Effects of Spatial Coherence on the Angular Distribution of Radiant Intensity Generated by Scattering on a Sphere

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In the analysis of light scattering on a sphere it is implicitly assumed that the incident field is spatially fully coherent. However, under usual circumstances the field is partially coherent. We generalize the partial waves expansion method to this situation and examine the influence of the degree of coherence of the incident field on the radiant intensity of the scattered field in the far zone. We show that when the coherence length of the incident field is comparable to, or is smaller than, the radius of the sphere, the angular distribution of the radiant intensity depends strongly on the degree of coherence. The results have implications, for example, for scattering in the atmosphere and colloidal suspensions.

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In the usual description of light scattering by a homogeneous sphere (the scalar analogue of the well-known Mie scattering) it is generally assumed that the incident field is spatially fully coherent [1–5]. In practice, this assumption is not always justified. Examples are fields generated by multimode lasers, and fields that have passed through a random medium such as the turbulent atmosphere. Hardly any studies have been devoted to this more general case (see, however, [6]). The extinguished power due to scattering of random fields on a random medium has been analyzed in [7,8], and certain reciprocity relations for cases of this kind were derived in [9]. The extinguished power from scattering a random field on deterministic media was discussed in [10,11]. However, the influence of the state of coherence of the incident field on the angular distribution of the scattered field seems to have been studied only in two publications [12,13].

In this Letter we analyze the scattering of a wide class of beams of any state of coherence on a homogeneous spherical scatterer, namely, beams of the well-known Gaussian Schell-model class (see [14], Sec. 5.6.4). We present numerical examples that show how the effective spectral coherence length (i.e., the coherence length at a fixed frequency) of the incident beam affects the angular distribution of the radiant intensity of the scattered field.

Let us first consider a plane, monochromatic scalar wave of unit amplitude, propagating in a direction specified by a real unit vector $\mathbf{u}_0$, incident on a deterministic, spherical scatterer occupying a volume V (see Fig. 1):

$$V^{(i)}(\mathbf{r}, t) = U^{(i)}(\mathbf{r}, \omega)\exp(-i\omega t), \tag{1}$$

where

$$U^{(i)}(\mathbf{r}, \omega) = \exp(ik\mathbf{u}_0 \cdot \mathbf{r}). \tag{2}$$

Here $\mathbf{r}$ denotes the position vector of a point in space, t the time, and ω the angular frequency. Also, $k = \omega/c = 2\pi/\lambda$ is the wave number, c being the speed of light in vacuum and λ denotes the wavelength. The time-independent part $U(\mathbf{r}, \omega)$ of the total field that results from scattering of the plane wave on a sphere may be expressed as the sum of the incident field $U^{(i)}(\mathbf{r}, \omega)$ and the scattered field $U^{(s)}(\mathbf{r}, \omega)$, viz.,

$$U(\mathbf{r}, \omega) = U^{(i)}(\mathbf{r}, \omega) + U^{(s)}(\mathbf{r}, \omega). \tag{3}$$

The scattered field in the far-zone of the scatterer, at an observation point $\mathbf{r} = r\mathbf{u}$ ($\mathbf{u}^2 = 1$) is given by the asymptotic formula

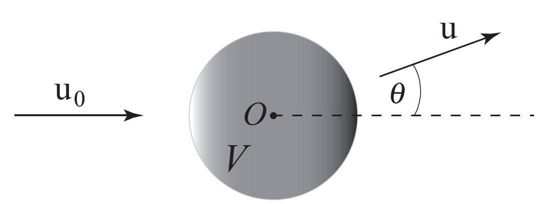

FIG. 1. Illustrating the notation. The origin O is taken at the center of the sphere.

$$U^{(s)}(r\mathbf{u}, \omega) \sim f(\mathbf{u}, \mathbf{u}_0, \omega)\frac{e^{ikr}}{r}, \qquad (kr \to \infty, \mathbf{u} \text{ fixed}), \tag{4}$$

where $f(\mathbf{u}, \mathbf{u}_0, \omega)$ denotes the *scattering amplitude*.

