Effective SNR for Space–Time Modulation Over a Time-Varying Rician Channel

Christian B. Peel and A. Lee Swindlehurst, *Senior Member, IEEE*

***Abstract*—Rapid temporal variations in wireless channels pose a significant challenge for space–time modulation and coding algorithms. This letter examines the performance degradation that results when time-varying flat fading is encountered when using trained and unitary space–time modulation. Performance is characterized for a channel having a constant specular component plus a time-varying diffuse component. A first-order autoregressive (AR) model is used to characterize diffuse channel coefficients that vary from symbol to symbol, and is shown to lead to an effective signal-to-noise ratio (SNR) that decreases with time. Differential modulation is shown to have an advantage in effective SNR over trained unitary modulation at high power. Simulation results are provided to support our analysis.**

***Index Terms*—Differential modulation, fading channels, multiple antennas, space–time modulation, time-varying channels, trained modulation, wireless communications.**

I. Introduction

THE exciting increase in capacity and diversity promised by multiple-antenna systems [1], [2] is derived under the assumption that the receiver knows the fading coefficient between each transmit and receive antenna. Knowledge of the channel coefficients at the receiver is a nontrivial assumption; often, a training signal is sent, from which the channel is estimated and used for decoding subsequent symbols, until the channel has changed enough to require training again. The number of channel uses over which the channel is approximately constant is known as the coherence interval. As the number of antennas used and the speed of fading increase, the fraction of the coherence interval that must be used for training increases. This obviously decreases the available data rate, and motivates interest in schemes that do not require explicit knowledge of the channel coefficients at the receiver.

Marzetta and Hochwald have studied situations where neither the transmitter nor receiver know the channel [3]. Assuming piecewise-constant Rayleigh fading, they proposed signal constellations composed of unitary matrices as a means to achieve capacity at high signal-to-noise ratio (SNR). These can be seen as multiple-antenna generalizations of phase-shift keying (PSK) for scalar channels. Hughes [4] and Hochwald *et al.* [5] apply these signals to the unknown channel by extending differential phase-shift keying (DPSK) ideas to the multiple-antenna case. Tarokh also discusses differential modulation with orthogonal signals in [6].

The quasi-statuc model for the time-varying channel coefficients assumed in these papers is useful for several reasons. It accurately describes the way a channel might appear in a time-division multiple access or frequency-hopping system, and its effects are simple to analyze. In other applications, however, its inability to account for the natural time variation of the channel make it less attractive.

To quantify the memory of the channel coefficients in our analysis, we adopt a first-order autoregressive (AR) model for their time evolution. While the time-autocorrelation function $r_{hh}(t)$ of a fading coefficient h is more often used to characterize the way a channel changes between coherence intervals, the AR model provides for a much simpler analysis. By choosing the AR coefficient so that it yields the same second-order statistics as $r_{hh}(t)$ at some specified t, excellent agreement is obtained using data generated by the autocorrelation model and analyzed using the AR assumption.

Our analysis approach is to combine the AR input to the changing channel and the error due to channel estimation (if any) together with the additive noise to effectively create an overall noise term with higher power. This higher *effective* SNR (ESNR) can then be used to calculate bit-error rate (BER) probabilities for the time-varying case using expressions derived for the static channel. The ESNR will be shown to accurately predict the performance error floor or SNR ceiling, beyond which increasing transmit power provides no benefit (due to channel estimation and modeling errors). Our expressions reduce to those of Korn [7] in the single-antenna case.

We focus on the performance of trained and differential modulation, and determine the conditions under which one approach outperforms the other when unitary signals are employed. The resulting performance breakpoint depends on a number of factors, including the rate at which the channel is changing, the actual SNR, the ratio of specular to diffuse energy in the channel, and the number of antennas on each end of the link. We consider only simple trained modulation schemes that do not attempt to track the channel in between training intervals. The performance of channel-tracking techniques such as those described in [8]–[10] will be reserved for future work, along with those which distribute training samples over time. We anticipate that these methods, though more complex, will perform better than the simple schemes described herein.

II. Channel Model

In what follows, we let $\mathcal{CN}(0,1)$ denote a zero-mean, unit-variance, circularly symmetric complex Gaussian distribution. The Frobenius norm will be represented by $\|\cdot\|$, and the expectation operator by $E[\cdot]$.

Paper approved by A. F. Naguib, the Editor for Wireless Communication of the IEEE Communications Society. Manuscript received March 22, 2001; revised August 2, 2002 and July 11, 2003. This work was supported by the National Science Foundation under Wireless Initiative Grant CCR-9979452 and Information Technology Grant CCR-0081476. This paper was presented at the IEEE International Conference on Communications, Helsinki, Finland, 2001.

The authors are with the Electrical and Computer Engineering Department, Brigham Young University, Provo, UT 84602 USA (e-mail: chris.peel@ieee.org; swindle@ee.byu.edu).

