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EFFICIENCY FOR SELF SEMI-DIRECT PRODUCTS OF THE FREE ABELIAN MONOID ON TWO GENERATORS

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ABSTRACT. Let A and K both be copies of the free abelian monoid on two generators. For any connecting monoid homomorphism $\theta : A \to \mathrm{End}\,(K)$, let $D = K \rtimes_\theta A$ be the corresponding monoid semi-direct product. We give necessary and sufficient conditions for the efficiency of a standard presentation for D in terms of the matrix representation for θ. Let p be a prime or 0. In [**4**], necessary and sufficient conditions were given for the standard presentation of the semi-direct product of any two monoids to be p-Cockcroft. We use that result to give more explicit conditions in the special case here.

1. Introduction. Let $\mathcal{P} = [X\ ;\ \mathbf{r}]$ be a monoid presentation where a typical element $R \in \mathbf{r}$ has the form $R_+ = R_-$. Here R_+, R_- are words on X, that is, elements of the free monoid X^* on X. The *monoid defined by* $[X\ ;\ \mathbf{r}]$ is the quotient of X^* by the smallest congruence generated by $\mathbf{r}$.

We have a (Squier) graph $\Gamma = \Gamma(X; \mathbf{r})$ associated with $[X\ ;\ \mathbf{r}]$, where the vertices are the elements of X^* and the edges are the 4-tuples $e = (U, R, \varepsilon, V)$ where $U, V \in X^*$, $R \in \mathbf{r}$ and $\varepsilon = \pm 1$. The initial, terminal and inversion functions for an edge e as given above are defined by $\iota(e) = UR_\varepsilon V$, $\tau(e) = UR_{-\varepsilon}V$ and $e^{-1} = (U, R, -\varepsilon, V)$. There is a two-sided action of X^* on Γ as follows. If $W, \overline{W} \in X^*$ then, for any vertex V of Γ, $W.V.\overline{W} = WV\overline{W}$ (product in X^*) and, for any edge $e = (U, R, \varepsilon, V)$ of Γ, $W.e.\overline{W} = (WU, R, \varepsilon, V\overline{W})$. This action can be extended to the paths in Γ.

Two paths π and π' in a 2-complex are equivalent if there is a finite sequence of paths $\pi = \pi_0, \pi_1, \cdots, \pi_m = \pi'$ where for $1 \leq i \leq m$ the path π_i is obtained from π_{i-1} either by inserting or deleting a pair ee^{-1} of inverse edges or else by inserting or deleting a defining path for one of

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the 2-cells of the complex. There is an equivalence relation, $\sim$, on paths in Γ which is generated by $(e_1.\iota(e_2))(\tau(e_1).e_2) \sim (\iota(e_1).e_2)(e_1.\tau(e_2))$ for any edges e_1 and e_2 of Γ. This corresponds to requiring that the closed paths $(e_1.\iota(e_2))(\tau(e_1).e_2)(e_1^{-1}.\tau(e_2))(\iota(e_1).e_2^{-1})$ at the vertex $\iota(e_1)\iota(e_2)$ are the defining paths for the 2-cells of a 2-complex having Γ as its 1-skeleton. This 2-complex is called the Squier complex of $\mathcal{P}$ and denoted by $\mathcal{D}(\mathcal{P})$, see, for example, [**9, 14, 15, 19**]. The paths in $\mathcal{D}(\mathcal{P})$ can be represented by geometric configurations, called *monoid pictures*. Monoid pictures and group pictures have been used in several papers by Pride and other authors. We assume here that the reader is familiar with monoid pictures. See [**9**, Section 4], [**14**, Section 1] or [**15**, Section 2]. Typically, we will use the following Euler Fraktur font, e.g. $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{P}$, as notation for monoid pictures. Atomic monoid pictures are pictures which correspond to paths of length 1. Write $[|U, R, \varepsilon, V|]$ for the atomic picture which corresponds to the edge (U, R, ε, V) of the Squier complex. Whenever we can concatenate two paths π and π' in Γ to form the path $\pi\pi'$, then we can concatenate the corresponding monoid pictures $\mathfrak{P}$ and $\mathfrak{P}'$ to form a monoid picture $\mathfrak{P}\mathfrak{P}'$ corresponding to $\pi\pi'$. The equivalence of paths in the Squier complex corresponds to an equivalence of monoid pictures. That is, two monoid pictures $\mathfrak{P}$ and $\mathfrak{P}'$ are equivalent if there is a finite sequence of monoid pictures $\mathfrak{P} = \mathfrak{P}_0, \mathfrak{P}_1, \cdots, \mathfrak{P}_m = \mathfrak{P}'$ where, for $1 \leq i \leq m$, the monoid picture $\mathfrak{P}_i$ is obtained from the picture $\mathfrak{P}_{i-1}$ either by inserting or deleting a subpicture $\mathfrak{A}\mathfrak{A}^{-1}$ where $\mathfrak{A}$ is an atomic monoid picture or else by replacing a subpicture $(\mathfrak{A}.\iota(\mathfrak{B}))(\tau(\mathfrak{A}).\mathfrak{B})$ by $(\iota(\mathfrak{A}).\mathfrak{B})(\mathfrak{A}.\tau(\mathfrak{B}))$ or vice versa, where $\mathfrak{A}$ and $\mathfrak{B}$ are atomic monoid pictures.