Next consider the situation where the incident field is not a plane wave but is of a more general form. Such a field may be represented as an *angular spectrum* of plane waves propagating into the half-space $z > 0$, i.e. ([14], Sec. 3.2)

$$U^{(i)}(\mathbf{r}, \omega) = \int_{|\mathbf{u}'_\perp|^2 \le 1} a(\mathbf{u}'_\perp, \omega) e^{ik\mathbf{u}'\cdot\mathbf{r}} d^2 u'_\perp, \tag{5}$$

where $\mathbf{u}'_\perp = (u'_x, u'_y)$ is a real two-dimensional vector, and evanescent waves have been omitted. The scattered field in the far zone can then be expressed in the form

$$U^{(s)}(r\mathbf{u}, \omega) = \frac{e^{ikr}}{r}\int_{|\mathbf{u}'_\perp|^2 \le 1} a(\mathbf{u}'_\perp, \omega) f(\mathbf{u}, \mathbf{u}', \omega) d^2 u'_\perp. \tag{6}$$

Let us next consider the case where the incident field is not deterministic but is stochastic. The *radiant intensity* of the scattered field in a direction specified by a real unit vector $\mathbf{u}$ is given by the formula ([14], Eq. (5.2–12)]

$$J_s(\mathbf{u}, \omega) \equiv r^2 \langle U^{(s)*}(r\mathbf{u}, \omega) U^{(s)}(r\mathbf{u}, \omega)\rangle \quad (kr \to \infty), \tag{7}$$

which, on using Eq. (6) becomes

$$J_s(\mathbf{u}, \omega) = \iint \mathcal{A}(\mathbf{u}', \mathbf{u}'', \omega) f^*(\mathbf{u}, \mathbf{u}', \omega) \times f(\mathbf{u}, \mathbf{u}'', \omega) d^2 u'_\perp d^2 u''_\perp, \tag{8}$$

where

$$\mathcal{A}(\mathbf{u}', \mathbf{u}'', \omega) = \langle a^*(\mathbf{u}'_\perp, \omega) a(\mathbf{u}''_\perp, \omega)\rangle \tag{9}$$

is the so-called *angular correlation function* [[14], Eq. (5.6–48)] of the stochastic field, and the angled brackets denote the ensemble average.

An important class of partially coherent beams (which includes the lowest-order Hermite-Gaussian laser mode) are the so-called *Gaussian Schell-model beams* (see [14], Sec. 5.6.4). For such beams the cross-spectral density function in the plane $z = 0$ (the plane which passes through the center of the sphere) has the form

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = [S^{(0)}(\boldsymbol{\rho}_1, \omega)]^{1/2} \times [S^{(0)}(\boldsymbol{\rho}_2, \omega)]^{1/2} \mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega), \tag{10}$$

with

$$S^{(0)}(\boldsymbol{\rho}, \omega) = \langle U^{(0)*}(\boldsymbol{\rho}, \omega) U^{(0)}(\boldsymbol{\rho}, \omega)\rangle, \tag{11}$$

representing the spectral density, and

$$\mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \frac{\langle U^{(0)*}(\boldsymbol{\rho}_1, \omega) U^{(0)}(\boldsymbol{\rho}_2, \omega)\rangle}{[S^{(0)}(\boldsymbol{\rho}_1, \omega) S^{(0)}(\boldsymbol{\rho}_2, \omega)]^{1/2}}, \tag{12}$$

representing the spectral degree of coherence of the field in the plane $z = 0$. Each of the functions on the right-hand side of Eq. (10) has a Gaussian form, i.e.,

$$S^{(0)}(\boldsymbol{\rho}, \omega) = A_0^2 \exp(-\rho^2/2\sigma_S^2), \tag{13}$$

$$\mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \exp[-(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2]/2\sigma_\mu^2. \tag{14}$$

In these formulas $\boldsymbol{\rho}_1 = (x_1, y_1)$ and $\boldsymbol{\rho}_2 = (x_2, y_2)$ are two-dimensional position vectors of points in the $z = 0$ plane, and A_0, σ_S, and σ_μ are positive constants that are taken to be independent of position, but may depend on frequency.

