_c = p_c(x, u)$  are shown in Figure 1.2. They relate to fine ( $k = k^-$ ) and to coarse ( $k = k^+$ ) material, where  $k^- < k^+$ .

We consider (1.7) in the domain  $\Sigma = \mathbb{R}$  and for  $t > 0$ , subject to the initial condition

$$(1.11) \quad u(x, 0) = u_0(x) \quad \text{for } x \in \Sigma.$$

When  $k$  is constant and  $u_0 : \Sigma \rightarrow [0, 1]$  is such that  $\int_0^{u_0} \lambda(s)J'(s)ds$  is uniformly Lipschitz continuous in  $\Sigma$ , problem (1.7), (1.11) admits a unique weak solution  $u : \Sigma \times [0, \infty) \rightarrow [0, 1]$ . This follows from the work of Alt and Luckhaus [1], van Duijn and Ye [15], Gilding [17], [18], or Benilan and Toure [3]. This weak solution is smooth whenever  $u \in (0, 1)$ , and has the usual regularity for degenerate equations across possible free boundaries near  $u = 0$  and  $u = 1$ . When  $k$  is piecewise constant, in

particular,

$$(1.12) \quad k(x) = \begin{cases} k^+, & x < 0, \\ k^-, & x > 0, \end{cases}$$

(1.7) cannot always be interpreted across the interface at which  $k$  is discontinuous. This is due to a possible discontinuity in capillary pressure. Using a regularization procedure, this was demonstrated by van Duijn, Molenaar, and de Neef [14] for (1.7), and rigorously proven by Bertsch, Dal Passo, and van Duijn [5] for a simplified equation. Instead, one considers (1.7) only in the subdomains where  $k$  is constant, together with matching conditions across  $k$ -discontinuities. For  $k$  given by (1.12), with  $k^- < k^+$ , the matching conditions read, for all  $t > 0$ ,

$$(1.13) \quad (i) \quad [F(t)] = 0,$$

$$(1.14) \quad (ii) \quad u(0+, t)[p_c(t)] = 0, \quad [p_c(t)] \geq 0,$$

where  $[F(t)] = F(0+, t) - F(0-, t)$  and  $[p_c(t)]$  likewise. The first condition expresses continuity of flux and is obvious. The second condition tells us that the capillary pressure is only continuous if both phases are present on both sides of the  $k$ -discontinuity. This is to be expected from Darcy law (1.2), since then both phase pressures are continuous. If oil is absent for  $x > 0$ , i.e., in the fine material, the entry pressure for oil leads to a discontinuous capillary pressure.

With reference to Figure 1.2, the pressure condition (1.14) can be formulated as

$$(1.15) \quad (ii') \quad \begin{cases} u(0-, t) < u^* \text{ implies } u(0+, t) = 0, \\ u(0-, t) \geq u^* \text{ implies } \frac{J(u(0-, t))}{\sqrt{k^+}} = \frac{J(u(0+, t))}{\sqrt{k^-}}, \end{cases}$$

where  $u^*$  is uniquely defined by

$$(1.16) \quad \frac{J(u^*)}{\sqrt{k^+}} = \frac{J(0)}{\sqrt{k^-}}.$$

The occurrence of oil trapping at the transition from coarse to fine material is directly explained by conditions (1.13), (1.15). Let  $k$  be given by (1.12), and consider a steady state flow ( $u = u(x)$ ) satisfying

$$(1.17) \quad u(\pm\infty) = 0,$$

i.e., injection and production of water, with oil possibly present near  $x = 0$ . Then, by (1.7a),

$$F = \text{constant} = 0 \text{ on } \mathbb{R}$$

or

$$f(u) \left\{ 1 - N_c \sqrt{k(x)} k_{r_w}(u) J'(u) \frac{du}{dx} \right\} = 0 \text{ on } \mathbb{R} \setminus \{0\},$$

with (1.15) at  $x = 0$ . Since  $f(u) > f(0) = 0$  for  $u > 0$ , we have

$$(1.18) \quad u = 0 \quad \text{or} \quad \frac{du}{dx} = \frac{1}{N_c \sqrt{k(x)} k_{r_w}(u) J'(u)} > 0$$

for  $x \in \mathbb{R} \setminus \{0\}$ . Since  $u(+\infty) = 0$ , we find

$$(1.19) \quad u(x) = 0 \quad \text{for all } x > 0$$

and, by (1.15),

$$(1.20) \quad u(0-) \leq u^*.$$

Using (1.20) as the initial condition for (1.18) on  $(-\infty, 0)$ , one easily constructs a family of nontrivial steady states describing the saturation of the trapped oil in the coarse material. The initial condition in an actual displacement process determines which of the steady states is selected. This is discussed by Bertsch, Dal Passo, and van Duijn [5].

Note that the nonuniqueness results from (1.19). Considering  $u(\pm\infty) = \hat{u} \in (0, 1]$ , one finds a unique steady state satisfying (1.15) (continuity of pressure) at  $x = 0$ . Such solutions were considered by Yortsos and Chang [29] for smooth  $k$ .

We now turn to the problem with microstructure, as indicated in Figure 1.1, where trapping occurs at all transitions from high to low permeability. As a result we expect to find a trapping-related threshold saturation (irreducible oil saturation) below which the oil becomes immobile. We consider the case of a periodic as well as a random medium. In section 2 we assume a periodic microstructure of coarse ( $k = k^+$ ) and fine ( $k = k^-$ ) material, each of length  $L_y \ll L_x$ ; see Figure 1.3. This leads to a natural choice of the small expansion parameter  $\varepsilon = L_y/L_x$ . We outline the homogenization procedure, study the resulting auxiliary problems, and derive the effective (upscaled or averaged) equations for the limit  $\varepsilon \searrow 0$ . In doing so, the magnitude of the capillary number  $N_c$  is important. We work out two cases, as follows.

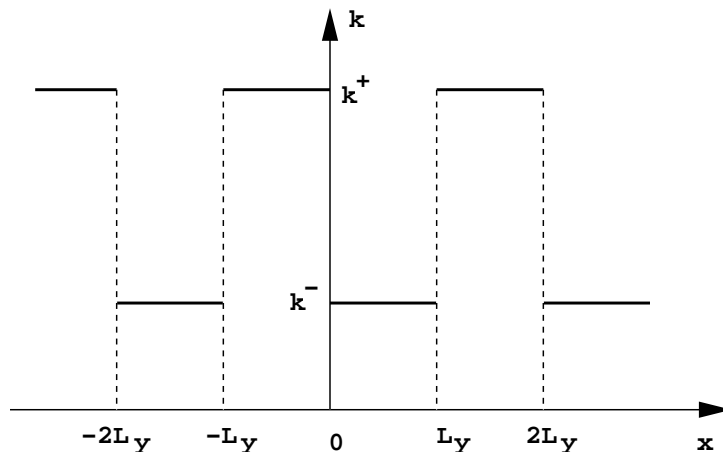


FIG. 1.3. Periodic permeability (unscaled coordinate).

*Capillary limit,  $N_c = 0(1)$ .* This case is relatively straightforward because the auxiliary problem has only constant state solutions (compare steady state solutions on  $(-\varepsilon, 0)$ ). As a consequence the effective equation is found explicitly. It is again of convection-diffusion type, and it involves weighted harmonic means of the fractional flow and capillary terms. Both convection and diffusion vanish from the equation if the averaged oil saturation drops below  $\frac{1}{2}u^*$ .

*Balance*,  $N_c = O(\varepsilon)$ . This case is much more involved. Now the diffusion term disappears in the homogenization procedure, and one is left with a first order conservation law of Buckley–Leverett type. This follows from a detailed study of the auxiliary problem. We show that the upscaled oil-fractional flow function is different from a  $k$ -weighted version of  $f$  and contains, quite surprisingly, some elements of the small scale diffusion. Again it vanishes if the averaged oil saturation drops below a certain value. This irreducible oil saturation is related to a specific solution of the auxiliary problem.

These cases correspond to different flow regimes. In section 3 we discuss their relevance and, in particular, the transition from one to the other, by considering  $N_c = O(\varepsilon^\gamma)$ ,  $\gamma \leq 1$ . Our approach fails when  $N_c = O(\varepsilon^\gamma)$  with  $\gamma > 1$ , yielding  $N_c = O(\varepsilon)$  as a critical case.

In section 4 we consider the case of a random microstructure with respect to both the location of the permeability jumps and the value of the permeability. The effective oil flux is obtained only for the capillary limit ( $N_c = O(1)$ ) and again involves the weighted harmonic means of the fractional flow and capillary pressure terms. The homogenized equation has coefficients depending on the realization, but we prove that average saturation, defined by the homogenized parabolic problem, is a deterministic function. Consequently, it is sufficient to solve the effective equation for a single realization.

Section 5 contains some numerical results. There we take power law relative permeabilities and a Brooks–Corey capillary pressure. We compute the effective fractional flow and diffusivity for the capillary limit  $N_c = O(1)$  and the effective fractional flow for the balance  $N_c = O(\varepsilon)$ .

Some concluding remarks are given in section 6.

Dale et al. [10] studied a similar multiphase flow problem. They considered steady state flow in a periodic porous column, allowing each periodicity cell to have more sub-layers with different relative permeabilities and Leverett functions. Without using the homogenization approach, they derived upscaled expressions for the relative permeabilities. In this paper we present a rigorous analysis of the auxiliary problems, resulting in a fairly complete description of the upscaled equations. In particular, the effect of microscopic trapping, as a result of the different entry pressures, is investigated explicitly.

**2. Homogenization procedure for periodic layers.** A simplified version of problem (1.7), (1.11), involving only a single permeability discontinuity (or trap), was studied by Bertsch, Dal Passo, and van Duijn [5]. They established the existence and uniqueness of a solution satisfying the usual porous-media equation regularity away from the trap. In particular, the solution is nonnegative and bounded. Moreover, the corresponding flux was shown to be continuous in  $x$  for almost all  $t > 0$ .

In this paper we silently assume the same properties for the saturation and flux in the case of multiple traps at arbitrary distances. In particular,  $0 \leq u \leq 1$ . In our problem we deal with two natural length scales: a macroscopic length scale  $L_x$  and a microscopic scale (the characteristic length scale of the layers)  $L_y$ . This disparity in length scales is what provides us with our expansion parameter  $\varepsilon = L_y/L_x$ . For fixed but small characteristic layer length  $L_y$ , the solutions will in general be complicated, having a different behavior on the two length scales. Closed-form solutions are unachievable, and numerical solutions will be nearly impossible to calculate. It is our object to derive a flow equation at the macro scale, keeping information about the trapping only through some averaged quantities.



To simplify our considerations we now suppose a periodic structure with the traps located at the points  $\{\varepsilon i : i \in \mathbb{Z}\}$ . The corresponding permeability  $k^\varepsilon(x)$  is defined by  $k^\varepsilon(x) = k(x/\varepsilon)$ , where

$$(2.1) \quad k = \begin{cases} k^+ & \text{on } (2i - 1, 2i), \\ k^- & \text{on } (2i, 2i + 1). \end{cases}$$

Without loss of generality we assume  $0 < k^- < k^+ < \infty$ . We distinguish two kinds of matching conditions: one going from  $k^+$  to  $k^-$ , and vice versa; see also (1.15).

At  $x = 2i\varepsilon$  we impose the following:

$$(2.2) \quad \begin{aligned} &\text{if } u(2i\varepsilon - 0) < u^*, \text{ then } u(2i\varepsilon + 0) = 0, \\ &\text{if } u(2i\varepsilon - 0) \geq u^*, \text{ then } \frac{J(u(2i\varepsilon - 0))}{\sqrt{k^+}} = \frac{J(u(2i\varepsilon + 0))}{\sqrt{k^-}}. \end{aligned}$$

At  $x = (2i + 1)\varepsilon$  we impose the following:

$$(2.3) \quad \begin{aligned} &\text{if } u((2i + 1)\varepsilon + 0) \geq u^*, \text{ then } \frac{J(u((2i + 1)\varepsilon + 0))}{\sqrt{k^+}} = \frac{J(u((2i + 1)\varepsilon - 0))}{\sqrt{k^-}}, \\ &\text{if } u((2i + 1)\varepsilon + 0) < u^*, \text{ then } u((2i + 1)\varepsilon - 0) = 0. \end{aligned}$$

We now replace  $k$  by  $k^\varepsilon$  in (1.7a–b). Clearly this equation holds in the domain  $\Sigma^\varepsilon = \mathbb{R} \setminus T_\varepsilon$ , where  $T_\varepsilon = \varepsilon \bigcup_{i \in \mathbb{Z}} i$ . Let  $u^\varepsilon$  be a solution of (1.7a) satisfying the matching conditions (2.2) and (2.3). Using the uniform  $L^\infty$  bound for  $u^\varepsilon$ , we consider the following two-scale asymptotic expansion:

$$(2.4) \quad u^\varepsilon(x, t) = u^0(x, y, t) + \varepsilon u^1(x, y, t) + \varepsilon^2 u^2(x, y, t) + \dots,$$

where functions  $u^j$  are periodic in  $y = x/\varepsilon$ , representing the fast scale. Substituting this expansion into (1.7) and equating terms of the same order of  $\varepsilon$  gives equations for  $u^0, u^1, \dots$ . As established for many linear problems containing periodic nonhomogeneities (see, for instance, Bensoussan, Lions, and Papanicolaou [4] or Sanchez-Palencia [27]), we expect that

$$(2.5) \quad U(x, t) = \frac{1}{2} \int_{-1}^{+1} u^0(x, y, t) dy$$

is the weak limit of  $u^\varepsilon$ , and that  $u^0(x, \frac{x}{\varepsilon}, t)$  is the approximation to  $u^\varepsilon$  in some norm. Proving convergence of the homogenization procedure for nonlinear flow problems in nonhomogeneous geometries poses difficulties, as shown by Hornung [19] and Mikelić [25]. Given the nonlinear nature of (1.7) and the matching conditions in (2.2)–(2.3), we shall therefore make no attempt at proving convergence as  $\varepsilon \searrow 0$ . The purpose of this paper is merely to derive the upscaled equations and to study the corresponding auxiliary problems.

