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Effective-medium theories for fluid-saturated materials with aligned cracks

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ABSTRACT

There is general agreement between different theories giving expressions for the overall properties of materials with dry, aligned cracks if the number density of cracks is small. There is also very fair agreement for fluid-filled isolated cracks. However, there are considerable differences between two separate theories for fluid-filled cracks with equant porosity. Comparison with recently published experimental data on synthetic sandstones gives a good fit with theory for dry samples. However, although the crack number density in the laboratory sample is such that first-order theory is unlikely to apply, expressions correct to second order (in the number density) provide a worse fit. It also appears that the ratio of wavelength to crack size is not sufficiently great for any detailed comparison with effective-medium theories, which are valid only when this ratio is large. The data show dispersion effects for dry cracks and scattering, neither of which will occur at sufficiently long wavelengths. Data from the water-saturated samples indicate that the effect of equant porosity is significant, although the two theories differ strongly as to just how significant. Once again, and in spite of the reservations mentioned above, a reasonable fit between theory and observation can be shown.

INTRODUCTION

Homogeneous material which has been rendered heterogeneous by the existence of a statistically homogeneous distribution of small-scale cracks appears still to be homogeneous, but with changed values of its material constants, to waves of wavelength large compared with the 'crack' size. If the cracks are partially or fully aligned, the overall properties as experienced by long-wavelength waves will be anisotropic, and observations of anisotropy have for some time been regarded as diagnostic of microfracturing in preferred directions (Crampin and Booth 1985). It is naturally very desirable to be able to extract more information from the observational data than the mere existence of fracturing. For a full characterization of underground reservoirs it is necessary to know crack size and number density, the nature of the crack infill (gas, liquid, gouge, etc.) and the degree of connectivity between cracks by fluid pathways (i.e. global and local permeability). As a result, a range of effective-medium theories have been introduced, including first-order theories (e.g. Eshelby 1957; Anderson, Minster and Cole 1974), the self-consistent method (Budiansky 1965; Hill 1965; O'Connell and Budiansky 1974), the method of smoothing (Karal and Keller 1964; Hudson 1986) and the differential method (Bruner 1976; Nishizawa 1982). Each of these theories predicts the overall wave speeds and polarizations of long-wavelength waves in cracked materials in terms of the properties of the uncracked solid and the parameters describing the cracks and their distribution. The crack shapes are idealized of course (usually circular) but the results have the advantage over those of empirical theories (e.g. Biot 1956) in that at least some inferences about the average microproperties of the cracks can be made.

The idealization of the crack model and, in some cases, the nature of the approximations made in order to cover a range of higher crack densities, means that confidence in the theoretical results would be greatly increased if there were experimental or numerical support. Two-dimensional evaluations for randomly orientated dry cracks using a boundary-element method (Kachanov 1992; Davis and Knopoff 1995) show agreement with all theories at low crack densities. At higher crack densities the closest fit was, rather surprisingly, the first-order result written in terms of compliances – but not the first-order result in terms of stiffnesses. A later numerical experiment of the same kind by Dahm and Becker (1998) showed closer agreement with the predictions of the differential method. The reason for the lack of agreement between these computational studies is probably the fact that, in general, an accurate description of the properties of materials with a high crack density depends on the high-order statistics of the crack distribution (Keller 1964). These statistics will no doubt vary from one experiment to another. Three-dimensional numerical studies of this kind remain out of reach at present owing to the computation time required.

In contrast to the oversimplification of the numerical models, experimental studies have generally contained features that have made direct comparisons difficult. One of the problems for the experimentalist is to construct a model with empty or liquid-filled cracks with known positions and orientations. Ass'ad, Tatham and McDonald (1992) substituted rubber discs for cracks in a model material built up with successive layers of epoxy resin. The results showed reasonable agreement with the formulae presented by Hudson (1980, 1981) for cracks containing weak material and crack densities up to 7%; the crack density ζ is calculated as (νa^3), where ν is the number of cracks per unit volume and a is the radius of a crack. Hudson's (1980, 1981) results were calculated by the method of smoothing, working to second order in ζ, which implies that pair-wise interactions between cracks had been taken into account. However, it is not clear from the description how Ass'ad *et al.* (1992) determined the values of the mechanical constants of the rubber discs nor what these values were.

Peacock *et al.* (1994) used samples of naturally occurring rock (Carrara marble) for a similar series of experiments. Their observations showed good agreement with Hudson's (1986) theoretical results for randomly orientated dry cracks for crack densities up to 8%. However, the authors expressed some doubt about the accuracy of measurements of the actual crack density in the sample. The agreement between theory and experiment was poor for water-saturated samples. The theoretical methods mentioned so far assume that each crack is isolated from the others and that the fluid is sealed in. This is not generally the case in sedimentary rocks although Peacock *et al.* (1994) used wave periods much shorter than the relaxation time for linear diffusion in the sample, so that pressure leakage from individual cracks should not have been significant.

The first experiments on a synthetic rock-like material containing a known distribution of cracks were performed by Rathore *et al.* (1995) and we discuss these in the following sections. We also review the different theories which predict the overall properties of cracked materials. Such theories largely agree with each other for isolated cracks at low crack densities but there are significant differences in the predictions of the effects of fluid-flow.

DRY CRACKS: EXPERIMENT AND THEORY

The experiments of Rathore *et al.* (1995) were performed on a synthetic sandstone with parallel circular cracks of aspect ratio (thickness/diameter) 0.004. The material was either completely dry or liquid-saturated. The crack density took the value 0.1 in the case of dry cracks; with $\zeta = 0.1$, theoretical results for the elastic parameters predicted by first-order theory, the method of smoothing (to second order), the differential method and the self-consistent approximation all agree to within 4.25% (Sayers and Kachanov 1991). Reasonable agreement was obtained between observations of the variation of wave speeds with angle in the dry sample and theoretical results obtained from Hudson (1980) and Thomsen (1995). Ironically, Rathore *et al.* (1995) found an even better fit if the second-order terms in Hudson's (1980) theory were dropped. If this were done, the results of the method of smoothing would be identical to those of first-order theories in which the stiffnesses are written as a linear expression in ζ, and also to the first iteration of the differential method and to the results of the self-consistent approximation expanded in the same way. (It was the expansion of the *compliances* up to $O(\zeta)$ that the two-dimensional numerical studies (Kachanov 1992; Davis and Knopoff 1995) found were accurate across a surprisingly large range of values of ζ.)

Effective-medium theories of the kind referred to here are long-wavelength theories where the response of the solid medium appears smooth and, in this case, homogeneous. The waveforms from transmission through the dry, uncracked, synthetic sandstone, which is itself heterogeneous and porous although on a scale much smaller than the cracks, are indeed similar to those for a homogeneous material with no evidence of either scattering or dispersion (Rathore *et al.* 1995, fig. 2). In contrast, the waveforms for transmission through the dry, cracked sandstone show evidence of both as noted by Thomsen (1995). For P-waves, the period of the first cycle of motion varies by up to 50% depending on the direction of travel, and the coda following the main pulse is considerably larger in amplitude than in the case of the uncracked sample. The P-wavelength corresponding to the central frequency of the emitting transducer (100 kHz) is 25 mm and the diameter of the cracks is 5.5 mm, i.e. between one-fifth and one-quarter of a wavelength. The value of $k_\alpha a$, where k_α is the wavenumber of P-waves in the uncracked material and a is the crack radius, is 0.69. Smyshlyaev, Willis and Sabina (1993), using a dynamic form of the self-consistent method, calculated that, for dry, circular, aligned cracks with $\zeta = 0.1$ and an angle of incidence of 45° with the crack normals, the phase velocity at $k_\alpha a = 0.69$ is over 3% different from the limiting value at long wavelengths. The corresponding attenuation of the mean wave may be calculated in terms of the imaginary part of the wavenumber and takes the theoretical value 0.01 mm^{-1}, which corresponds to an amplitude reduction of e^{-1} over the 10 cm path-length of the pulse. There is no mechanism of energy loss in the dry samples except for the scattering which is clearly significant here from the evidence of the recorded codas. These comparisons indicate that the data are somewhat outside the range for which direct quantitative support for the results derived from effective-medium theory can be obtained. It would be more appropriate to use a theory which is valid at higher frequencies (e.g. Smyshlyaev *et al.* 1993).