The angular correlation function of such a beam may be expressed as a four-dimensional Fourier transform of its cross-spectral density in the plane $z = 0$, viz. [[14], Eq. (5.6–49)]

$$\mathcal{A}(\mathbf{u}', \mathbf{u}'', \omega) = \left(\frac{k}{2\pi}\right)^4 \iint_{-\infty}^{+\infty} W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) \times \exp[-ik(\mathbf{u}''_\perp \cdot \boldsymbol{\rho}_2 - \mathbf{u}'_\perp \cdot \boldsymbol{\rho}_1)] d^2\rho_1 d^2\rho_2. \tag{15}$$

On substituting from Eqs. (13) and (14) into Eq. (15), one obtains for the angular correlation function of a Gaussian Schell-model beam the expression

$$\mathcal{A}(\mathbf{u}', \mathbf{u}'', \omega) = \left(\frac{k^2 A_0 \sigma_S \sigma_{\mathrm{eff}}}{2\pi}\right)^2 \exp\left\{-\frac{k^2}{2}\left[(\mathbf{u}'_\perp - \mathbf{u}''_\perp)^2 \sigma_S^2 + (\mathbf{u}'_\perp + \mathbf{u}''_\perp)^2 \frac{\sigma_{\mathrm{eff}}^2}{4}\right]\right\}, \tag{16}$$

where

$$\frac{1}{\sigma_{\mathrm{eff}}^2} = \frac{1}{\sigma_\mu^2} + \frac{1}{4\sigma_S^2}. \tag{17}$$

In order for the incident field to be beamlike, the parameters σ_S and σ_μ must satisfy the so-called *beam condition* ([14], Eq. 5.6–73)

$$\frac{1}{\sigma_\mu^2} + \frac{1}{4\sigma_S^2} \ll \frac{k^2}{2}. \tag{18}$$

The scattering amplitude $f(\mathbf{u}', \mathbf{u}'', \omega)$ of the field arising from scattering on a sphere centered on the axis of the beam has the form

$$f(\mathbf{u}', \mathbf{u}'', \omega) = \text{function}\,(\mathbf{u}' \cdot \mathbf{u}'', \omega) = \text{function}\,(\cos\theta, \omega), \tag{19}$$

where θ denotes the angle between the directions of incidence and scattering (see Fig. 1). For a homogeneous spherical scatterer of radius a and of refractive index n, the scattering amplitude can be expressed as [[15], Eq. (4.66)]

$$f(\cos\theta, \omega) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\exp[i\delta_l(\omega)] \times \sin[\delta_l(\omega)] P_l(\cos\theta), \tag{20}$$

where P_l is a Legendre polynomial, and the phase shifts $\delta_l(\omega)$ are given by the expressions (see Secs. 4.3.2 and 4.4.1 of Ref. [15])

$$\tan[\delta_l(\omega)] = \frac{\bar{k} j_l(ka) j_l'(\bar{k}a) - k j_l(\bar{k}a) j_l'(ka)}{\bar{k} j_l'(\bar{k}a) n_l(ka) - k j_l(\bar{k}a) n_l'(ka)}. \quad (21)$$

Here j_l and n_l denote spherical Bessel functions and spherical Neumann functions, respectively, of order l. Furthermore,

$$\bar{k} = nk, \quad (22)$$

and

$$j_l'(ka) = \frac{d j_l(x)}{dx}\bigg\rfloor_{x=ka}, \quad (23)$$

$$n_l'(ka) = \frac{d n_l(x)}{dx}\bigg\rfloor_{x=ka}. \quad (24)$$

On substituting from Eqs. (20) and (16) into Eq. (8) we obtain for the radiant intensity of the scattered field the expression

$$J_s(\mathbf{u}, \omega) = \left(\frac{kA_0\sigma_S\sigma_{\mathrm{eff}}}{2\pi}\right)^2 \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(2l+1)(2m+1)e^{i[\delta_m(\omega)-\delta_l(\omega)]}\sin[\delta_l(\omega)]\sin[\delta_m(\omega)] \times \iint \exp\left\{-\frac{k^2}{2}\left[(\mathbf{u}'_\perp - \mathbf{u}''_\perp)^2\sigma_S^2 + (\mathbf{u}'_\perp + \mathbf{u}''_\perp)^2\frac{\sigma_{\mathrm{eff}}^2}{4}\right]\right\} P_l(\mathbf{u}\cdot\mathbf{u}')P_m(\mathbf{u}\cdot\mathbf{u}'')d^2u'_\perp d^2u''_\perp. \quad (25)$$

