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# EFFECTS OF MASSIVE CENTRAL BLACK HOLES ON DENSE STELLAR SYSTEMS

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SUMMARY

We discuss some dynamical and astrophysical consequences of the presence of a massive black hole, of mass $M_h$ in a dense stellar system, applying our results to the cores of globular clusters and to galactic nuclei. The black hole would dominate the dynamics of stars out to a radius $r_h \simeq GM_h/v_c^2$, $v_c$ being the velocity dispersion in the core. Within $r_h$, the stellar velocity dispersion is proportional to $r^{-1/2}$ and the stellar density $n(r)$ may be enhanced. A quasi-steady state can be established, involving a steady influx of stars which are swallowed or disrupted near the hole. We define and calculate a 'critical radius' $r_{crit}$ such that most stars on orbits with $r \lesssim r_{crit}$ diffuse into low-angular momentum 'loss-cone' orbits (and are swallowed) in the 'reference time' $t_R$; whereas outside $r_{crit}$, loss-cone effects are negligible and a typical star diffuses inward into a more tightly-bound orbit on a time scale $\sim t_R(r/r_{crit})$. The density profile in the cusp is $n(r) \propto r^{-7/4}$ for $r_{crit} \lesssim r \lesssim r_h$; and somewhat flatter inside $r_{crit}$. Generally $r_{crit}$ is larger than either the tidal radius $r_T$ of the hole or the 'collision radius' $r_{coll}$ at which $GM_h/r_{coll}$ equals the binding energy per unit mass of a typical star: indeed, in some cases $r_{crit} \gtrsim r_h$.

The swallowing or disruption rate of stars varies as $M^{4/3}$ when $r_{crit} \gtrsim r_h$, $M^{61/27}$ when $r_h \lesssim r_{crit} \lesssim r_{coll}$ and $M^3$ when $r_{crit} \lesssim r_{coll}$. We discuss some consequences of stellar disruption and tidal capture by black holes of $10^3$–$10^4\,M_\odot$ in globular clusters: X-ray emission, possible optical or ultraviolet 'flares', and the likelihood of there being a captured star in orbit near $r_T$.

Finally, we briefly apply our considerations to Hills' quasar model, which invokes $\gtrsim 10^7\,M_\odot$ black holes in galactic nuclei.

## 1. INTRODUCTION

Studies of dynamical evolution of dense stellar systems and the search for the mechanism powering galactic nuclei and QSOs have led many people to the idea that a massive black hole could be sitting at the centre of evolved stellar systems. Recent X-ray data (Clark, Markert & Li 1975) have prompted the suggestion that some globular clusters may contain central black holes of $\sim 10^3\,M_\odot$. We here discuss some dynamical and astrophysical consequences of this idea. Although we focus attention mainly on globular clusters, we conclude by applying our results to galactic nuclei, with particular reference to the recent model of Hills (1975).

The stellar distribution in the cores of globular clusters (and perhaps galactic nuclei as well) can be described by a density $n_c$ and a 'core radius' $r_c$. The virial theorem then tells us that the characteristic (one-dimensional) velocity dispersion is $v_c \simeq (Gm_* n_c r_c^2)^{1/2}$, $m_*$ being the stellar mass. If there is a central point mass $M_h$ such that $n_c r_c^3 m_* \gg M_h \gg m_*$, then its potential well will affect the stellar velocity field out to a distance

$$r_h = \frac{GM_h}{v_c^2}.$$

The effect on the density distribution is less straightforward. However, if the central mass has been present for a time comparable with the stellar relaxation time (or ' reference time ') in the core, we expect some kind of stationary state to be established, involving a slow, inward drift of the stars. (In fact the whole core will evolve on a time scale $\sim 10t_R$ so the situation will never be an exactly stationary one.) The standard ' reference time ' (Spitzer & Harm 1958) is

$$t_R = \frac{\sqrt{2}v_c^3}{\pi G^2 m_*^2 n \log(0.5N)},$$

where $N = \frac{4}{3}\pi m_c r_c^3$.

For the cores of some globular clusters, $t_R$ is only $10^8$ yr, implying that a quasi-stationary stellar distribution could indeed have been established around a central mass. The low values of $t_R$ also suggest that a runaway process may already have occurred in some clusters, which lends support to the conjecture that massive, central black holes may indeed sometimes exist (Wyller 1970; Ostriker, Spitzer & Chevalier 1972). In galactic nuclei, $t_R$ is quite uncertain: Wolfe & Burbidge (1970) considered cases when it was $\gtrsim 10^{10}$ yr; but others have envisaged extremely concentrated regions in galactic nuclei where $t_R \lesssim 10^{10}$ yr (e.g. Spitzer 1971; Saslaw 1974; Hills 1975 and references cited therein). Note that $t_R$ is $\sim N/\log N$ times the crossing time $\sim r_c/v_c$.

A central black hole provides an effective ' sink ' for stars approaching too close to it—such stars will be swallowed or disrupted, by processes which we discuss later. There is therefore no possibility of establishing an ' isothermal ' distribution where the density $n(r)$ rises exponentially within $r_h$. This point was first emphasized by Peebles (1972). He conjectured that the distribution of stars in bound orbits followed a power law $N(E) \propto E^{-p}$ in binding energy $E$. There is then a *power law* cusp in the stellar density:

$$n(r) \simeq n_c \left(\frac{r_h}{r}\right)^q \quad (r_h \gtrsim r \gtrsim r_{min}), \text{ with } q = \tfrac{3}{2}+p, \tag{1}$$

and the velocity dispersion within the cusp scales as $r^{-1/2}$.

Peebles suggested, specifically, that $n(r)$ should be such that $n(r)\, r^3 (t_R(r))^{-1}$ was independent of $r$, and derived a law of form (1) with $q = \frac{9}{4}$. This solution would apparently correspond to a constant inflow of stars at a rate determined by the conditions at $r_h$. However, it is unacceptable since it does not yield an energy outflow rate independent of $r$; the correct solution ought also to yield a swallowing rate that depends on the inner-boundary condition, because the energy that has to be transported away per star swallowed is proportional to $r_{min}^{-1}$. Bahcall & Wolf (1976) propose a law of form (1) with $q = 0$, which has the property that the outward energy flux $\sim n(r)\, r^3 v^2 (t_R(r))^{-1}$ is independent of $r$. They call this a ' zero flow solution ' because, even though the stars in the cusp do move into more tightly-bound orbits as energy is carried away, the time scale for a star at radius $r$ to drift inward is *longer* than $t_R(r)$ by a factor $\sim r/r_{min}$. Bahcall & Wolf (1976) exhaustively discuss the validity of the various approximations involved in their solution (isotropic velocity distribution, etc.) and we refer to their paper for the details. We follow earlier authors in assuming that the stars in the cusp have a more or less isotropic distribution (except near $r_{\rm min}$), and that the orbital energy of a typical star at radius $r$ is $\sim GM_{\rm h}m_{*}r^{-1}$. Each star moves in an elliptical orbit around the central hole, the orbital energy and angular momentum being gradually changed by encounters with other stars. Our primary aim is to estimate the rate at which stars in the cusp are swallowed or disrupted, and to consider probably observable manifestations of these processes. This involves estimating the appropriate value of $r_{\rm min}$.