Clearly our results depend strongly on the scaling of the capillary number  $N_c$ . The main cases of interest are  $N_c = O(1)$  and  $N_c = O(\varepsilon)$ . We will deal with each of them separately.

**2.1. Capillary limit:  $N_c = O(1)$ .** Introducing the oil flux

$$(2.6) \quad F^\varepsilon = f(u^\varepsilon) - N_c \sqrt{k^\varepsilon(x)} D(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x},$$

where

$$(2.7) \quad D(u^\varepsilon) = k_{rw}(u^\varepsilon) f(u^\varepsilon) J'(u_\varepsilon),$$

equation (1.7a) becomes

$$(2.8) \quad \frac{\partial u^\varepsilon}{\partial t} + \frac{\partial F^\varepsilon}{\partial x} = 0 \quad \text{in } \Sigma^\varepsilon \times (0, \infty).$$

We now apply expansion (2.4) to  $F^\varepsilon$ , which gives

$$(2.9) \quad \begin{aligned} F^\varepsilon &= -N_c D(u^0) \frac{\partial u^0}{\partial y} \sqrt{k} \varepsilon^{-1} \\ &\quad + f(u^0) - N_c \sqrt{k} \left\{ D(u^0) \left( \frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x} \right) + D'(u^0) u^1 \frac{\partial u^0}{\partial y} \right\} \\ &\quad + \left\{ f'(u^0) u^1 - N_c \sqrt{k} \left[ D(u^0) \left( \frac{\partial u^2}{\partial y} + \frac{\partial u^1}{\partial x} \right) \right. \right. \\ &\quad \quad \left. \left. + D'(u^0) u^1 \left( \frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x} \right) \right. \right. \\ &\quad \quad \left. \left. + \left( D''(u^0) \frac{(u^1)^2}{2} + D'(u^0) u^2 \right) \frac{\partial u^0}{\partial y} \right] \right\} \varepsilon + O(\varepsilon^2) \\ &=: F^0 \varepsilon^{-1} + F^1 + F^2 \varepsilon + O(\varepsilon^2). \end{aligned}$$

Using this in (2.8) results in the following equations:

$$\varepsilon^{-2} : -N_c \frac{\partial}{\partial y} \left( \sqrt{k} D(u^0) \frac{\partial u^0}{\partial y} \right) = 0;$$

thus, by continuity of  $F^\varepsilon$ ,

$$(2.10) \quad -N_c \sqrt{k} D(u^0) \frac{\partial u^0}{\partial y} = F^0 = F^0(x, t),$$

which holds for every  $x, y \in \mathbb{R}$  and for all  $t > 0$ . Note that this observation is expected because of the continuity of the flux. We also have the following:

$$(2.11) \quad \begin{aligned} \varepsilon^{-1} : 0 &= \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = \frac{\partial}{\partial x} \left\{ -N_c \sqrt{k} D(u^0) \frac{\partial u^0}{\partial y} \right\} \\ &\quad + \frac{\partial}{\partial y} \left\{ f(u^0) - N_c \sqrt{k} \left[ D(u^0) \left( \frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x} \right) + D'(u^0) u^1 \frac{\partial u^0}{\partial y} \right] \right\}, \end{aligned}$$

$$(2.12) \quad \varepsilon^0 : 0 = \frac{\partial u^0}{\partial t} + \frac{\partial F^2}{\partial y} + \frac{\partial F^1}{\partial x}.$$

We look for  $y$ -periodic solutions of (2.10) satisfying (2.2) and (2.3), with  $x$  and  $t$  as given parameters. Our goal is to prove that  $F^0 = 0$ . We argue by contradiction.

Suppose  $F^0 < 0$ . Let

$$w(y) := J(u^0(y)), \quad \lambda(w) := k_{rw}(J^{-1}(w))f(J^{-1}(w))$$

and

$$\Lambda(w) = \int_{J(0)}^w \lambda(s)ds,$$

the last function being strictly increasing. Then (2.10) reads

$$\lambda(w)\sqrt{k} \frac{dw}{dy} = -\frac{F^0}{N_c} =: F > 0.$$

Hence, for  $-1 < y < 0$ ,

$$\Lambda(w(0-)) - \Lambda(w(-1+0)) = \frac{F}{\sqrt{k^+}},$$

giving

$$(2.13) \quad w(0-) \geq w(-1+0) + \frac{F}{\sqrt{k^+}} \frac{1}{\|\lambda\|_\infty}.$$

Similarly, for  $0 < y < 1$ ,

$$(2.14) \quad w(1-0) \geq w(0+) + \frac{F}{\sqrt{k^-}} \frac{1}{\|\lambda\|_\infty}.$$

Now we apply matching conditions (2.2) and (2.3) in terms of  $w$ . First, suppose  $w(0-) \leq J(u^*)$ . Then  $w(0+) = J(0)$  and, by (2.14),  $w(1-0) > J(0)$ . Hence  $w(-1+0) > J(u^*)$ , giving—by (2.13)— $w(0-) > J(u^*)$ , which contradicts the assumption. Next suppose  $w(0-) > J(u^*)$ . In this case we obtain  $w(0+) = \sqrt{k^-/k^+} w(0-) < w(0-)$ . By (2.14) and (2.13) we have

$$(2.15) \quad \begin{aligned} w(1-0) &\geq \sqrt{\frac{k^-}{k^+}} w(0-) + \frac{F}{\sqrt{k^-}} \frac{1}{\|\lambda\|_\infty} \\ &\geq \sqrt{\frac{k^-}{k^+}} w(-1+0) + \frac{F}{\|\lambda\|_\infty} \left\{ \frac{\sqrt{k^-}}{k^+} + \frac{1}{\sqrt{k^-}} \right\}. \end{aligned}$$

If  $w(-1+0) > J(u^*)$ , then  $w(-1+0) = \sqrt{k^+/k^-} w(1-0)$ . Substituting this into (2.15) yields  $w(1-0) > w(-1-0)$ , which contradicts the periodicity. If  $w(-1+0) \leq J(u^*)$ , then  $w(1-0) = J(0)$ , which contradicts (2.14). Hence  $F^0 \geq 0$ . A similar argument gives  $F^0 \leq 0$ , implying

$$F^0 = 0.$$

This conclusion allows us to solve (2.10) with the matching conditions. We find

$$(2.16) \quad u^0(y) = \begin{cases} C > u^* & \text{for } -1 < y < 0, \\ \bar{C} & \text{for } 0 < y < 1, \end{cases}$$

where  $\bar{C} = J^{-1}(\sqrt{k^-/k^+}J(C))$ , or

$$(2.17) \quad u^0(y) = \begin{cases} C \leq u^* & \text{for } -1 < y < 0, \\ 0 & \text{for } 0 < y < 1. \end{cases}$$

Now we consider the  $\varepsilon^{-1}$ -equation (2.11). Since  $F^0 = 0$  and the flux is continuous, we find

$$F^1 = F^1(x, t).$$

Using (2.16) and (2.17), the local form of  $F^1$  is

$$(2.18) \quad F^1 = f(C) - N_c \sqrt{k^+} D(C) \left\{ \frac{\partial C}{\partial x} + \frac{\partial u^1}{\partial y} \right\}$$

for  $-1 < y < 0$ , and

$$(2.19) \quad F^1 = \begin{cases} f(\bar{C}) - N_c \sqrt{k^-} D(\bar{C}) \left\{ \frac{\partial \bar{C}}{\partial x} + \frac{\partial u^1}{\partial y} \right\} & \text{for } C > u^*, \\ 0 & \text{for } C \leq u^*, \end{cases}$$

for  $0 < y < 1$ . Clearly we have to consider only the nontrivial case  $C > u^*$ . From (2.18) and (2.19) we deduce

$$\frac{\partial u^1}{\partial y} = \begin{cases} \frac{f(C) - F^1}{\sqrt{k^+} N_c D(C)} - \frac{\partial C}{\partial x} =: B_1(x, t) & \text{for } -1 < y < 0, \\ \frac{f(\bar{C}) - F^1}{\sqrt{k^-} N_c D(\bar{C})} - \frac{\partial \bar{C}}{\partial x} =: B_2(x, t) & \text{for } 0 < y < 1. \end{cases}$$

After integration we observe that  $B_1 + B_2 = 0$ . Hence we can solve for  $F^1$  to find

$$F^1 = \frac{\frac{f(C)}{\sqrt{k^+} D(C)} + \frac{f(\bar{C})}{\sqrt{k^-} D(\bar{C})}}{\frac{1}{\sqrt{k^+} D(C)} + \frac{1}{\sqrt{k^-} D(\bar{C})}} - N_c \frac{\frac{\partial C}{\partial x} + \frac{\partial \bar{C}}{\partial x}}{\frac{1}{\sqrt{k^+} D(C)} + \frac{1}{\sqrt{k^-} D(\bar{C})}}.$$

Finally we use the  $\varepsilon^0$ -equation in (2.12). Since  $F^2$  is continuous in the fast scale, we find for the averaged oil saturation  $U = \frac{1}{2}(C + \bar{C})$  the effective convection-diffusion equation

$$(2.20) \quad \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left\{ \mathcal{F}(U) - N_c \mathcal{D}(U) \frac{\partial U}{\partial x} \right\} = 0,$$

where  $-\infty < x < \infty$  and  $t > 0$ . One easily verifies

$$\mathcal{F}(U) = \begin{cases} 0 & \text{for } 0 \leq U \leq \frac{1}{2} u^*, \\ \text{strictly increasing} & \text{for } \frac{1}{2} u^* < U < 1, \\ 1 & \text{for } U = 1 \end{cases}$$

and

$$\mathcal{D}(U) = \begin{cases} 0 & \text{for } 0 \leq U \leq \frac{1}{2} u^*, \\ > 0 & \text{for } \frac{1}{2} u^* < U < 1, \\ 0 & \text{for } U = 1. \end{cases}$$

In section 5 we show the graphs of  $\mathcal{F}$  and  $\mathcal{D}$  based on power law relative permeabilities and a Brooks–Corey capillary pressure.

The effective equation (2.20) is written in terms of the averaged oil saturation  $U = U(x, t)$ . The oscillatory zeroth order approximation  $u^0(x, \frac{x}{\varepsilon}, t)$  in the asymptotic expansion (2.4) can be reconstructed from  $U$  in a straightforward way, by using  $U = \frac{1}{2}(C + \bar{C})$  and expressions (2.16) and (2.17).

**2.2. Balance:  $N_c = O(\varepsilon)$ .** Writing  $N_c := N_c\varepsilon$ , the oil flux (2.6) becomes

$$(2.21) \quad F^\varepsilon = f(u^\varepsilon) - N_c\varepsilon\sqrt{k^\varepsilon(x)}D(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x}.$$

Clearly expansion (2.9) changes due to the additional  $\varepsilon$  factor. It now takes the form

$$(2.22) \quad \begin{aligned} F^\varepsilon &= f(u^0) - N_c\sqrt{k}D(u^0)\frac{\partial u^0}{\partial y} \\ &+ \left\{ f'(u^0)u^1 - N_c\sqrt{k} \left[ D(u^0) \left( \frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) + D'(u^0)u^1 \frac{\partial u^0}{\partial y} \right] \right\} \varepsilon + O(\varepsilon^2) \\ &=: F^0 + F^1\varepsilon + O(\varepsilon^2). \end{aligned}$$

Using this expansion in (2.8) gives

$$\frac{\partial u^0}{\partial t} + \frac{1}{\varepsilon} \frac{\partial F^0}{\partial y} + \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = O(\varepsilon),$$

resulting in the equations

$$\varepsilon^{-1} : \frac{\partial F^0}{\partial y} = 0,$$

or, by the continuity of  $F^\varepsilon$ ,

$$(2.23) \quad f(u^0) - N_c\sqrt{k}D(u^0)\frac{\partial u^0}{\partial y} = F^0 = F^0(x, t),$$

which holds for every  $x, y \in \mathbb{R}$  and for all  $t > 0$ , and

$$(2.24) \quad \varepsilon^0 : \frac{\partial u^0}{\partial t} + \frac{\partial F^0}{\partial x} + \frac{\partial F^1}{\partial y} = 0.$$

First (2.23) needs to be considered. It leads to the following auxiliary problem.

*Problem  $A_u$ .* Given  $F \in \mathbb{R}$ , find  $u : [-1, 0) \cup (0, 1] \rightarrow [0, 1]$  satisfying

$$(2.25) \quad f(u) - N_c\sqrt{k}k_{rw}(u)f(u)J'(u)\frac{du}{dy} = F \quad \text{in } (-1, 0) \cup (0, 1)$$

subject to the matching condition ( $y = 0$ )

$$(2.26) \quad \begin{cases} \text{if } u(0-) < u^*, & \text{then } u(0+) = 0, \\ \text{if } u(0-) \geq u^*, & \text{then } \frac{J(u(0-))}{\sqrt{k^+}} = \frac{J(u(0+))}{\sqrt{k^-}}, \end{cases}$$

and the periodicity condition ( $y = \pm 1$ )

$$(2.27) \quad \begin{cases} \text{if } u(-1+0) < u^*, & \text{then } u(1-0) = 0, \\ \text{if } u(-1+0) \geq u^*, & \text{then } \frac{J(u(-1+0))}{\sqrt{k^+}} = \frac{J(u(1-0))}{\sqrt{k^-}}. \end{cases}$$

This problem is considered in detail in the following sections. We prove existence for  $0 \leq F \leq 1$  and uniqueness for  $F > 0$ . Moreover, we show the monotone dependence and differentiability of  $u$  with respect to  $F$ . After that, (2.24) is averaged over the cell  $(-1, 0) \cup (0, 1)$  to obtain the scaled-up (macroscopic) transport equation. This equation turns out to be of Buckley–Leverett type.