Let us restrict ourselves to the commonly occurring situation where the beam width is much greater than the transverse spectral coherence length of the beam, i.e., $\sigma_S \gg \sigma_\mu$. One may then use the asymptotic approximation $k\sigma_S \to \infty$ in two of the four integrations (those over $\mathbf{u}''_\perp$), and apply Laplace's method [3,16,17], which asserts that for two well-behaved functions $h(x, y)$ and $g(x, y)$

$$\iint_\Omega e^{-ph(x,y)}g(x,y)dxdy \sim \frac{\pi g(x_0, y_0)}{p\sqrt{\mathrm{Det}\{\mathcal{H}[h(x_0,y_0)]\}}} \times e^{-ph(x_0,y_0)}, \quad \text{as } p \to \infty, \quad (26)$$

where (x_0, y_0) is the point at which $h(x, y)$ attains its smallest value, and $\mathcal{H}[h(x_0, y_0)]$ is the Hessian matrix of $h(x, y)$, evaluated at the point (x_0, y_0), i.e.,

$$\mathcal{H}[h(x_0, y_0)] = \begin{pmatrix} \frac{\partial^2 h(x,y)}{\partial x^2} & \frac{\partial^2 h(x,y)}{\partial x \partial y} \\ \frac{\partial^2 h(x,y)}{\partial y \partial x} & \frac{\partial^2 h(x,y)}{\partial y^2} \end{pmatrix}_{x=x_0, y=y_0}. \quad (27)$$

Let us make use of Eq. (26) with the choices

$$g(\mathbf{u}, \mathbf{u}'_\perp, \mathbf{u}''_\perp) = \left(\frac{kA_0\sigma_S\sigma_{\mathrm{eff}}}{2\pi}\right)^2 \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(2l+1)(2m+1) \times e^{i[\delta_m(\omega)-\delta_l(\omega)]}\sin[\delta_l(\omega)]\sin[\delta_m(\omega)] \times \exp\left\{-\frac{k^2\sigma_{\mathrm{eff}}^2}{8}(\mathbf{u}'_\perp + \mathbf{u}''_\perp)^2\right\} \times P_l(\mathbf{u}\cdot\mathbf{u}')P_m(\mathbf{u}\cdot\mathbf{u}''), \quad (28)$$

$$h(\mathbf{u}'_\perp, \mathbf{u}''_\perp) = \frac{1}{2}(\mathbf{u}'_\perp - \mathbf{u}''_\perp)^2, \quad (29)$$

$$p = (k\sigma_s)^2. \quad (30)$$

The minimum of $h(\mathbf{u}'_\perp, \mathbf{u}''_\perp)$ as a function of $\mathbf{u}''_\perp$ occurs at $u''_x = u'_x$ and $u''_y = u'_y$. The determinant of the Hessian matrix evaluated at this point is readily found to have the value unity. Expression (25) for the radiant intensity then reduces to

$$J_s(\mathbf{u}, \omega) = \frac{A_0^2\sigma_{\mathrm{eff}}^2}{4\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(2l+1)(2m+1)e^{i(\delta_m-\delta_l)} \times \sin\delta_l \sin\delta_m \int P_l(\mathbf{u}\cdot\mathbf{u}')P_m(\mathbf{u}\cdot\mathbf{u}') \times e^{-k^2\sigma_{\mathrm{eff}}^2 u'^2_\perp/2}d^2u'_\perp, \quad (31)$$

where $\mathbf{u}\cdot\mathbf{u}' = \sin\theta\sin\theta'\cos\phi' + \cos\theta\cos\theta'$ in spherical coordinates, and we have made use of the fact that the radiant intensity is rotationally symmetric about the beam axis.