For applications to real systems such as globular clusters, it will often prove convenient to work in terms of $n_{\rm c}$ and $r_{\rm c}$, since these are better determined observationally than (say) $v_{\rm c}$. We then have

$$\frac{v_{\rm c}}{10\ {\rm km/s}} \simeq 1{\cdot}76\left(\frac{n_{\rm c}}{5\times 10^4\ {\rm pc}^{-3}}\right)^{1/2}\left(\frac{r_{\rm c}}{1\ {\rm pc}}\right), \tag{2}$$

and

$$\frac{r_{\rm h}}{1\ {\rm pc}} \simeq 1{\cdot}5\times 10^{-2}\left(\frac{M_{\rm h}}{10^3\ M_\odot}\right)\left(\frac{n_{\rm c}}{5\times 10^4\ {\rm pc}^{-3}}\right)^{-1}\left(\frac{r_{\rm c}}{1\ {\rm pc}}\right)^{-2}. \tag{3}$$

The dynamical time scale is then

$$t_{\rm dyn}(r) \simeq 10^3\ {\rm yr}\left(\frac{M_{\rm h}}{10^3\ M_\odot}\right)\left(\frac{n_{\rm c}}{5\times 10^4\ {\rm pc}^{-3}}\right)^{-3/2}\left(\frac{r_{\rm c}}{1\ {\rm pc}}\right)^{-3}\times\begin{cases}(r/r_{\rm h})^{3/2} & r\lesssim r_{\rm h}\\ (r/r_{\rm h}), & r\gtrsim r_{\rm h}\end{cases} \tag{4}$$

or, in terms of a fixed distance unit,

$$t_{\rm dyn}(r) \simeq \begin{cases}5\times 10^5\ {\rm yr}\ (M_{\rm h}/10^3\ M_\odot)^{-1/2}(r/1\ {\rm pc})^{3/2}, & r\lesssim r_{\rm h}\\ 6\times 10^4\ {\rm yr}\ (n_{\rm c}/5\times 10^4\ {\rm pc}^{-3})^{-1/2}(r_{\rm c}/1\ {\rm pc})^{-1}(r/1\ {\rm pc}), & r\gtrsim r_{\rm h}.\end{cases} \tag{5}$$

The reference time in the core is

$$t_{\rm R} \simeq 2\times 10^8\ {\rm yr}\left(\frac{n_{\rm c}}{5\times 10^4\ {\rm pc}^{-3}}\right)^{1/2}\left(\frac{r_{\rm c}}{1\ {\rm pc}}\right)^3,\quad r\gtrsim r_{\rm h}. \tag{6}$$

If the $r^{-7/4}$ cusp solution applies, the value of $t_{\rm R}$ inside $r_{\rm h}$ is shorter by a factor $\sim(r/r_{\rm h})^{1/4}$. The logarithmic term (which involves the ratio of the maximum and minimum relevant impact parameters) and a further correction resulting from the density gradient (Bahcall & Wolf 1976) introduces a slow additional $r$-dependence, but these refinements amount to less than a factor $\sim 2$ and are unimportant for our present purposes.

The quantities $t_{\rm dyn}$, $t_{\rm R}$, $v_{\rm c}$ and $r_{\rm h}$ depend on the stellar-dynamics alone, and thus only on the assumption that the stars behave as gravitationally interacting point masses. However, the following radii, which depend more explicitly on the physical properties of the system, are also relevant to the problem:

(i) The Schwarschild radius of the hole

$$r_{\rm S} \simeq 3\times 10^5\left(\frac{M_{\rm h}}{M_\odot}\right)\ {\rm cm}. \tag{7}$$

(ii) The tidal radius (or 'Roche radius') within which a star would be disrupted. This obviously depends on the type of star (and to some extent on the shape of its orbit around the hole) but for solar-type stars it is

$$r_{\rm T} \simeq 1{\cdot}4\times 10^{11}\left(\frac{M_{\rm h}}{M_\odot}\right)^{1/3}\ {\rm cm}. \tag{8}$$

For other types of stars it scales as $(r_*/R_\odot)(m_*/M_\odot)^{-1/3}$. There will be an intermediate range of radii around $r_T$ at which tidal effects would partially disrupt or merely distort the star rather than destroying it completely. These processes may still be able to reduce the orbital energy by an amount sufficient to remove stars from the cusp, and capture them into very tightly-bound orbits passing close to the tidal radius.

(iii) The 'collision radius' $r_{coll}$ at which the velocity dispersion $\sim(GM_h/r)^{1/2}$ is comparable with the escape velocity from typical stars. This is significant because the stellar encounters responsible for the relaxation of the velocity distribution, energy diffusion, etc., can be treated as elastic Coulomb-type encounters only outside $r_{coll}$: when $r \lesssim r_{coll}$, two stars cannot deflect each other's velocities