Digital Object Identifier 10.1109/TCOMM.2003.822148

A. Fading Channel Model

Assume a flat-fading communications environment with M transmit and N receive antennas. A complex channel coefficient describes the effect of the propagation between each pair of transmit and receive antennas. These channel coefficients are assumed to be independent from element to element across the antenna array, but possibly correlated in time. At each receive antenna, interference and other disturbances add temporally and spatially independent noise to the signal.

These statements are formalized as follows. For $m = 1, \ldots, M$ transmit, and $n = 1, \ldots, N$ receive antennas, at time instants $t = 0, 1, \ldots, T-1$, the channel coefficient is $h_{m,n,t}$, with the signal transmitted from antenna m at time t denoted by $s_{t,m}$. We assume that the $M \times N$ matrix H_t formed from $h_{m,n,t}$ is normalized so that $E[||H_t||^2] = MN$, and the $T \times M$ matrix S formed from $s_{t,m}$ is normalized so that $E[||S||^2] = TM$. With these definitions, the data at receive antenna n is written

$$x_{t,n} = \sqrt{\frac{\rho}{M}} \sum_{m=1}^{M} h_{m,n,t} s_{t,m} + v_{t,n} \tag{1}$$

where we assume that the noise $v_{t,n}$ is $\mathcal{CN}(0,1)$. The values in this expression are normalized so that ρ represents the SNR expected at each receive antenna, and does not depend on the number of transmit antennas.

The channel equation in (1) allows for arbitrary channel coefficients at every time instant. One common simplification is to assume that the channel is constant for T consecutive samples, and express the operation of the channel in matrix form

$$X_\tau = \sqrt{\frac{\rho}{M}} S_\tau H_\tau + V_\tau \tag{2}$$

where X_τ and V_τ are $T \times N$ matrices constructed from $x_{t,n}$ and $v_{t,n}$, S_τ is a $T \times M$ matrix constructed from $s_{t,m}$, H_τ is an $M \times N$ matrix formed from $h_{m,n,t} = h_{m,n}$, and τ indexes the current symbol block of T samples. We refer to S_τ as the space–time symbol transmitted at symbol time τ, and the subscript on H_τ indicates that the channel will, in general, be different from symbol to symbol.

In what follows, we will separate the channel into specular and diffuse components, writing:

$$H_\tau = \sqrt{1-\beta} H^s + \sqrt{\beta} H_\tau^d$$

with known time-invariant specular channel H^s, and diffuse component H_τ^d, which we assume has elements distributed as $\mathcal{CN}(0,1)$. Though a distribution is not specified for H^s, we do require $||H^s||^2 = MN$ to maintain the power relationship in (1).

We will also often separate ρ into specular power ρ^s and diffuse power ρ^d terms

$$\rho^s = (1-\beta)\rho \tag{3a}$$

$$\rho^d = \beta\rho \tag{3b}$$

so that $\rho = \rho^s + \rho^d$. If $\beta = 1$, then the Rayleigh channel assumed in most space–time coding research [2], [11] is obtained. If $\beta = 0$, a strong specular or line-of-sight signal arriving at the receiver is obtained, and for $0 < \beta < 1$, we have a combination of the two.

B. An Innovations Fading Channel Model

In Section III, we analyze the performance of space–time modulation under the assumption that the current channel matrix occurs t samples after a reference (or estimated) channel H_r. We assume that the dispersive part of the channel H_{r+t}^d varies from the reference channel according to the following first-order AR model:

$$H_{r+t}^d = \sqrt{\alpha_t} H_r^d + \sqrt{1-\alpha_t} W_{r+t} \tag{4}$$

where H_r^d and W_{r+t} have independent, identically distributed (i.i.d.) Gaussian elements, W_{r+t} is independent from symbol to symbol, $\alpha_t \in \mathbb{R}$ and $0 \le \alpha_t \le 1$. Under this model, H_t^d is i.i.d. Gaussian, and thus, (4) represents MN first-order AR processes. Note that $\alpha_t = 1$ produces a time-invariant channel, and $\alpha_t = 0$ indicates a completely random (from symbol to symbol) time-varying channel. With differential coding, $t \approx M$ and demodulation is based on the previous symbol of length $T = M$, while typically for trained modulation $t \approx KT$, with demodulation based on a channel estimate obtained $K \ge 1$ symbols in the past.

The AR parameter α_t can be chosen, for example, to match the second-order statistics of models based on the mechanisms of physical propagation. Let $r_{hh}(t)$ denote the autocorrelation function of an element of H_t^d. For example, Jakes' model of the land mobile fading channel [12] predicts $r_{hh}(t) = J_0(2\pi f t)$, where $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind, $f = f_d T_s$, f_d is the maximum Doppler frequency in the fading environment, and T_s is the sampling period. Solving the Yule–Walker equations for α_t in the first-order AR process (4) gives

$$\alpha_t = \left[\frac{r_{hh}(t)}{r_{hh}(0)}\right]^2 = J_0(2\pi t f)^2 \tag{5}$$

which provides a reasonable choice for α_t. This AR model is an appropriate approximation to Jakes' model when using the maximum-likelihood (ML) decoders of [3] that depend on a single reference channel. This fact is borne out by the simulation results of Section V, where excellent agreement is obtained with data generated according to Jakes' model, but analyzed with the AR model using (5). Note that any other model for $r_{hh}(t)$ could be fit to the AR processes as in (5).