In a reworking of the data of Rathore *et al.* (1995), Thomsen (1995) arrived at slightly different conclusions although it still remains doubtful whether the experiments represent a satisfactory check on theory. Whereas Rathore *et al.* (1995) measured phase velocities using the first cross-over point of the traces, Thomsen (1995) used the first break, which he took to represent wave speeds of the highest observable frequencies (80 kHz – a wavelength of 31 mm). These raised values of the wave speeds are shown to fit well with Thomsen's (1995) theoretical predictions in which the properties of the porous matrix were adjusted to fit the theoretical wave speeds in the crack-parallel direction. Hudson's (1981) first-order theory, when fitted in this way, gave poor predictions. However, Thomsen (1995) pointed out that, in view of the considerable anisotropy, second-order terms should be used.

Thomsen (1995) also calculated 'low'-frequency wave speeds (30 kHz) from the data by estimating dispersion corrections from the measured attenuation. These corrections are of the order of 5% in value and the wave speeds remain satisfactorily close to both theoretical predictions; since the cracks are dry neither theory is affected by frequency. However, it is the existence of attenuation and dispersion in the system which presents the problem when comparing the data with the results of effective-medium (long wavelength, no dispersion) theory, particularly at the higher frequencies. In addition, at the lower frequency of 30 kHz, the P-wavelength in the uncracked sample is 8.4 cm, almost the width of the sample. Although the ratio of wavelength to crack diameter is now over 15 and therefore quite satisfactory for the application of effective-medium theory, the use of plane-wave theory to estimate dispersion is not satisfactory.

ISOLATED FLUID-SATURATED CRACKS

As for dry cracks, theoretical formulae for isolated fluid-filled cracks generally agree when reduced to first-order expressions. With dry cracks there are in fact only two non-dimensional structural variables; that is, we can write the wave speed V_P say, of the quasi-P-wave in the effective medium as

$$\left(\frac{V_\mathrm{P}}{\alpha}\right)^2 = f\left(\zeta, \frac{\alpha}{\beta}, \theta, \phi\right), \tag{1}$$

where f is an arbitrary function, α, β are the wave speeds in the (isotropic) matrix material and θ, ϕ are directional angles of the wave normal. If the matrix solid were anisotropic there would be little difference, just one or two more ratios of material constants. The frequency of the wave does not enter into (1) since these theories are calculated in the limit of long wavelength ($k_\alpha a \to 0$). The aspect ratio c/a of the cracks does not appear either and the results are valid in the limit $c/a \to 0$. Theoretical calculations by Smyshlyaev *et al.* (1993) indicated that the long wavelength approximation is valid to within 1% if $k_\alpha a <$ 0.3 and $\zeta = 0.1$. Douma's (1988) study of the effects of varying the aspect ratio indicated that the limiting value for small c/a is valid for $c/a < 0.1$. (In the experiments of Rathore *et al.* (1995), $c/a = 0.004$.)

Expansion of (1) as an ascending series in ζ gives

$$\left(\frac{V_\mathrm{P}}{\alpha}\right)^2 = 1 - f_1\left(\frac{\alpha}{\beta}, \theta, \phi\right)\zeta + f_2\left(\frac{\alpha}{\beta}, \theta, \phi\right)\zeta^2 - \ldots, \tag{2}$$

for suitable functions f_1, f_2, etc. First-order theory ignores all terms except the first two. Equation (2) is equivalent to expanding the stiffnesses in terms of ζ. If the compliances are expanded to O(ζ), we get (to first order)

$$\left(\frac{V_{\rm P}}{\alpha}\right)^2 = \left[1 + f_1\left(\frac{\alpha}{\beta}, \theta, \phi\right)\zeta\right]^{-1}. \tag{3}$$

Equation (3) is, in principle, the same as the first two terms of (2) for small enough ζ. However, for larger values of ζ, a truncated version of (2) clearly becomes non-physical, whereas (3) has been shown to fit the results of two-dimensional numerical calculations quite well (see above). In general, first-order theory is much more suitable for use in the form of (3) (Liu, Hudson and Pointer 2000).

The earliest theories for fluid-filled cracks (Walsh 1969; Garbin and Knopoff 1975) ignored the fluid compressibility and assumed that any crack-opening displacement could be discounted. If, on the other hand, the compressibility of the fluid is taken into account, the theoretical expressions collect another non-dimensional material parameter, namely the ratio ($\kappa_{\rm f}/\kappa$) of the bulk modulus of the fluid to that of the solid. In addition, this ratio appears in the formulae coupled with c/a, so Hudson (1981), for instance, retained terms in ($c\kappa/a\kappa_{\rm f}$) and ignored ($c^2\kappa/a^2\kappa_{\rm f}$) with the result, for aligned cracks with normals in the direction $\theta = 0$, that the dependence on ϕ disappeared, and

$$f_1\left(\frac{\alpha}{\beta}, \theta\right) = P\left(\frac{\alpha^2}{2\beta^2} - \sin^2\theta\right)^2 + Q\sin^2\theta\cos^2\theta \tag{4}$$

where

$$P = \frac{16\mu}{3(\lambda+\mu)}(1+K)^{-1},$$

$$Q = \frac{64\mu}{3(3\lambda+4\mu)}, \tag{5}$$

$$K = \frac{\alpha^2}{\pi\beta^2}\frac{a\kappa_{\rm f}}{c(\lambda+\mu)},$$

and λ, μ are the Lamé parameters for the matrix material.

Substituting into (2) and retaining the first two terms only, we obtain

$$\frac{V_{\rm P}^2}{\alpha^2} = 1 - P\left(\frac{\alpha^2}{2\beta^2}\right)^2\zeta - \left\{P\left(1 - \frac{\alpha^2}{\beta^2}\right)\sin^4\theta + \left(Q - \frac{P\alpha^2}{\beta^2}\right)\sin^2\theta\cos^2\theta\right\}\zeta. \tag{6}$$

The wave speed α_0 of qP along the axis of symmetry ($\theta = 0$) is given by

$$\left(\frac{\alpha_0}{\alpha}\right)^2 = 1 - P\left(\frac{\alpha^2}{2\beta^2}\right)^2\zeta, \tag{7}$$

and so, retaining terms of order ζ only, we have

$$\left(\frac{V_{\rm P}}{\alpha_0}\right)^2 = 1 + 2\delta\sin^2\theta\cos^2\theta + 2\epsilon\sin^4\theta, \tag{8}$$

where we have used Thomsen's (1986) notation with

$$\begin{aligned}\epsilon &= \zeta\frac{P}{2}\left(\frac{\alpha^2}{\beta^2} - 1\right),\\ &= \frac{8\zeta}{3}(1+K)^{-1},\\ \delta &= \frac{\zeta}{2}\left(\frac{P\alpha^2}{\beta^2} - Q\right),\\ &= 2(1-\sigma)\epsilon - 2\left(\frac{1-2\sigma}{1-\sigma}\right)\gamma,\\ \gamma &= \frac{8\zeta}{3}\left(\frac{1-\sigma}{2-\sigma}\right),\end{aligned} \tag{9}$$

and σ is Poisson's ratio of the matrix material. Hudson's (1981) expressions for the wave speeds of quasi-S-waves polarized parallel to the plane of the cracks ($V_{\rm S\parallel}$) and the quasi-S-waves polarized perpendicular to these ($V_{\rm S\perp}$) are

$$\left(\frac{V_{\rm S\perp}}{\beta}\right)^2 = 1 - \frac{16}{3}\zeta\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right) - \frac{16}{3}\zeta(\lambda+2\mu) \times\left\{\frac{(1+K)^{-1}}{\lambda+\mu} - \frac{4}{3\lambda+4\mu}\right\}\sin^2\theta\cos^2\theta, \tag{10}$$

$$\left(\frac{V_{\rm S\parallel}}{\beta}\right)^2 = 1 - \frac{16}{3}\zeta\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right) + \frac{16}{3}\zeta\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right)\sin^2\theta. \tag{11}$$