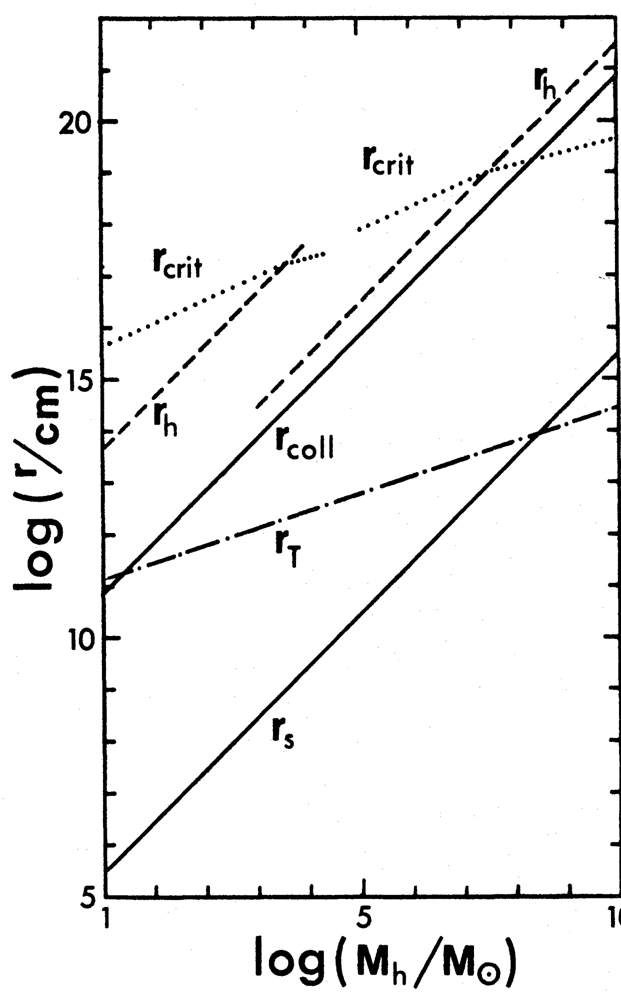


FIG. 1. *The different radii defined in the text are plotted as a function of the mass $M_h$ of the hypothetical central black hole for parameters adequate for globular clusters and galactic nuclei. The Schwarzschild radius is denoted by $r_s$. The tidal radius $r_T(\propto M_h^{1/3})$ and the collision radius $r_{coll}$ $(\propto M_h)$ are plotted for stars of solar type. The velocity cusp radius $r_h$ $(\propto M_h$, broken line), within which the gravitational potential is dominated by the central point mass, and the critical radius $r_{crit}$ (dotted line), outside which angular diffusion is negligible, are plotted for two typical situations: globular clusters ($n_c = 5 \times 10^4\ pc^{-3}$, $r_c = 1$ pc, $M_h \lesssim 10^4\ M_\odot$); and galactic nuclei ($n_c = 10^7\ pc^{-3}$, $r_c = 1$ pc, $M_h \gtrsim 10^7\ M_\odot$). Two different situations arise depending on whether $r_{crit} > r_h$ or $r_{crit} < r_h$. In the first case $r_{crit} \propto M_h^{4/9}$ and the swallowing rate (SR) is $\propto M_h^{4/3}$; in the second case a density cusp exists, $r_{crit} \propto M_h^{7/27}$ and SR $\propto M_h^{61/27}$.*

through a large angle without coming so close that they actually collide. For solar-type stars, we have

$$r_{\rm coll} \simeq 7\times 10^{10}\left(\frac{M_{\rm h}}{M_\odot}\right)\ {\rm cm}, \tag{9}$$

while for other types of stars $r_{\rm coll}$ scales as $(r_*/R_\odot)(m_*/M_\odot)^{-1}$.

It is clear that a stellar-dynamical discussion of the cusp is only strictly applicable at radii exceeding both $r_{\rm T}$ and $r_{\rm coll}$. Fig. 1 shows these quantities as a function of $M_{\rm h}$, for parameters appropriate to globular cluster cores. Note that $r_{\rm coll}$ greatly exceeds $r_{\rm T}$ for any interesting value of $M_{\rm h}$.

We shall now argue, however, that other considerations severely limit the applicability of the Bahcall–Wolf (1976) 'zero-flow' solution, giving a value of $r_{\rm min}$ that can be much larger than either $r_{\rm coll}$ or $r_{\rm T}$. This is because, even though stars may be changing their orbital energy on a time scale much slower than $t_{\rm R}$, they can still change their angular momentum on a time scale $\sim t_{\rm R}$. A star may thus—without changing its orbital energy—diffuse into a very eccentric (low angular momentum) orbit which allows it to pass so close to the hole that it gets captured or disrupted. Such a star is then, in effect, lost from the cusp. The Bahcall–Wolf solution would need modifying if this process removed stars faster than the time scale $\sim(r/r_{\rm min})\,t_{\rm R}$ on which a star's orbital binding energy can be significantly increased.

## 2. THE CENTRAL CUSP AND THE SWALLOWING RATE

### 2.1 *Loss-cone diffusion*

If a star at distance $r \gg r_{\rm T}$ is in orbit of such low angular momentum that its peribothron* is $\lesssim r_{\rm T}$, it must be moving nearly radially, its velocity vector lying within a small 'loss-cone' of semi-angle $\theta_{\rm lc}$ given by:

$$\theta_{\rm lc} = f\begin{cases}(2r_{\rm T}/3r)^{1/2} & r \leqslant r_{\rm h}\\ (\tfrac{2}{3}r_{\rm T}r_{\rm h})^{1/2}/r & r \gtrsim r_{\rm h}.\end{cases} \tag{10}$$

The factor $f$ takes account of the fact that stars with slightly larger impact parameters can be lost from the cusp owing to tidal capture. We take $f \simeq 2$ in the numerical estimates below.

We assume that the stars have velocities $\sim v_{\rm c}(r/r_{\rm h})^{-1/2}$ for $r \lesssim r_{\rm h}$ and $\sim v_{\rm c}$ for $r \gtrsim r_{\rm h}$. The numerical value, for solar-type stars, is

$$\theta_{\rm lc} \simeq 8{\cdot}1\times 10^{-3}\left(\frac{M_{\rm h}}{10^3\,M_\odot}\right)^{-1/3}\left(\frac{n_{\rm c}}{5\times 10^4\ {\rm pc}^{-3}}\right)^{1/2}\left(\frac{r_{\rm c}}{1\ {\rm pc}}\right)\times\begin{cases}(r/r_{\rm h})^{-1/2}, & r \leqslant r_{\rm h}\\ (r/r_{\rm h})^{-1}, & r \gtrsim r_{\rm h}\end{cases} \tag{11}$$

and for other types of stars it scales as $(r_*/R_\odot)^{1/2}(m_*/M_\odot)^{-1/6}$.

The cumulative effect of distant encounters causes a star's velocity to diffuse through a small angle $\theta_{\rm D}$ in each dynamical time scale, where

$$\theta_{\rm D} \simeq \left(\frac{t_{\rm dyn}}{t_{\rm R}}\right)^{1/2} \simeq 2{\cdot}1\times 10^{-3}\left(\frac{M_{\rm h}}{10^3\,M_\odot}\right)^{1/2}\left(\frac{n_{\rm c}}{5\times 10^4\ {\rm pc}^{-3}}\right)^{-1}\left(\frac{r_{\rm c}}{1\ {\rm pc}}\right)^{-3}\times\begin{cases}(r/r_{\rm h})^{5/8}, & r \leqslant r_{\rm h}\\ (r/r_{\rm h})^{1/2}, & r \gtrsim r_{\rm h}\end{cases} \tag{12}$$

* We are grateful to W. R. Stoeger for suggesting this word, derived from the Greek *bothros*, a pit.

(i.e. the orbital angular momentum would change by an amount $\sim \theta_D rv$ in a time $t_{dyn}$). This quantity is independent of $m_*$.

The ratio $R = \theta_{lc}/\theta_D$ decreases monotonically with $r$, and there is a critical radius $r_{crit}$ at which it is unity.