C. Channel Estimation

Space–time coding algorithms often assume that the receiver either knows the channel, or has an estimate obtained by means of known pilot symbols embedded in the data. The channel estimate is used to decode several subsequent symbols, over which the receiver assumes the channel to be the same as that during training. This approach will be referred to as trained modulation.

We will consider the ML estimate $\hat{H}_r$ of the channel

$$\hat{H}_r = \sqrt{\frac{M}{\rho}} \left(S_r^H S_r\right)^{-1} S_r^H X_r \tag{6}$$

where S_r is the training signal, X_r is the received training data, and all parameters are assumed to be known except the diffuse component of the channel H_r^d. It has been found [13] that for training over the quasi-static channels that our decoders will assume, the optimal training signals have orthogonal columns

$$S_r^H S_r = T_\tau I_M$$

where T_τ is the length of the training signal. Because the specular part of the channel is known, we may remove it from our data, and estimate the diffuse part of the channel only. Assuming (6), the ML estimator then becomes

$$\hat{H}_r^d = \sqrt{\frac{M}{\rho^d T_\tau}} S_r^H \left(X_r - \sqrt{\frac{\rho^s}{M}} S_r H^s \right) = H_r^d + \sqrt{\frac{M}{\rho^d T_\tau}} \hat{V}_r \tag{7}$$

where H_r^d and $\hat{V}_r$ are, respectively, the diffuse part of the channel and the receiver noise seen during training.

We will discuss techniques such as differential modulation that do not explicitly require an estimate of the channel, and we will also be interested in the performance bound provided by perfect channel estimation. To enable the derivation of a single expression for all cases, we use the factor γ

$$\hat{H}_r^d = H_r^d + \sqrt{\frac{\gamma M}{\rho^d T_\tau}} \hat{V}_r. \tag{8}$$

When $\gamma = 1$, we include the effects of channel estimation in the results, otherwise $\gamma = 0$.

D. Differential versus Trained Modulation

Differential unitary space–time modulation [4], [5] assumes a channel that is constant over each pair of consecutive square symbols. This scheme uses data at the current and previous time instants for encoding and decoding. The channel matrices are assumed to be equal at symbols τ and $\tau - 1$, and are denoted below without subscript by H. The current signal matrix is a unitary rotation of the previous signal, $S_\tau = \Phi_{z_\tau} S_{\tau-1}$, where $z_\tau \in 0, \ldots, L-1$ indexes the unitary constellation and selects the matrix to be transmitted, and S_τ is $\sqrt{M}$ times a unitary matrix [3]. Using these definitions, and working with the current received data X_τ, the following expressions are obtained in [5]:

$$X_\tau = \sqrt{\frac{\rho}{M}} \Phi_{z_\tau} S_{\tau-1} H + V_\tau + (1-1)\Phi_{z_\tau} V_{\tau-1} \tag{9}$$

$$= \Phi_{z_\tau} X_{\tau-1} + V_\tau - \Phi_{z_\tau} V_{\tau-1} \tag{10}$$

$$= \Phi_{z_\tau} X_{\tau-1} + \sqrt{2}\hat{V}_\tau. \tag{11}$$

In (9), $\Phi_{z_\tau} V_{\tau-1}$ is added and subtracted from (2), resulting in (10), which does not explicitly depend on H. Finally, because the noise matrices are statistically invariant to multiplication by unitary matrices, (11) is obtained, in which $\hat{V}_\tau$ is i.i.d. Gaussian (like V_τ). Equation (11) is called the "fundamental differential receiver equation" in [5].

Because the effective channel $(X_{\tau-1})$ has signal strength ρ, the system has an ESNR of $\rho/2$. This factor of two is the same well-known 3-dB loss in performance seen when using DPSK versus coherent PSK. With the identification of $X_{\tau-1}$ as the effective channel, (11) is simply (2) with half the signal strength that would be seen with coherent detection.

III. Performance of Trained Modulation: Arbitrary Signals

We now look at the performance of trained space–time modulation using the innovations model of Section II to relate the speed of a mobile to the correlation between samples of the time-varying channel. In this section, we assume that the channel is constant over each symbol, and use the time-autocorrelation function to characterize the variation of the diffuse component of the channel as described in Section II-B.