**2.3. Auxiliary problem.** To simplify the analysis, we introduce, as in section 2.1, the function  $w = J(u)$  and set

$$\gamma(w) = k_{rw}(J^{-1}(w)) \quad \text{and} \quad \varphi(w) = f(J^{-1}(w)).$$

In terms of  $w$ , the auxiliary problem  $A_u$  becomes the following.

*Problem  $A_w$ .* Given  $F \in \mathbb{R}$ , find  $w : [-1, 0) \cup (0, 1] \rightarrow [J(0), \infty)$  satisfying

$$(2.28) \quad \varphi(w) \left\{ 1 - N_c \sqrt{k} \gamma(w) \frac{dw}{dy} \right\} = F \quad \text{in } (-1, 0) \cup (0, 1)$$

such that (at  $y = 0$ )

$$(2.29) \quad \begin{cases} \text{if } w(0-) < J(u^*), & \text{then } w(0+) = J(0), \\ \text{if } w(0-) \geq J(u^*), & \text{then } w(0+) = \sqrt{\frac{k^-}{k^+}} w(0-), \end{cases}$$

and (at  $y = \pm 1$ )

$$(2.30) \quad \begin{cases} \text{if } w(-1+0) < J(u^*), & \text{then } w(1-0) = J(0), \\ \text{if } w(-1+0) \geq J(u^*), & \text{then } w(1-0) = \sqrt{\frac{k^-}{k^+}} w(-1+0). \end{cases}$$

We first demonstrate existence and some qualitative properties for  $0 = f(0) \leq F \leq f(1) = 1$ .

**LEMMA 2.1.** *Let  $F > 1$ . Then there are no solutions to Problem  $A_w$ .*

*Proof.* Since  $f$  is strictly increasing, we have

$$\frac{F}{\varphi(w)} - 1 \geq \frac{F}{f(1)} - 1 > 0,$$

and consequently, by (2.28),  $dw/dy < 0$  on  $(-1, 0) \cup (0, 1)$ . Now suppose  $w(0-) < J(u^*)$ . Then  $w(0+) = J(0)$ , and thus  $w < J(0)$  on  $(0, 1)$ , contradicting  $w \geq J(0)$  from the definition. If  $w(0-) \geq J(u^*)$ , then clearly  $w(-1+0) > w(0-) \geq J(u^*)$ , yielding

$$w(0+) = \sqrt{\frac{k^-}{k^+}} w(0-), \quad w(1-0) = \sqrt{\frac{k^-}{k^+}} w(-1+0).$$

This implies  $w(1-0) > w(0+)$ , contradicting  $dw/dy < 0$  on  $(0, 1)$ .  $\square$

**LEMMA 2.2.** *Let  $F < 0$ . Then there are no solutions to Problem  $A_w$ .*

*Proof.* Equation (2.28) now gives  $dw/dy > 0$  on  $(-1, 0) \cup (0, 1)$ . Now suppose  $w(0-) \leq J(u^*)$ . Then  $w(0+) = J(0)$  and  $w(1-0) > J(0)$ . Hence  $w(-1+0) > J(u^*)$ , contradicting  $w(0-) \leq J(u^*)$ . Next let  $w(0-) > J(u^*)$ . Then  $w(0+) = \sqrt{k^-/k^+}w(0-)$ , and  $w(1-0) > w(0+) = \sqrt{k^-/k^+}w(0-) > \sqrt{k^-/k^+}J(u^*) = J(0)$ . Thus  $w(-1+0) \geq J(u^*)$  and, from the  $w$ -monotonicity,  $\sqrt{k^-/k^+}w(0-) > \sqrt{k^-/k^+}w(-1+0)$  or  $w(0+) > w(1-0)$ , contradicting  $dw/dy > 0$  on  $(0, 1)$ .  $\square$

**COROLLARY 2.3.** *Let  $F = 1$ . Then  $u = 1$  uniquely solves Problem  $A_u$ .*

*Proof.* We use the  $u$ -formulation in Problem  $A_u$ . Clearly  $u = 1$  is a solution. To show uniqueness, suppose there exists a solution  $u$  such that  $u(y_0) < 1$  for some  $y_0 \in (-1, 0) \cup (0, 1)$ . Since  $du/dy < 0$  whenever  $u < 1$ , we have the following two possibilities: either we have  $u < 1$  everywhere and strictly decreasing, or there exists  $y_1 < y_0$  such that  $u(y_1) = 1$ . The first possibility leads to a contradiction, using the monotone relations imposed by the matching conditions, since  $u(0+) > u(1)$  implies  $u(0-) > u(-1)$ . The second possibility implies  $u(y) = 1$  for all  $y \leq y_1$ , in particular  $u(-1) = 1$ , which contradicts the periodicity.  $\square$

**LEMMA 2.4.** *Let  $F = 0$ . Then Problem  $A_u$  admits the following family of solutions (for all  $0 \leq l \leq u^*$ ):*

$$\phi(u(y)) = \begin{cases} \left[ \frac{y}{N_c \sqrt{k^+}} + \phi(l) \right]_+ & \text{for } -1 < y < 0, \\ 0 & \text{for } 0 < y < 1, \end{cases}$$

where

$$\phi(s) = \int_0^s k_{rw}(v)J'(v)dv.$$

*Proof.* Equation (2.27) implies that any solution must be a combination of

$$(2.31) \quad u \equiv 0 \quad \text{and} \quad \frac{d}{dy}\phi(u(y)) = \frac{1}{N_c \sqrt{k}}.$$

One immediately deduces that  $u(y) = 0$  for  $0 < y < 1$  is the only possibility. Any other choice contradicts the periodicity. Then clearly  $u(0-) \leq u^*$ , and (2.31) provides the required structure.  $\square$

Now we consider the case  $0 < F < 1$ . To understand the structure of the solutions of Problem  $A_w$ , we first introduce the following.

**DEFINITION 2.5.** *Given  $F \in (0, 1)$ , let  $\xi_0(F) \in (J(0), \varphi^{-1}(F))$  be the unique root of*

$$(2.32) \quad \int_{J(0)}^{\xi_0(F)} V(s, F)ds = \frac{1}{N_c \sqrt{k^-}},$$

where  $V(\cdot, F) : (J(0), \varphi^{-1}(F)) \cup (\varphi^{-1}(F), \infty) \rightarrow \mathbb{R}^+$  is given by

$$(2.33) \quad V(s, F) = \frac{\gamma(s)\varphi(s)}{|F - \varphi(s)|}.$$

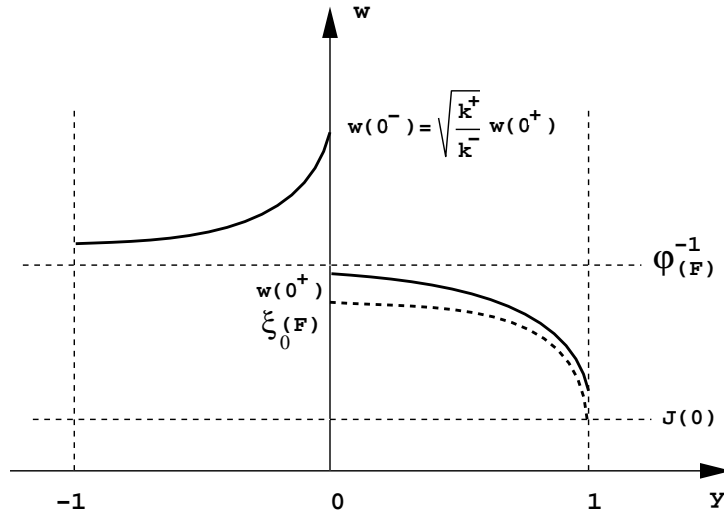


FIG. 2.1. Sketch of the behavior of solution  $w$ .

Clearly  $\xi_0(0+) = J(0)$ ,  $\xi_0 \in C^1((0, 1))$ , and  $d\xi_0/dF > 0$  for  $F > 0$ . We are now in a position to prove the following structure for solutions of Problem  $A_w$  (see also Figure 2.1).

PROPOSITION 2.6. *Let  $0 < F < 1$ . Then any solution of Problem  $A_w$  satisfies*

- (i)  $\frac{dw}{dy} < 0$  on  $(0, 1)$ , with  $\xi_0(F) \leq w(0+) < \varphi^{-1}(F)$ ;
- (ii)  $\frac{dw}{dy} > 0$ ,  $w > \varphi^{-1}(F)$  on  $(-1, 0)$ .

*Proof.* By a uniqueness argument for (2.28), we note that either  $w \equiv \varphi^{-1}(F)$  or  $w \neq \varphi^{-1}(F)$  on the intervals  $(-1, 0)$  and  $(0, 1)$ . Furthermore,  $w \leq \varphi^{-1}(F)$  implies  $dw/dy \leq 0$ . Using this monotonicity and conditions (2.29), (2.30), the result  $w(0+) < \varphi^{-1}(F)$  follows directly, giving  $dw/dy < 0$  on  $(0, 1)$ . Integrating (2.28) on  $(0, 1)$  gives

$$\int_{w(1)}^{w(0+)} V(s, F) ds = \frac{1}{N_c \sqrt{k^-}}.$$

Since  $w(1) \geq J(0)$ , we find

$$\int_{J(0)}^{w(0+)} V(s, F) ds \geq \frac{1}{N_c \sqrt{k^-}},$$

implying  $w(0+) \geq \xi_0(F)$ . Since  $w(0+) > w(1)$ , conditions (2.29), (2.30) give  $w(0-) > w(-1)$ , proving the second statement of the proposition.  $\square$

We shall now demonstrate solvability for Problem  $A_w$ . We start with the simplest case, where a solution satisfies  $w(1) = J(0)$  and  $w(0+) = \xi_0(F)$ . By Definition 2.5, such local solutions exist on  $(0, 1)$ . Using again (2.29), (2.30), we find for the left interval

$$(2.34) \quad w(-1) \leq J(u^*) \quad \text{and} \quad w(0-) = \sqrt{\frac{k^+}{k^-}} \xi_0(F).$$



By Proposition 2.6(ii) we need  $\varphi^{-1}(F) < J(u^*)$ , or

$$F < f(u^*),$$

for such solutions to exist. Integrating (2.28) over  $(-1, 0)$  and using (2.34) once more yields the nonlinear algebraic equation

$$(2.35) \quad \int_{w(-1)}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} V(s, F) ds = \frac{1}{N_c \sqrt{k^+}},$$

where  $\sqrt{k^+/k^-}\xi_0(F) > \sqrt{k^+/k^-}J(0) = J(u^*)$ .

If this equation can be solved for  $w(-1) \in (\varphi^{-1}(F), J(u^*))$ , we have found a solution of Problem  $A_w$  satisfying  $w(1) = J(0)$ . To investigate the solvability we define, for  $0 \leq F < f(u^*)$ ,

$$(2.36a) \quad G(F) = \int_{J(u^*)}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} V(s, F) ds.$$

One easily verifies

$$G(0) = 0, \quad G(f(u^*)) = \infty, \quad \text{and } dG/dF > 0 \quad \text{on } (0, f(u^*)).$$

Hence there exists a unique  $F^* \in (0, f(u^*))$  such that

$$(2.36b) \quad G(F^*) = \frac{1}{N_c \sqrt{k^+}}.$$

As a consequence, integral equation (2.35) can be uniquely solved, provided  $0 < F \leq F^*$ : the left-hand side of (2.35) decreases monotonically in  $w(-1)$ , becomes unbounded when  $w(-1) \searrow \varphi^{-1}(F)$ , and attains a value  $\leq \frac{1}{N_c \sqrt{k^+}}$  when  $w(-1) \nearrow J(u^*)$ . Thus we have shown the following (see also Figure 2.2).

**THEOREM 2.7.** *Let  $0 < F \leq F^* < f(u^*)$ , where  $F^*$  is defined by (2.36b). Further, let  $\xi_0(F)$  be given by Definition 2.5. Then Problem  $A_w$  admits a solution  $w$  satisfying*

$$w(1) = J(0), \quad w(0+) = \xi_0(F), \quad \text{and } w(0-) = \sqrt{\frac{k^+}{k^-}}\xi_0(F).$$

Next we consider  $F^* < F < 1$ . Since now  $G(F) > \frac{1}{N_c \sqrt{k^+}}$ , there are no solutions possible in the class  $w(1) = J(0)$ . For convenience we introduce

$$(2.37) \quad \zeta := w(1) \in (b, \varphi^{-1}(F)),$$

where  $b = \max\{J(0), \sqrt{k^-/k^+}\varphi^{-1}(F)\}$  and  $z := w(0+) \in (\zeta, \varphi^{-1}(F))$ . Below we construct solutions satisfying  $w(1) > b$  and  $w(-1) > J(u^*)$ . Then the problem of existence for Problem  $A_w$  is reduced to the following system of algebraic equations (integrating (2.28) on  $(-1, 0)$  and on  $(0, 1)$ , and using (2.29), (2.30), and (2.33)):

$$(2.38a) \quad \int_{\zeta}^z V(s, F) ds = \frac{1}{N_c \sqrt{k^-}},$$

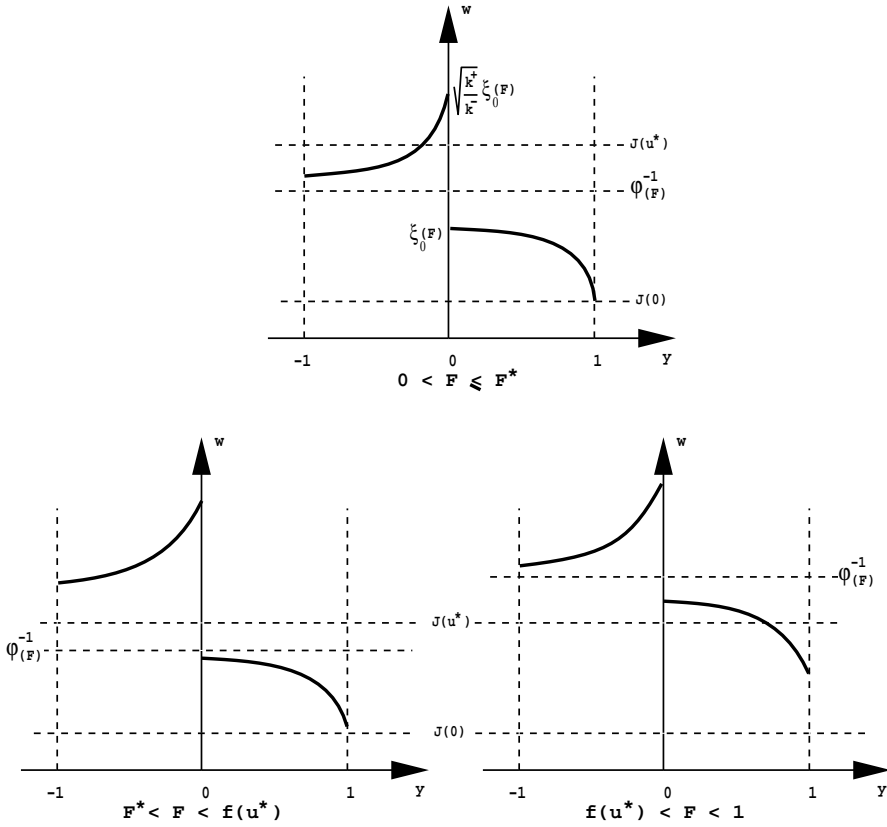


FIG. 2.2. Sketch of behavior of  $w$  for three ranges of  $F$ .