Inside $r_{crit}$ (where $R > 1$) we expect a deficit of stars on loss-cone orbits, since stars are removed on a time $t_{dyn}$, but take $\sim R^2$ longer to replenish by angular diffusion. (Repopulation by energy diffusion takes a time $\gtrsim t_R$ which is longer by a further factor $\theta_D^{-2}$). The diffusion approximation can then be applied to estimate the rate at which stellar encounters repopulate the 'loss-cone' orbits by angular-momentum diffusion.

The situation is analogous to a simple problem of heat conduction in a hemispherical shell, when the equator is maintained at one temperature $T_1$, and a small ring at co-latitude $\theta$ is maintained at temperature $T_2 < T_1$. The heat conduction rate is then proportional to $(T_1 - T_2)/\log(2/\theta)$. If we imagine the angle diffusion to be represented by the diffusion of the tips of the velocity vectors on a hemisphere until they disappear into the loss cone, we conclude that inside $r_{crit}$, the stars can be swallowed in a characteristic 'angular diffusion' time $\sim t_R \log(2/\theta_{lc})$. Of course the diffusion is superposed on a basic elliptical rather than linear motion. It is therefore better (though this does not affect the conclusion) to visualize the process as diffusion of the vector representing the magnitude and orientation of the *latus rectum* of the orbit. Note that, provided the diffusion approximation applies (i.e. $R > 1$), the swallowing time only depends logarithmically on $\theta_{lc}$ and is always of order $t_R$. A more elaborate discussion by Lightman & Shapiro (1976), which takes explicit account of the distribution of eccentricities, etc., confirms these estimates.

Outside $r_{crit}$ ($R < 1$) the loss-cone loses its significance, because a given star can drift in and out of it within $t_{dyn}$. The loss-cone orbits are therefore depleted only by a factor $\sim(1 - R^2)$, and the heat conduction analogy then suggests that the diffusion rate into these orbits is reduced by a factor $R^2$. Thus a star whose orbit is initially entirely outside $r_{crit}$ may have time to diffuse within $r_{crit}$ by energy diffusion (i.e. reducing the size of its orbit), even if this takes $\gg t_R$ before it gets swallowed.

### 2.2 *Estimates of the swallowing rate*

The main characteristic feature of the 'zero-flow', $q = 7/4$ solution—that stars remain in the cusp for a time limited only by the slow energy-diffusion time scale $\sim (r/r_{min})\, t_R$—will strictly apply only at radii where the 'loss-cone' diffusion is less rapid than this. We must therefore, for consistency, take the appropriate $r_{min}$ for this solution as

$$r_{min} \simeq r_{crit}\{\log(2/\theta_{lc}(r_{crit}))\}^{-1}. \tag{13}$$

If this radius turns out to be smaller than $\max\{r_T, r_{coll}\}$, then the latter of course determines the actual value for $r_{min}$.

If $R < 1$ at $r_h$ and we assume that the 'zero flow' ($q = 7/4$) solution applies in the range $r_{crit} < r < r_h$, then

$$\frac{r_{crit}}{r_h} \simeq 3{\cdot}1 \left(\frac{M_h}{10^3\, M_\odot}\right)^{-20/27} \left(\frac{n_c}{5 \times 10^4\ \mathrm{pc}^{-3}}\right)^{4/3} \left(\frac{r_c}{1\ \mathrm{pc}}\right)^{32/9}, \quad r_{crit} \lesssim r_h. \tag{14}$$

For other types of star this scales as $(r_*/R_\odot)^{4/9}(m_*/M_\odot)^{-4/27}$. Under some conditions, $R$ may be $> 1$ at $r_h$, implying that $r_{crit} > r_h$. The appropriate expression is

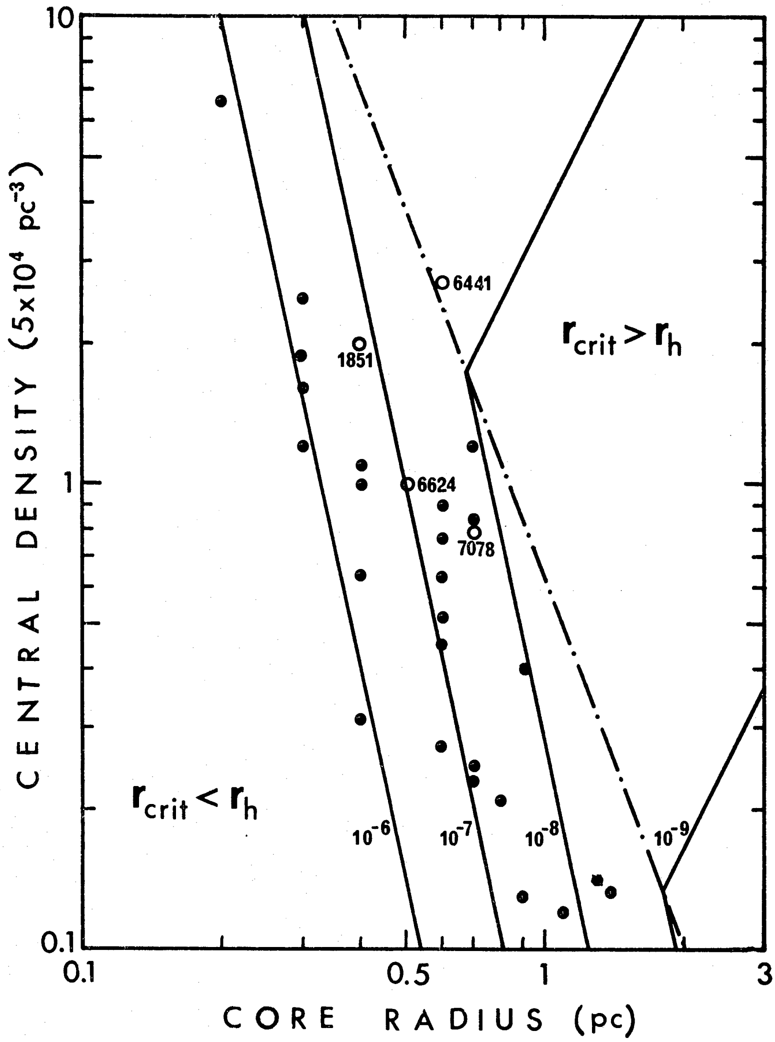


FIG. 2. *A plot of all globular clusters with central densities larger than* $5 \times 10^3$ *pc*$^{-3}$ *on a coordinate grid defined by two observable parameters: the core radius in parsecs and the central density in units of* $5 \times 10^4$ *stars pc*$^{-3}$. *Each globular cluster is represented by a full circle, except the four represented by open circles and labelled by their NGC number, with which X-ray emission has been associated. The relevant data for all of them are given in Table I. The broken line with slope* $-8/3$ *separates the two different swallowing regimes with* $r_h \gtrless r_{crit}$. *A cusp with density as steep as* $r^{-7/4}$ *is expected only when* $r_{crit} < r_h$. *The line is plotted for the case when the hypothetical central black hole has a mass* $M_h \simeq 10^3 M_\odot$. *The full lines represent loci of constant swallowing rate, and are labelled in yr*$^{-1}$. *These rates scale with* $M_h$ *according to* (16*a*) *and* (16*b*).

then

$$\frac{r_{\rm crit}}{r_{\rm h}} \simeq 2{\cdot}35 \left(\frac{M_{\rm h}}{10^3 M_\odot}\right)^{-5/9} \left(\frac{n_{\rm c}}{5 \times 10^4 \,{\rm pc}^{-3}}\right) \left(\frac{r_{\rm c}}{1 \,{\rm pc}}\right)^{8/3} \tag{15}$$

and scales as $(r_*/R_\odot)^{1/3}(m_*/M_\odot)^{-1/9}$.