Most training-based modulation techniques (those that do not employ channel tracking) implicitly assume that the channel is piecewise constant; i.e., it is assumed that the training-based channel estimate is "good" until the next training interval. Of course, most wireless channels are not truly constant over any time period, and the longer the interval since the last training data was sent, the more the channel estimate will differ from the truth. To more accurately model the effects of a time-varying channel, we assume in this section that the channel is constant over each symbol, but varies between symbols according to the first-order AR model introduced in Section II. *Lemma 1* below quantifies the reduction in ESNR that results under this time-varying channel model. We will look at performance t time samples after an initial reference channel estimate is obtained. By letting $t = KT$, this models the performance of trained modulation K symbols (of length T) after training.

Lemma 1 (ESNR for Trained Modulation): Given the data model of Section II, assume that the channel varies from a reference channel at each symbol according to (4) with AR parameter α_t. If space–time modulation is implemented with ML channel estimation $(\gamma = 1)$ or with a perfect channel estimate $(\gamma = 0)$, the system can be described by

$$X_{r+t} = \sqrt{\frac{\rho_t^s}{M}} S_{r+t} H_r^s + \sqrt{\frac{\rho_t^d}{M}} \alpha_t S_{r+t} \hat{H}_r^d + V'_{r+t}$$

where ESNR values for the specular and dispersive parts of the channel t time samples after the reference are

$$\rho_t^s = \frac{\rho(1-\beta)}{1 + \gamma\alpha_t \frac{M}{T_\tau} + (1-\alpha_t)\rho\beta} \leq \rho^s \tag{12a}$$

$$\rho_t^d = \frac{\rho\beta\alpha_t}{1 + \gamma\alpha_t \frac{M}{T_\tau} + (1-\alpha_t)\rho\beta} \leq \rho^d \tag{12b}$$

and the columns of the effective noise matrix are identically distributed, with covariance matrix

$$R_{V'_{r+t}|S_{r+t}} = N\frac{\gamma}{T_\tau}\alpha_t S_{r+t} S_{r+t}^H + N\frac{\rho^d}{M}(1-\alpha_t) S_{r+t} S_{r+t}^H + N I_T \tag{13}$$

where S_{r+t} is the transmitted space–time signal.

Proof: The AR model of (4) is used to describe how the dispersive component of the channel has changed since the training data was transmitted, with the effect of channel estimation modeled using (8)

$$X_{r+t} = \sqrt{\frac{\rho^s}{M}} S_{r+t} H_r^s + \sqrt{\frac{\rho^d}{M}} \alpha_t S_{r+t} \hat{H}_r^d \underbrace{- \sqrt{\gamma \alpha_t T_\tau} S_{r+t} V_r + \sqrt{(1-\alpha_t)\frac{\rho^d}{M}} S_{r+t} W_{r+t} + V_{r+t}}_{V'_{r+t}}. \quad (14)$$

The covariance matrix of the columns of the effective noise term V'_{r+t} is given by

$$R_{V'_{r+t}|S_{r+t}} = NI_T + N\gamma \frac{\alpha_t}{T_\tau} S_{r+t} S_{r+t}^H + \frac{\rho^d}{M}(1-\alpha_t) S_{r+t} S_{r+t}^H$$

and the variance of the effective noise term V'_{r+t} is calculated as

$$\begin{aligned}\sigma^2_{V'_{r+t}} &= \frac{1}{TN} \mathrm{tr} E\left[R_{V'_{r+t}|S_{r+t}}\right] \\ &= 1 + \gamma \alpha_t \frac{M}{T_\tau} + \rho^d (1-\alpha_t).\end{aligned}$$

The ESNR values in (12a) and (12b) are found by dividing the effective signal strength by $\sigma^2_{V'_{r+t}}$. ■

Combining the effects of noise and channel time-variation into a single SNR parameter provides a straightforward link with previous work that assumes a piecewise-constant channel. In particular, for purposes of analysis, we can treat the trained time-varying case using a time-invariant channel model with a lower ESNR. This ESNR is time-varying; it decreases with time, until a new reference channel is obtained.

Let $\rho_t = \rho_t^s + \rho_t^d$. In the limit, as the channel becomes constant ($\alpha_t \to 1$), the ESNR ρ_t converges from below to the original SNR due only to the additive noise (and channel estimation), as desired

$$\lim_{\alpha_t \to 1} \rho_t = \frac{\rho}{1 + \gamma \frac{M}{T_\tau}}.$$

When using ML channel estimation with $T_\tau = M$ we are left with a 3-dB penalty for imperfect channel estimation. If our channel estimate is perfect ($\gamma = 0$), then the ESNR converges to the original SNR.