A monoid picture is a *spherical* monoid picture when the corresponding path in the Squier complex is a closed path. Suppose $\mathbf{Y}$ is a collection of spherical monoid pictures over $\mathcal{P}$. Two monoid pictures $\mathfrak{P}$ and $\mathfrak{P}'$ are *equivalent relative to* $\mathbf{Y}$ if there is a finite sequence of monoid pictures $\mathfrak{P} = \mathfrak{P}_0, \mathfrak{P}_1, \cdots, \mathfrak{P}_m = \mathfrak{P}'$ where, for $1 \leq i \leq m$, the monoid picture $\mathfrak{P}_i$ is obtained from the picture $\mathfrak{P}_{i-1}$ either by the insertion, deletion and replacement operations of the previous paragraph or else by inserting or deleting a subpicture of the form $W.\mathfrak{Y}.V$ or of the form $W.\mathfrak{Y}^{-1}.V$ where $W, V \in X^*$ and $\mathfrak{Y} \in \mathbf{Y}$. By definition, a set $\mathbf{Y}$ of spherical monoid pictures over $\mathcal{P}$ is a *trivializer of* $\mathcal{D}(\mathcal{P})$ if every spherical monoid picture is equivalent to an empty picture relative to $\mathbf{Y}$. By [**15**, Theorem 5.1], if $\mathbf{Y}$ is a trivializer for the Squier complex, then the elements of $\mathbf{Y}$ generate the first homology group of the Squier complex. The trivializer is also called a set of generating pictures. Some examples and more details of the trivializer can be found in [**3, 6, 10, 14, 15, 19**] and [**20**].

For any word W on X and $x \in X$, we use the notation $L(W)$ for the *length* of W and the notation $L_x(W)$ for the *length of W with respect to x*, the number of occurrences of x in W. If $R_+ = R_-$ is a relator R in $\mathbf{r}$, then $\exp_x(R)$ is defined by $\exp_x(R) = L_x(R_+) - L_x(R_-)$.

For any monoid picture $\mathfrak{P}$ over $\mathcal{P}$ and for any $R \in \mathbf{r}$, $\exp_R(\mathfrak{P})$ denotes the *exponent sum* of R in $\mathfrak{P}$ which is the number of positive discs labeled by R_+, minus the number of negative discs labeled by R_-. For a nonnegative integer n, $\mathcal{P}$ is said to be *n-Cockcroft* if $\exp_R(\mathfrak{P}) \equiv 0\colon \pmod{n}$, where congruence $\pmod{0}$ is taken to be equality, for all $R \in \mathbf{r}$ and for all spherical pictures $\mathfrak{P}$ over $\mathcal{P}$. Then a monoid $\mathcal{M}$ is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation.

We note that to verify the n-Cockcroft property, it is enough to check for pictures $\mathfrak{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a trivializer, see [**14, 15**]. The 0-Cockcroft property is usually just called Cockcroft. In general we take n to be equal to 0 or a prime p. Examples of monoid presentations with Cockcroft and p-Cockcroft properties can be found in the author's thesis [**3**].

In group theory, the homological concept of efficiency has been widely studied. In [**2**], Ayık, Campbell, O'Connor and Ruškuc, defined efficiency for finite semi-groups and hence for finite monoids. The following definition for not necessarily finite monoids follows [**3**] and is equivalent to the definition in [**2**] when the monoids are finite. For an abelian group G, $rk_{\mathfrak{Z}}(G)$ denotes the $\mathfrak{Z}$-rank of the torsion free part of G and $d(G)$ means the minimal number of generators of G. Suppose that $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is a finite presentation for a monoid $\mathcal{M}$. Then the *Euler characteristic*, $\chi(\mathcal{P})$ is defined by $\chi(\mathcal{P}) = 1 - |\mathbf{x}| + |\mathbf{r}|$ and $\delta(\mathcal{M})$ is defined by $\delta(\mathcal{M}) = 1 - rk_{\mathfrak{Z}}(H_1(\mathcal{M})) + d(H_2(\mathcal{M}))$. In unpublished work, Pride has shown that $\chi(\mathcal{P}) \geq \delta(\mathcal{M})$. With this background, we define the finite monoid presentation $\mathcal{P}$ to be *efficient* if $\chi(\mathcal{P}) = \delta(\mathcal{M})$ and we define the monoid $\mathcal{M}$ to be *efficient* if it has an efficient presentation.

The following result is also an unpublished result by Pride. We will use this result rather than making more direct computations of homology for monoids. Kilgour and Pride prove the analogous result for groups in [**12**] and credit an earlier proof by Epstein, [**8**].

Theorem 1.1. *Let $\mathcal{P}$ be a monoid presentation. Then $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

The definition for the semi-direct product of two monoids can be found in [**3, 13, 17, 18**] or [**20**]. Our presentation below for this semi-direct product can be found in [**3, 17, 18**] or [**20**]. Let A and K be monoids with associated presentations $\mathcal{P}_A = [X\,;\,\mathbf{r}]$ and $\mathcal{P}_K = [Y\,;\,\mathbf{s}]$, respectively. Let $D = K \rtimes_\theta A$ be the corresponding semi-direct product of these two monoids where θ is a monoid homomorphism from A to $\mathrm{End}\,(K)$. (Note that the reader can find some examples of monoid endomorphisms in [**7**].) The elements of D can be regarded as ordered pairs (a, k) where $a \in A$, $k \in K$ with multiplication given by $(a, k)(a', k') = (aa', (k\theta_{a'})k')$. The monoids A and K are identified with the submonoids of D having elements $(a, 1)$ and $(1, k)$, respectively. We want to define standard presentations for D. For every $x \in X$ and $y \in Y$, choose a word, which we denote by $y\theta_x$, on Y such that $[y\theta_x] = [y]\theta_{[x]}$ as an element of K. To establish notation, let us denote the relation $yx = x(y\theta_x)$ on $X \cup Y$ by T_{yx} and write $\mathbf{t}$ for the set of relations T_{yx}. Then, for any choice of the words $y\theta_x$, $\mathcal{P}_D = [X, Y\,;\,\mathbf{r}, \mathbf{s}, \mathbf{t}]$ is a standard monoid presentation for the semi-direct product D.

If $W = y_1 y_2 \cdots y_m$ is a positive word on Y, then for any $x \in X$, we denote the word $(y_1\theta_x)(y_2\theta_x)\cdots(y_m\theta_x)$ by $W\theta_x$. If $U = x_1 x_2 \cdots x_n$ is a positive word on X, then for any $y \in Y$, we denote the word $(\cdots((y\theta_{x_1})\theta_{x_2})\theta_{x_3}\cdots)\theta_{x_n})$ by $y\theta_U$.