Both of these equations can be set into Thomsen's (1986) format:

$$\left(\frac{V_{\rm S\perp}}{\beta_0}\right)^2 = 1 + 2\frac{\alpha_0^2}{\beta_0^2}(\epsilon - \delta)\sin^2\theta\cos^2\theta,$$

$$\left(\frac{V_{\rm S\parallel}}{\beta_0}\right)^2 = 1 + 2\gamma\sin^2\theta, \tag{12}$$

to first order in ζ, where β_0 is the common wave speed of both quasi-S-waves in the symmetry direction, and

$$\left(\frac{\beta_0}{\beta}\right)^2 = 1 - \frac{16\zeta}{3}\left(\frac{\lambda+2\mu}{3\lambda+4\mu}\right). \tag{13}$$

Expressions (8) and (12) are only accurate to first order in ζ and the factors $(\alpha/\alpha_0)^2$ and $(\beta/\beta_0)^2$ were taken to be unity when multiplying the terms in ϵ, δ and γ since all terms of second or higher powers in ζ have been ignored. However, $(\alpha_0/\alpha)^2$ is given by (7) and takes the value 0.4 for dry cracks ($K = 0$) when $\zeta = 0.1$ (and $\lambda = \mu$). This clearly shows that for crack densities of 10% or more – and, as noted by Thomsen (1995) for the experiments of Rathore *et al.* (1995) – first-order theories are not, strictly speaking, valid even though they may fit the data. In these circumstances, second-order terms (Hudson 1980, 1986) or some other theory with a claim to validity at higher crack densities must be used.

One such theory is the self-consistent approximation and Hoenig (1979) has given the corresponding formulae for aligned isolated cracks, either dry or fluid-filled. Thomsen (1995) took these results, but linearized in ζ so that the result was once again a first-order theory. However, he then extended the formulae to apply to the case where the matrix material is porous on a smaller length scale than the cracks. His expressions for the anisotropy parameters are, for low equant porosity (less than 10%),

$$\epsilon = \frac{8\zeta}{3}\left(1-\frac{\kappa_f}{\kappa_s}\right)D_{cp},$$

$$\delta = 2(1-\sigma_s)\epsilon - 2\left(\frac{1-2\sigma_s}{1-\sigma_s}\right)\gamma, \tag{14}$$

$$\gamma = \frac{8\zeta}{3}\left(\frac{1-\sigma_s}{2-\sigma_s}\right),$$

where σ_s, κ_s are Poisson's ratio and the bulk modulus of the solid part of the porous matrix and D_{cp} is the 'fluid influence factor'.

If the matrix rock is not porous, $\kappa_s = \kappa$ and $\sigma_s = \sigma$ and equations (14) for δ and γ are identical to (9). The equations for ϵ are the same except that $(1+K)^{-1}$ in (9) has been replaced in (14) by $(1-\kappa_f/\kappa_s)D_{cp}$ in Thomsen's formula. For moderately high frequencies, that is, high enough for diffusion of fluid into the matrix from the cracks to be neglected (isolated cracks) but not so high that wavelengths cease to be long compared with the size of cracks, Thomsen (1995) gives

$$\left(1-\frac{\kappa_f}{\kappa_s}\right)D_{cp} = \left\{1+\frac{1}{\pi}\left(\frac{\alpha_s}{\beta_s}\right)^2\frac{a\kappa_f}{c(\lambda_s+\mu_s)}\left(1-\frac{\kappa_f}{\kappa_s}\right)^{-1}\right\}^{-1}, \tag{15}$$

where the subscript s once again refers to the solid part of the matrix material. For a non-porous matrix, therefore, (14) corresponds exactly to (9) except for the factor $(1-\kappa_f/\kappa_s)^{-1}$ appearing on the right-hand side of (15). However, the difference between this expression with and without the factor $(1-\kappa_f/\kappa_s)^{-1}$ is of order c/a, which is why the factor does not appear in Hudson's (1981) expressions since he was working consistently to the limit $c/a \to 0$. On the other hand, Thomsen's (1995) expression (equation (15)) has the attractive property that, when $\kappa_f = \kappa_s, \epsilon = 0$; that is, when the bulk modulus of the liquid infill is the same as that of the matrix material, there is no effect due to the reduced compliance of the crack, as would be expected. Hudson's (1981) expressions (equation (9)) give, in the same circumstances, an effect of order $\zeta c/a$.

Apart from this point, first-order expressions for the wave speeds in non-porous materials with isolated fluid-filled cracks agree. It may be noted here that we have taken no account of fluid viscosity. This can, of course, be incorporated into the model in exactly the same way as the fluid compressibility (O'Connell and Budiansky 1977; Hudson 1981).

If the matrix material is porous but the frequencies are sufficiently high that there is no fluid-flow between cracks and matrix, the cracks are effectively isolated and Hudson's (1981) results (equations (9)) remain valid as long as the material parameters λ, μ, α, β are taken to be those for the porous fluid-filled matrix (see Rathore *et al.* 1995). This is on the assumption that the scale of the pore structure is smaller than that of the cracks so that, as far as the cracks are concerned, the porous material acts like a continuous solid. If now we put $\kappa_f = 0$ (the case of dry cracks in dry porous material), both Hudson's (1981) and Thomsen's (1995) theories agree in the expressions for ϵ; that is, the last of equations (5) gives $K = 0$ and (15) gives $(1-\kappa_f/\kappa_s)D_{cp} = 1$. However, Thomsen's (1995) expressions for γ and δ (equation (14)) differ from those of Hudson (1981) in using the parameters for the solid material of the matrix rather than those for the dry porous material. Similarly, if κ_f is not zero, the two theories give different results in that, where Hudson's (1981) theory leads to the use of the overall parameters λ, μ, κ, etc. of the fluid-filled matrix, Thomsen (1995) once again uses the parameters of the solid material.

Expressions given by Thomsen (1995) appropriate for an equant porosity higher than 10% and once again at moderately high frequencies (no fluid interchange occurs) are

$$\epsilon = \frac{8\zeta}{3}\frac{(1-\sigma^{*2})}{(1-\sigma^2)}\frac{E}{E^*}\left(1-\frac{\kappa_f}{\kappa_s}\right)D_{ci},$$

$$\delta = 2(1-\sigma)\epsilon - 2\left(\frac{1-2\sigma}{1-\sigma}\right)\gamma, \tag{16}$$

$$\gamma = \frac{8\zeta}{3}\left(\frac{1-\sigma^*}{2-\sigma^*}\right),$$

where E is Young's modulus of the saturated porous matrix and σ^*, E^* are Poisson's ratio and Young's modulus of the dry matrix rock; D_{ci} is a second 'fluid influence factor' given by

$$\left(1-\frac{\kappa_f}{\kappa_s}\right)D_{ci}=\left\{1+\frac{1}{\pi}\left(\frac{\alpha^*}{\beta^*}\right)^2\frac{a\kappa_f}{c(\lambda^*+\mu^*)}\left(1-\frac{\kappa_f}{\kappa}\right)^{-1}\right\}^{-1}. \tag{17}$$

With $\kappa_f = 0$ we should again obtain an expression appropriate for a dry porous material containing cracks. In addition, it is clear that $\sigma = \sigma^*$, $E = E^*$ and this time we get

$$\epsilon=\frac{8\zeta}{3},$$

$$\delta=(1-\sigma)\epsilon-2\left(\frac{1-2\sigma}{1-\sigma}\right)\gamma, \tag{18}$$

$$\gamma=\frac{8\zeta}{3}\left(\frac{1-\sigma}{2-\sigma}\right),$$

which is identical to Hudson's (1981) result (equations (9) with $K = 0$) although, as we have seen above, Thomsen's (1995) results for dry cracks with low equant porosity only agree if the porosity is sufficiently small that the elastic properties of the dry porous rock are almost the same as those of the solid material.