For a fast-fading channel that varies randomly from symbol to symbol, we obtain[1]

$$\lim_{\alpha_t \to 0} \rho_t = \frac{\rho^s}{1 + \rho^d}.$$

As the SNR increases, the ESNR becomes a function only of the fading parameters, and is independent of ρ

$$\tilde{\rho}_t^s \triangleq \lim_{\rho \to \infty} \rho_t^s = \frac{1-\beta}{\beta} \frac{1}{1-\alpha_t} \quad (15a)$$

$$\tilde{\rho}_t^d \triangleq \lim_{\rho \to \infty} \rho_t^d = \frac{\alpha_t}{1-\alpha_t}. \quad (15b)$$

This confirms the intuition that as we increase the power to the system, errors due to thermal and other noise will become less significant, and performance will be dominated by errors induced by the changing channel. This will happen when the true SNR is greater than $\tilde{\rho}_t \triangleq \tilde{\rho}_t^s + \tilde{\rho}_t^d$. We note also that $\lim_{\alpha_t \to 0} \tilde{\rho}^d = 0$, but

$$\lim_{\alpha_t \to 0} \tilde{\rho}^s = \frac{1-\beta}{\beta}. \quad (16)$$

At high SNR in fast fading, all effective signal power is due to the specular component.

Longer training data results in better ESNR

$$\lim_{T_\tau \to \infty} \rho_t^s = \frac{\rho^s}{1 + (1-\alpha_t)\rho^d} \quad (17a)$$

$$\lim_{T_\tau \to \infty} \rho_t^d = \frac{\rho^d \alpha_t}{1 + (1-\alpha_t)\rho^d} \quad (17b)$$

but lowers the time available for data transmission. If $\alpha_t = 1$, then there are no time variations in the diffuse component of the channel, and we have $\rho_t^s = \rho^s$ and $\rho_t^d = \rho^d$. This means that it is always better in terms of ESNR to train more frequently, but this, of course, ignores the loss in effective bit rate due to training.

For the case $M = N = 1$, and a constellation of $L = 2$ unitary symbols, we have binary PSK (BPSK) with bit error probability of $(1/2)(1 - \sqrt{\rho/(1+\rho)})$. Assuming a Rayleigh channel, and substituting (15a) for the SNR, we obtain the high-SNR error floor

$$P_e = \frac{1}{2}\left[1 - r_{hh}(KT)\right].$$

When $K = T = 1$, this is equivalent to the high-SNR error floor derived by Korn [7] for binary DPSK. The 3-dB penalty that DPSK suffers in comparison with PSK is due to the effective doubling of the additive noise power, so it is reasonable to expect that the same error floor holds for PSK and DPSK at high SNR, where the changing channel dominates additive noise.

IV. Performance With Unitary Modulation

Lemma 1 applies to general signals; we now examine the case of square ($T = M$) unitary signals. This class of signals is used for differential space–time modulation and was motivated by the discovery that unitary signals maximize capacity for the quasi-static channel [11]. Differential space–time modulation has received much attention recently for its excellent behavior in a time-varying channel [15], [16]. We give a corollary which states the effect of the time-varying channel on trained modulation with perfect channel estimates and with ML channel estimation, as well as on differential modulation. Differential modulation is shown to have good performance; it has higher ESNR than trained modulation in all cases except with perfect channel estimates at low SNR.

The derivation of the fundamental differential receiver (11) assumes that the channel is constant for overlapping periods of $2T$ time instants. In this section, we use our channel model to obtain a more realistic result for differential modulation. Finally, we compare trained and differential modulation. We begin by using the AR model (4) to express the effect of the time-varying channel as an ESNR.

[1]For a rapidly changing channel ($\alpha_M \to 0$) there would be significant variations of the channel within each symbol (and possibly within each time sample), which we do not take into account in our analysis. In [14], we present a performance analysis that allows channel variation at each sample within a symbol.

A. ESNR for Unitary Modulation

Corollary 1: Given the data model of (2), assume that the diffuse portion of the channel varies according to (4) with AR parameter α_t. If unitary space–time modulation is implemented, then the ESNR is

$$\rho_U^s(\rho^s, \rho^d, \alpha_t) = \frac{\rho^s}{1+\gamma\alpha_t + (1-\alpha_t)\rho^d} \leq \rho^s \quad \text{(18a)}$$

$$\rho_U^d(\rho^d, \alpha_t) = \frac{\rho^d \alpha_t}{1+\gamma\alpha_t + (1-\alpha_t)\rho^d} \leq \rho^d \quad \text{(18b)}$$

where the subscript U indicates unitary modulation. For differential modulation, $t = M$ and $\gamma = 1$. For "trained" modulation with a perfect channel estimate, $t = KM$ and $\gamma = 0$, and for trained modulation with an ML channel estimate, $t = KM$ and $\gamma = 1$.

Proof: For trained modulation, this corollary is a straightforward application of *Lemma 1*, where for square unitary signals, the covariance matrix for the effective noise becomes a scaled identity matrix. The proof for differential modulation is also a straightforward extension. ■

Similar limiting expressions to those found in Section III apply for differential modulation as well. For example, as the true SNR increases, we find that performance is dominated by the changing channel

$$\tilde{\rho}_D^s \triangleq \lim_{\rho\to\infty} \rho_D^s = \frac{1-\beta}{\beta}\frac{1}{1-\alpha_M} \quad \text{(19a)}$$

$$\tilde{\rho}_D^d \triangleq \lim_{\rho\to\infty} \rho_D^d = \frac{\alpha_M}{1-\alpha_M}. \quad \text{(19b)}$$

These equations indicate that increasing signal power does not always give better performance. In fact, for true SNR values greater than $\tilde{\rho}_D \triangleq \tilde{\rho}_D^s + \tilde{\rho}_D^d$, performance no longer depends on signal power, but on the effect of the changing channel.