In [**20**], Wang constructs a finite trivializer set for the standard presentation $\mathcal{P}_D = [X, Y\,;\,\mathbf{r}, \mathbf{s}, \mathbf{t}]$ for the semi-direct product D. We will essentially follow [**3**] in describing this trivializer set using spherical pictures and certain non-spherical subpictures of these.

Let W be any word on Y and $x \in X$. By induction on $n = L(W)$, we define a nonspherical picture $\mathfrak{D}_{W,x}$ over the presentation $[X \cup Y\,;\,\mathbf{t}]$ with $\iota(\mathfrak{D}_{W,x}) = Wx$, $\tau(\mathfrak{D}_{W,x}) = x(W\theta_x)$, $\exp_{T_{yx}}(\mathfrak{D}_{W,x}) = L_y(W)$

$\mathbb{A}_{U,y}$

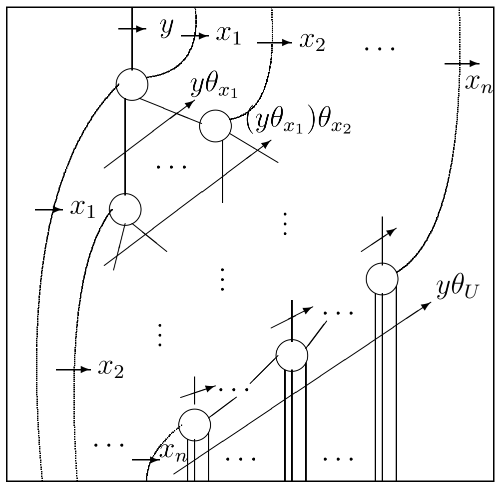

FIGURE 1.

and $\exp_{T_{y\hat{x}}}(\mathfrak{D}_{W,x}) = 0$ for $x \neq \hat{x}$. When $L(W) = 1$ and $W = y$, let $\mathfrak{D}_{W,x} = [|1, T_{yx}, 1, 1|]$. When $L(W) > 1$, write $W = y_n \cdots y_2 y_1$, $W' = y_{n-1} \cdots y_2 y_1$ and let $\mathfrak{D}_{W,x} = (y_n.\mathfrak{D}_{W',x})([|1, T_{y_n x}, 1, W'\theta_x|]$.

Let U be any word on X and $y \in Y$. By induction on $n = L(U)$, we define a non-spherical picture $\mathfrak{A}_{U,y}$ over the presentation $[X \cup Y \ ; \ \mathbf{t}]$ with $\iota(\mathfrak{A}_{U,y}) = yU$ and $\tau(\mathfrak{A}_{U,y}) = U(y\theta_U)$. When $L(U) = 1$ and $U = x$, let $\mathfrak{A}_{U,y} = [|1, T_{yx}, 1, 1|]$. When $L(U) > 1$, write $U = x_1 x_2 \cdots x_n$ and $U' = x_1 x_2 \cdots x_{n-1}$. Then we define $\mathfrak{A}_{U,y}$ to be $(\mathfrak{A}_{U',y}.x_n)(U'.\mathfrak{D}_{y\theta_{U'},x_n})$. See Figure 1.

For $y \in Y$ and the relation $R_+ = R_-$ in $\mathbf{r}$, we have the two important special cases, $\mathfrak{A}_{R_+,y}$ and $\mathfrak{A}_{R_-,y}$, of this construction.

Let $S \in \mathbf{s}$ and $x \in X$. Since $[S_+\theta_x] = [S_-\theta_x]$ as elements of K, there is a nonspherical picture over $\mathcal{P}_K$ which we denote by $\mathfrak{B}_{S,x}$ with $\iota(\mathfrak{B}_{S,x}) = S_+\theta_x$ and $\tau(\mathfrak{B}_{S,x}) = S_-\theta_x$.

Let $R_+ = R_-$ be a relation $R \in \mathbf{r}$ and $y \in Y$. Since θ is a homomorphism, by our definition for $y\theta_U$, we have that $y\theta_{R_+}$ and $y\theta_{R_-}$ must represent the same element of K. Hence there is a nonspherical picture over $\mathcal{P}_K$ which we denote by $\mathfrak{C}_{R,y}$ with $\iota(\mathfrak{C}_{R,y}) = y\theta_{R_+}$ and $\tau(\mathfrak{C}_{R,y}) = y\theta_{R_-}$.

We have not, at this point, made any restrictions upon the set $\mathbf{s}$ of relations for the presentation of K. Hence, there may be many different ways to construct the pictures $\mathfrak{B}_{S,x}$ and $\mathfrak{C}_{R,y}$. These pictures must exist, but they need not be unique. The pictures $\mathfrak{A}_{U,y}$ and $\mathfrak{D}_{W,x}$ will depend upon our choices for words $y\theta_x$, but they are unique once these choices are made.

For $S \in \mathbf{s}$, $x \in X$, $R \in \mathbf{r}$ and $y \in Y$, we define pictures $\mathfrak{P}_{S,x}$ and $\mathfrak{P}_{R,y}$ by

$$\mathfrak{P}_{S,x} = ([|1, S, 1, x|])(\mathfrak{D}_{S_-,x})(x.\mathfrak{B}_{S,x}^{-1})(\mathfrak{D}_{S_+,x}^{-1})$$

and

$$\mathfrak{P}_{R,y} = (\mathfrak{A}_{R_+,y})([|1, R, 1, y\theta_{R_+}|])((R_-).\mathfrak{C}_{R,y})(\mathfrak{A}_{R_-,y}^{-1})([|y, R, -1, 1|]).$$

We note that $\mathfrak{P}_{R,y}$ is equivalent to

$$(\mathfrak{A}_{R_+,y})((R_+).\mathfrak{C}_{R,y})([|1, R, 1, y\theta_{R_-}|])(\mathfrak{A}_{R_-,y}^{-1})([|y, R, -1, 1|]).$$

See Figure 2 for illustrations of $\mathfrak{P}_{S,x}$ and $\mathfrak{P}_{R,y}$.