$$(2.38b) \quad \int_{\sqrt{\frac{k^+}{k^-}}\zeta}^{\sqrt{\frac{k^+}{k^-}}z} V(s, F) ds = \frac{1}{N_c \sqrt{k^+}}.$$

To study the solvability of this system, we introduce

$$\psi : (b, \varphi^{-1}(F)) \cup \left( \varphi^{-1}(F), \sqrt{\frac{k^+}{k^-}}\varphi^{-1}(F) \right) \rightarrow \mathbb{R},$$

$$\psi(v) = \begin{cases} \int_b^v V(s, F) ds & \text{for } b < v < \varphi^{-1}(F), \\ \int_v^{\sqrt{\frac{k^+}{k^-}}\varphi^{-1}(F)} V(s, F) ds & \text{for } \varphi^{-1}(F) < v < \sqrt{\frac{k^+}{k^-}}\varphi^{-1}(F). \end{cases}$$

Note that  $\psi$  is strictly increasing on  $(b, \varphi^{-1}(F))$ , respectively strictly decreasing on  $(\varphi^{-1}(F), \sqrt{k^+/k^-}\varphi^{-1}(F))$ ; see Figure 2.3 for a sketch. By the monotonicity of  $\psi$ , the function

$$(2.39a) \quad z = z(\zeta) = \psi^{-1} \left( \psi(\zeta) + \frac{1}{N_c \sqrt{k^-}} \right)$$

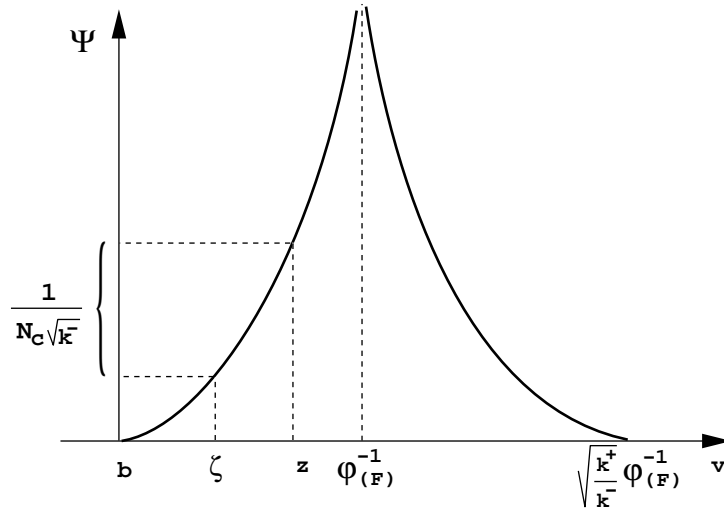


FIG. 2.3. Sketch of  $\psi$  and construction of  $z = z(\zeta)$ .

is well defined on  $(b, \varphi^{-1}(F))$ , satisfying  $dz/d\zeta > 0$ . Now system (2.38) reduces to the map  $W : (b, \varphi^{-1}(F)) \rightarrow \mathbb{R}$  given by

$$(2.39b) \quad W(\zeta) = \psi\left(\sqrt{\frac{k^+}{k^-}}\zeta\right) - \psi\left(\sqrt{\frac{k^+}{k^-}}z\right) - \frac{1}{N_c\sqrt{k^+}}.$$

We first formulate the theorem.

**THEOREM 2.8.** *For  $F^* < F < 1$ , there exists a solution to (2.38); i.e., the auxiliary Problem  $A_w$  admits a solution.*

*Proof.* Since  $z(\varphi^{-1}(F)-) = \varphi^{-1}(F)$ , we have

$$W(\varphi^{-1}(F)-) = -\frac{1}{N_c\sqrt{k^+}} < 0.$$

To investigate the behavior near  $\zeta = b$ , we use  $z > \xi_0(F)$  and consider

$$\int_{\sqrt{\frac{k^+}{k^-}}b}^{\sqrt{\frac{k^+}{k^-}}\xi_0(F)} V(s, F)ds = \begin{cases} +\infty & \text{for } f(u^*) \leq F < 1, \\ > \frac{1}{N_c\sqrt{k^+}} & \text{for } F^* < F < f(u^*). \end{cases}$$

The first follows from  $\sqrt{k^+/k^-}b = \varphi^{-1}(F)$  for  $F \geq f(u^*)$ , the second from  $\sqrt{k^+/k^-}b = J(u^*)$  and (2.36a) for  $F^* < F < f(u^*)$ . As a consequence we find  $W(\zeta) > 0$  for  $\zeta$  close to  $b$ . Since  $W$  is continuous, the equation  $W(\zeta) = 0$  has at least one root, which provides the existence for (2.38).  $\square$

**2.4. Continuity, monotonicity, and uniqueness.** To construct an effective equation for  $U$ , we need to show that the solution of the auxiliary problem is unique, continuous, and monotone in  $F$  for  $0 < F \leq 1$ . The  $F$ -dependence is denoted by  $u = u(y, F)$ ,  $w = w(y, F)$ , or simply  $u(F), w(F)$ . We treat  $F \in (0, F^*)$  and  $F \in (F^*, 1)$  first, and then consider the behavior near  $F = 0+$ ,  $F = F^*$ , and  $F = 1$ .

$F \in (0, F^*)$ . Since uniqueness has not yet been demonstrated, we consider here the solution  $w(F)$  given by Theorem 2.7. It satisfies

$$(2.40a) \quad \int_{J(0)}^{w(y,F)} V(s, F) ds = \frac{1-y}{N_c \sqrt{k^-}} \quad \text{for } 0 < y \leq 1,$$

$$(2.40b) \quad \int_{w(y,F)}^{\sqrt{\frac{k^+}{k^-}} \xi_0(F)} V(s, F) ds = -\frac{y}{N_c \sqrt{k^+}} \quad \text{for } -1 \leq y < 0.$$

The smoothness of  $\xi_0$  and  $V(s, \cdot)$  implies  $w(y, \cdot) \in C^1((0, F^*))$  for each  $y \in [-1, 0) \cup (0, 1]$ . Let  $\xi(F) = dw/dF$ . Differentiating (2.40a) with respect to  $F$  yields

$$-\int_{J(0)}^{w(y,F)} \frac{\gamma(s)\varphi(s)}{(F-\varphi(s))^2} ds + V(w(y, F), F)\xi(y, F) = 0.$$

Hence

$$(2.41) \quad \xi(y, F) > 0 \quad \text{for } 0 < y < 1$$

and

$$\xi(0+, F) = \frac{d\xi_0}{dF} > 0, \quad \xi(1, F) = 0.$$

From (2.40b) we find

$$\begin{aligned} & \int_{w(y,F)}^{\sqrt{\frac{k^+}{k^-}} \xi_0(F)} \frac{\gamma(s)\varphi(s)}{(\varphi(s)-F)^2} ds + V\left(\sqrt{\frac{k^+}{k^-}} \xi_0(F), F\right) \sqrt{\frac{k^+}{k^-}} \frac{d\xi_0}{dF} \\ & = V(w(y, F), F)\xi(y, F), \end{aligned}$$

implying

$$(2.42) \quad \xi(y, F) > 0 \quad \text{for } -1 \leq y < 0$$

with

$$\xi(0-, F) = \sqrt{\frac{k^+}{k^-}} \frac{d\xi_0}{dF} > 0.$$

$F \in (F^*, 1)$ . Then any solution of Problem  $A_w$  satisfies

$$\int_{w(1,F)}^{w(y,F)} V(s, F) ds = \frac{1-y}{N_c \sqrt{k^-}} \quad \text{for } 0 < y \leq 1,$$

with  $w(1, F) > J(0)$ . Hence

$$(2.43) \quad \begin{aligned} & - \int_{w(1,F)}^{w(y,F)} \frac{\varphi(s)\gamma(s)}{(F - \varphi(s))^2} ds + V(w(y, F), F)\xi(y, F) \\ & = V(w(1, F), F)\xi(1, F), \end{aligned}$$

implying the following statements:

$$(2.44a) \quad \begin{aligned} & \text{if } \xi(1, F) > 0, \text{ then } \xi(y, F) > 0 \text{ for } 0 < y < 1, \\ & \text{if } \xi(0+, F) < 0, \text{ then } \xi(1, F) < 0. \end{aligned}$$

Similarly we deduce on  $(-1, 0)$  the following:

$$(2.44b) \quad \begin{aligned} & \text{if } \xi(0-, F) > 0, \text{ then } \xi(y, F) > 0 \text{ for } -1 < y < 0, \\ & \text{if } \xi(-1, F) = 0, \text{ then } \xi(0-, F) < 0. \end{aligned}$$

The conditions at  $y = 0\pm$  and  $y = \pm 1$  translate into

$$(2.45) \quad \begin{aligned} \xi(0-, F) &= \sqrt{\frac{k^+}{k^-}} \xi(0+, F), \\ \xi(-1, F) &= \sqrt{\frac{k^+}{k^-}} \xi(1, F). \end{aligned}$$

Next we combine (2.44) and (2.45). Suppose there exists  $\hat{F} \in (F^*, 1)$  such that  $\xi(1, \hat{F}) = 0$ . Then  $\xi(-1, \hat{F}) = 0$ ,  $\xi(0-, \hat{F}) < 0$ ,  $\xi(0+, \hat{F}) < 0$ , giving  $\xi(1, \hat{F}) < 0$ , a contradiction.

Hence either  $\xi(1, F) > 0$  or  $\xi(1, F) < 0$  for all  $F \in (F^*, 1)$ . We rule out the second possibility. By (2.45),  $\xi(1, F) < 0$  gives  $\xi(-1, F) < 0$ , implying that  $w(-1, F)$  is strictly decreasing in  $(F^*, 1)$ . However, Proposition 2.6 gives  $w(-1, F) > \varphi^{-1}(F) \rightarrow \infty$  as  $F \rightarrow 1$ , a contradiction. Hence  $\xi(1, F) > 0$ , and by (2.44)

$$(2.46) \quad \xi(y, F) > 0 \quad \text{for } y \in [-1, 0) \cup (0, 1].$$

*Remark 2.1.* Note that the monotonicity result (2.46) applies to any solution of Problem  $A_w$  satisfying  $w(1, F) > J(0)$ . We use this to show uniqueness for Problem  $A_w$  and hence for Problem  $A_u$ .

**THEOREM 2.9.** *The auxiliary problem  $(A_u)$  has a unique solution  $u(F)$  for each  $F \in (0, 1]$ . We have*

- (i)  $u(1) = 1$ ;
- (ii)  $u(F) = J^{-1}(w(F))$ , where  $w(F)$  is given by

$$\begin{aligned} & \int_{w(1,F)}^{w(y,F)} V(s, F) ds = \frac{1-y}{N_c \sqrt{k^-}} \quad \text{for } 0 < y \leq 1, \\ & \int_{w(y,F)}^{\sqrt{\frac{k^+}{k^-}} w(0+, F)} V(s, F) ds = -\frac{y}{N_c \sqrt{k^+}} \quad \text{for } -1 \leq y < 0, \end{aligned}$$

with  $w(1, F) = J(0)$ ,  $w(0+, F) = \xi_0(F)$  for  $0 < F \leq F^*$ , and  $w(1, F) > J(0)$  satisfying  $W(w(1, F), F) = 0$  for  $F^* < F < 1$ .

*Proof.* In section 2.3 we have shown that for  $F^* < F < 1$  no solutions are possible with  $w(1, F) = J(0)$ . Furthermore for  $0 < F \leq F^*$ , Problem  $A_w$  is uniquely solvable in the class  $w(1, F) = J(0)$ . What remains is to rule out solutions satisfying  $w(1, F) > J(0)$  for  $0 < F \leq F^*$  and to show uniqueness for  $F^* < F < 1$  in the class  $w(1, F) > J(0)$ .