If the cracks and pores are fluid-filled, we find once again, apart from the factor $(1 - \kappa_f/\kappa_s)$, a fundamental difference between Thomsen's (1995) and Hudson's (1981) results in that Thomsen uses parameters corresponding to the dry porous matrix and Hudson has those of the fluid-filled matrix. At high porosities these two sets of parameters are likely to take very different values. However, in circumstances where the porous matrix appears as a continuum when viewed at the scale of the cracks, the physical principle of using the formulae for a non-porous matrix, about which both Hudson and Thomsen agree, with the properties of the effective continuum seems strong.

The expressions (8) and (12) for the wave speeds in terms of the parameters ϵ, ζ and γ may be rewritten as

$$\left(\frac{V_P}{\alpha_0}\right)^2=\left(1+\frac{3\epsilon}{4}+\frac{\delta}{4}\right)-\epsilon\cos 2\theta+\left(\frac{\epsilon-\delta}{4}\right)\cos 4\theta,$$

$$\left(\frac{V_{S\perp}}{\beta_0}\right)^2=\left\{1+\frac{\alpha_0^2}{4\beta_0}(\epsilon-\delta)\right\}-\frac{\alpha_0^2}{4\beta_0^2}(\epsilon-\delta)\cos 4\theta, \tag{19}$$

$$\left(\frac{V_{S\parallel}}{\beta_0}\right)^2=(1+\gamma)-\gamma\cos 2\theta.$$

The variation with θ in the form of $\cos 2\theta$ and $\cos 4\theta$ is evident from the curves calculated by Rathore *et al.* (1995). For instance, $(V_{S\perp})^2$ varies as $\cos 4\theta$ and $(V_{S\parallel})^2$ as $\cos 2\theta$. If ϵ is small, $(V_P)^2$ varies as $\cos 4\theta$ and this is what happens for isolated fluid-filled cracks when K is large (see equation (5)) or, equivalently, when D_{cp} is small (see equation (15)), i.e. when the compressibility of the fluid can be ignored. This leads to characteristically different shapes of curve for the variation of V_P with θ for dry and fluid-filled isolated cracks (see Rathore *et al.* 1995).

As remarked above, the porosity of the fabric of sedimentary rocks means that cracks are in general connected by fluid pathways which, if the permeability is large enough, invalidate the assumption, made in almost all the theoretical models mentioned so far, that the cracks are isolated. This is of no importance for dry cracks where there is no pressure build-up within the cracks and it explains the agreement between theory and experiment in this case. The synthetic sandstone of Rathore *et al.* (1995) has porosity 0.346 and clearly the cracks cannot be assumed to be pressure sealed. Accordingly, the fit between the predictions of Hudson's (1980, 1981) results and the observations on the liquid-saturated model is very poor. In particular, the $\cos 4\theta$ variation for $(V_P)^2$ is not reproduced in the data which show a $\cos 2\theta$ variation.

LIQUID-SATURATED CRACKS IN POROUS MATERIAL

Two developments of the effective-medium theories, which take account of diffusion into a porous matrix, have been published recently. The first, referred to above, is by Thomsen (1995) and the other is by Hudson, Liu and Crampin (1996). In contrast to the partial agreement found for the expressions for the overall properties of materials with a dilute distribution of *isolated* cracks, results from these two investigations are quite different.

At low frequencies, when pressure leakage out of the cracks into the porous matrix has to be taken into account, Thomsen's (1995) expressions for the parameters ϵ, δ and γ at porosities greater than 10% are exactly the same as (16) but with

$$\left(1-\frac{\kappa_f}{\kappa_s}\right)D_{ci}=\{\{1+\frac{\kappa_f}{\kappa^*\phi}[(\frac{\kappa_s-\kappa^*}{\kappa_s-\kappa_f})+\frac{4}{3}\zeta(\frac{\alpha^*}{\beta^*})^2\frac{\kappa^*}{(\lambda^*+\mu^*)}\cdot(1-\frac{\kappa_f}{\kappa_s})^{-1}]\}^{-1}, \tag{20}$$

where ϕ is the overall porosity including both cracks and pores. Thomsen (1995) also gives a low-frequency equivalent to (15) for low porosities.

The expressions given by Hudson *et al.* (1996) are the same

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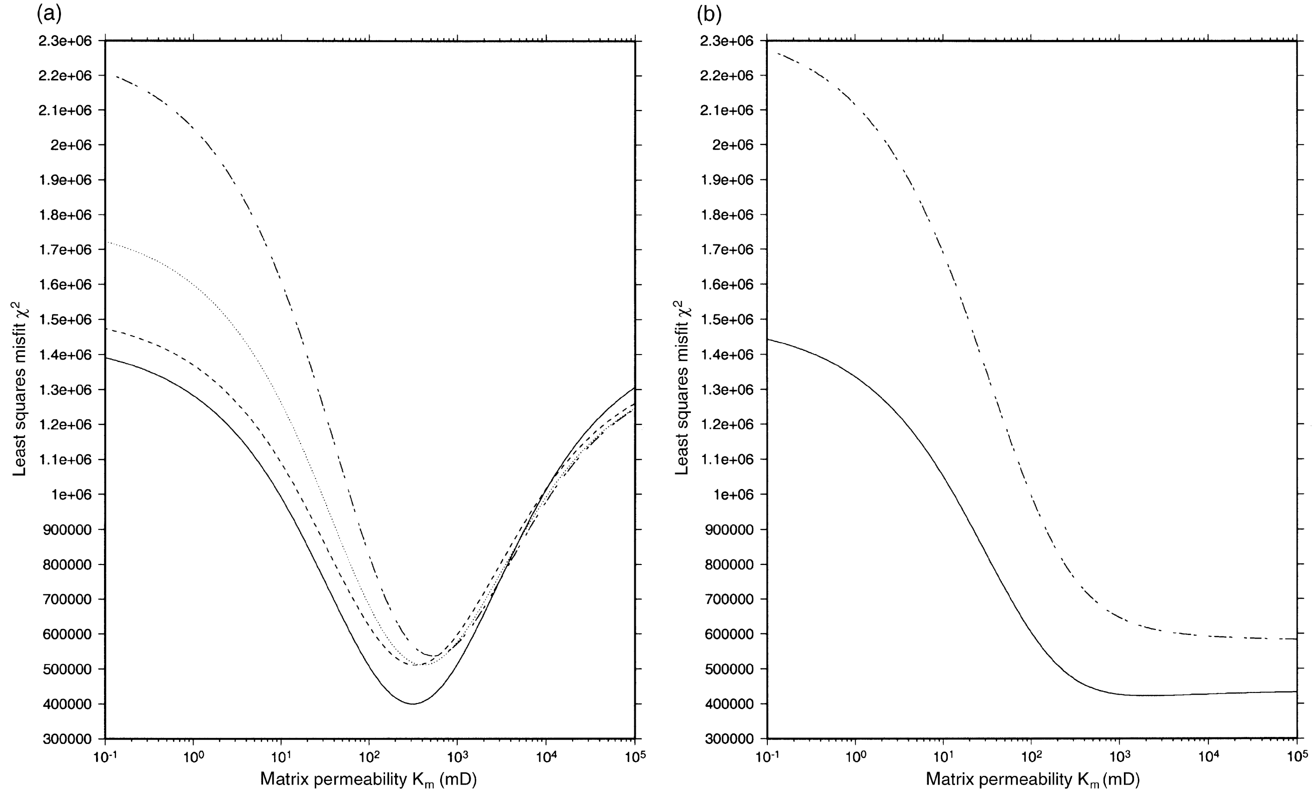