From Fig. 2 we see that the existence of a 'zero-flow' cusp ($r_{\rm min} \lesssim r_{\rm h}$) in a globular cluster core is rather marginal for $M_{\rm h} = 10^3 M_\odot$. We also note, that, for all cases of interest, $r_{\rm coll}$ is very much smaller than $r_{\rm crit}$, so that it is the conditions at $r_{\rm crit}$ which determine the swallowing rate. The swallowing rate (SR) due to diffusion into loss-cone orbits follows directly from the angle-diffusion time scale and star density at $r_{\rm crit}$. If $r_{\rm crit} < r_{\rm h}$ we estimate $n(r_{\rm crit})$ from the $q = 7/4$

cusp solution matched on to the core at $r_h$, obtaining approximately (for solar-type stars)

$$SR \simeq \frac{1}{4\cdot3\times10^8\,\mathrm{yr}}\left(\frac{M_h}{10^3\,M_\odot}\right)^{61/27}\left(\frac{n_c}{5\times10^4\,\mathrm{pc}^{-3}}\right)^{-7/6}\left(\frac{r_c}{1\,\mathrm{pc}}\right)^{-49/9}. \qquad (16a)$$

In order to obtain a simple power-law solution we have taken the log term in (13) as $\sim 5$.

In the case when $r_{crit} > r_h$, we can use the same argument, taking $n = n_c$. This gives

$$SR \simeq \frac{1}{2\cdot1\times10^8\,\mathrm{yr}}\left(\frac{M_h}{10^3\,M_\odot}\right)^{4/3}\left(\frac{n_c}{5\times10^4\,\mathrm{pc}^{-3}}\right)^{1/2}\left(\frac{r_c}{1\,\mathrm{pc}}\right)^{-1}. \qquad (16b)$$

Note that, even though the loss-cone diffusion rate at a given $r$ depends only logarithmically on $\theta_{lc}$ (and therefore on $r_T$), the swallowing rate is still sensitive to $r_T$ because $r_{crit}$ (the maximum radius at which the diffusion approximation applies) itself depends on a power of $r_T$.

The result (16a) can also be obtained in two alternative ways: (i) we can derive the appropriate $r_{min}$ for the Bahcall–Wolf solution in terms of $r_{crit}$ from (13) and assume that stars at $r_{min}$ are drifting in on a time scale $t_R$, or (ii) we can assume that the velocities are almost isotropic at $r_{min}$ and estimate the inward flux of stars with orbits such that $\theta \leqslant \theta_{lc}(r_{min})$, so that

$$SR \simeq n(r_{min})\, r_{min}^2 v(r_{min})\, \pi\theta_{lc}^2(r_{min}). \qquad (17)$$

By substituting the value of $\theta_{lc}$ from (10) into (17) we get

$$SR \simeq \pi n(r_{min})\, r_{min}^2 v(r_{min}) \times \begin{cases} \dfrac{2r_T}{r_{min}} & (r_{min} \lesssim r_h) \\ \dfrac{2r_T r_h}{r_{min}^2} & (r_{min} \gtrsim r_h). \end{cases} \qquad (18)$$

This shows that for $r_{min} > r_h$ the swallowing rate reduces to

$$SR \simeq 2\pi n_c v_c r_T r_h \quad (r_{min} \gtrsim r_h), \qquad (19)$$

which agrees with the value given by Hills (1975) from a straightforward '$n\sigma v$' argument. It also shows that when $r_{min} < r_h$ the presence of the cusp enhances the swallowing rate by a factor $(r_h/r_{min})^{5/4}$, i.e.

$$SR \simeq 2\pi n_c v_c r_T r_h \left(\frac{r_h}{r_{min}}\right)^{5/4}. \qquad (20)$$

As $M_h$ increases, $r_h$ increases faster than $r_{crit}$ (see equation 14) so the enhancement becomes progressively more important. This accounts for the difference between the $M_h^{61/27}$ and $M_h^{4/3}$ dependence of the swallowing rates in (16a) and (16b) respectively. (Note that if $M_h$ grows so large that $r_{crit}$ eventually becomes less than $r_{coll}$, then $r_{min}$ must be taken equal to $r_{coll}$. Equation (16a) then no longer holds, and the stellar inflow rate becomes $\propto M^3$. The stars would then suffer disruption or coalescence before getting into loss-cone orbits, and the subsequent fate of the debris seems quite uncertain.)

In Fig. 2 we have plotted the loci of the most centrally-condensed clusters in terms of $r_c$ and $n_c$, these being the most directly-observable quantities (see also

TABLE I

*Swallowing rates (SR) for globular clusters with central densities larger than 5000 $pc^{-3}$ if they contained a central black hole with $M_h = 10^3 M_\odot$ (scaling laws discussed in text)*