With  $W$  given by (2.39b), let us consider the equation

$$W(\zeta(F), F) = 0 \quad \text{with } \zeta(F) = w(1, F) > J(0).$$

Differentiating with respect to  $F$  and denoting  $\partial/\partial\zeta$  by a prime gives

$$W' \frac{d\zeta}{dF} + \frac{\partial W}{\partial F} = 0.$$

Since  $\partial\zeta/\partial F > 0$ , as explained in Remark 2.1, we have

$$(2.47) \quad W'(\zeta(F), F) < 0$$

whenever  $\partial W/\partial F > 0$ . The definition of  $W$  involves  $z = z(\zeta, F)$ , given by

$$\psi(z, F) = \psi(\zeta, F) + \frac{1}{N_c \sqrt{k^-}}.$$

Hence

$$\psi'(z, F) \frac{\partial z}{\partial F} = \frac{\partial}{\partial F} (\psi(\zeta, F) - \psi(z, F)),$$

implying  $\partial z/\partial F > 0$ . Using this we find directly

$$\begin{aligned} \frac{\partial W}{\partial F} &= \frac{\partial}{\partial F} \left( \psi \left( \sqrt{\frac{k^+}{k^-}} \zeta, F \right) - \psi \left( \sqrt{\frac{k^+}{k^-}} z, F \right) \right) \\ &\quad - \sqrt{\frac{k^+}{k^-}} \psi' \left( \sqrt{\frac{k^+}{k^-}} z, F \right) \frac{\partial z}{\partial F} > 0. \end{aligned}$$

Thus (2.47) holds for any solution of  $(A_w)$  with  $\zeta(F) = w(1, F) > J(0)$ .

Next we consider  $W(b, F)$ . In section 2.3 we showed  $W(b, F) > 0$  for  $F > F^*$  and  $W(\varphi^{-1}(F), F) = -\frac{1}{N_c \sqrt{k^-}} < 0$ . In fact, for  $F < f(u^*)$  we have

$$(2.48) \quad \begin{aligned} W(b, F) &= \int_{J(u^*)}^{\sqrt{\frac{k^+}{k^-}} \xi_0(F)} V(s, F) ds - \frac{1}{N_c \sqrt{k^-}} \\ &= G(F) - \frac{1}{N_c \sqrt{k^-}} \quad (\text{see 2.36a}). \end{aligned}$$

Hence

$$W(b, F) = \begin{cases} > 0 & \text{for } F > F^*, \\ 0 & \text{for } F = F^*, \\ < 0 & \text{for } F < F^*. \end{cases}$$

Combining these inequalities with (2.47) gives uniqueness for  $F > F^*$  and non-existence for  $F \leq F^*$ .  $\square$

Let  $\bar{u} : [-1, 0) \cup (0, 1] \rightarrow [0, 1]$ , defined by (see Lemma 2.4)

$$\phi(\bar{u}(y)) = \begin{cases} \left[ \frac{y}{N_c \sqrt{k^+}} + \phi(u^*) \right]_+ & \text{for } -1 \leq y < 0, \\ 0 & \text{for } 0 < y \leq 1, \end{cases}$$

denote the maximal solution corresponding to  $F = 0$ .

We are now in a position to formulate the following continuity and monotonicity results.

**THEOREM 2.10.** *The solution  $u(F)$  satisfies the following:*

- (i)  $u(\cdot) \in C^1((0, F^*) \cup (F^*, 1))$  and  $\frac{\partial u}{\partial F}(\cdot, F) > 0$  on  $[-1, 0) \cup (0, 1]$ , except for  $0 < F < F^*$ , where  $\frac{\partial u}{\partial F}(1, F) = 0$ ;
- (ii)  $\lim_{F \nearrow 1} u(y, F) = 1$ ;
- (iii)  $\lim_{F \nearrow F^*} u(y, F) = \lim_{F \searrow F^*} u(y, F) = u(y, F^*)$ ;
- (iv)  $\lim_{F \searrow 0} u(y, F) = \bar{u}(y)$ .

The convergence in (ii)–(iv) is uniform in the subintervals  $[-1, 0)$  and  $(0, 1]$ .

*Proof.* Monotonicity follows directly from the previous results. Therefore we only need to demonstrate the continuity properties (ii)–(iv).

(ii) By Proposition 2.6 we have

$$w(y, F) > \varphi^{-1}(F) \quad \text{for } -1 \leq y < 0,$$

and consequently

$$w(y, F) \geq w(1, F) = \sqrt{\frac{k^-}{k^+}} w(-1, F) > \sqrt{\frac{k^-}{k^+}} \varphi^{-1}(F)$$

for  $0 < y \leq 1$  and  $F > F^*$ . Since  $\varphi^{-1}(F) \rightarrow \infty$  as  $F \nearrow 1$ , the uniform convergence of  $u(\cdot, F)$  follows.

(iii). The result for  $F \nearrow F^*$  is a direct consequence of the continuity of  $\xi_0(F)$ . To establish the result for  $F \searrow F^*$ , we consider the function  $W(\zeta, F)$  for  $F$  near  $F^*$  and  $\zeta$  near  $b = J(0)$ . Direct computation shows

$$(2.49) \quad W'(b, F) = -\sqrt{\frac{k^+}{k^-}} \frac{f(u^*) k_{rw}(u^*)}{f(u^*) - F} < 0.$$

Since  $W(\zeta, F)$  and  $W'(\zeta, F)$  are uniformly continuous in  $\{(\zeta, F) : b \leq \zeta \leq b + \delta, F^* \leq F \leq F^* + \delta\}$  for  $\delta$  sufficiently small, we use (2.48) and (2.49) to find

$$\zeta(F) = w(1, F) \searrow J(0) \quad \text{as } F \searrow F^*.$$

The uniform convergence on both intervals now follows from the  $w(y, F)$  expressions in Theorem 2.9.

(iv). The uniform convergence in  $(0, 1]$  results from  $\xi_0(F) \searrow 0$  as  $F \searrow 0$ . To establish the result in  $[-1, 0)$ , we note that the monotonicity and boundedness of  $u(\cdot, F)$  imply

$$\lim_{F \searrow 0} u(y, F) = \tilde{u}(y), \quad \text{pointwise in } [-1, 0),$$

with  $\tilde{u}(0-) = u^*$ . Moreover, since

$$0 < N_c \sqrt{k^+} k_{rw}(u) f(u) J'(u) \frac{du}{dy} = f(u) - F < 1$$

on  $[-1, 0)$ ,  $u(\cdot, F)$  is uniformly continuous in  $F$ . Hence, by Dini's theorem, the convergence is uniform in  $[-1, 0)$  and  $\tilde{u} \in C([-1, 0))$ . Let  $y_0 \in [-1, 0)$  with  $\tilde{u}(y_0) > 0$ . For  $F > 0$ , the integral equation for  $u(F)$  can be written as

$$\phi(u(0-, F)) - \phi(u(y, F)) + F \int_{u(y, F)}^{u(0-, F)} \frac{k_{rw}(s) J'(s)}{f(s) - F} ds = -\frac{y}{N_c \sqrt{k^+}}.$$

Let  $y = y_0$ . Then, for  $F$  sufficiently small,

$$0 < F \int_{u(y_0, F)}^{u(0-, F)} \frac{k_{rw}(s) J'(s)}{f(s) - F} ds < F \text{ Const} \int_{\tilde{u}(y_0)}^{u(0-, F)} \frac{1}{f(s) - F} ds \rightarrow 0$$

as  $F \searrow 0$ . Hence

$$\phi(u^*) - \phi(\tilde{u}(y_0)) = -\frac{y}{N_c \sqrt{k^+}},$$

implying  $\tilde{u}(y_0) = \bar{u}(y_0)$ . □

**2.5. The effective equation.** Let  $u = u(F)$  denote the unique solution of Problem  $A_u$ . As in section 2.2, we write  $F^0 = F^0(x, t)$  and set

$$u^0(x, y, t) = u(y, F^0(x, t))$$

for  $x \in \mathbb{R}$ ,  $y \in [-1, 0) \cup (0, 1]$ , and  $t > 0$ . The equation for the averaged saturation

$$U(x, t) = \frac{1}{2} \int_{-1}^1 u^0(x, y, t) dy$$

results from (2.24). Integrating this equation in  $y$  and using the continuity of  $F^1(x, \cdot, t)$ , we find

$$(2.50) \quad \frac{\partial U}{\partial t} + \frac{\partial F^0}{\partial x} = 0 \quad \text{for } x \in \mathbb{R}, t > 0.$$

From here on we drop the superscript and write  $F = F^0$ . As a consequence of Theorem 2.10, we note that the cell-averaged saturation  $U = U(F)$  satisfies

$$U \in C([0, 1]) \cap C^1((0, F^*) \cup (F^*, 1)),$$

with

$$\frac{dU}{dF} > 0 \quad \text{on } (0, F^*) \cup (F^*, 1).$$

Moreover,

$$U(0+) = \bar{U}, \quad U(1) = 1,$$



where

$$\bar{U} = \frac{1}{2} \int_{-1}^0 \bar{u}(y) dy.$$

The continuity and monotonicity allow us to define the inverse  $F : [0, 1] \rightarrow [0, 1]$  satisfying, with  $F(U^*) = F^*$ ,

$$F \in C([0, 1]) \cap C^1((\bar{U}, U^*) \cup (U^*, 1))$$

and

$$\frac{dF}{dU} > 0 \quad \text{on } (\bar{U}, U^*) \cup (U^*, 1).$$

Further,

$$F(U) = 0 \quad \text{for } 0 \leq U \leq \bar{U} \text{ and } F(1) = 1.$$

Taking  $F = F(U)$  as a nonlinear flux function in (2.50) results in an effective equation that is a first order conservation law for  $U$ , with  $\bar{U}$  as macroscopic irreducible oil saturation.

Under additional (but usual) assumptions on  $k_{ro}, k_{rw}$ , and  $J$ , we show that (2.50) is of Buckley–Leverett type in the following sense.

**THEOREM 2.11.** *For  $\alpha_o, \alpha_w > 1$  and  $\beta > 0$ , let*

$$\frac{k_{ro}(s)}{s^{\alpha_o}} = O(1), \quad \frac{k_{rw}(s)}{(1-s)^{\alpha_w}} = O(1),$$

and

$$(1-s)^\beta J(s) = O(1).$$

Then  $F \in C^1([0, 1])$  (implying  $F'(\bar{U}) = 0$ ) and  $F'(1) = 0$ .

*Proof.* We first consider the behavior near  $U = \bar{U}$ . Writing (2.40a) in terms of  $u = J^{-1}(w)$  and differentiating with respect to  $F$  yields

$$\frac{\partial u}{\partial F} = \frac{F - f(u)}{k_{rw}(u)f(u)J'(u)} \int_0^u \frac{k_{rw}(s)f(s)J'(s)}{(F - f(s))^2} ds.$$

We now use (2.25) twice to rewrite this expression into

$$\frac{\partial u}{\partial F} = -\frac{\partial u}{\partial y} \int_y^1 \frac{1}{F - f(u(s, F))} ds.$$

Next we integrate in  $y$ . Setting  $U^+(F) = \int_0^1 u(y, F) dy$  and  $a(F) = J^{-1}(\xi_0(F))$ , we find

$$\frac{dU^+}{dF} = \int_0^1 \frac{a(F) - u(s, F)}{F - f(u(s, F))} ds > \frac{1}{F} (a(F) - U^+(F)).$$

Thus

$$\frac{d}{dF}(F U^+(F)) > a(F),$$

implying

$$U^+(F) > \frac{1}{F} \int_0^F a(s) ds \quad \text{for } 0 < F \leq F^*.$$

Since

$$U(F) > \bar{U} + \frac{1}{2}U^+(F),$$

we have

$$(2.51) \quad U(F) > \bar{U} + \frac{1}{2F} \int_0^F a(s) ds \quad \text{for } 0 < F \leq F^*.$$

We need to estimate  $a(F) = u(0+, F)$  from below. For this we use Definition 2.5, i.e.,

$$\int_0^{a(F)} \frac{k_{rw}(s)f(s)J'(s)}{F - f(s)} ds = \frac{1}{N_c \sqrt{k^-}},$$

which gives

$$\frac{1}{F - f(a(F))} \int_0^{a(F)} k_{rw}(s)f(s)J'(s) ds > \frac{1}{N_c \sqrt{k^-}},$$

and further

$$0 < F - f(a(F)) < \mathcal{C}a(F)f(u(F)) \quad \text{for } 0 < F < F^*,$$

where  $\mathcal{C}$  (here and below) denotes a suitably chosen positive constant.

Now using  $f(s)/s^{\alpha_0} = O(1)$  (implied by the asymptotic behavior of  $k_{ro}$ ), we find, for small  $F$ ,

$$a(F) > \mathcal{C}F^{1/\alpha_0}.$$

Combining this with (2.51) gives

$$F(U) < \mathcal{C}(U - \bar{U})^{\alpha_0}$$

in a right neighborhood of  $\bar{U}$ .

Next we consider the differentiability of  $F(U)$  at  $U = U^*$ . For  $F < F^*$  we use (2.40). Differentiating the equations with respect to  $F$  and using the continuity of  $w(y, F)$  gives the existence of  $\xi(y, F^* -)$  directly for each  $y \in [-1, 0) \cup (0, 1]$ . For  $F > F^*$  we first observe that  $\xi(1, F)$  is bounded in a right neighborhood of  $F^*$ . This follows from the proof of Theorem 2.10(iii). Hence, in (2.43),

$$V(w(1, F), F)\xi(1, F) \rightarrow 0 \text{ as } F \searrow F^*,$$

and thus, again using (2.43),  $\xi(y, F^*+) = \xi(y, F^*-)$  for  $y \in (0, 1)$ . A similar argument holds in  $(-1, 0)$ . As a consequence,  $F$  is differentiable at  $U^*$ .