Figure 1 (a) The least-squares misfit versus matrix permeability K_{m}, derived from fitting the laboratory wave-speed data recorded by Rathore *et al.* (1995) for aligned water-saturated cracks (central frequency 100 kHz), using the equant porosity theory of Hudson *et al.* (1996). The different curves correspond to the estimated and measured rock matrix wave speeds listed in Table 2: the dummy matrix: measured values from the dummy sample (Rathore *et al.* 1995) (solid line); matrix A: application of equant porosity model (Thomsen 1995) by Rathore *et al.* (1995) (broken dashed line); matrix B: application of isolated fluid-filled cracks model (Hudson 1981) by Rathore *et al.* (1995) (dotted line); matrix C: application of equant porosity model (Thomsen 1995) (dashed line). The results are generated using first-order theory. (b) As (a) except that the results are generated using second-order theory for the dummy matrix (solid line) and matrix A (broken dashed line).

as equations (9) except that K is now given by

$$K = \frac{1}{\pi}\frac{a}{c}\left(\frac{\alpha}{\beta}\right)^2 \frac{\kappa_f}{\lambda+\mu}[1+3(1-i)\mathcal{J}/2c]^{-1}, \tag{21}$$

where

$$\mathcal{J}^2 = \phi_m \kappa_f D_m / 2\omega \tag{22}$$

and D_m, ϕ_m are, respectively, the coefficients of diffusion and porosity of the porous matrix material and ω is the angular frequency. (It should be noted here that there is an error in the expression (equation (74) of Hudson *et al.* 1996) originally given for J. Equation (22) is correct.) The quantity D_m may be replaced by K_m/η_f (Sheriff 1991) where K_m is the permeability of the matrix and η_f the viscosity of the fluid. The parameters relating to the matrix rock (e.g. $\alpha, \beta, \lambda, \mu$) in Hudson's expressions (equations (5) and (9)) now take values appropriate to the saturated porous rock.

Whereas Thomsen (1995) has two separate expressions for the cases of isolated cracks (moderately high frequencies – equation (16)) and diffusing cracks (low frequencies – equation (20)), Hudson *et al.* (1996) have a single expression for a whole range of frequencies (equation (21)) which reverts to the equivalent expression (the last of equations (5)) when the frequency is sufficiently large that the cracks are isolated, i.e. when $\mathcal{J}/c \ll 1$. Dispersion and attenuation are implicit over the range where $\mathcal{J}/c$ is not small. For instance, if $\mathcal{J}/c \gg 1, K$ becomes (from (21))

$$K \simeq \left(\frac{1+i}{3}\right)\left(\frac{\alpha}{\beta}\right)^2\left(\frac{\kappa_f}{\lambda+\mu}\right)\frac{a}{\mathcal{J}}, \tag{23}$$

and the critical parameter is now $a\kappa_f/\mathcal{J}(\lambda+\mu)$. At very low frequencies this parameter is small, the quantity K is therefore also small and the equations give a response which is equivalent to that of dry cracks. This is a physically appealing result for the model adopted by Hudson *et al.* (1996), that of a simple crack in unbounded porous material. However, in material with a population of cracks, the flow associated with separate cracks will begin to interfere at relatively low frequencies, a feature which is not taken into account by the model.

Table 1 Crack, fluid and matrix parameters taken from the laboratory experiments of Rathore *et al.* (1995)

Parameter	Formula	Value
Crack half-thickness	c	10^{-5} m
Crack aspect ratio	c/a	0.0036
Crack density	ζ	0.1
Fluid bulk modulus	κ_f	2.16×10^9 Pa
Fluid viscosity	η_f	10^{-3} Pa s
Matrix porosity	ϕ_m	0.346
Matrix density	ρ_m	1712 kg/m^3

Table 2 Estimated and measured matrix wave speeds: dummy matrix: measured values from dummy sample (Rathore *et al.* 1995); matrix A: application of equant porosity model (Thomsen 1995) by Rathore *et al.* (1995); matrix B: application of isolated fluid-filled cracks model (Hudson 1981) by Rathore *et al.* (1995); matrix C: application of equant porosity model (Thomsen 1995)

	Dummy matrix	Matrix A	Matrix B	Matrix C
V_P (m/s)	2678	2786	2715	2670
V_S (m/s)	1384	1408	1408	1410

At the lowest frequencies, we expect the pressure to be equalized in the cracks and pores, and in the limit the material will behave as if static and undrained (Xu 1998). The elastic constants for this are given by the extension of the Gassmann equations presented by Brown and Korringa (1975):

$$s^u_{ijkl} = s^d_{ijkl} + \frac{(s^d_{ijpp} - s^0_{ijpp})(s^d_{qqk\ell} - s^0_{qqk\ell})}{\phi(1/\kappa - 1/\kappa_f) - (1/\kappa_d - 1/\kappa)}, \tag{24}$$

where $\mathbf{s}^0$, $\mathbf{s}^d$ and $\mathbf{s}^u$ are the compliances for the solid matrix, the dry system (matrix plus pores plus cracks) and the undrained system respectively, ϕ is the porosity of the system and

$$1/\kappa = s^0_{jjkk},$$

$$1/\kappa_d = s^d_{jjkk}. \tag{25}$$

Unfortunately the forms of (20) and (25) are sufficiently different to make any comparison rather difficult.

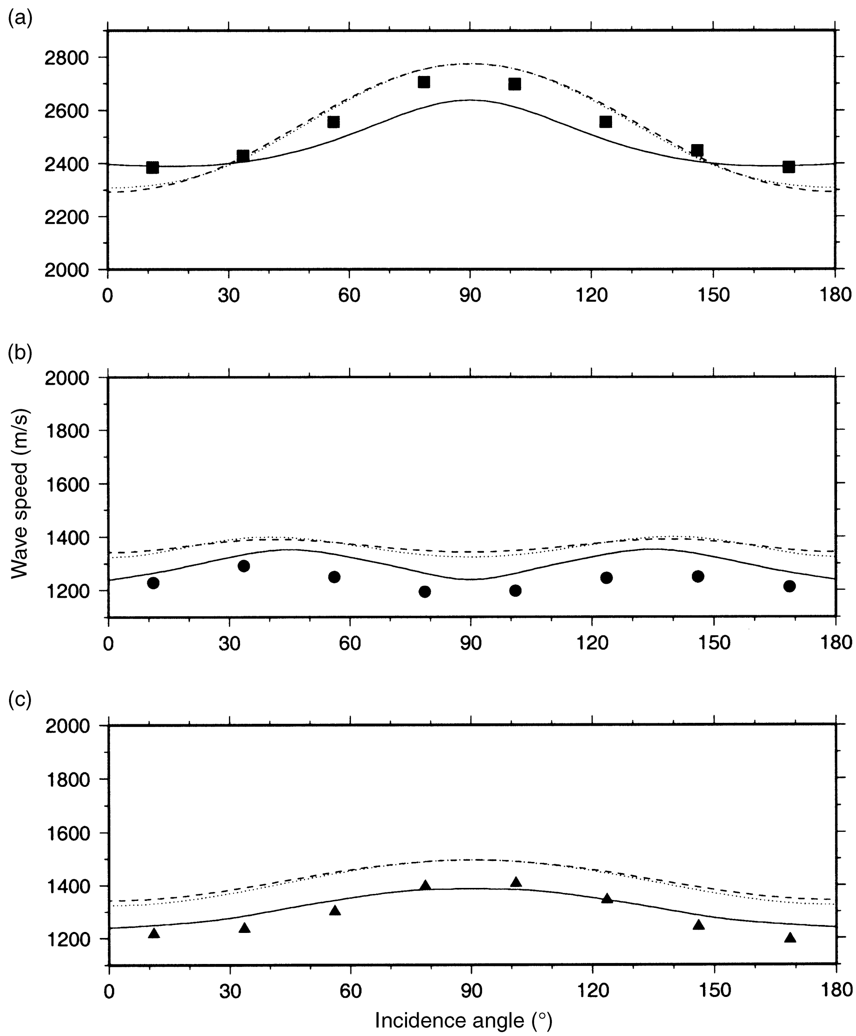
We may calculate the point at which the theory given by Hudson *et al.* (1996) becomes unreliable; it is when the diffusion length is of the same order of magnitude as the crack spacing. From Hudson *et al.* (1996), the diffusion length is approximately $\left(\kappa_f K_m/\omega\phi_m\eta_f\right)^{1/2}$. Thus, when the frequency is of the order of