Let $\mathbf{X}_A$ and $\mathbf{X_K}$ be trivializer sets for $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$, respectively. Let $\mathbf{C_1} = \{\mathfrak{P}_{S,x} : S \in \mathbf{s},\ x \in X\}$ and $\mathbf{C_2} = \{\mathfrak{P}_{R,y} : R \in \mathbf{r},\ y \in Y\}$.

We will use the following result of Wang. See [**20**] for a proof.

Theorem 1.2. *Suppose that the monoids A and K have respective monoid presentations $\mathcal{P}_A = [X\ ;\ \mathbf{r}]$ and $\mathcal{P}_K = [Y\ ;\ \mathbf{s}]$. If $D = K \rtimes_\theta A$ is the semi-direct product with standard presentation $\mathcal{P}_D = [X \cup Y\ ;\ \mathbf{r} \cup \mathbf{s} \cup \mathbf{t}]$ then $\mathbf{X_A} \cup \mathbf{X_K} \cup \mathbf{C_1} \cup \mathbf{C_2}$ is a trivializer set for the Squier complex $\mathcal{D}(\mathcal{P}_D)$.*

Several different notions of asphericity have been introduced and examined for groups, monoids and semi-groups. In this paper, we will define a presentation for a monoid to be *aspherical* if every spherical picture over the presentation is equivalent to a trivial picture. Aspherical presentations are therefore Cockcroft and then p-Cockcroft. For discussions of other forms of asphericity, see [**3, 5, 9**, Section 12], [**10**, Section 5], [**11, 12**], [**15**, Section 7] and [**16**].

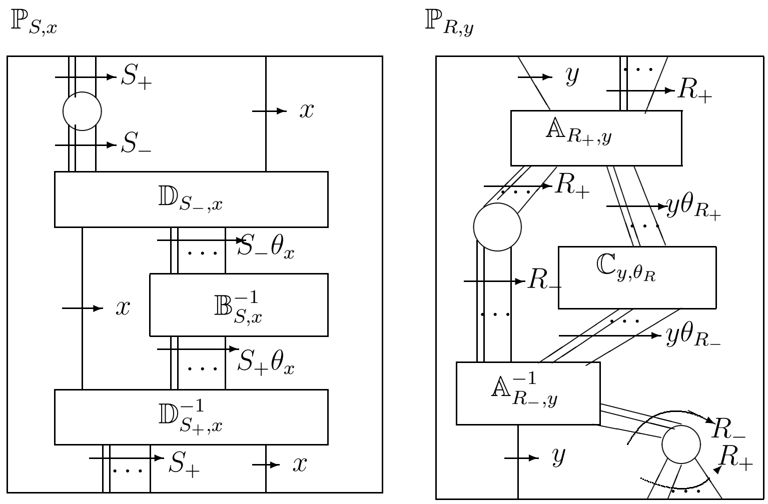

FIGURE 2.

Lemma 1.3. *The monoid presentation* $[a, b\ ;\ ab = ba]$ *is aspherical.*

Proof. This result follows from [**1**], but it can also be proved directly. □

We will also use the following special case of the main result in [**4**].

Theorem 1.4. *Let* p *be a prime or* 0 *and let* $\mathcal{P}_D = [X \cup Y\ ;\ \{R\} \cup \{S\} \cup \mathbf{t}]$ *be a standard presentation for the monoid semi-direct product* $K \rtimes_\theta A$ *where the presentations for* A *and for* K *are aspherical one-relator presentations with relators* R *and* S*, respectively. Then the presentation* $\mathcal{P}_D$ *is* p*-Cockcroft if and only if*

(i) $\exp_y(S) \equiv 0 \pmod p$ *for all* $y \in Y$,

(ii) $\exp_S(\mathfrak{B}_{S,x}) \equiv 1 \pmod p$ *for all* $x \in X$,

(iii) $\exp_S(\mathfrak{C}_{R,y}) \equiv 0 \pmod p$ *for all* $y \in Y$*, and*

(iv) $\exp_{T_{yx}}(\mathfrak{P}_{R,\hat{y}}) \equiv 0 \pmod p$ *for all* $y, \hat{y} \in Y$*,* $x \in X$*.*

Proof. Suppose that the presentation $\mathcal{P}_D$ is p-Cockcroft where p is a prime or 0. Recall that $\mathfrak{P}_{S,x} = ([|1,S,1,x|])(\mathfrak{D}_{S_-,x})(x.\mathfrak{B}_{S,x}^{-1})(\mathfrak{D}_{S_+,x}^{-1})$ where $\mathfrak{B}_{S,x}$ contains only S-discs and the subpictures $\mathfrak{D}_{S_-,x}$ and $\mathfrak{D}_{S_+,x}$ contain only T_{yx} discs. Reviewing the construction of $\mathfrak{D}_{W,x}$ we see that $\exp_{T_{yx}}(\mathfrak{D}_{W,x}) = L_y(W)$, so $\exp_{T_{yx}}(\mathfrak{P}_{S,x}) = L_y(S_-) - L_y(S_+)$, which is the negative of $\exp_y(S)$, by definition. Hence (i) must hold. Furthermore, we see that $\exp_S(\mathfrak{P}_{S,x}) = 1 - \exp_S(\mathfrak{B}_{S,x})$ so (ii) must hold also.