For the case $M = N = 1$, $L = 2$, and $\beta = 1$, we have binary DPSK in Rayleigh flat fading with bit-error probability $1/(2 + 2\rho)$. Substituting (19) for the SNR, we obtain the high-SNR error floor

$$P_e = \frac{1}{2}\frac{[1-r_{hh}(1)][1+r_{hh}(1)]}{1+r_{hh}(1)^2}.$$

For many values of $f (f \leq .05)$, the autocorrelation function satisfies $r_{hh}(1) \approx r_{hh}(1)^2$, and we obtain $P_e \approx 0.5[1 - r_{hh}(1)]$, which is the high-SNR error floor derived by Korn [7] for DPSK.

The ESNR ρ_D can be used in place of the true SNR in the probability of error expressions given below. In contrast to *Lemma 1*, in the differential case, the ESNR is not time varying. The pairwise probability of error expressions given below extends the results of [3] derived for Rayleigh channels to channels with a rank-one specular component. The ESNR given above allows this result to be applied to time-varying channels as well.

Lemma 2: Given the quasi-static Rician channel model of (2) with specular parameter β (assuming a rank-one specular component), SNR ρ, M transmit antennas, N receive antennas, and assuming the ML decoder of [3] for a known channel

$$\arg\max_{l\in\{0,1,\ldots,L-1\}} \left\| X - \sqrt{\frac{\rho}{M}} S_l H \right\|$$

the pairwise probability of error for square unitary signals is

$$P_e = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+\frac{1}{4}} \prod_{m=1}^{M}\left\{1+\left(\omega^2+\frac{1}{4}\right)\rho\beta d_m^2\right\}^{-N} \times \exp\left\{-N\sum_{m=1}^{M}\frac{1-\beta}{\beta}\frac{\left(\omega^2+\frac{1}{4}\right)\rho\beta d_m^2}{1+\left(\omega^2+\frac{1}{4}\right)\rho\beta d_m^2}\right\} d\omega \quad (20)$$

where d_m are the singular values of the difference matrix of the two signals in question $(S_1 - S_2)$.

Proof: The proof involves expressing the pairwise probability of error in terms of the trace of a quadratic form, then integrating over the characteristic function [14] of this value to obtain the final result. To conserve space, and because it is a straightforward extension of techniques used in [3] and [14], we omit details of the proof. ■

We now turn to the case of modulation for the unknown channel. We allow an arbitrary-rank specular channel in this case to illustrate the potential of our analysis. A similar expression for the known channel case is left as an exercise for the reader.

Lemma 3: Given the quasi-static Rician channel model of (2) with specular parameter β, SNR ρ, M transmit antennas, N receive antennas, and assuming the ML decoder of [3] for an unknown channel

$$\arg\max_{l\in\{0,1,\ldots,L-1\}} \left\| S_l^H X \right\|_F^2 \quad (21)$$

the pairwise probability of error is

$$P_e = \frac{1}{2\pi j}\int_{-\infty-j\epsilon}^{\infty-j\epsilon} \frac{1}{\omega}|I + j\omega RQ|^{-N} \times \exp\left(-\bar{Y}^H R^{-1}\left[I - (I+j\omega RQ)^{-1}\right]\bar{Y}\right) d\omega \quad (22)$$

where

$$R = \begin{bmatrix} \rho^s \frac{T}{M} I & \rho^s \frac{T}{M} D \\ \rho^s \frac{T}{M} D & \frac{\rho^s \frac{T}{M}}{\sqrt{\rho^d \frac{T}{M}+1}}\left(\rho^d \frac{T}{M} D^2 + I\right) \end{bmatrix} \quad (23)$$

$$Q = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \quad \bar{Y} = \frac{\rho^s \frac{T}{M}}{\sqrt{\rho^d \frac{T}{M}+1}}\begin{bmatrix} \Gamma H^s \\ D\Gamma H^s \end{bmatrix} \quad (24)$$

and $\Phi_2^H \Phi_1 = \Theta D \Gamma^H$ is the singular value decomposition of the product of the unitary matrices Φ_1 and Φ_2 (where $S_i = \sqrt{T}\Phi_i$).

Proof: Similar techniques are used to prove this *Lemma* as for the previous, with a major difference being in the lack of a rank-one constraint on H^s, which results in a slightly more complex expression in this case. ■

The ESNR values derived previously may be used directly in (20) and (22) to characterize the error performance of unitary codes in time-varying fading. In our simulation results below, we show that the above probability of error expressions give excellent agreement with simulation when used with the ESNR values of *Corollary 1*.