$$\frac{\kappa_f K_m}{\phi_m \eta_f l^2} \tag{26}$$

or less, (21) should not be used.

When the compressibility of the fluid is the same as that of the solid, $\kappa_f = \kappa_s$ and Thomsen's (1995) expression for $(1 - \kappa_f/\kappa_s)D_{ci}$ in (20) becomes zero, so that $\epsilon = 0$ once again. On the other hand, the expression for $(1+K)^{-1}$, which is the equivalent quantity in the formula for ϵ (equation (9))

Figure 2 Comparison of measured wave-speed data (for (a) qP – solid squares, (b) $qS_\perp$ – solid circles, (c) $qS_\parallel$ – solid triangles, after Rathore *et al.* (1995), central frequency 100 kHz) with (i) the first-order Hudson *et al.* (1996) equant porosity model (dotted line), (ii) the second-order Hudson *et al.* (1996) equant porosity model (dashed line) and (iii) the Thomsen (1995) equant porosity model (solid line) (Rathore *et al.* 1995). The matrix wave speeds measured from the dummy sample were used for the model calculations, $K_m = 313$ mD for first-order theory (see Table 3a) and $K_m = 2110$ mD for second-order theory (see Table 3b).



proposed by Hudson *et al.* (1996), may be written as

$$(1+K)^{-1} = \{c/a + 3(1-\mathrm{i})\mathcal{J}/2a\}$$

$$\left\{\frac{1}{\pi}\left(\frac{\alpha}{\beta}\right)^2 \frac{\kappa_f}{\lambda+\mu} + \frac{c}{a} + \frac{3(1-\mathrm{i})\mathcal{J}}{2a}\right\}^{-1}, \tag{27}$$

using (21) for K and recognizing that, if $\kappa_f = \kappa_s$, then $\kappa_f = \kappa$, the bulk modulus of the saturated porous material, as well. If $\mathcal{J}/a \ll 1$, this expression is of order $c/a(\ll 1)$ as in the case of isolated cracks. However, if $\mathcal{J}/a$ is not very small, neither is $(1+K)^{-1}$ and the cracks respond as though they have low compliance. This is the case if the permeability of the matrix rock is sufficiently high and the thin cracks are of high compliance compared with the pores; under pressure the cracks collapse, driving fluid into the relatively undistorted pores. The condition $\kappa_f = \kappa$ for *isolated* cracks means that the cracks are non-compliant under compression; fluid-flow from the cracks into a porous matrix allows the cracks to be compliant.

Hudson *et al.* (1996) showed that the condition, $\mathcal{J}/c \ll 1$, is simply that the volume of fluid within the matrix material within a diffusion length $l = (\kappa_f D_m/\omega\phi_m)^{\frac{1}{2}}$ is small compared with the volume of the crack. The condition on frequency, that the cracks can be considered as isolated, is

$$\omega \gg \omega_c = \frac{\phi_m \kappa_f K_m}{c^2 \eta_f}. \tag{28}$$

O'Connell and Budiansky (1977) gave an alternative expression for the critical angular frequency ω_c for fluid diffusion into a porous matrix. This is

$$\omega_c = \frac{8\kappa K_m}{a^2 \eta_f}, \tag{29}$$

but it is based on a model of steady flow of an incompressible fluid.

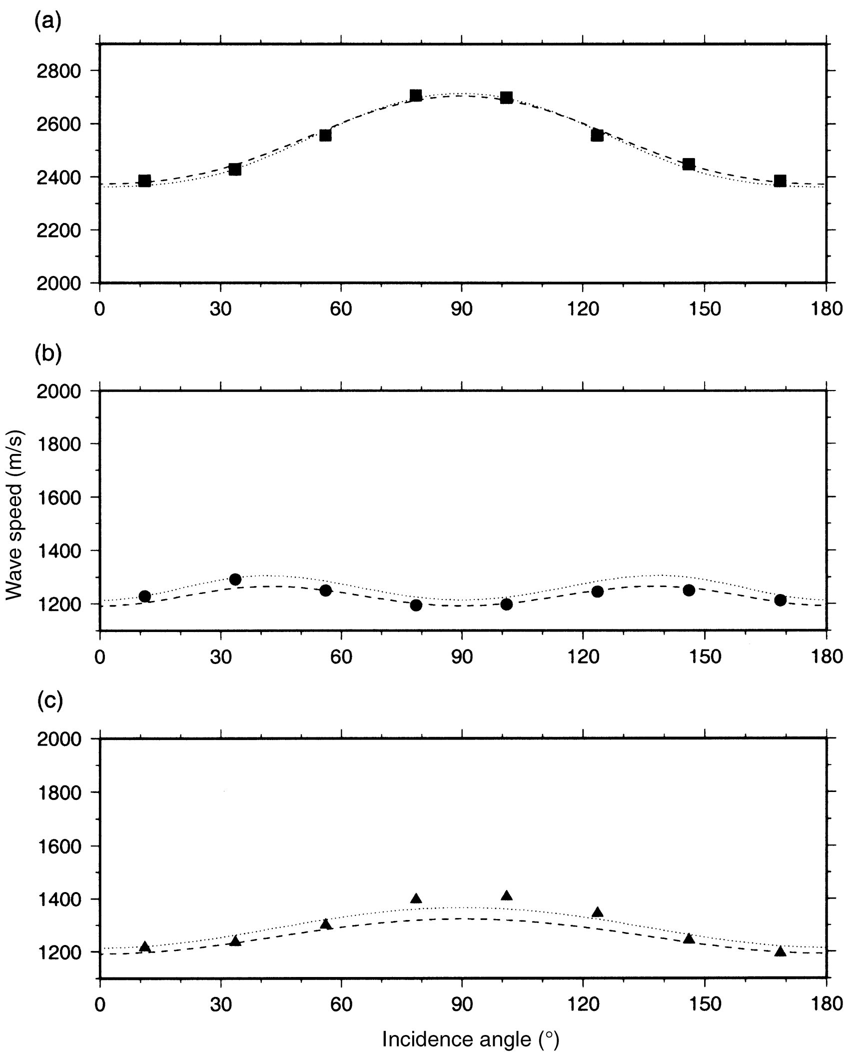


Figure 3 Comparison of measured wave-speed data (for (a) qP – solid squares, (b) $qS_{\perp}$ – solid circles, (c) $qS_{\parallel}$ – solid triangles, after Rathore *et al.* (1995), central frequency 100 kHz) with theory using the least-squares fit of matrix permeability K_m and matrix wave speeds with the Hudson *et al.* (1996) equant porosity model (Table 4); (i) first-order theory (dotted line), (ii) second-order theory (dashed line).