Recall that $\mathfrak{P}_{R,y} = (\mathfrak{A}_{R_+,y})([|1,R,1,y\theta_{R_+}|])((R_-).\mathfrak{C}_{R,y})(\mathfrak{A}_{R_-,y}^{-1}) \times ([|y,R,-1,1|])$ where $\mathfrak{C}_{R,y}$ contains only S-discs and the subpictures $\mathfrak{A}_{R_+,y}$ and $\mathfrak{A}_{R_-,y}$ contain only T_{yx} discs. Then $\exp_S(\mathfrak{P}_{R,y}) = \exp_S(\mathfrak{C}_{R,y})$, so (iii) must hold. Condition (iv) obviously must hold because $\mathfrak{P}_{R,y}$ is a spherical monoid picture.

Conversely, suppose that the four conditions hold. Since the presentations for A and K are aspherical, it will suffice, by Lemma 1.3, to show that $\exp_Q(\mathfrak{P}_{S,x})$ and $\exp_Q(\mathfrak{P}_{R,y})$ are equivalent to 0, modulo p, whenever $x \in X$, $y \in Y$ and Q is S, R or some $T_{\hat{y}\hat{x}}$. We see that $\exp_R(\mathfrak{P}_{S,x})$ and $\exp_R(\mathfrak{P}_{R,y})$ are both always equal to 0, while $\exp_S(\mathfrak{P}_{S,x}) \equiv 0 \pmod p$ follows from the condition (ii) and $\exp_S(\mathfrak{P}_{R,y}) \equiv 0 \pmod p$ follows from the condition (iii). For any word W on Y, $\exp_{T_{y\hat{x}}}(\mathfrak{D}_{W,x}) = 0$ if $x \neq \hat{x}$ and $\exp_{T_{yx}}(\mathfrak{D}_{W,x}) = L_y(W)$. Since $\mathfrak{P}_{S,x} = ([|1,S,1,x|])(\mathfrak{D}_{S_-,x})(x.\mathfrak{B}_{S,x}^{-1})(\mathfrak{D}_{S_+,x}^{-1})$ we have $\exp_{T_{y\hat{x}}}(\mathfrak{P}_{S,x}) = 0$ if $x \neq \hat{x}$ and $\exp_{T_{yx}}(\mathfrak{P}_{S,x}) = -\exp_y(S)$. Thus, we will have $\exp_{T_{\hat{y}\hat{x}}}(\mathfrak{P}_{S,x}) \equiv 0 \pmod p$ provided that condition (i) is satisfied. Condition (iv) assures us that we have $\exp_{T_{\hat{y}\hat{x}}}(\mathfrak{P}_{R,y}) \equiv 0 \pmod p$. □

2. The main result. Let both A and K be free abelian monoids having rank 2, with respective presentations, $\mathcal{P}_A = [a, b\ ;\ ab = ba]$ and $\mathcal{P}_K = [c, d\ ;\ cd = dc]$. If we regard the elements $[c^m d^n]_K$ of K as 1×2 matrices $[m, n]$ then we can represent endomorphisms of K by 2×2 matrices with nonnegative integer entries. We will represent endomorphisms $\theta_{[a]}$ and $\theta_{[b]}$ of K, respectively, by the matrices

$$\mathcal{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}.$$

For the general case of the previous section, we were allowed to choose, for each $x \in X$ and $y \in Y$, a word $y\theta_x$ on Y with $[y\theta_x]_K = [y]_K\theta_{[x]_A}$. In this section, we will restrict ourselves to the following choice for these:

$$\begin{aligned} c\theta_a &= c^{\alpha_{11}}d^{\alpha_{12}} & \qquad c\theta_b &= c^{\beta_{11}}d^{\beta_{12}} \\ d\theta_a &= c^{\alpha_{21}}d^{\alpha_{22}} & \qquad d\theta_b &= c^{\beta_{21}}d^{\beta_{22}}. \end{aligned}$$

For the function $\theta : A \to \mathrm{End}\,(K)$ to be a well-defined homomorphism, we need also to require that $\theta_{[a]}\theta_{[b]} = \theta_{[b]}\theta_{[a]}$ or equivalently that $\mathcal{AB} = \mathcal{BA}$. The three equations in the following lemma will be used in the proof of our main result.

Lemma 2.1. *The function $\theta : A \to \mathrm{End}\,(K)$ defined by $[a] \mapsto \theta_{[a]}$, $[b] \mapsto \theta_{[b]}$ is a well-defined monoid homomorphism if and only if*

(i) $\alpha_{12}\beta_{21} = \alpha_{21}\beta_{12}$

(ii) $\alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} = \alpha_{12}\beta_{11} + \alpha_{22}\beta_{12}$, *and*

(iii) $\alpha_{21}\beta_{11} + \alpha_{22}\beta_{21} = \alpha_{11}\beta_{21} + \alpha_{21}\beta_{22}$.

Proof. This follows immediately from $\mathcal{AB} = \mathcal{BA}$. $\square$

The main result of this paper is the following.

Theorem 2.2. *Let p be a prime. Suppose that $\theta : A \to \mathrm{End}\,(K)$ is a monoid homomorphism represented by 2×2 matrices $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{P}_D$ be the resulting standard presentation*

$$\begin{aligned} &[a, b, c, d; ab = ba, cd = dc, ca = ac^{\alpha_{11}}d^{\alpha_{12}}, \\ &\qquad\qquad da = ac^{\alpha_{21}}d^{\alpha_{22}}, cb = bc^{\beta_{11}}d^{\beta_{12}}, db = bc^{\beta_{21}}d^{\beta_{22}}] \end{aligned}$$

for the semi-direct product $D = K \rtimes_\theta A$. Then $\mathcal{P}_D$ is p-Cockcroft if and only if

$$\mathcal{A} \equiv I_{2\times 2} \pmod{p} \quad \text{and} \quad \mathcal{B} \equiv I_{2\times 2} \pmod{p}.$$

Proof. We apply Lemma 1.3 to both $\mathcal{P}_A$ and $\mathcal{P}_K$ to argue that these are aspherical and then we use Theorem 1.4.

Since the relator S here is $cd = dc$, we see that condition (i) of Theorem 1.4 is satisfied.