| NGC | $n_c$ ($5\times10^4$ $pc^{-3}$) | $r_c$ (pc) | $r_{crit}/r_h$ | $N(r_{crit})$ | $N(r_h)$ | SR ($10^{-8}$ $yr^{-1}$) |
|---|---|---|---|---|---|---|
| 104 | 0·84E 00 | 0·7 | 0·4 | 2·6 | 20·7 | 2·0 |
| 362 | 0·75E 00 | 0·6 | 0·2 | 3·4 | 65·1 | 5·3 |
| 1851(*) | 0·20E 01 | 0·4 | 0·2 | 4·6 | 101·7 | 15·0 |
| 2808 | 0·12E 01 | 0·7 | 0·7 | 2·3 | 9·6 | 1·3 |
| 5824 | 0·19E 01 | 0·3 | 0·0 | 7·4 | 669·7 | 78·9 |
| 5904 | 0·12E 00 | 1·1 | 0·1 | 2·5 | 69·4 | 1·7 |
| 6093 | 0·66E 01 | 0·2 | 0·07 | 9·2 | 606·1 | 163·8 |
| 6266 | 0·63E 00 | 0·6 | 0·2 | 3·6 | 91·2 | 6·4 |
| 6273 | 0·27E 00 | 0·6 | 0·05 | 4·8 | 485·3 | 17·0 |
| 6284 | 0·11E 01 | 0·4 | 0·08 | 5·7 | 346·7 | 30·7 |
| 6293 | 0·12E 01 | 0·3 | 0·03 | 8·7 | 1727·1 | 137·2 |
| 6304 | 0·13E 00 | 0·9 | 0·08 | 3·3 | 184·7 | 4·4 |
| 6333 | 0·21E 00 | 0·8 | 0·1 | 3·3 | 141·6 | 4·7 |
| 6341 | 0·45E 00 | 0·6 | 0·1 | 4·1 | 181·5 | 9·6 |
| 6356 | 0·14E 00 | 1·3 | 0·3 | 1·8 | 17·0 | 0·5 |
| 6397 | 0·31E 00 | 0·4 | 0·01 | 8·7 | 4432·4 | 135·8 |
| 6441(*) | 0·27E 01 | 0·6 | 1·1 | 1·8 | 4·8 | 1·3 |
| 6522 | 0·25E 01 | 0·3 | 0·09 | 6·8 | 379·1 | 56·6 |
| 6528 | 0·64E 00 | 0·4 | 0·04 | 6·8 | 1006·9 | 57·2 |
| 6541 | 0·52E 00 | 0·6 | 0·1 | 3·9 | 133·7 | 8·0 |
| 6624(*) | 0·10E 01 | 0·5 | 0·2 | 4·1 | 108·8 | 10·1 |
| 6626 | 0·10E 01 | 0·4 | 0·07 | 5·8 | 382·2 | 32·5 |
| 6637 | 0·12E 00 | 1·1 | 0·1 | 2·5 | 67·3 | 1·6 |
| 6715 | 0·90E 00 | 0·6 | 0·3 | 3·2 | 45·0 | 4·2 |
| 6752 | 0·25E 00 | 0·7 | 0·08 | 3·9 | 238·8 | 8·3 |
| 6760 | 0·23E 00 | 0·7 | 0·07 | 4·0 | 282·9 | 9·2 |
| 6864 | 0·40E 00 | 0·9 | 0·4 | 2·2 | 20·0 | 1·2 |
| 7078(*) | 0·79E 00 | 0·7 | 0·4 | 2·7 | 23·3 | 2·1 |
| 7089 | 0·14E 00 | 1·4 | 0·4 | 1·6 | 11·5 | 0·4 |
| 7099 | 0·16E 01 | 0·3 | 0·05 | 7·8 | 893·0 | 93·3 |

*Notes*

Detected X-ray emission is indicated by (*).
Data for central densities and core radii from Peterson & King (1975).

Table I). We have also plotted curves representing different swallowing rates for $M_h = 10^3 M_\odot$ and the division between clusters with $r_{crit} < r_h$ and $r_{crit} > r_h$.

## 2.3 *The applicability of diffusion concepts*

The discussion in terms of the 'diffusion angle' $\theta_D$ (*cf.* equation 12) is applicable only if the total effect on a stellar orbit can be envisaged as the integrated effect of many small influences each contributing a deflection less than $\theta_D$. Outside $r_h$, where the central mass does not dominate the dynamics, each of the $N$ stars of the core at typical distances $\sim r_c$ produces a deflection through an angle $\sim N^{-1}$ in one crossing time, and therefore through a still smaller angle $\sim (r/r_c)\, N^{-1}$ in a time $t_{dyn}(r)$. In the cusp, $t_{dyn}$ is determined by the central mass, and the angular

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deflection which one star in the cusp can produce on another in $t_{\rm dyn}$ is $\sim(m_*/M_{\rm h})$. The fact that these are indeed smaller than the relevant values of $\theta_{\rm lc}$ reassures us that a diffusion treatment is approximately valid.

However, the cumulative effect of distant encounters never dominates the time-averaged effect of occasional encounters by more than a logarithmic factor. No diffusive arguments—nor even a detailed Fokker–Planck analysis—can ever, therefore, be very exact.

Another feature of the situation is the small total number

$$N\left(\frac{r_{\rm h}}{r_{\rm c}}\right)^3 \simeq \left(\frac{n_{\rm c}}{5\times 10^4\ {\rm pc}^{-3}}\right)^{-2}\left(\frac{M_{\rm h}}{10^3\ M_\odot}\right)^3\left(\frac{r_{\rm c}}{1\ {\rm pc}}\right)^{-6} \qquad (21)$$

of stars in the hypothetical cusp in globular clusters cores. For appropriate parameters (*cf.* Fig. 2) this number would never be more than $50(M_{\rm h}/10^3\ M_\odot)^3$. If $r_{\rm min}<r_{\rm h}$, the number of stars near $r_{\rm min}$ is lower by a further factor $\sim(r_{\rm min}/r_{\rm h})^{3-q}$, so the inner part of the cusp solution may be valid only in some 'ensemble-averaged' sense. This difficulty is somewhat eased if $r_{\rm min}$ has the (larger) value given by (13): there are then several stars with $r\simeq r_{\rm min}$ if $M_{\rm h}$ lies in the range $10^3$–$10^4\ M_\odot$.

The fact that the number of relevant stars in the cusp is small has, nevertheless, the gratifying corollary that $N$-body experiments, properly incorporating the effects of close encounters, may be practicable (though there is the difficulty that, to be of interest, these computations would have to extend over very many dynamical time scales). One could then study other interesting effects which a diffusion treatment ignores. For instance, the occasional close encounters within the cusp would eject stars with velocities up to $\sim(GM/r_{\rm crit})^{1/2}$, which can exceed the escape velocity from the whole cluster. The rate at which stars in the cusp get flung out of the cluster should differ by only a logarithmic factor from the swallowing rate. One could also include a range of masses, calculating whether the proportion of stars with different masses depends on $r$, and study the effects of binary stars. The outward energy flux from the cusp constitutes a 'heat input' into the cluster core, although this will not be a dominant effect on the core's overall evolution unless $M_{\rm h}$ becomes comparable with the total mass of the stars in the core. The inner part of the cusp may effectively reduce to a three-body system, and it is thus perhaps interesting that numerical studies of the effects of a third body orbiting around a binary system (Heggie 1975) do indeed show that the eccentricity (i.e. angular momentum) of the binary orbit changes more rapidly than its energy.