Xu (1998) also fitted the data of Rathore *et al.* (1995) to a first-order effective-medium theory. In the theory, both the pores and the cracks were modelled by ellipsoids and the aspect ratio of the pores was chosen to get the best data match. Two versions of the theory were used, one in which the cracks were isolated and another in which the pressure was equilibrated locally between cracks and pores. Xu (1998) found that the latter model (of cracks fully connected to the pores) fitted the data well.

COMPARISON WITH DATA

In comparing the data of Rathore *et al.* (1995) (see Table 1) with the theory given by Hudson *et al.* (1996), two problems arise. The first is that no value for the permeability of the porous matrix rock is given and so we use the data to estimate it. The second is that there are four sets of parameter values for the mechanical properties of the matrix (see Table 2). The properties of the first set were measured by Rathore *et al.* (1995) on a ‘dummy’ sample, i.e. a section of the model in which no cracks were inserted. The second set (matrix A) was calculated by Rathore *et al.* (1995) by fitting the data to the theory of Thomsen (1995) in the crack-parallel direction. The third set (matrix B) was also calculated by Rathore *et al.* (1995) by fitting the theory for isolated cracks (Hudson 1981) in the same way. Matrix C was given by Thomsen (1995) based on a fit similar to that which produced matrix A.

We have used the first-order theory for aligned saturated cracks in a matrix material with equant porosity given by Hudson *et al.* (1996) to fit the data of Rathore *et al.* (1995) for the variation of all three wave speeds with angle. Figure

Table 3 (a) Results from a least-squares fit of the matrix permeability K_m. The least-squares misfit, matrix permeability K_m and the corresponding quantities $\mathcal{J}/c$ and $\alpha\kappa_f/\mathcal{J}(\lambda+\mu)$, determined from fitting the equant porosity model of Hudson *et al.* (1996) to the laboratory wave-speed data measured by Rathore *et al.* (1995), for each of the rock matrices given in Table 1 (see Fig. 1a). The frequency is taken as 100 kHz. The results shown were obtained from the application of first-order theory. (b) As (a) except that the results are generated from the application of second-order theory for the dummy matrix and matrix A. There was no minimum value for the misfit over the range 0.1 mD–100 D for matrix A (see Fig. 1b)

(a)	Dummy matrix	Matrix A	Matrix B	Matrix C
χ^2	4.00×10^5	5.37×10^5	5.11×10^5	5.11×10^5
K_m (mD)	313	541	410	339
$\mathcal{J}/c$	42.9	56.4	49.1	44.7
$\alpha\kappa_f/\mathcal{J}(\lambda+\mu)$	1.54	1.07	1.31	1.51

(b)	Dummy matrix	Matrix A
χ^2	4.23×10^5	—
K_m (mD)	2110	—
$\mathcal{J}/c$	111	—
$\alpha\kappa_f/\mathcal{J}(\lambda+\mu)$	0.594	—

1(a) shows the least-squares misfit as a function of the permeability K_m of the porous matrix for all four sets of matrix wave speeds. The best fits lie between values of 350 and 700 mD and the dummy matrix gives the best fit of all. It is encouraging that the permeability values are realistic for a porous sandstone (see Table 3a).

Figure 1(b) shows similar curves fitting the data to the theory of Hudson *et al.* (1996) taken to second order. This time we have used only the dummy matrix and matrix A, as the other two sets of matrix values are based on isolated-crack theory which is clearly not applicable. The χ^2 values for the best fit are somewhat worse for both sets of matrix properties than for first-order theory (see Table 3b), a result which parallels a similar observation made by Rathore *et al.* (1995) for dry cracks. The curves do not give a clear minimum value for best fit; in fact with matrix A, there is no minimum at all. The value of K_m at the minimum point using the dummy matrix is considerably larger than the values given by first-order theory (and is slightly too high to be realistic) although the shape of the curve in Fig. 1(b) shows that there is a range of values of K_m which give almost the same standard of fit.

Tables 3(a,b) also list the corresponding values of $\mathcal{J}/c$, which are all large, showing once again that the model of isolated cracks is inapplicable. They also show values of $\alpha\kappa_f/\mathcal{J}(\lambda+\mu)$ which are not far from unity (although second-order theory gives a significantly different result from that obtained from the first-order expressions). This indicates that neither the high-frequency limit of isolated cracks nor the low-frequency limit with the pressure uniform throughout the cracks and pores is applicable. The value of the critical frequency (26), below which diffusion from each crack begins to affect that from other cracks, is approximately 2 kHz, where we have used values from Table 1 and taken $K_m = 200$ mD and $\ell = 1$ cm. The frequencies used in the experiments (Rathore *et al.* 1995) were centred on 100 kHz, confirming that the low-frequency limit is not strictly applicable. The value of frequency used in calculating J in Table 3(a,b) is also 100 kHz.

Figure 2 shows the experimental and theoretical variation with wave direction of the qP, $S_\perp$ and $S_\parallel$ waves. The angle of incidence is measured from the crack normal direction. The experimental values of Rathore *et al.* (1995) are compared with theoretical curves using Thomsen's (1995) expressions for low frequencies and high porosity (equations (16) and (20)) and the first- and second-order results of Hudson *et al.* (1996). The matrix wave speeds that were used were those of the dummy matrix, and the value of the matrix permeability is that given by minimizing the misfit to the theory of Hudson *et al.* (1996). The first point to notice is that there is little difference in these figures between first-order and second-order theories. The second point is that the $\cos 4\theta$ dependence shown by isolated-crack theory has given way to a $\cos 2\theta$ variation using the equant porosity model, which matches that displayed by the data. In general, the fit between

Table 4 Results from a least-squares fit of the matrix permeability K_m, matrix P-wave speed and matrix S-wave speed. The least-squares misfit, matrix permeability K_m, matrix P-wave speed, matrix S-wave speed, the quantity $\mathcal{J}/c$ and the quantity $\alpha\kappa_f/\mathcal{J}(\lambda+\mu)$ are determined by fitting the equant porosity model of Hudson *et al.* (1996) to the laboratory wave-speed data measured by Rathore *et al.* (1995). The frequency is taken as 100 kHz. The results shown were obtained from the application of first-order theory. (b) As (a) except that the results are generated from the application of second-order theory

	First-order	Second-order
χ^2	1.26×10^5	1.18×10^5
K_m (mD)	180	314
V_P	2638	2638
V_S	1264	1224
$\mathcal{J}/c$	32.5	43.0
$\alpha\kappa_f/\mathcal{J}(\lambda+\mu)$	1.99	1.48