The following pictures $\mathfrak{E}_{m,n}$ and $\mathfrak{F}_{q,m,n}$ will be useful to show that conditions (ii), (iii) and (iv) of Theorem 1.4 are satisfied.

Using induction on m and n, construct nonspherical pictures $\mathfrak{E}_{m,n}$ over $\mathcal{P}_K$ as follows, observing that $\iota(\mathfrak{E}_{m,n}) = c^m d^n$, $\tau(\mathfrak{E}_{m,n}) = d^n c^m$ and $\exp_S(\mathfrak{E}_{m,n}) = mn$. Let $\mathfrak{E}_{1,1}$ consist of a single S-disc. For $m > 1$, let $\mathfrak{E}_{m,1} = (c.\mathfrak{E}_{m-1,1})[|1, S, 1, c^{m-1}|]$ and, for $n > 1$, let $\mathfrak{E}_{m,n} = ((\mathfrak{E}_{m,n-1}).d)(d^{n-1}.\mathfrak{E}_{m,1})$.

For $q \geq 2$, $m \geq 1$ and $n \geq 1$, we define, by induction on q, a nonspherical picture $\mathfrak{F}_{q,m,n}$ over $\mathcal{P}_K$ with $\iota(\mathfrak{F}_{q,m,n}) = (c^m d^n)^q$, $\tau(\mathfrak{F}_{q,m,n}) = c^{mq} d^{nq}$ and $\exp_S(\mathfrak{F}_{q,m,n}) = -(1/2)q(q-1)mn$. As a base step, let $\mathfrak{F}_{2,m,n} = c^m.\mathfrak{E}^{-1}_{m,n}.d^n$. Inductively, for $q > 2$, let $\mathfrak{F}_{q,m,n} = ((\mathfrak{F}_{q-1,m,n}).c^m d^n)((c^{mq-m}).(\mathfrak{E}^{-1}_{m,nq-n}).d^n)$.

We want to show next that the second condition of Theorem 1.4 is satisfied if $\mathcal{A} \equiv I \pmod p$ and $\mathcal{B} \equiv I \pmod p$. Recall that we choose $\mathfrak{B}_{S,x}$ to be a nonspherical picture over $\mathcal{P}_K$ with $\iota(\mathfrak{B}_{S,x}) = (S_+)\theta_x$ and $\tau(\mathfrak{B}_{S,x}) = (S_-)\theta_x$. With our current hypotheses, we have $S_+ = cd$ and $S_- = dc$, so $\iota(\mathfrak{B}_{S,x}) = (c\theta_x)(d\theta_x)$ and $\tau(\mathfrak{B}_{S,x}) = (d\theta_x)(c\theta_x)$. We will consider only the case $x = a$. The case where $x = b$ is parallel. Since we have made the choices $c\theta_a = c^{\alpha_{11}} d^{\alpha_{12}}$ and $d\theta_a = c^{\alpha_{21}} d^{\alpha_{22}}$, we need $\iota(\mathfrak{B}_{S,a}) = c^{\alpha_{11}} d^{\alpha_{12}} c^{\alpha_{21}} d^{\alpha_{22}}$ and $\tau(\mathfrak{B}_{S,a}) = c^{\alpha_{21}} d^{\alpha_{22}} c^{\alpha_{11}} d^{\alpha_{12}}$. We will accomplish this if we let

$$\mathfrak{B}_{S,a} = (c^{\alpha_{11}}.(\mathfrak{E}^{-1}_{\alpha_{21},\alpha_{12}}).d^{\alpha_{22}})(c^{\alpha_{21}}.(\mathfrak{E}_{\alpha_{11},\alpha_{22}}).d^{\alpha_{12}}).$$

Then $\exp_S(\mathfrak{B}_{S,a}) = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = \det \mathcal{A}$ and whenever $\mathcal{A} \equiv I \pmod p$, we will have $\exp_S(\mathfrak{B}_{S,a}) \equiv 1 \pmod p$. Similarly, we can always find pictures $\mathfrak{B}_{S,b}$ with $\exp_S(\mathfrak{B}_{S,b}) = \det \mathcal{B}$, so condition (ii) of Theorem 1.4 will always be satisfied if $\mathcal{A} \equiv I \pmod p$ and $\mathcal{B} \equiv I \pmod p$.

We consider condition (iii) of Theorem 1.4. We will discuss only the case for $\mathfrak{C}_{R,c}$. The case for $\mathfrak{C}_{R,d}$ is parallel. Recall that R is $ab = ba$ and that $\mathfrak{C}_{R,c}$ is a nonspherical picture over $\mathcal{P}_K$ with $\iota(\mathfrak{C}_{R,c}) = c\theta_{ab}$ and $\tau(\mathfrak{C}_{R,c}) = c\theta_{ba}$. Since we have made the choices $c\theta_a = c^{\alpha_{11}} d^{\alpha_{12}}$, $d\theta_a = c^{\alpha_{21}} d^{\alpha_{22}}$, $c\theta_b = c^{\beta_{11}} d^{\beta_{12}}$ and $d\theta_b = c^{\beta_{21}} d^{\beta_{22}}$, we need to construct $\mathfrak{C}_{R,c}$ with $\iota(\mathfrak{C}_{R,c}) = (c\theta_a)\theta_b = (c^{\alpha_{11}} d^{\alpha_{12}})\theta_b = (c\theta_b)^{\alpha_{11}}(d\theta_b)^{\alpha_{12}} = (c^{\beta_{11}} d^{\beta_{12}})^{\alpha_{11}}(c^{\beta_{21}} d^{\beta_{22}})^{\alpha_{12}}$ and similarly, $\tau(\mathfrak{C}_{R,c}) =$

$(c^{\alpha_{11}}d^{\alpha_{12}})^{\beta_{11}}(c^{\alpha_{21}}d^{\alpha_{22}})^{\beta_{12}}$. Define intermediate pictures $\mathfrak{C}^{\iota}_{R,c}$ and $\mathfrak{C}^{\tau}_{R,c}$ as follows.