### 2.4 *Observable properties of the stellar cusp*

The above arguments do not enable us to determine the form of $n(r)$ inside $r_{\rm crit}$. To do this one requires to calculate the diffusion in both angular momentum and energy (Lightman & Shapiro 1976). However, we can immediately see that the fraction of stars in the cusp that diffuse into orbits of major axis $r\ll r_{\rm crit}$ before being swallowed must be $\leqslant r/r_{\rm crit}$. Otherwise the corresponding energy outflow rate would be larger than can be transported between $r_{\rm crit}$ and $r_{\rm h}$. To maintain a steady state at $r\lesssim r_{\rm crit}$, dynamical friction must drag stars inward on a time scale $t_{\rm R}$ to replace those disappearing into the loss-cone. The energy outflow (which is independent of $r$ for $q=\frac{7}{4}$) must therefore increase with $r$, implying that the cusp flattens off.

Although the observational consequences of stellar disruption and accretion on to a central black hole could (as we discuss in Section 3) be more conspicuous, it is interesting to consider the observability of the hole's *direct* effects on the distribution and velocity of the ordinary stars in the globular cluster. The total angular extent of the expected cusp is only $\sim (M_h/10^3 M_\odot)$ arcsec for a typical globular cluster at distance $\sim 10$ kpc, and the enhancement in the stellar *surface* density at a projected distance $r_{crit} \lesssim r \leqslant r_h$ from the centre is $\sim (r_h/r_c)(r/r_h)^{-3/4}$ (this would be an overestimate inside $r_{crit}$, where $q = 7/4$). Thus it is only the *inner* part of the cusp which stands out as a significant surface-density enhancement. The prospects of observing either the higher velocity dispersion in the cusp or individual runaway stars would also seem dim (see Bahcall & Wolf 1976, for further discussion of these points).

## 3. THE FATE OF STARS AND GAS

### 3.1 *Stellar disruption on loss-cone orbits*

Any star whose orbit around the hole has such low angular momentum that it passes within $\sim r_T$ will be tidally disrupted. The energy needed to unbind the star comes from its orbital kinetic energy, and therefore (as described by Hills 1975) the gaseous debris would never get out beyond $\sim r_{coll}$ even if the incoming star had an orbit of much larger major axis. The subsequent fate of the gas, and the associated observable effects, are difficult to estimate, and we shall restrict ourselves here to some general order-of-magnitude comments. (If $r_{crit} > r_{coll}$, disruption or coalescence due to stellar collisions within the cusp will be less frequent than loss-cone swallowing by a factor $\sim r_{coll}/r_{crit}$.)

Even if the debris were spread uniformly through a sphere or disc of radius $\sim r_{coll}$, the optical depth for electron scattering $\sim 10^{33}/r_{coll}^2$ would still typically be $\gg 1$, at least when $M_h$ is in the range $\sim 10^4 M_\odot$ appropriate to globular clusters. However, if its radius were as large as $r_{coll}$, it would have to be supported by radiation pressure, since its angular momentum would suffice to provide rotational support only at a radius $\sim r_T$.

If one could be sure that: (a) all the debris was eventually swallowed and (b) the radiative output per unit mass swallowed was as large as $\sim 0.1c$ (this being the value typical of ' standard ' accretion discs), then we could immediately conclude that the resultant luminosity would, at least in a time-averaged sense, be important compared with the ordinary stellar luminosity of the cluster. Each disrupted star would then provide enough fuel to maintain a luminosity of $\sim 10^{41}$ erg s$^{-1}$ for $\sim 10^5$ yr (or a lower luminosity for a correspondingly longer time); but it would still be unclear in what part of the electromagnetic spectrum this energy should emerge. However, we wish to emphasize that there is no good reason for believing either (a) or (b).

The initial binding energy of the debris is only $\sim (r_s/r_{coll})\, c^2$. Thus it is possible in principle for a fraction $r_s/r_{coll}(10^{-5})$ of the mass falling into the hole to liberate enough energy to blow the remainder away. The high optical depth would allow radiation pressure to be effective. This would mean that the energy output per star swallowed would be only $\sim 10^{49}$ erg. Moreover, this energy would be partly transformed into kinetic energy of expelled material; and the emitted radiation, being effectively produced by a ' photosphere ' at $r \gtrsim r_{coll}$, would be in the optical or ultraviolet rather than the X-ray band.

As regards (b), the expected densities (and hence the optical depths $\tau$) are much higher than in the steady-state accretion flows customarily considered. The photons emitted near the hole cannot diffuse relative to the plasma at a velocity greater than $\sim c/\tau$. If this is less than the inward velocity of the bulk flow, the photons will themselves be swallowed and cannot contribute to the luminosity observed. Thus the efficiency of accretion may automatically become low when the density is high (Pringle 1976 in preparation).

## 3.2 *Tidal capture*

There will be a range of impact parameters for which the tidal effects, though insufficient to disrupt the star, cause enough distortion and subsequent dissipation to reduce the orbital energy appreciably. This process (familiar in other branches of astronomy) has been proposed by Fabian, Pringle & Rees (1975) as a way of trapping compact stellar-mass bodies in close binary systems. A star originating at $r \gtrsim r_{\min}$, but with an eccentric orbit passing within some critical peribothron of (3–4) $r_T$, would dissipate enough to prevent it from again getting out to $\sim r_{\min}$. It would then evolve independently of the rest of the cusp: further energy would be dissipated after each peribothron passage, and it could eventually settle into a circular orbit at $\sim r_T$.* This process is rather more probable than complete tidal disruption, because the appropriate cone angle is somewhat larger.

The only circumstance capable of suddenly dislodging a star from such a small and tightly-bound orbit would be the tidal capture of a *second* star into a similar orbit. The two stars would then very quickly suffer a very high-velocity collision (since typically $r_T \ll r_{\text{coll}}$) which would disrupt them both. The debris would then either go down the hole or leave the system. Because this debris is produced in a more tightly-bound orbit, the lower limit on the fraction swallowed is higher, i.e. $\sim r_s/r_T$ rather than $\sim r_s/r_{\min}$.

The time scale for a solar-type star in a circular orbit at $r \simeq r_T$ to spiral inward owing to the effects of gravitational radiation is

$$\tau_{\text{GR}} \simeq 5 \times 10^8 \left(\frac{M_h}{10^3 M_\odot}\right)^{-2/3} \text{yr}. \tag{22}$$

For typical parameters $\tau_{\text{GR}}^{-1}$ does not exceed the swallowing rate. Therefore, unless other dissipative effects (e.g. interaction with gas) cause a star in such an orbit to spiral inward in a much shorter time than $\tau_{\text{GR}}$, we expect it to remain there until another star from the cusp is injected into the loss-cone. The interesting conclusion is then that, in any globular cluster containing a central black hole, there is a $\sim 50$ per cent chance that there will be a star in such an orbit. If this were an ordinary main-sequence star, its orbital period would be $\sim 6$ hr, and its orbital velocity $\sim 6000(M_h/10^3 M_\odot)^{1/3}$ km s$^{-1}$. A giant or horizontal branch star (for which $r_T$ is larger) would be in a somewhat bigger and slower orbit. If such a binary could be detected, it would provide gratifyingly unambiguous evidence for a massive black hole.