In Fig. 2 the radiant intensity (normalized to the radiant intensity in the forward direction), calculated from Eq. (31), as a function of the angle of scattering θ is shown for selected values of the spectral coherence length σ_μ of the incident field. (For a method to determine σ_μ, see

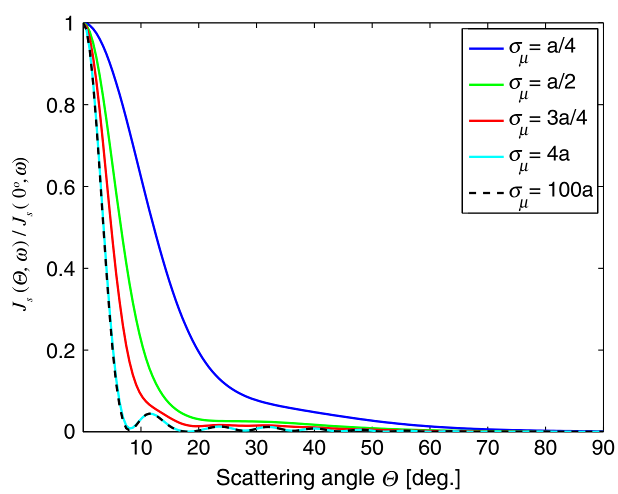

FIG. 2 (color online). The angular distribution of the normalized radiant intensity $J_s(\theta, \omega)/J_s(0°, \omega)$ of the scattered field for selected values of the transverse spectral coherence length σ_μ of the incident beam, with the choices $a = 4\lambda$ and $n = 1.5$.

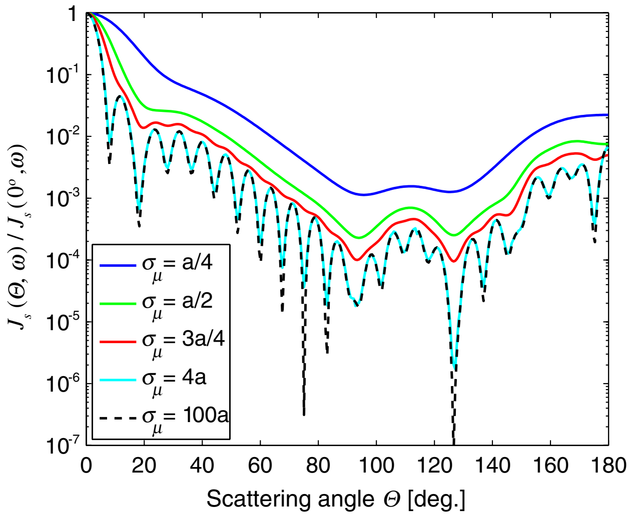

FIG. 3 (color online). The normalized radiant intensity $J(\theta, \omega)/J_s(0°, \omega)$ of the scattered field for selected values of the transverse spectral coherence length σ_μ, plotted on a logarithmic scale. The sphere radius a has been taken to be 4λ, and the refractive index $n = 1.5$.

Sec. 4.3.2 of [14]). It is seen that the scattered field becomes less diffuse as the parameter σ_μ increases. If the coherence length of the incident beam is comparable to or is larger than the radius of the sphere (i.e., when $\sigma_\mu > a$), secondary maxima occur. For $\sigma_\mu = 4a$ the radiant intensity can hardly be distinguished from that generated by an almost spectrally fully coherent beam with $\sigma_\mu = 100a$. The displayed scattering angle θ is restricted to the range $0° \leq \theta \leq 90°$, because for larger values the curves essentially coincide with the horizontal axis. In Fig. 3 the results are shown on a logarithmic scale, for the full range of the scattering angle, i.e., $0° \leq \theta \leq 180°$. It is seen that in all cases there is some backscattering, i.e., $J_s(\theta = 180°, \omega) > 0$, with the largest amount occurring when $\sigma_\mu = a/4$.

We can summarize our results by saying that we have studied the effects of spatial coherence of the incident beam on the angular distribution of the intensity of the field scattered by a small homogeneous sphere; and we found that when the transverse spectral coherence length of the incident beam is smaller than the radius of the scatterer, the radiant intensity is rather diffuse and exhibits no secondary maxima. Our results may find useful application in, for example, determining scattering effects in the atmosphere and colloidal suspensions.

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