B. Comparing Differential and Trained Unitary Modulation

In this section, we compare the ESNR for differential versus trained unitary modulation. Comparison of differential modulation against trained modulation with nonunitary and/or nonsquare signals may give significantly different results than those given below.

Corollary 2 (ESNR Comparison): Given the channel model of Section II, assume that the parameters α_t and α_M for trained and differential modulation are related according to $\alpha_t < \alpha_M$, and that $T = M$ and $t > T$ are fixed. If ML channel estimation $(\gamma = 1)$ is used, then $\rho_D > \rho_t$. If perfect channel estimates are used $(\gamma = 1)$, then $\rho_D > \rho_t$ if $\rho > \hat{\rho}$, where

$$\hat{\rho} = \frac{\frac{\alpha_M}{\beta} + \alpha_t + \alpha_M\alpha_t - 2\alpha_M}{\alpha_M - \alpha_t} \tag{25}$$

otherwise, $\rho_D < \rho_t$. If $t = M$, then for ML channel estimation $\rho_t = \rho_D$ for all ρ.

Proof: Recall that as ρ goes to ∞, $\rho_D \to \tilde{\rho}_D$, and $\rho_t \to \tilde{\rho}_t$. Since $\alpha_M > \alpha_t$, then $\tilde{\rho}_D > \tilde{\rho}_t$. Because $\hat{\rho}$ is the only solution to the equation $\rho_D(\rho) = \rho_t(\rho)$, we know that $\rho_D > \rho_t$ for $\rho > \hat{\rho}$. ■

In simpler language than stated in the lemma, differential modulation is better than trained modulation (in terms of ESNR) in a changing channel if the SNR is high enough, even if perfect channel estimates are available. If ML channel estimation is used, then differential modulation always gives higher ESNR than trained modulation. This is in contrast to a constant channel, where trained modulation is always better. In particular, as the channel approaches a constant channel, $\alpha_M \to 1$, and thus, $\lim_{\alpha_M \to 1, \alpha_t \to 1} \hat{\rho} = \infty$.

V. Simulation Results

We have presented analytic results quantifying performance for a continuously varying fading channel; we now present simulation results that support our analysis. In the figures that follow, a square or circle indicates a simulation result for that SNR value. We generated channel coefficients that obey Jakes' channel model, and simulated them with 10^5 symbols at each SNR value to calculate the probability of error results shown. When a specular component is present, it is rank one.

Fig. 1 illustrates the utility of the model presented in Section III. In this figure, we show results for $M = 2$ transmit antennas, $N = 2$ receive antennas, training interval $K = 2$, specular parameter $\beta = 1$, and for fading parameters, $f = 0.01$ and $f = 0.02$. This corresponds to the Doppler shift obtained at 20 and 40 mi/h with a carrier-to-bandwidth ratio of about 333 000/1. Although our analysis is based on an AR model for the time variation of the channel, we simulated with channel coefficients that obey Jakes' model [12], using (5) to reconcile the two techniques. Probability of symbol-error results are shown for simulations of the "vertical Bell Labs layered space–time" (VBLAST) algorithm [17], [18] with quaternary PSK (QPSK) symbols. In this figure, we also show results for simulation with a slow channel (shown with a dashed line) which will be used

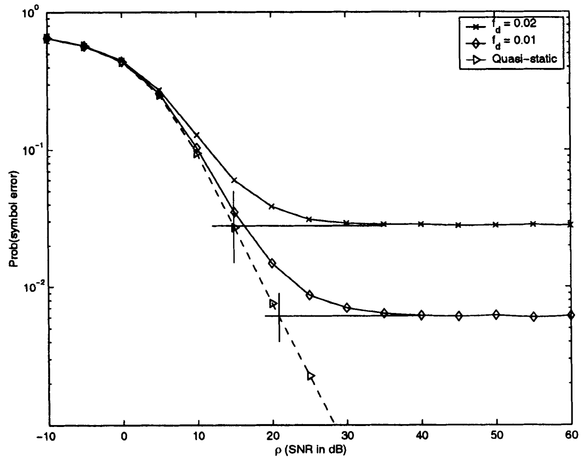

Fig. 1. Results using VBLAST with QPSK symbols, $M = N = 2$ antennas, and specular parameter $\beta = 1$. Simulation with fading parameter $f = 0.01$ is indicated by diamonds, $f = 0.02$ with x's, and a quasi-static channel by a dashed line. Vertical lines show the SNR from *Lemma 1*, at which a high-SNR ceiling should occur; these values agree with simulation.