To prove  $F'(1) = 0$ , we construct an upper bound for  $U(F)$  near  $F = 1$ . For  $-1 < y < 0$  we have, as in (2.43),

$$\int_{w(y,F)}^{w(0-,F)} \frac{\gamma(s)\varphi(s)}{(\varphi(s) - F)^2} ds + V(w(0-, F), F)\xi(0, F) = V(w(y, F), F)\xi(y, F).$$

Hence

$$\frac{\partial u}{\partial F} > \frac{f(u) - F}{k_{rw}(u)f(u)J'(u)} \int_{u(y,F)}^{u(0-,F)} \frac{k_{rw}(s)f(s)J'(s)}{(f(s) - F)^2} ds,$$

which can be written as

$$\frac{\partial u}{\partial F} > \frac{\partial u}{\partial y} \int_y^0 \frac{1}{f(u(s, F)) - F} ds.$$

Consequently,  $U^-(F) = \int_{-1}^0 u(y, F)dy$  satisfies

$$\frac{dU^-}{dF} > \int_{-1}^0 \frac{u(s, F) - u(-1, F)}{f(u(s, F)) - F} ds > \frac{1}{1 - F} \{U^- - u(-1, F)\},$$

which implies

$$(2.52) \quad U^-(F) < \frac{1}{1 - F} \int_F^1 u(-1, s)ds.$$

Next we estimate  $u(-1, F)$  from above near  $F = 1$ . Since  $u(1, F) < f^{-1}(F)$ , the periodicity condition implies

$$u(-1, F) < J^{-1} \left( \sqrt{\frac{k^+}{k^-}} J(f^{-1}(F)) \right).$$

Using  $\frac{1-f(s)}{(1-s)^{\alpha_w}} = O(1)$  and  $(1-s)^\beta J(s) = O(1)$ , we find

$$u(-1, F) < 1 - C(1 - F)^{\frac{1}{\alpha_w}} \text{ near } F = 1.$$

Substituting this estimate into (2.52) and using  $U^+(F) < 1$ , we deduce

$$F(U) > 1 - C(1 - U)^{\alpha_w}$$

in a left neighborhood of  $U = 1$ . □

**3. Different scaling for  $N_c$ .** In this section we explain why the general case  $N_c = O(\varepsilon^\gamma)$  reduces to the special cases  $N_c = O(1)$  and  $N_c = O(\varepsilon)$ .

The effective behavior of two-phase flow with trapping on the micro scale is determined by the size of the capillary number  $N_c$ . It is analogous to studying filtration of a viscous fluid through a porous medium. It is known that for moderate Reynolds numbers the effective filtration velocity is given by Darcy's law. For high Reynolds numbers inertia effects are important, and the filtration laws are nonlinear. Finally, for very high Reynolds numbers effective flow is turbulent. Rigorous studies of the effective filtration laws of viscous flows through porous media, with Reynolds numbers being a power of the characteristic pore size  $\varepsilon$ , are carried out in [24], [6], and [23]. Review [25] contains a detailed discussion of the effective behavior. If the power of  $\varepsilon$  exceeds a critical value, the effective filtration is described by Darcy's law. In the critical case, a nonlinear and nonlocal effective filtration law arises; see [24] and [23]. Below the critical power, the viscosity forces are negligible compared to the inertial term, and the effective filtration becomes turbulent. Furthermore, when the power of  $\varepsilon$  approaches the critical value from above, there is an important nonlinear correction to Darcy's law. Then the effective filtration law is polynomial [6], and the transition from linear to nonlinear filtration is continuous [23]. In our situation, all possible cases would be covered if we studied  $N_c$  as a function of  $\varepsilon$ . Therefore we suppose that the capillary number takes the form  $N_c\varepsilon^\gamma$ . Analogous to the discussion above, we identify the following regimes.

$\gamma < 0$ . This means that capillary effects, caused by the surface tension, dominate transport. In this case the expression for  $\frac{\partial u^1}{\partial y}$  doesn't contain  $f$ , and (2.20) changes to

$$(3.1) \quad \frac{\partial U}{\partial t} - N_c \frac{\partial}{\partial x} \left\{ \mathcal{D}(U) \frac{\partial U}{\partial x} \right\} = 0.$$

When  $\gamma$  approaches 0 from below, transport becomes important, and for  $\gamma \approx 0-$  we end up with

$$(3.2) \quad \frac{\partial U}{\partial t} - N_c \frac{\partial}{\partial x} \left\{ \mathcal{D}(U) \frac{\partial U}{\partial x} \right\} + \varepsilon^{-\gamma} \frac{\partial}{\partial x} \mathcal{F}(U) = O(\varepsilon).$$

Here  $\mathcal{D}(U)$  and  $\mathcal{F}(U)$  are calculated as in the case  $N_c = O(1)$ .

$0 < \gamma < 1$ . In this regime capillary effects are still dominating, but the influence of transport increases with  $\gamma$ . The resulting effective equation is

$$(3.3) \quad \frac{\partial U}{\partial t} - N_c \varepsilon^\gamma \frac{\partial}{\partial x} \left\{ \mathcal{D}(U) \frac{\partial U}{\partial x} \right\} + \frac{\partial}{\partial x} \mathcal{F}(U) = O(\varepsilon).$$

Again  $\mathcal{D}(U)$  and  $\mathcal{F}(U)$  are determined as in the case  $N_c = O(1)$ . In deriving (3.3) we used the asymptotic expansion

$$(3.4) \quad u^\varepsilon = u^0 \left( x, \frac{x}{\varepsilon}, t \right) + \varepsilon^{1-\gamma} u^1 \left( x, \frac{x}{\varepsilon}, t \right) + \varepsilon u^2 \left( x, \frac{x}{\varepsilon}, t \right) + \dots,$$

where  $u^0$  is obtained as mentioned at the end of section 2.1, but now (2.18) for  $\frac{\partial u^1}{\partial y}$  reduces to

$$(3.5) \quad F^1 = f(C) - N_c \sqrt{k^+} D(C) \frac{\partial u^1}{\partial y}, \quad -1 < y < 0.$$

When  $\gamma \approx 1-$ , we approach the asymptotic expansion corresponding to the balance case,  $N_c = O(\varepsilon)$ .

$\gamma > 1$ . Now transport is the dominating mechanism. Asymptotic expansion (2.4) leads to an auxiliary problem which has no solution. In analogy with the theory of the effective filtration laws [25], we call this case *turbulent trapping*.

Based on the above observations, we conclude the following. Interpreting  $N_c = O(\varepsilon^\gamma)$ , we find for  $\gamma < 1$  an effective equation of degenerate parabolic type, in which diffusion dominates when  $\gamma < 0$  and convection dominates when  $\gamma > 0$ . In section 2.1 we analyze the typical case  $\gamma = 0$ . When  $\gamma > 1$  (turbulent trapping), no effective equation can be obtained. The critical case  $\gamma = 1$ , in which viscous and capillary forces balance on the micro scale, leads to a nonlinear conservation law. This situation is analyzed in section 2.2. Under the assumption  $N_c = O(\varepsilon^\gamma)$ , this exhausts all flow regimes.

**4. Randomly layered media in the capillary limit.** In this section we drop the periodicity assumption and suppose a stationary ergodic geometrical structure. It is characterized by a probability space  $(\Omega, \mu)$ , with an ergodic dynamical system  $T(x)$ ,  $x \in \mathbb{R}$  (see, e.g., [26] or [12] for details). For a  $\mu$ -measurable subset  $\mathcal{P} \subset \Omega$ , we introduce  $P = P(\omega) \subset \mathbb{R}$  by

$$(4.1) \quad P(\omega) = \{x \in \mathbb{R} : T(x)\omega \in \mathcal{P}\},$$

and we call it a random stationary set.

In our application we suppose that  $P(\omega)$  has the following form:

$$(4.2) \quad P(\omega) = \bigcup_{i \in \mathbb{Z}} (y_{2i-1}, y_{2i}),$$

where the random variables  $y_i \in \mathbb{R}$  are strictly increasing with respect to  $i$ .

A representative example is a Poisson process  $\Pi$  in  $\mathbb{R}$  with constant rate  $\gamma > 0$ . In this case the number of points of  $\Pi$  in an interval  $A = (a, b)$  has expectation  $\gamma(b - a)$ . The number of points of  $\Pi$  in any bounded interval is then finite with probability 1, and  $\Pi$  has no finite limit points. On the other hand, the number in  $(0, +\infty)$  is infinite, so that the points in  $(0, +\infty)$  can be written in order as

$$0 < y_1 < y_2 < y_3 < \dots$$

Similarly the points in  $(-\infty, 0)$  can be written in order as

$$\dots < y_{-3} < y_{-2} < y_{-1} < 0.$$

These exhaust the points of  $\Pi$ , since the probability that  $0 \in \Pi$  is equal to 0. The  $y_n$  are random variables, and the subsequences  $\{y_n, n \leq -1\}$  and  $\{y_n, n \geq 1\}$  are independent, with the same joint distributions. Furthermore, the random variables  $\ell_1 = y_1$ ,  $\ell_n = y_n - y_{n-1}$  ( $n \geq 2$ ),  $\ell_{-1} = -y_{-1}$ ,  $\ell_{-n} = y_{-n+1} - y_{-n}$  ( $n \geq 2$ ) are independent, and each has probability density  $g(y) = \gamma e^{-\gamma|y|}$ . The number of points  $N(0, t]$  of  $\Pi$  in  $(0, t]$  satisfies the law of large numbers:

$$\lim_{t \rightarrow +\infty} \frac{1}{t} N(0, t] = \gamma \quad \text{with probability 1.}$$

Finally, the process of Poisson is ergodic. Another example is that of hardcore processes (Gibbs processes, Matérn processes, etc.). We construct them from a Poisson

point process by eliminating all points having a distance to their neighbors smaller than a prescribed value. They satisfy the mixing property, and the ergodicity is assured.

By Birkhoff's ergodic theorem there exists a density (fraction of high permeability layers) of  $P$ , given by

$$(4.3) \quad \varphi^+ := \mu(P) = \lim_{N, M \rightarrow \infty} \frac{1}{y_{2M+1} - y_{-2N-1}} \sum_{-N}^M |y_{2i}(\omega) - y_{2i-1}(\omega)|$$

for almost all  $\omega \in \Omega$ , satisfying

$$0 \leq \varphi^+ \leq 1.$$

The corresponding random permeability is given by

$$(4.4) \quad k(x, \omega) = k(T(x)\omega) = \begin{cases} k^+(\omega) & \text{if } x \in P, \\ k^-(\omega) & \text{if } x \in \mathbb{R} \setminus \overline{P}, \end{cases}$$

and it is a stationary random variable. Then

$$(4.5) \quad k^\varepsilon(x, \omega) = k(T(x/\varepsilon)\omega).$$

As a consequence,  $u^* = u^*(\omega)$  through (1.16).

Next we turn to the two-scale expansion for the saturation and the flux, adapted to the stochastic case. We write

$$(4.6) \quad F^\varepsilon = \varepsilon^{-1}F^0 + \varepsilon^0F^1 + \varepsilon F^2 + \dots,$$

where  $F^k$  are stationary ergodic random fields and

$$(4.7) \quad u^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots$$

From these expansions and (2.8) we obtain directly

$$(4.8) \quad \frac{dF^0}{dy} = 0, \quad \text{implying } F^0 = F^0(x, t),$$

with the random variable  $u^0$  satisfying (2.10). We reconsider this equation for a given realization  $\omega$ ; see Figure 4.1. As before, we want to show  $F^0 = 0$ . Suppose  $F^0 < 0$ . Introducing  $w$  and  $\lambda$  as in section 2.1, we again obtain

$$(4.9) \quad \lambda(w)\sqrt{k} \frac{dw}{dy} = -\frac{F^0}{N_c} =: F > 0.$$

We argue below that this inequality does not permit us to construct a global nonnegative solution satisfying the matching conditions at the interfaces. Suppose  $w(y_{2i} - 0) \leq J(u^*)$ . By (4.9), implying strict monotonicity of  $w$ , we have  $w(y_{2i-1} + 0) < J(u^*)$ , giving  $w(y_{2i-1} - 0) = J(0)$ . This contradicts the monotonicity of  $w$  in  $(y_{2i-2}, y_{2i-1})$ . Next suppose  $w(y_{2i} - 0) > J(u^*)$ . Then  $w(y_{2i} + 0) = \sqrt{k^-/k^+}w(y_{2i} - 0)$  and  $w(y_{2i+1} - 0) \geq w(y_{2i} + 0) + \frac{F}{\|\lambda\|_\infty} \frac{1}{\sqrt{k^-}} |y_{2i+1} - y_{2i}|$ . Therefore we have  $w(y_{2i+1} + 0) = \sqrt{k^+/k^-}w(y_{2i+1} - 0) \geq w(y_{2i-1} + 0) + \frac{F}{\|\lambda\|_\infty} \left\{ \frac{\sqrt{k^+}}{k^-} |y_{2i+1} - y_{2i}| + \frac{1}{\sqrt{k^+}} |y_{2i} - y_{2i-1}| \right\}$ .

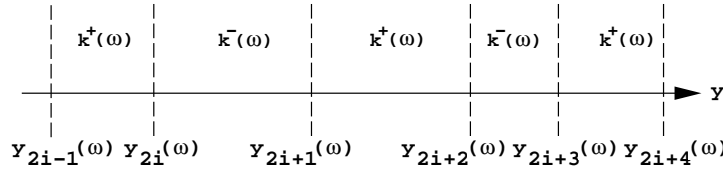


FIG. 4.1. Random distribution of layers.

Repeating this reasoning backwards in  $i$  shows that  $w$  will drop below  $J(u^*)$  at the right side of a certain transition, again yielding a contradiction. Hence  $F^0 \geq 0$ . A similar argument gives  $F^0 \leq 0$ , and so  $F^0 = 0$ .