$$\begin{aligned}
\mathfrak{C}^{\iota}_{R,c} &= ((\mathfrak{F}_{\alpha_{11},\beta_{11},\beta_{12}}).(c^{\beta_{21}}d^{\beta_{22}})^{\alpha_{12}})((c^{\alpha_{11}\beta_{11}}d^{\alpha_{11}\beta_{12}}).(\mathfrak{F}_{\alpha_{12},\beta_{21},\beta_{22}}))\\
&\quad ((c^{\alpha_{11}\beta_{11}}).(\mathfrak{E}^{-1}_{\alpha_{12}\beta_{21},\alpha_{11}\beta_{12}}).(d^{\alpha_{12}\beta_{22}}))\\
\mathfrak{C}^{\tau}_{R,c} &= ((\mathfrak{F}_{\beta_{11},\alpha_{11},\alpha_{12}}).(c^{\alpha_{21}}d^{\alpha_{22}})^{\beta_{12}})((c^{\alpha_{11}\beta_{11}}d^{\alpha_{12}\beta_{11}}).(\mathfrak{F}_{\beta_{12},\alpha_{21},\alpha_{22}}))\\
&\quad ((c^{\alpha_{11}\beta_{11}}).(\mathfrak{E}^{-1}_{\alpha_{21}\beta_{12},\alpha_{12}\beta_{11}}).(d^{\alpha_{22}\beta_{12}})).
\end{aligned}$$

We observe that

$$\begin{aligned}
\iota(\mathfrak{C}^{\iota}_{R,c}) &= (c^{\beta_{11}}d^{\beta_{12}})^{\alpha_{11}}(c^{\beta_{21}}d^{\beta_{22}})^{\alpha_{12}}\\
\tau(\mathfrak{C}^{\iota}_{R,c}) &= c^{\alpha_{11}\beta_{11}+\alpha_{12}\beta_{21}}d^{\alpha_{11}\beta_{12}+\alpha_{12}\beta_{22}}\\
\iota(\mathfrak{C}^{\tau}_{R,c}) &= (c^{\alpha_{11}}d^{\alpha_{12}})^{\beta_{11}}(c^{\alpha_{21}}d^{\alpha_{22}})^{\beta_{12}}\\
\tau(\mathfrak{C}^{\tau}_{R,c}) &= c^{\alpha_{11}\beta_{11}+\alpha_{21}\beta_{12}}d^{\alpha_{12}\beta_{11}+\alpha_{22}\beta_{12}}.
\end{aligned}$$

Using equations (i) and (ii) from Lemma 2.1, we see that $\tau(\mathfrak{C}^{\iota}_{R,c}) = \tau(\mathfrak{C}^{\tau}_{R,c})$. We define $\mathfrak{C}_{R,c}$ to be $(\mathfrak{C}^{\iota}_{R,c})(\mathfrak{C}^{\tau}_{R,c})^{-1}$. Suppose that we have $\mathcal{A} \equiv I \pmod p$ and $\mathcal{B} \equiv I \pmod p$. Then α_{12}, α_{21}, β_{12} and β_{21} are all divisible by p. Since $\exp_S(\mathfrak{E}_{m,n})$ and $\exp_S(\mathfrak{F}_{q,m,n})$ are divisible by p whenever either of m or n is divisible by p, it follows that the values for $\exp_S(\mathfrak{F}_{\alpha_{11},\beta_{11},\beta_{12}})$, $\exp_S(\mathfrak{F}_{\alpha_{12},\beta_{21},\beta_{22}})$, $\exp_S(\mathfrak{E}_{\alpha_{12}\beta_{21},\alpha_{11}\beta_{12}})$, $\exp_S(\mathfrak{F}_{\beta_{11},\alpha_{11},\alpha_{12}})$, $\exp_S(\mathfrak{F}_{\beta_{12},\alpha_{21},\alpha_{22}})$, $\exp_S(\mathfrak{E}_{\alpha_{21}\beta_{12},\alpha_{12}\beta_{11}})$ and then $\exp_S(\mathfrak{C}^{\iota}_{R,c})$ and $\exp_S(\mathfrak{C}^{\tau}_{R,c})$ as well, are divisible by p and hence that $\exp_S(\mathfrak{C}_{R,c}) \equiv 0 \pmod p$.

Finally, we want to show that the fourth condition of Theorem 1.4 is satisfied if and only if $\mathcal{A} \equiv I \pmod p$ and $\mathcal{B} \equiv I \pmod p$. Recall that $\mathfrak{P}_{R,\hat{y}}$ is defined by

$$\mathfrak{P}_{R,\hat{y}} = (\mathfrak{A}_{R_+,\hat{y}})([|1,R,1,\hat{y}\theta_{R_+}|])((R_-).\mathfrak{C}_{R,\hat{y}})(\mathfrak{A}^{-1}_{R_-,\hat{y}})([|\hat{y},R,-1,1|])$$

where T_{yx} discs occur only in $\mathfrak{A}_{R_+,\hat{y}}$ and $\mathfrak{A}_{R_-,\hat{y}}$. It will suffice then to show that we have $\exp_{T_{yx}}(\mathfrak{A}_{ab,\hat{y}}) - \exp_{T_{yx}}(\mathfrak{A}_{ba,\hat{y}}) \equiv 0 \pmod p$, for all $x \in \{a,b\}$ and all $y,\hat{y} \in \{c,d\}$, if and only if $\mathcal{A} \equiv I \pmod p$ and $\mathcal{B} \equiv I \pmod p$. By the definition of $\mathfrak{A}_{U,\hat{y}}$ we have $\mathfrak{A}_{ab,\hat{y}} = ([|1,T_{\hat{y}a},1,b|])(\mathfrak{D}_{\hat{y}\theta_a,b})$ and $\mathfrak{A}_{ba,\hat{y}} = ([|1,T_{\hat{y}b},1,a|])(\mathfrak{D}_{\hat{y}\theta_b,a})$. Recall also that for words W on $\{c, d\}$, we have $\exp_{T_{yx}}(\mathfrak{D}_{W,x}) = L_y(W)$ and $\exp_{T_{y\hat{x}}}(\mathfrak{D}_{W,x}) = 0$ for $x \neq \hat{x}$. Using these, we calculate