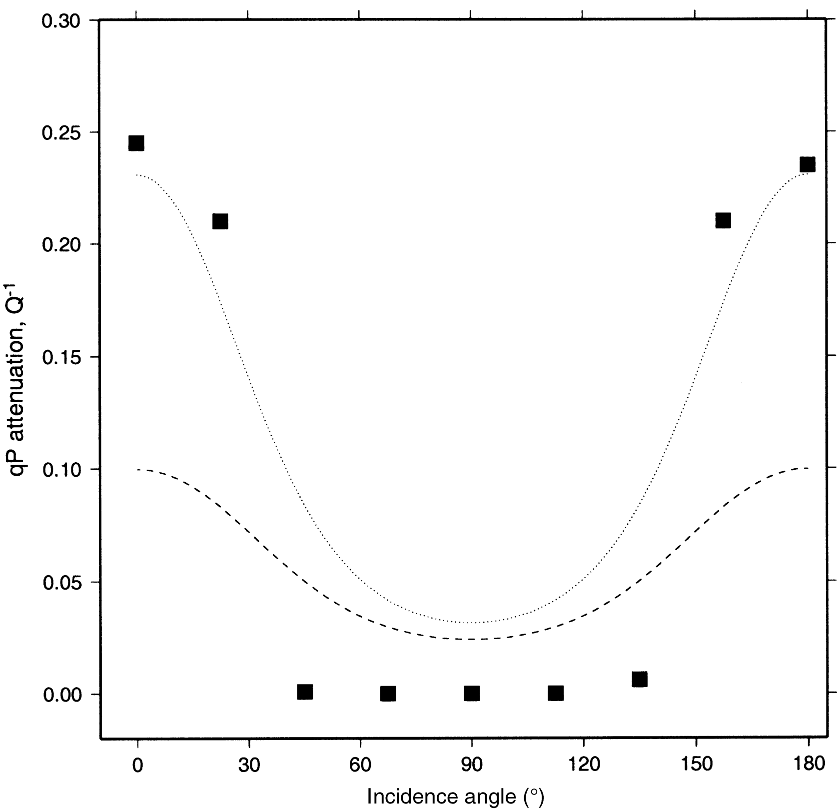


Figure 4 Comparison of qP attenuation data (solid squares) determined by Thomsen (1995) (central frequency 55 kHz) from the laboratory data recorded by Rathore *et al.* (1995), with the predictions of the equant porosity model of Hudson *et al.* (1996). The dotted curve is given by first-order theory using the matrix wave speeds measured from the dummy sample (Table 2) and $K_m = 313$ mD (see Table 3a). The dashed curve is given by second-order theory using a least-squares fit to matrix permeability K_m and wave speeds (Table 4).

theory and data is not very good, the closest being achieved by Thomsen's (1995) theoretical results.

Rathore *et al.* (1995) suggested that the failure to match theory to data arises from use of the dummy sample for values of the wave speeds in the uncracked matrix. They therefore constructed new values by fitting the values of the wave speeds in the crack-parallel direction with, in the first place, Thomsen's (1995) theory (matrix A, Table 2) and, secondly, the theory for isolated cracks given by Hudson (1981) (matrix B, Table 2). We follow a similar path here and choose values for the matrix wave speeds and permeability by minimizing the misfit with the equant porosity model (Hudson *et al.* 1996) rather than the isolated-crack model which is clearly inapplicable. Table 4 gives the results for both first-order and second-order theory. This time the value of K_m from second-order theory is not so far out of line with the others.

A comparison of the data with the two theoretical results is shown in Fig. 3. Not unexpectedly, the fit is much better than before. This time first-order theory appears to be as successful in accounting for the data as second-order theory.

Thomsen (1995) has calculated average Q values in the frequency band 30–80 kHz for the P-wave from the data of Rathore *et al.* (1995). These are shown in Fig. 4, together with the predictions of the equant porosity model of Hudson *et al.* (1996) at the central frequency of 55 kHz. The first-order curve is calculated using the matrix wave speeds from the dummy matrix and the permeability given by a least-squares fit to the data (Table 3a). The second-order curve uses matrix wave speeds and permeability values found by a least-squares fit to the data (Table 4). Once again, first-order theory appears to be slightly more successful in fitting the data.

CONCLUSIONS

The experiments of Rathore *et al.* (1995) provide a useful check on effective-medium theories for materials containing cracks. However, it appears that the wavelengths used are not long enough for effective-medium results to apply accurately; some attenuation and dispersion due to scattering can be observed in the experimental results on dry samples and are, indeed, predicted by theoretical considerations. For dry cracks there is qualitative agreement on the azimuthal dependence of the three wave speeds between theory and experiment; also, the difference in the predictions of a range of separate theories is small. In fact, they all agree to first order in the parameter $\zeta = \nu a^3$, where ν is the number density of cracks and a is the crack radius. However, it is clear that first-order theories are unreliable at $\zeta = 0.1$, as is the case in the experiments.

Theoretical results given by Hudson (1981) and Thomsen (1995) for isolated liquid-filled cracks agree to first order except that Thomsen (1995) retains a factor $(1 - \kappa_f/\kappa)$, where κ_f, κ are the bulk moduli of the fluid and matrix material respectively; this factor is taken to be unity in Hudson's (1981) expressions. The difference between the two results is small overall (of order c/a, the aspect ratio of the cracks) but Thomsen's (1995) expression gives the correct result in the limit $\kappa_f \to \kappa$, while Hudson's (1981) does not.

In the experiments (Rathore *et al.* 1995) on liquid-saturated samples, the effect of the porosity of the matrix rock – a synthetic sandstone – is clearly very strong since there is not even qualitative agreement with theoretical results for isolated cracks. The two theories for saturated porous material containing cracks differ markedly in their predictions. Thomsen (1995) found a fit between experiment and theory both at 'moderately high' frequencies (80 kHz) and at 'low' frequencies (30 kHz). At moderately high frequencies it is assumed that diffusion into the pores may be neglected and the fit is to the theory for isolated cracks. At low frequencies, the theory used takes account of equant porosity. On the other hand, the criteria for the effect of porosity given by Hudson *et al.* (1996) and here (equation (26)) indicate that diffusion out of the cracks is significant at both frequencies, although the limit of effectively undrained cracks is not reached.

Comparison of theory and observations shows that a reasonable fit can be achieved, especially if the wave speeds in the uncracked material are regarded as parameters to be adjusted to minimize the misfit. It is encouraging to note that theory provides a reasonable qualitative fit to the attenuation data as well as to the wave speeds. A surprising feature of this comparison is that, on the whole, first-order theory provides a better fit to the observations than second-order theory. However, high accuracy is not to be expected in view of the fact that the conditions necessary for the validity of an effective-medium theory were not strictly complied with in the experiments.

The theory used here has one additional parameter to that needed by Thomsen (1995), namely the matrix permeability K_m. Since Thomsen (1995) gave expressions for the high- and low-frequency limits, this is not surprising. However, it does mean that we have an additional parameter to help fit the data. It is remarkable that both Thomsen (1995) and Xu (1998) were able to fit the data with (apparently different) low-frequency expressions, although Xu (1998) also fitted an additional parameter; this was the apparent aspect ratio of the pores, which he needed to calculate their compliance.

The laboratory results might be extrapolated for use in field observations if scaled to the correct proportions. If we extend the expression (1) for one of the wave speeds (quasi-P, say) as a function of non-dimensional variables to the case of a saturated, porous, cracked material, we have

$$\left(\frac{V_P}{\alpha}\right)^2 = f(\zeta, \alpha/\beta, \kappa_f/\kappa, c/a, \phi_m, \mathcal{J}/c), \tag{30}$$

where $\mathcal{J} = (\phi_m \kappa_f K_m/\omega\eta_f)^{\frac{1}{2}}$ and ϕ_m, K_m are the porosity and permeability of the matrix material respectively, and η_f is the viscosity of the liquid infill. Equation (7) shows f as a function of all the available non-dimensional variables, although Hudson *et al.* (1996) coupled κ_f/κ and c/a together as $(a\kappa_f/c\kappa)$ and did not have ϕ_m separate from J; i.e.

$$\left(\frac{V_P}{\alpha}\right)^2 = f\left(\zeta, \frac{\alpha}{\beta}, \frac{a}{c}\frac{\kappa_f}{\kappa}, \mathcal{J}/c\right). \tag{31}$$

In order to extrapolate from the results of laboratory experiments to field observations it is necessary for each of these non-dimensional quantities to take approximately the same value in field conditions as in the experiment. In particular the product of rock porosity and permeability relative to the square of the crack thickness must give the same value to $\mathcal{J}/c$.

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