This implies for  $u^0$ , with  $i \in \mathbb{Z}$ ,

$$u^0(y, \omega) = \begin{cases} C(\omega) > u^* & \text{for } y_{2i-1}(\omega) < y < y_{2i}(\omega), \\ J^{-1}\left(\sqrt{\frac{k^-}{k^+}} J(C(\omega))\right) & \text{for } y_{2i}(\omega) < y < y_{2i+1}(\omega), \end{cases}$$

or

$$u^0(y, \omega) = \begin{cases} C(\omega) \leq u^* & \text{for } y_{2i-1}(\omega) < y < y_{2i}(\omega), \\ 0 & \text{for } y_{2i}(\omega) < y < y_{2i+1}(\omega). \end{cases}$$

Now consider the  $\varepsilon^{-1}$ -equation (2.11). Since  $F^0 = 0$ , the ergodicity of  $F^1$  implies  $F^1 = F^1(x, t)$ , which is given by

$$F^1 = f(u^0) - N_c \sqrt{k(\omega)} D(u^0) \left( \frac{\partial u^1}{\partial y} + \frac{\partial u^0}{\partial x} \right).$$

Suppose  $C(\omega) \leq u^*$ . Then  $F^1 = 0$  on  $(y_{2i-1}(\omega), y_{2i}(\omega))$  implies  $F^1 = 0$  for all  $x \in \mathbb{R}$  and  $t > 0$ . If  $C(\omega) > u^*$ , then we have on  $(y_{2i-1}(\omega), y_{2i}(\omega))$

$$\frac{\partial u^1}{\partial y} = \frac{f(C(\omega)) - F^1}{\sqrt{k^+(\omega)} N_c D(C(\omega))} - \frac{\partial C(\omega)}{\partial x} =: B_1(\omega).$$

On  $(y_{2i}(\omega), y_{2i+1}(\omega))$  we have

$$\frac{\partial u^1}{\partial y} = \frac{f(\bar{C}(\omega)) - F^1}{\sqrt{k^-(\omega)} D(\bar{C}(\omega))} - \frac{\partial \bar{C}(\omega)}{\partial x} =: B_2(\omega),$$

with  $\bar{C}$  as in (2.16). Since  $\frac{\partial u^1}{\partial y}$  is the local representation of a stationary random variable with zero mean, we have that the mean value of

$$\chi_{\{k=k^+\}} B_1(\omega) + \chi_{\{k=k^-\}} B_2(\omega)$$

is zero. Here  $\chi$  denotes the characteristic function. Hence

$$\begin{aligned} & \frac{\varphi^+}{\sqrt{k^+(\omega)}} \frac{f(C(\omega)) - F^1}{N_c D(C(\omega))} + \frac{1 - \varphi^+}{\sqrt{k^-(\omega)}} \frac{f(\bar{C}(\omega)) - F^1}{N_c D(\bar{C}(\omega))} \\ &= \varphi^+ \frac{\partial C(\omega)}{\partial x} + (1 - \varphi^+) \frac{\partial \bar{C}(\omega)}{\partial x}. \end{aligned}$$

Solving for  $F^1$  gives (dropping the  $\omega$ -dependence), for  $C(\omega) > u^*$ ,

$$(4.10) \quad F^1 = \frac{\frac{\varphi^+}{\sqrt{k^+}} \frac{f(C)}{D(C)} + \frac{1-\varphi^+}{\sqrt{k^-}} \frac{f(\bar{C})}{D(\bar{C})}}{\frac{\varphi^+}{\sqrt{k^+}} \frac{1}{D(C)} + \frac{1-\varphi^+}{\sqrt{k^-}} \frac{1}{D(\bar{C})}} - N_c \frac{\varphi^+ \frac{\partial C}{\partial x} + (1-\varphi^+) \frac{\partial \bar{C}}{\partial x}}{\frac{\varphi^+}{\sqrt{k^+}} \frac{1}{D(C)} + \frac{1-\varphi^+}{\sqrt{k^-}} \frac{1}{D(\bar{C})}}.$$

Averaging (2.12) and using the ergodicity of  $F^2$  yields the effective transport equation

$$(4.11) \quad \frac{\partial U}{\partial t} + \frac{\partial F^1}{\partial x} = 0 \quad \text{for } -\infty < x < \infty, t > 0,$$

where  $U$  denotes the averaged oil saturation

$$U = \varphi^+ C + (1 - \varphi^+) \bar{C}.$$

We note that for  $\varphi^+ u^* \geq U > 0$ ,  $F^1 = 0$ . In the periodic case  $\varphi^+ = \frac{1}{2}$ . Hence for each realization  $\omega$  we obtain an equation of the “periodic” form (2.20).

We stress that the averaged saturation  $U$  is deterministic. This is implied by (4.11) and by the deterministic initial condition. Consequently, it suffices to consider only one realization to determine  $U$  and its corresponding flux  $F^1$ .

*Remark 4.1.* In the case of the balance  $N_c = O(\varepsilon)$ , we could proceed analogously. Now we should solve the problem (2.25)–(2.26) on the real line for every realization. The periodicity condition (2.27) is replaced by the condition that  $u$  take values between 0 and 1 on  $\mathbb{R}$ . We note that the matching condition is now posed at every point  $y_i$ ,  $i \in \mathbb{Z}$ . Solving the auxiliary problem  $A_u$  in the stochastic case is much more complicated than in the periodic case. The analysis of the periodic case was already quite lengthy, and in the proofs of Proposition 2.6 and Theorem 2.7 periodicity was essential. Also the unboundedness of  $J$  complicates proofs. Using arguments from this section we are able to conclude that  $F \in [0, 1]$ , but the complete construction is still an open problem. We expect to consider randomly layered media in the limit  $N_c = O(\varepsilon)$  in a future publication.

**5. Numerical results.** Since we have no convergence proof, we are going to verify the homogenization procedure numerically for the periodic case. Both the capillary limit and the balance will be considered. We will use the Leverett model with Corey relative permeabilities and Brooks–Corey capillary pressure [9], [7]. Specifically, the following functions and parameters are used:

$$k_{ro}(u) = u^2, \quad k_{rw}(u) = (1 - u)^2, \quad J(u) = 10(1 - u)^{-\frac{1}{2}}, \quad M = 1, \quad k^+ = 1, \quad k^- = 0.5,$$

with  $N_c$  being either 1 or  $\varepsilon$ , depending on the case. In the asymptotic expansion, we assumed that both the fractional flow  $f$  and the diffusivity  $D$  were of order 1. To maintain this assumption in the numerical experiments, we include the factor 10 in the Leverett function.

Tests are done on the interval  $(-1, 1)$ , i.e.,  $L_x = 1$ . For both cases we compute the full problem with a periodic micro structure, as shown in Figure 1.3; i.e.,  $k(x) = k^+$  in the coarse layers, and  $k(x) = k^-$  in the fine layers. The thickness  $L_y$  of the layers is related to the number of cells and determines the expansion parameter  $\varepsilon = L_y/L_x$ . The matching conditions defined in (1.13) and (1.14) or (1.15) are imposed at the interfaces separating the two types of materials. The resulting solution is averaged on each micro cell consisting of two adjacent layers. This average is compared with the numerical solution of the effective equations.



In the tests, we consider a medium originally saturated by oil ( $u(x, 0) = 1, x \in (-1, 1)$ ), with water injection from the left ( $u(-1, t) = 0$ ). At  $x = 1$  the Neumann condition is chosen not to affect the flow. To demonstrate convergence, we take for  $\varepsilon$  the values  $1/20, 1/40$ , and  $1/80$ . In the capillary limit the differences are only noticeable near the injection point  $x = -1$ , which is a direct consequence of the thickness of the layers. Away from the injection point the averaged saturations are nearly indistinguishable. Therefore we present in Figure 5.2 the capillary limit computations only for  $\varepsilon = 1/40$ , showing excellent agreement with the upscaled saturation. In the balance case, letting  $\varepsilon \searrow 0$ , we observe a more significant improvement. Therefore we compare in Figure 5.5 the upscaled saturation with the averaged saturations obtained for the three selected values of  $\varepsilon$ .

**5.1. Capillary limit ( $N_c = O(1)$ ).** In this case we take  $N_c = 1$ . Inside each layer of constant permeability we apply a first order explicit discretization scheme with upwind finite volumes. With  $u_i^n$  denoting the approximate oil saturation at  $t_n = n\tau$  inside the volume centered at  $x_i = (i - 1/2)h$  ( $\tau$  being the time-step and  $h$  the grid size), the solution at the next time-step follows from

$$(5.1) \quad u_i^{n+1} = u_i^n - \frac{\tau}{h} \left( F_{i+1/2}^n - F_{i-1/2}^n \right).$$

Here  $F_{i+1/2}^n$  approximates the flux at  $t = t_n$  and  $x = x_i + h/2 = ih$ , the edge between the volumes centered in  $x_i$  and  $x_{i+1}$ . Likewise,  $F_{i-1/2}^n$  approximates the flux at  $t = t_n$  and  $x = x_i - h/2 = (i - 1)h$ .

Assume first that  $x = ih$  lies inside a homogeneous microlayer. Following [16], we rewrite (1.7a) by means of the Kirchhoff transform

$$(5.2) \quad \beta(u) = \int_0^u k_{rw}(v) f(v) J'(v) dv.$$

Note that  $\beta$  is strictly increasing and smooth due to the properties of  $k_{rw}$ ,  $f$ , and  $J$ . In general, the integration cannot be carried out explicitly. Therefore, we need to construct a table of pairs  $(u, \beta(u))$ . In doing so we apply an adaptive quadrature method.

By this transform, the flux in (1.7b) becomes

$$(5.3) \quad F = f(u) - N_c k(x) \frac{\partial}{\partial x} \beta(u).$$

Since the flow is from left to right, a first order upwind approximation is

$$(5.4) \quad F_{i+1/2}^n = f(u_i^n) - N_c \sqrt{k(x_i)} \frac{\beta(u_{i+1}^n) - \beta(u_i^n)}{h},$$

where the permeability  $k(x_i)$  is either  $k^+$  or  $k^-$ , depending on the type of the material.

Computing the flux at a position where the permeability and saturation are discontinuous requires more attention. Let us assume that this position is located at  $x = ih$ , thus separating the control volumes centered in  $x_i$  and  $x_{i+1}$ . Moreover, let  $k(x_i) = k^+$  and  $k(x_{i+1}) = k^-$ . As in [8] and [14], we introduce two sets of dummy variables at  $ih$ ,  $u_{i+}^n$ , and  $u_{i-}^n$  for all  $n = 0, 1, 2, \dots$ , which satisfy the pressure condition in (1.15),

$$(5.5) \quad \begin{aligned} u_{i-}^n < u^* &\text{ implies } u_{i+}^n = 0, \text{ or} \\ u_{i-}^n \geq u^* &\text{ implies } \frac{J(u_{i-}^n)}{\sqrt{k^+}} = \frac{J(u_{i+}^n)}{\sqrt{k^-}}, \end{aligned}$$

and for which the numerical flux is continuous at  $x_{i+1/2} = ih$ .

Given a pair  $u_{i\pm}^n$  as above, the flux at the interface is calculated analogously to (5.4): in the coarse sublayer

$$(5.6) \quad F_{i+1/2}^n = f(u_i^n) - N_c \sqrt{k^+} \frac{\beta(u_{i-}^n) - \beta(u_i^n)}{h/2},$$

and in the fine sublayer

$$(5.7) \quad F_{i+1/2}^n = f(u_{i+}^n) - N_c \sqrt{k^-} \frac{\beta(u_{i+1}^n) - \beta(u_{i+}^n)}{h/2}.$$

These fluxes are equal due to the assumptions on  $u_{i\pm}^n$ . Knowing all fluxes, the oil saturation  $u_i^{n+1}$  at all interior points follows from (5.1).

To determine the dummy variables  $u_{i\pm}^{n+1}$  at  $t = t_{n+1}$ , we proceed as follows. Knowing  $u_i^{n+1}$  and  $u_{i+1}^{n+1}$ , we use (5.6) and (5.7) to obtain

$$f(u_i^{n+1}) - N_c \sqrt{k^+} \frac{\beta(u_{i-}^{n+1}) - \beta(u_i^{n+1})}{h/2} = f(u_{i+}^{n+1}) - N_c \sqrt{k^-} \frac{\beta(u_{i+1}^{n+1}) - \beta(u_{i+}^{n+1})}{h/2}.$$

Defining  $g : [0, 1]^2 \rightarrow \mathbb{R}$  by

$$g(u, v) = \frac{h}{2} f(v) + N_c \sqrt{k^+} \beta(u) + N_c \sqrt{k^-} \beta(v),$$

we write equivalently

$$(5.8) \quad g(u_{i-}^{n+1}, u_{i+}^{n+1}) = \frac{h}{2} f(u_i^{n+1}) + N_c \sqrt{k^+} \beta(u_i^{n+1}) + N_c \sqrt{k^-} \beta(u_{i+1}^{n+1}) =: T.$$

The CFL restriction implies  $0 \leq T \leq h/2 + N_c(\sqrt{k^-} + \sqrt{k^-})\beta(1)$ . Note that  $g$  is continuous and strictly increasing in both variables.

To solve (5.8) subject to the pressure condition (5.5), we distinguish two cases. Assume first that  $0 \leq T \leq (h/2)f(u^*) + N_c \sqrt{k^-} \beta(u^*)$ . Clearly there exists a unique  $u \in [0, u^*]$  so that  $g(u, 0) = T$ . Since  $0 \leq u \leq u^*$ ,  $u_{i-}^{n+1} = u$  and  $u_{i+}^{n+1} = 0$  are choices fulfilling both continuity of flux and the pressure condition. For any pair  $u_{i\pm}^{n+1}$  satisfying (5.5) with  $u_{i-}^{n+1} > 0$  we have  $g(u_{i-}^{n+1}, u_{i+}^{n+1}) > (h/2)f(u^*) + N_c \sqrt{k^-} \beta(u^*) \geq T$ ; hence no other solutions are possible.

If  $(h/2)f(u^*) + N_c \sqrt{k^-} \beta(u^*) < T \leq h/2 + N_c(\sqrt{k^-} + \sqrt{k^-})\beta(1)$ , the same argument shows that (5.8) has no solution of the form  $(u, 0)$ , with  $0 \leq u \leq u^*$ . Recalling (5.5), the pressure becomes continuous in this case and

$$(5.9) \quad u_{i+}^{n+1} = J^{-1}(\sqrt{k^-/k^+} J(u_{i-}^{n+1})) > 0.$$

Monotonicity and continuity of  $g$  and  $J$  ensure the existence of a unique  $u_{i-}^{n+1} \in (u^*, 1]$  satisfying  $g(u_{i-}^{n+1}, u_{i+}^{n+1}) = T$ , with  $u_{i+}^{n+1} > 0$  given by (5.9).

In this way we obtain a unique pair of dummy variables at transitions from coarse to fine. A similar procedure is applied at fine to coarse transitions. Details are omitted.