* The total amount of energy that a star must dispose of during this process may be several times larger than its gravitational binding energy $Gm_*^2/r_*$. The later stages of circularization, occurring after the star had spun up to the orbital angular velocity at peribothron, would therefore have to occur on a time scale longer than the star's thermal time scale (unless the star could reform again after being disrupted).

### 3.3 *Preferential capture of giants, etc.*

One or two per cent of the stars in globular clusters are giants with radii 10–100 times larger than the main sequence stars. The appropriate value of $r_T$ for a star of given mass scales with its radius $r_*$, and so $\theta_{1c}$ also scales with $r_*$. (Because such stars are more centrally condensed than main sequence stars, there will be a larger range of intermediate impact parameters for which the envelope would be torn off leaving the core intact.) If there is no cusp in the stellar distribution, the capture probability is proportional to $r_T$, and therefore $\gtrsim 20$ per cent of capture stars might be giants. When the cusp exists, the situation is less straightforward: $\theta_{1c}$ has to be defined separately for each type of star, but each type interacts dynamically with all other types so that the diffusion of giants into their loss-cone is inhibited by the presence of the other stars; the effective $r_{crit}$ is therefore the same as for main sequence stars. The net result is that there is still a tendency for preferential capture of horizontal branch or giant stars. The proportions of different types of stars that are swallowed would of course be affected by any mass-segregation that had occurred within the core, since this would cause the heavier stars to be over-represented near $r_h$.

White dwarfs (for which $r_T$ is comparable with $r_s$) or neutron stars (which can be swallowed whole) are unlikely to be captured until they have drifted into very tightly-bound orbits. Energy-outflow argument then tells us that the swallowing rate for these compact objects is much less than that for ordinary stars. If these were numerically dominant in globular cluster cores, their presence would inhibit the loss-cone diffusion of ordinary stars and thus reduce our estimates in (16a) and (16b)—which were based on the assumption that all the stars were the same. However, if compact stars comprise only a small fraction $x$ of the total, the swallowing rate for ordinary stars is reduced only by an extra factor $(1-x)$. This is because even when $R \gg 1$ the loss-cone orbits are still populated $x$ times as much as the other orbits, and the diffusion rate (from the 'temperature gradient' analogy) is lowered by a factor $(1-x)$.

### 3.4 *Origin of X-rays*

Even though the dominant supply of gas in globular-cluster cores may come from disrupted stars, this gas is produced in circumstances that make it unlikely to produce efficient X-ray emission. If the X-ray emission from some globular clusters does involve accretion on to massive black holes (Bahcall & Ostriker 1975; Silk & Arons 1975), a slower and steadier gas supply would be more efficient. Two such possibilities are: (i) gas supplied by conventional stellar mass-loss processes occurring throughout the core; or (ii) gradual Roche-lobe-overflow—perhaps at a rate controlled by the gravitational radiation time scale (22)—from a star in orbit near $r_T$.

It would be interesting to discover evidence (possibly by optical or ultraviolet observations) for non-stellar luminosity or flaring activity due to accretion of disrupted stars (interactions of stars and gas near the hole may also have observable effects). Our estimates suggest that such disruptions may occur as often as once every $10^7$ yr. Even if the effects were short-lived, the chances of observing them would be greatly enhanced if they were conspicuous enough to be detected in globular clusters as far away as the Virgo Cluster.

## 4. APPLICATIONS TO GALACTIC NUCLEI AND QUASARS(?)

Hills (1975) has proposed that galactic nuclei may contain a black hole of mass $\gtrsim 10^7 M_\odot$ surrounded by a region of high stellar density; and that tidal disruption of stars, and swallowing of the resultant gaseous debris, initiates a chain of events that may explain many properties of QSOs and Seyfert Nuclei. 'Typical' parameters are $n_c \simeq 10^7 \text{ pc}^{-3}$, $r_c \simeq 1$ pc ($v_c \simeq 225(r_c/1 \text{ pc}) \text{ km s}^{-1}$). He argues that a 'seed' black hole of $\sim 10^3 M_\odot$ could have grown by accretion to $10^7$–$10^8 M_\odot$ in the Hubble time, and that the resulting luminosity would be high enough to account for quasars. The Eddington limit for $\sim 10^8 M_\odot$ is $\sim 10^{46} \text{ erg s}^{-1}$, and Hills suggests that his model sets a natural upper limit of this order to quasar luminosities. This is because, when $M_h$ exceeds $10^8 M_\odot$, ordinary stars can be swallowed whole (i.e. $r_T < r_s$) and do not then generate a conspicuous outflow of gas or radiation.

Our foregoing arguments impinge on Hills' conclusions in several ways:

(i) When $M_h$ grows above a certain value, which for Hills' choice of $n_c$ and $r_c$ turns out to be $3 \times 10^7 M_\odot$, then $r_h$ starts to exceed $r_{crit}$, and a cusp forms. The accretion rate thereafter starts to rise as $M \propto M^{61/27}$ rather than as $M^{4/3}$. (But note that the discussion in terms of $r_h$ and $r_c$ needs modification when the black-hole mass is larger than the whole core. Also, the formation of a cusp is rather sensitive to $r_c$, and there are choices of parameters for which a cusp would never form.)

(ii) The gas production rate due to swallowing of giant stars would continue even if $M_h > 10^8 M_\odot$. (The critical mass for which $r_T \simeq r_s$ is $3 \times 10^9 M_\odot$ for stars with $r_* = 10 R_\odot$ and $10^{11} M_\odot$ for stars with $r_* = 100 R_\odot$.) Since the swallowing rate varies as $M_h^{4/3}$ until a cusp forms and as $M_h^{61/27}$ thereafter, luminosities above $10^{46} \text{ erg s}^{-1}$ are possible. This may indeed be to the advantage of the model.

(iii) If a cusp forms, (14) tells us that $r_{crit}/r_h$ varies as $M_h^{-20/27}$. When $r_{crit}$ becomes $\lesssim r_{coll}$ (which is $\lesssim 10$ times smaller than $r_h$ if the core velocity dispersion is $\gtrsim 225 \text{ km s}^{-1}$) it will actually be $r_{coll}$ which sets the inner boundary of the cusp. The inflow rate of stars then rises as $M_h^3$ and all the stars undergo collisions, leading to coalescence or disruption, at $r \simeq r_{coll}$. The further consequences are very complex and speculative: it is unclear what fraction of the debris from stellar impacts eventually gets swallowed. However, it seems relatively clear that the violent activity would continue until the black hole had swallowed or disrupted all the stars in the original dense core, and was surrounded by less closely-packed stars for which $t_R \gg 10^{10}$ yr. The steady-state diffusion arguments would then no longer apply, and the swallowing rate would be limited by the inefficient relaxation.



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