$$\begin{aligned}
&\exp_{T_{ca}}(\mathfrak{A}_{ab,c}) - \exp_{T_{ca}}(\mathfrak{A}_{ba,c}) = 1 - \beta_{11}\\
&\exp_{T_{cb}}(\mathfrak{A}_{ab,c}) - \exp_{T_{cb}}(\mathfrak{A}_{ba,c}) = \alpha_{11} - 1\\
&\exp_{T_{da}}(\mathfrak{A}_{ab,c}) - \exp_{T_{da}}(\mathfrak{A}_{ba,c}) = 0 - \beta_{12}\\
&\exp_{T_{db}}(\mathfrak{A}_{ab,c}) - \exp_{T_{db}}(\mathfrak{A}_{ba,c}) = \alpha_{12} - 0\\
&\exp_{T_{ca}}(\mathfrak{A}_{ab,d}) - \exp_{T_{ca}}(\mathfrak{A}_{ba,d}) = 0 - \beta_{21}\\
&\exp_{T_{cb}}(\mathfrak{A}_{ab,d}) - \exp_{T_{cb}}(\mathfrak{A}_{ba,d}) = \alpha_{21} - 0\\
&\exp_{T_{da}}(\mathfrak{A}_{ab,d}) - \exp_{T_{da}}(\mathfrak{A}_{ba,d}) = 1 - \beta_{22}\\
&\exp_{T_{db}}(\mathfrak{A}_{ab,d}) - \exp_{T_{db}}(\mathfrak{A}_{ba,d}) = \alpha_{22} - 1. \qquad \square
\end{aligned}$$

Corollary 2.3. *Suppose that $\theta : A \to \mathrm{End}\,(K)$ is a monoid homomorphism represented by 2×2 matrices $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{P}_D$ be the standard presentation*

$$\begin{aligned}[][a, b, c, d; ab = ba, cd = dc, ca = ac^{\alpha_{11}}d^{\alpha_{12}},\\
da = ac^{\alpha_{21}}d^{\alpha_{22}}, cb = bc^{\beta_{11}}d^{\beta_{12}}, db = bc^{\beta_{21}}d^{\beta_{22}}]\end{aligned}$$

for the semi-direct product $D = K \rtimes_\theta A$. Then $\mathcal{P}_D$ is efficient if and only if there is a prime p for which $\mathcal{A} \equiv I_{2\times 2} \pmod p$ and $\mathcal{B} \equiv I_{2\times 2} \pmod p$.

Proof. This is an immediate consequence of Theorem 1.1 and Theorem 2.2. $\square$

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REFERENCES

1. S.I. Adian, *Transformations of words in a semigroup presented by a system of defining relations*, Algebra i Logika **15** (1976), 611–621; Algebra Logic **15** (1976), 379–386 (in English).

2. H. Ayık, C.M. Campbell, J.J. O'Connor and N. Ruškuc, *Minimal presentations and efficiency of semigroups*, Semigroup Forum **60** (2000), 231–242.

3. A.S. Çevik, *Minimality of group and monoid presentations*, Ph.D. Thesis, University of Glasgow, 1997.

4. ———, *The p-Cockcroft property of the semi-direct products of monoids*, Internat. J. Algebra Comput. **13** (2003), 1–16.

5. J.R. Cho and S.J. Pride, *Embedding semigroups into groups, and asphericity of semigroups*, Internat. J. Algebra Comput. **3** (1993), 1–13.

6. R. Cremanns and F. Otto, *Finite derivation type implies the homological finiteness condition* FP_3, J. Symbolic Comput. **18** (1994), 91–112.

7. V. Dlab and B.H. Neumann, *Semigroups with few endomorphisms*, J. Austral. Math. Soc. **10** (1969), 162–168.

8. D.B.A. Epstein, *Finite presentations of group and* 3*-manifolds*, Quart. J. Math. Oxford **12** (1961), 205–212.

9. V. Guba and M. Sapir, *Diagram groups*, Memoirs Amer. Math. Soc. **620**, Amer. Math. Soc., Providence, 1997.

10. S.V. Ivanov, *Relation modules and relation bimodules of groups, semigroups and associative algebras*, Internat. J. Algebra Comput. **1** (1991), 89–114.

11. C.W. Kilgour, *Relative monoid presentations*, University of Glasgow, 1995, preprint.

12. C.W. Kilgour and S.J. Pride, *Cockcroft presentations*, J. Pure Appl. Algebra **106** (1996), 275–295.

13. G. Lallement, *Semigroups and combinatorial applications*, John Wiley & Sons, New York, 1979.

14. S.J. Pride, *Geometric methods in combinatorial semigroup theory*, in *Semigroups, formal languages and groups* (J. Fountain, ed.), Kluwer Academic Publishers, Dordrecht, 1995, pp. 215–232.

15. ———, *Low-dimensional homotopy theory for monoids*, Internat. J. Algebra Comput. **5**, (1995), 631–649.

16. S.J. Pride and J. Wang, *Relatively aspherical monoids*, University of Glasgow, 1996, preprint.

17. N. Ruškuc, *Semigroup presentations*, Ph.D. Thesis, University of St. Andrews, 1996.

18. T. Saito, *Orthodox semi-direct products and wreath products of monoids*, Semigroup Forum **38** (1989), 347–354.

19. C.C. Squier, *Word problems and a homological finiteness condition for monoids*, J. Pure Appl. Algebra **49** (1987), 201–216.

20. J. Wang, *Finite derivation type for semi-direct products of monoids*, Theoret. Comput. Sci. **191**, (1998), 219–228.

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