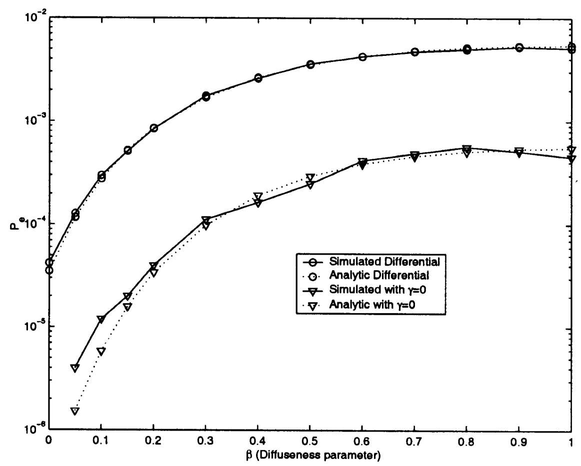

Fig. 2. Probability of error as function of β with $M = N = 2$ antennas, $L = 2$ constellation points, at an SNR of $\rho = 5$ dB, and $f_d = 0.003$. The solid lines show the results of simulation, which agree well with analysis (dotted lines) obtained using (20).

as if they were for a quasi-static channel. *Lemma 1* predicts that at high SNR, we should obtain performance equivalent to that at $\tilde{\rho} = 14.9$ dB SNR for $f_d = 0.02$ and $\tilde{\rho} = 20.9$ dB SNR for $f_d = 0.01$. The performance of the constant channel at these SNR values is indeed that obtained with the changing channel, as indicated by the intersection of the horizontal and vertical lines.

Fig. 2 shows performance as a function of the specular parameter β. Square diagonal unitary codes are used with $L = 2$. Simulation and analysis results are shown for differential modulation and genie-aided trained modulation $(\gamma = 0)$; trained modulation with an ML channel estimate gives results similar to differential modulation. Two transmit and two receive antennas were used at a SNR of $\rho = 5$ dB, and $f_d = 0.003$. In this scenario, a specular channel $(\beta = 0)$ gives better results than a Rayleigh $(\beta = 1)$ channel. At all values of the specular parameter, the analytic results match simulation very well.

In Fig. 3, we illustrate the performance of trained modulation as a function of the training period K. We used a fading

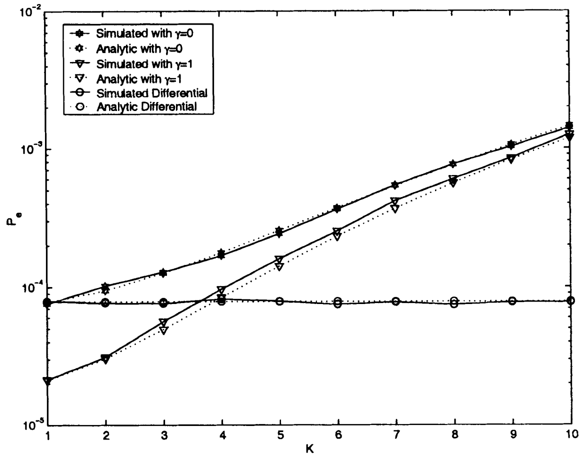

Fig. 3. Performance as a function of the training period K with $M = 2$ transmit antennas, $N = 1$ receive antenna, a fully diffuse channel $\beta = 1$, and $L = 2$ signals. Analysis and simulation agree that though there is a 3-dB penalty for differential and trained modulation at $K = 1$, for higher values of K, the penalty is greater for trained modulation.

parameter of $f = 0.003$, $M = 2$ transmit antennas, $N = 1$ receive antennas, a diagonal constellation with $L = 2$ space–time symbols, no specular component ($\beta = 1$), and a training period of $t = KM$ for $K = \{1, 2, \cdots, 10\}$. Solid lines indicate the results of simulation, while the dashed lines indicate analytic results using ESNR values from Section III in place of the true SNR in the probability of error expressions from [3]. Results for training using ML channel estimation ($\gamma = 1$), modulation with perfect channel estimates ($\gamma = 0$), and differential modulation are shown. The analytic and simulation results match well in all cases. As expected, increasing the training period increases the probability of error. We note that for longer training intervals, differential performs significantly better than trained modulation, even with a perfect channel estimate.

VI. Conclusions

Previous research in space–time modulation has typically assumed channels that are constant for two or more symbol periods. In this letter, we have examined the performance degradation that results when this assumption is violated. We considered the case of a time-varying, temporally correlated diffuse channel component, with a temporally invariant specular component. AR modeling of the diffuse channel variations allowed us to derive expressions for ESNR that combine the effects of the changing channel and the additive noise into a single scalar value when using unitary signal matrices. The ESNR can be used in place of the noise-only SNR to analyze the effects of a time-varying channel using expressions derived assuming the channel to be constant. Comparing ESNR expressions for trained and differential modulation, we are able to determine the SNR above which differential modulation outperforms trained unitary modulation. Using probability of symbol error as our metric, we validated our analysis with several simulations. The capacity of trained space–time modulation is a well-studied problem [19]–[21]. The capacity for differential modulation, on the other hand, is an open problem [22], making a comparison between the two schemes difficult. There is some reason to believe that the capacity for differential modulation is significantly smaller than that for coherent space–time modulation [22]. This may offset some or all of the advantage in ESNR that differential modulation has. We leave this as a topic for future research.

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