As explained in section 2.1, the effective equation is known explicitly in the capillary limit. Figure 5.1 shows the effective diffusivity  $\mathcal{D}$  and convection  $\mathcal{F}$  in terms of the cell-averaged oil saturation  $U$ . Here we use the relative permeabilities and Leverett function as proposed for this section. This equation is of degenerate parabolic

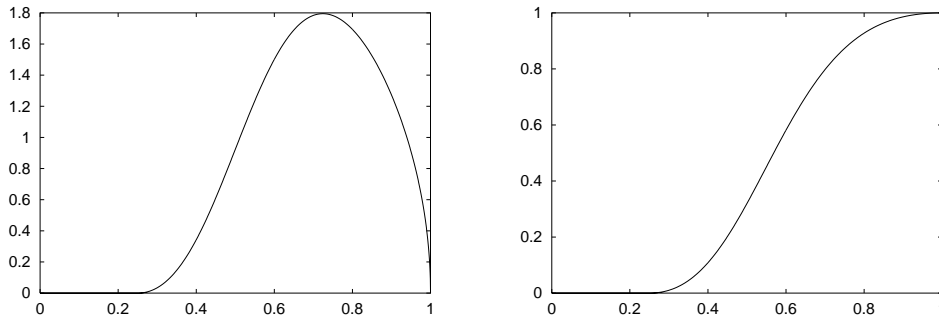


FIG. 5.1. *Effective diffusion (left) and convection (right) for the Brooks–Corey model. Note that  $\mathcal{D}(U) = \mathcal{F}(U) = 0$  for  $0 \leq U \leq 1/2u^* = 0.25$ .*

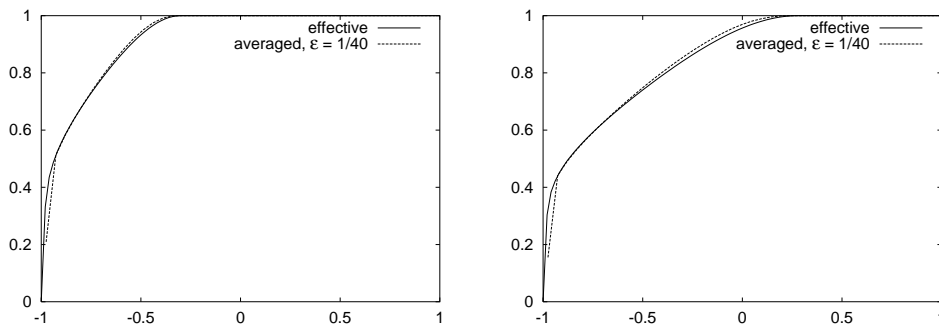


FIG. 5.2. *Effective and averaged oil saturation at  $t = 0.3$  and  $t = 1.0$ .*

type, since the effective diffusion  $\mathcal{D}(U)$  vanishes for  $0 \leq U \leq \frac{1}{2}u^*$  and at  $U = 1$ . Several numerical methods can be applied to this kind of problem. Here we use the explicit upwind scheme (see [28]) for the convergence analysis:

$$U_i^{n+1} = U_i^n - \frac{\tau}{h} \left( F_{i+1/2}^n - F_{i-1/2}^n \right),$$

$$F_{i+1/2}^n = \mathcal{F}(U_i^n) - \mathcal{D} \left( \frac{U_i^n + U_{i+1}^n}{2} \right) \frac{U_{i+1}^n - U_i^n}{h}.$$

Figure 5.2 shows the solution of the effective equation (solid line) and the average of the solution of the full problem (dashed line) at  $t = 0.3$  and  $t = 1.0$ . Here we used  $\varepsilon = 1/40$ . Smaller values give results that cannot be distinguished on the scale of the figure. Since oil is being displaced from the column, both solutions are above the macroscopic irreducible oil saturation corresponding to the maximum amount of trapped oil:  $U = 1/2u^* = 0.25$ . Since diffusion dominates, a long time is needed for reaching this value.

The solution of the original problem is shown in Figure 5.3, together with its cell-average. A part of the flow domain is enlarged in the graph on the right. Note the good agreement with the theoretical results: the profile is highly oscillatory on the macro scale and quite flat within the micro structure. Further note that even though the original problem is of a degenerate type, free boundaries do not occur inside the homogeneous sublayers. As a consequence the solution behaves in a fairly

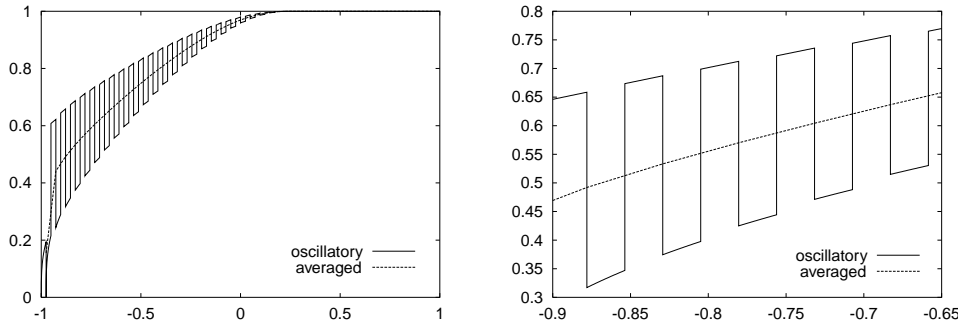


FIG. 5.3. Full problem; averaged and oscillatory oil saturation at  $t = 1.0$ , full (left) and zoomed view (right).

nondegenerate manner and thus smoothly. Therefore a relatively coarse grid was used:  $h = L_y/20 \equiv \varepsilon L_x/20$ . Since the numerical method is explicit, the time-step  $\tau$  is restricted by a CFL condition.

**5.2. Balance ( $N_c = O(\varepsilon)$ ).** To compute the solution for the full problem with  $N_c = \varepsilon$ , we proceed as in the previous section. However, the effective equation requires more attention. As shown in section 2.5, this equation is of the Buckley–Leverett type, but the fractional flow function is not known explicitly. In this case a table of values for the pairs  $(U, \mathcal{F})$  has to be constructed, where  $\mathcal{F}$  ranges from 0 to 1. For a given  $\mathcal{F}$  value from this table, we compute the solution  $u(\mathcal{F})$  of the auxiliary problem ( $A_u$ ) defined in section 2.2 and calculate its cell-average as the corresponding  $U$  value in the table. For the purpose of this paper we took  $\mathcal{F}_i = i\Delta\mathcal{F}$ , with  $\Delta\mathcal{F} = 10^{-3}$  and  $i = 0, 10^3$ . As stated in Lemma 2.4, we take  $\mathcal{F}(U) = 0$  for all  $U \in [0, \bar{U}]$ ,  $\bar{U}$  being the average of the maximal steady state solution corresponding to  $l = u^*$ .

To find accurate solutions of Problem  $A_u$ , we first modify the differential equation through the Kirchhoff transform defined in (5.2). Thus, instead of solving Problem  $A_u$ , we consider the equivalent, as given next.

*Problem  $A_\theta$ .* Given  $\mathcal{F}_i$ , find  $\theta : [-1, 0) \cup (0, 1] \rightarrow \mathbb{R}$  satisfying

$$(5.10) \quad f(\beta^{-1}(\theta)) - N_c \sqrt{k} \frac{d\theta}{dy} = \mathcal{F}_i \text{ in } (-1, 0) \cup (0, 1),$$

together with the corresponding matching and periodicity conditions defined in (2.26) and (2.27).

The matching and periodicity conditions can be viewed as boundary conditions for (5.10) on the two subintervals. To find a solution  $u(\mathcal{F}_i)$  we have applied the following shooting procedure. Choose  $\theta(1) \geq 0$  and use this value as the initial condition for (5.10) on  $(0, 1)$ . This yields the corresponding  $\theta(0+)$  and, by the matching conditions,  $\theta(0-)$ . Use this value as the initial condition for (5.10) on  $(-1, 0)$ . Then adjust  $\theta(1)$  so that  $\theta(1)$  and  $\theta(-1)$  satisfy the periodicity condition. In carrying out this shooting procedure several technical difficulties had to be resolved. We omit the details in this paper.

Figure 5.4 shows the effective oil fractional flow function for the specific model considered in this section. Observe that indeed we have recovered a Buckley–Leverett model in which the fractional flow has only one inflection point. Note that the theoretical analysis resulted only in  $\mathcal{F}'(\bar{U}) = \mathcal{F}'(1) = 0$ . No statements about inflection

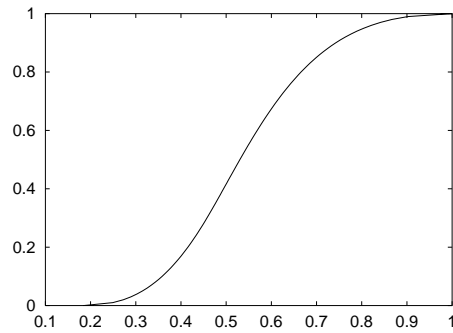


FIG. 5.4. *Effective oil fractional flow for the Brooks-Corey model. Note that  $\mathcal{F}(U) = 0$  for  $0 \leq U \leq \bar{U} \approx 1.846 \cdot 10^{-1}$ .*

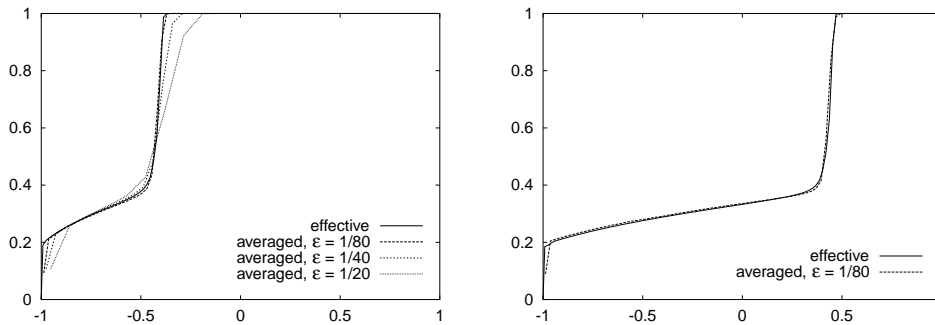


FIG. 5.5. *Effective and averaged oil saturation at  $t = 0.4$  and  $t = 1.0$ .*

points could be given. Also note that the upscaled fractional flow contains details of the small scale capillary forces. This effect does not appear explicitly but it is present due to Problem  $A_u$ . Finally, note that the macroscopic irreducible oil saturation  $\bar{U}$  is smaller than the one obtained in the capillary limit. This is to be expected because the capillary forces are now  $O(\varepsilon)$ , leading to nonconstant steady states.

Once the effective convection is known, the oil saturation equation is solved by the first order explicit upwind scheme

$$U_i^{n+1} = U_i^n - \frac{\tau}{h} (\mathcal{F}(U_i^n) - \mathcal{F}(U_{i-1}^n)).$$

Figure 5.5 shows the solution of the effective equation (solid line) and the cell-average of the solution of the full problem (dashed lines) at  $t = 0.4$  and  $t = 1.0$ . The graph on the left provides numerical evidence of convergence as  $\varepsilon \searrow 0$ . As expected, a rarefaction part is followed by a shock in the structure of the solutions. Note the good agreement between the effective solution and the averaged one computed for  $\varepsilon = 1/80$ .

The solution of the full problem together with its average is shown in Figure 5.6, with a zoomed view in the graph on the right. Note the highly oscillatory profile on both scales. Large gradients occur inside almost every homogeneous sublayer as a consequence of the small diffusivity ( $O(\varepsilon)$ ). In this case the computational grid has to be sufficiently fine to approximate accurately both the local solution and macroscopic

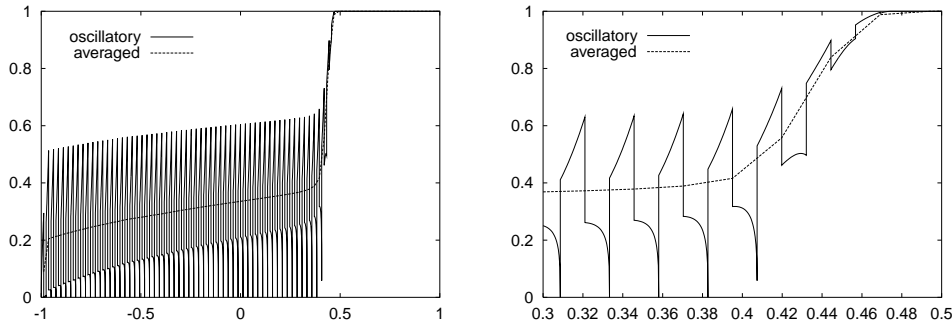


FIG. 5.6. Full problem; averaged and oscillatory oil saturation at  $t = 1.0$ , full (left) and zoomed view (right).

irreducible oil saturation. We used  $h = L_y/30 \equiv \varepsilon L_x/30$  and  $\tau$  again restricted by a CFL condition.

## 6. Conclusions.

- The results of this paper lead to the following conclusions.
- For  $N_c = O(1)$  (capillary limit) the effective equation is explicitly known and of degenerate parabolic type. The diffusion and convection vanish up to the macroscopic irreducible oil saturation  $\bar{U} = \frac{1}{2}u^*$ .
  - For  $N_c = O(\varepsilon)$  (balance) the scaled-up equation is of Buckley–Leverett type, with effects of the local capillary forces in the fractional flow function.
  - The macroscopic irreducible oil saturation depends strongly on the value of the capillary number.
  - The solution of the auxiliary problem in the capillary limit has two constant states connected by the pressure condition at the interface.
  - The solution of the auxiliary problem in the balance is unique and can be classified completely.
  - The choice of the characteristic values in (1.6) is important for deciding which of the two cases (capillary limit or balance) applies in a real situation.
  - Random layers are considered only in the capillary limit. The effective equation is similar to the periodic one.
  - The method used in this paper can be applied to heterogeneous media in which the porosity, relative permeabilities, and Leverett function are periodic as well.

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